



HAL
open science

Phase portrait of a matter bounce in HoavaLifshitz cosmology

E Czuchry

► **To cite this version:**

E Czuchry. Phase portrait of a matter bounce in HoavaLifshitz cosmology. *Classical and Quantum Gravity*, 2011, 28 (8), pp.85011. 10.1088/0264-9381/28/8/085011 . hal-00690081

HAL Id: hal-00690081

<https://hal.science/hal-00690081>

Submitted on 21 Apr 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Phase portrait of a matter bounce in Hořava-Lifshitz cosmology

E. Czuchry

¹Instytut Problemów Jądrowych, ul. Hoża 69, 00-681 Warszawa, Poland

E-mail: eczuchry@fuw.edu.pl

Abstract.

The occurrence of a bounce in the FRW cosmology requires modifications of general relativity. An example of such a modification is the recently proposed Hořava-Lifshitz theory of gravity, which includes a “dark radiation” term with a negative coefficient in the analog of the Friedmann equation. This paper describes a phase space analysis of models of this sort with the aim of determining to what extent bouncing solutions can occur. It is found that they are possible, but not generic in models under consideration. Apart from previously known bouncing solutions some new ones are also described. Other interesting solutions found include ones which describe a novel sort of quasi stationary, oscillating universes.

1. Introduction

The standard Λ CDM model has solved many issues in cosmology. However, in spite of all this success, it also leaves a number of issues unaddressed. Perhaps the most significant is the problem of initial singularity, where general relativity breaks down. There have been many attempts to modify Einstein’s theory to avoid this singularity. Some are made at classical level, some involve quantum effects. Examples include the ekpyrotic/cyclic model ([1, 2, 3, 4, 5, 6, 7, 8, 9]) and loop quantum cosmology ([10, 11, 12, 13, 14, 15]), which replace the Big Bang with the Big Bounce. Attempts to address these issues at the classical level include braneworld scenarios ([16, 17]), where the universe goes from an era of accelerated collapse to an expanding era without any divergences or singular behavior. There are also higher order gravitational theories and theories with scalar fields (see [18] for a review of bouncing cosmologies). It is however fair to say that the issue of the initial singularity remains one of the key questions of the early Universe cosmology, and the idea that it is avoided due to a bounce is an elusive (and promising) notion. As discussed below, it is clear that close to the singularity the Friedmann equation has to be modified for a bounce to be possible.

In the recent year much effort has been devoted to studies of a proposal for a UV complete theory of gravity due to Hořava [19, 20, 21] and modifications of the theory [22, 23, 24, 25, 26, 27] (for a recent review see [28, 29]). Because in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space, this theory is referred to as the Hořava-Lifshitz gravity. Soon after this theory was proposed many specific solutions have been found, including cosmological ones ([30, 31, 32, 22, 33, 34, 35, 36]). It was also realized that the analog of the Friedmann equation in the HL gravity contains a term which scales in the same way as dark radiation in braneworld scenarios [30, 31, 32] and gives a negative contribution to the energy density. Thus, at least in principle it is possible to obtain non-singular cosmological evolution within the Hořava theory, as it was pointed out in [30, 32, 37]. Propagation of linear cosmological perturbations through the bounce was studied in [38], and it was shown that their evolution remains non-singular throughout, despite a singularity in perturbations’ equation of motion at the bounce point. The scale invariance of the perturbation spectrum is preserved during the bounce – without the need for inflation. Thus, the HL gravity can provide a realization of the “matter bounce” scenario.

The Hořava theory evolved in many aspects attempting to solve problems and inconsistencies appearing in the original formulation [25, 39, 40, 41]. Numerous sophisticated versions contain new terms added to the original Lagrangian with attempt to make the proposal more general [25] and to solve the so called strong coupling problem [39, 42, 43, 44]. The latter one seems to be the key problem of Hořava theory. It concerns the extra scalar degree of freedom arising from the explicit breaking of the general covariance and appearing when expanding about flat space-time [25, 39, 41]. While approaching the classical IR limit this mode becomes strongly coupled [39]. Blas *et al* proposed an extension of Hořava theory [44] (the so called *healthy* extension), showing in [26, 45] that fluctuations about Minkowski space-time can be made well-behaved and this extra mode does not lead to any phenomenological inconsistencies. Another solution to this problem was proposed by Hořava and Melby-Thompson. In [27] they investigate the possibility of extending the gauge group by a local $U(1)$ symmetry, which eliminates scalar graviton. The presence of this new symmetry forces the coupling constant λ to be equal to one (however this result was questioned in [48], where an action with the extended gauge symmetry and $\lambda \neq 1$ is formulated). On the other hand it was shown e.g. in [28, 26] that to avoid instabilities of the scalar graviton λ has to be very close to 1^+ at low energies. The question on the value of λ – whether it has to be fixed at 1 – is still a matter of debate.

Nevertheless the influence of the dark radiation term on the existence and stability of a cosmological bounce remains an interesting issue. This contribution to the modified Friedmann equations appears both in the projectable formulation [25] and in the non-projectable *healthy* extension of the HL theory [29]. The dynamics of the version exhibiting $U(1)$ symmetry has not been adequately studied yet (though one expects that for cosmological solutions the terms suggested in [27] will not contribute to the equations of motion). In this paper we are going to analyze how this specific modification of equations of motion affects the dynamics of the system. As a starting point we will take the Friedmann equations of the projectable HL theory with detailed balance. This allows us to explore implications of the “dark radiation” contribution and this formulation is convenient if one is interested in homogeneous and isotropic solutions. The dynamics of the system in the IR limit will be visualized with the help of the phase portrait techniques described in [60, 61] and compared to the standard GR cosmology. For the purpose of illustration we will assume that matter in the pre-bounce epoch is described by a scalar field φ with a quadratic potential. In order to concentrate on modifications created by the “dark radiation” terms and to simplify the analysis we will take the cosmological constant $\Lambda \rightarrow 0$ limit. Such scenario is as an approximation to a general case with non-vanishing Λ , valid in the regime of small scale factor a , when standard curvature and Λ terms are not relevant. Thus, the present analysis can be regarded as an exploration of the cosmologies with modified equations of motion, where the modifications considered are inspired by Hořava cosmology.

Related analysis of Hořava-Lifshitz cosmology have also appeared in [46] and [47], which we become aware of while this work was being typed. Those papers address a somewhat different set of issues from what we have pursued. The analysis presented in [46] and [47] consider the full 4-dimensional phase space of HL cosmology. The results presented here focus on the region close to where the scale factor vanishes, which admits a critical simplification: the number of dynamical equations under study can be reduced from 4 to 3 (as discussed in more detail in Section 3). This makes it possible to visualize the possible phase space trajectories in a 3-dimensional space.

The structure of this note is following: in Section 2. we briefly sketch the Hořava-Lifshitz gravity and cosmology. In Section 3. the possibility of bounce is discussed. In Section 4 we discuss phase portraits of the discussed system of equations and describe different families of phase trajectories.

2. Hořava-Lifshitz cosmology

The metric of Hořava-Lifshitz theory, due to anisotropy in UV, is written in the (3+1)-dimensional ADM formalism:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \quad (1)$$

where N , N_i and g_{ij} are dynamical variables. The general action of the Hořava-Lifshitz theory of gravity is ([20])

$$I_g = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} N \{ (K_{ij} K^{ij} - \lambda K^2 - V_g(g_{ij}, N)) \}, \quad (2)$$

where $K_{ij} = \frac{1}{N} \left[\frac{1}{2} \dot{g}_{ij} - \nabla_{(i} N_{j)} \right]$ is an extrinsic curvature of a spacelike hypersurface with a fixed time, covariant derivatives are defined with respect to the spatial metric g_{ij} , λ is a dimensionless running coupling, κ is a gravitational coupling.

Potential V_g is a function of the metric g_{ij} , lapse N and their spatial derivatives. It shall contain terms at least sixth order in spatial derivatives due to requirements set by power counting renormalizability. There are many possible forms of V_g , different choices leading do different version of the theory. In the original proposal [20] the potential V was written in the so-called ‘‘detailed balance’’ form, derivable from a superpotential W :

$$V_g = E^{ij} \mathcal{G}_{ijkl} E^{kl}, \quad (3)$$

where

$$E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W}{\delta g_{ij}}, \quad (4)$$

and \mathcal{G}^{ijkl} is a generalized De Witt metric

$$\mathcal{G}^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk} - \lambda g^{ij} g^{kl}). \quad (5)$$

This leads to the gravitational action in the form:

$$S_{\text{DB}} = \int dt d^3x N \sqrt{g} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda R - 3\Lambda^2)}{8(1-3\lambda)} + \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 - \frac{\kappa^2}{2\omega^4} Z_{ij} Z^{ij} \right\}, \quad (6)$$

where

$$Z_{ij} = C_{ij} - \frac{\mu\omega^2}{2} R_{ij}, \quad (7)$$

μ , ω and Λ are constant parameters and the Cotton tensor, C_{ij} , is defined by

$$C^{ij} = \epsilon^{ikl} \nabla_k \left(R^j_l - \frac{1}{4} R \delta^j_l \right) = \epsilon^{ikl} \nabla_k R^j_l - \frac{1}{4} \epsilon^{ikj} \partial_k R. \quad (8)$$

Imposing detailed balance condition reduces highly the number of terms in the potential part V of the action (2) and additionally introduces a superpotential W which may simplify quantization of the system. Although there is nothing fundamental in this formulation [25, 28, 29] it is convenient if one is interested in homogeneous and isotropic solutions.

In the HL gravity, opposite to GR, there is no direct prescription on how to couple matter to gravitational field. In GR there is the Lorentz invariance that provides arguments on how to couple matter, but in the Hořava theory there are no such guides for choosing a particular type from among the general family of couplings between the gravity and matter sectors. In some approaches the minimal coupling is assumed ([44, 49]) to make more easy contact with GR, there are also considered particular couplings [50, 51, 52, 53, 54, 55]. This issue has been recently addressed in [56]. We follow the procedure described in [30, 31, 32], and for the scalar field take the same anisotropic scaling as the one chosen for the gravity sector. Thus the action for matter is

$$I_m = \int dt d^3x \sqrt{g} N \mathcal{L}_m, \quad (9)$$

where

$$\mathcal{L}_m = \frac{3\lambda-1}{2} \left[\frac{1}{N^2} (\dot{\varphi} - N^i \partial_i \varphi)^2 - F(\partial_i \varphi, \varphi) \right]. \quad (10)$$

The potential F can in principle contain arbitrary combinations of φ and its spatial derivatives. The UV renormalizability indicates that F should contain up to six derivatives, but otherwise arbitrary powers of the scalar field.

In the IR, the terms with less number of spatial derivatives in the gravitational action (6) dominate. Thus the leading terms would be the kinetic one $I_k = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} N \{ (K_{ij} K^{ij} - \lambda K^2)$, the cosmological constant Λ term and the Ricci scalar term, the forth and six spatial derivative terms being irrelevant. Therefore the following action describes the IR behavior of the theory:

$$S_{\text{IR}} = \int dt d^3x N \sqrt{g} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 \Lambda R}{8(1-3\lambda)} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(1-3\lambda)} \right\}. \quad (11)$$

Comparing the action of the Hořava-Lifshitz theory near its IR limit to the Einstein-Hilbert action of General Relativity, one can see that the speed of light, Newton's constant and the cosmological constant are

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1-3\lambda}}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \Lambda_E = -\frac{3\kappa^4 \mu^2 \Lambda^2}{3\lambda - 1} \frac{1}{32}, \quad (12)$$

respectively. Setting dynamical constant $\lambda = 1$, reduces the terms in (11) to the usual ones of Einstein's relativity.

The equations for Hořava-Lifshitz cosmology are obtained by imposing conditions of homogeneity and isotropy of the metric. The associated ansatz is $N = N(t)$, $N_i = 0$, $g_{ij} = a^2(t) \gamma_{ij}$ where $a(t)$ is a scale factor and γ_{ij} is a maximally symmetric constant curvature metric, with a curvature $k = \{-1, 0, 1\}$. On this background

$$K_{ij} = \frac{H}{N} g_{ij}, \quad R_{ij} = \frac{2k}{a^2} g_{ij}, \quad C_{ij} = 0, \quad (13)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter.

The non-linear function F in the scalar action (9) reduces effectively to a potential $V(\varphi)$ and matter Lagrangian \mathcal{L}_m takes the form:

$$\mathcal{L}_m = \frac{3\lambda - 1}{2} \left(\frac{\dot{\varphi}^2}{2N^2} - V(\varphi) \right) \quad (14)$$

The energy density and pressure of the scalar field may be defined in the following way:

$$\rho = \frac{3\lambda - 1}{4} \dot{\varphi}^2 + V(\varphi), \quad (15)$$

$$p = \frac{3\lambda - 1}{4} \dot{\varphi}^2 - V(\varphi) \quad (16)$$

In numerical calculations presented further on a specific form of the scalar potential will be assumed (see eq. (25)).

The total action becomes:

$$S_{\text{FRW}} = \int dt d^3x N a^3 \left\{ \frac{3(1-3\lambda)}{2\kappa^2} \frac{H^2}{N^2} + \frac{3\kappa^2 \mu^2 \Lambda}{4(1-3\lambda)} \left(\frac{k}{a^2} - \frac{\Lambda}{3} \right) - \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \frac{k^2}{a^4} \right\} \\ + \int d^3x dt N a^3 \frac{3\lambda - 1}{2} \left\{ \frac{\dot{\varphi}^2}{2N^2} - V(\varphi) \right\}. \quad (17)$$

The equations of motion are obtained by varying the action (17) with respect to N , a and φ , then setting $N = 1$ at the end of the calculations.

$$H^2 = \frac{\kappa^2 \rho}{6(3\lambda - 1)} + \frac{\kappa^4 \mu^2 \Lambda}{8(3\lambda - 1)^2} \frac{k}{a^2} - \frac{\kappa^4 \mu^2}{16(3\lambda - 1)^2} \left(\Lambda^2 + \frac{k^2}{a^4} \right), \quad (18)$$

$$\dot{H} = -\frac{\kappa^2(\rho + p)}{4(3\lambda - 1)} - \frac{\kappa^4 \mu^2 \Lambda}{8(3\lambda - 1)^2} \frac{k}{a^2} + \frac{\kappa^4 \mu^2}{32(3\lambda - 1)^2} \frac{k^2}{a^4}, \quad (19)$$

and also equation of motion for the scalar field:

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{2}{3\lambda - 1} V' = 0, \quad (20)$$

where $H = \dot{a}/a$, a prime denotes the derivative with respect to scalar field φ .

Equations (18-19) are the non-relativistic Friedmann equations. They are derived from the general HL action (not the one in IR limit) and thus they differ from the GR Friedmann equations by the presence of the parameter λ and the $(1/a^4)$ terms. Taking the IR limit first would only reproduce the standard Friedmann equations, losing additional structure. The $(1/a^4)$ terms, produced by the R^2 term in the HL action, are reminiscent of the dark radiation term in braneworld cosmology [57] and are present only if the spatial curvature of the metric is non-vanishing.

The coupling constant λ is dimensionless. In general, it runs (logarithmically in the UV) and may eventually reach one of the three IR fixed points ([19]): $\lambda = 1/3$, $\lambda = 1$ or $\lambda = \infty$. The range $1 > \lambda > 1/3$ leads to ghost instabilities in the IR limit of the theory [58]. However, this range of λ is exactly the flow-interval between the UV and IR regimes. The only physically interesting case that remains, allowing for a possible flow towards GR – at $\lambda = 1$ – is the regime $\lambda \geq 1$. Region $\lambda \leq 1/3$ is disconnected from $\lambda = 1$ and cannot be included in realistic considerations. Moreover, to avoid instabilities of the scalar graviton λ has to be very close to 1^+ at low energies. Additionally, in one of the recent extensions of Hořava theory [27] λ is only a coefficient in the action, forced by an imposed $U(1)$ symmetry to be equal to 1 (though this result was questioned in [48]).

Nonetheless the phenomenologically relevant range is $\infty > \lambda \geq 1$. In this range the qualitative description of the above system does not depend on the specific value of λ : equations (18) and (19) show different behavior for $\lambda > 1/3$ and $\lambda < 1/3$. Therefore if we want to stay within the IR limit we may simplify further calculations and set the value $\lambda = 1$. This leads to:

$$H^2 = \frac{\kappa^2 \rho}{12} + \frac{\kappa^4 \mu^2 \Lambda}{32} \frac{k}{a^2} - \frac{\kappa^4 \mu^2}{64} \left(\Lambda^2 + \frac{k^2}{a^4} \right), \quad (21)$$

$$\dot{H} = -\frac{\kappa^2(\rho + p)}{8} - \frac{\kappa^4 \mu^2 \Lambda}{32} \frac{k}{a^2} + \frac{\kappa^4 \mu^2}{128} \frac{k^2}{a^4}, \quad (22)$$

and an equation of motion for the scalar field:

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0. \quad (23)$$

where in this limit $\rho = \dot{\varphi}^2/2 + V(\varphi)$ and $p = \dot{\varphi}^2/2 - V(\varphi)$.

3. Existence of bounce

New terms in the cosmological equations introduce the possibility of a bounce. The form of (21), with $k = \pm 1$ implies that it is possible that $H = 0$ at some moment of time. This is a necessary condition for the realization of the bounce. It was pointed out in [30], that it may happen in the presence of matter, at the critical time t_* , $a = a_*$, when the critical energy density is equal to

$$\rho = \rho_* = \frac{3\kappa^2 \mu^2}{2} \left(-\frac{\Lambda}{4} \frac{k}{a_*^2} + \frac{\Lambda^2}{8} + \frac{1}{8} \frac{k^2}{a_*^4} \right), \quad (24)$$

which is determined by the couplings of the theory.

From the continuity equation it follows that at the bounce point $\dot{H} > 0$. Therefore a transition from a contracting to an expanding phase may be possible. It was shown in [37] that the necessary condition for a cosmological bounce is that the energy density of regular matter increases less fast than a^{-4} as the scale factor decreases and $(\frac{\rho}{12} - p) > 0$.

We begin our considerations during a contracting phase. At the beginning the scale factor is quite large and the contribution of dark radiation to the total energy density is quite small. As the universe contracts, the energy density increases and the scale factor decreases rapidly. When a critical density is achieved, a bounce is about to take place.

One would expect that near the bounce, when the scale factor a is sufficiently small, the leading term in (21) and (22) would be the dark radiation one, with curvature and cosmological constant terms neglectable small. Specifically, assuming for a moment an equation of state of the form $p = w\rho$ with constant w , it is well known that H^2 , \dot{H} and ρ scale as $a^{-3(1+w)}$. Therefore we may keep the density term and omit the curvature term $\sim 1/a^2$ if $w > -\frac{1}{3}$. In the case of

a quadratic potential considered below (for which $w \neq \text{const}$, so the above argument does not directly apply) we have checked numerically that in all bounce scenarios discussed in this paper, this approximation is valid near bounce point (up to 10^{-7}).

We will model the matter sector in this pre-bounce epoch by assuming it is described by a scalar field φ with a potential

$$V(\varphi) = \frac{1}{2}m^2\varphi^2. \quad (25)$$

For calculational simplicity we put $m = 1$.

This way we have the following equations modeling the bounce in the Hořava-Lifshitz cosmology.

$$\dot{H} = -\frac{\kappa^2}{8}\dot{\varphi}^2 + \frac{\kappa^4\mu^2 k^2}{32 a^4}, \quad (26)$$

$$H^2 = \frac{\kappa^2}{24}(\dot{\varphi}^2 + \varphi^2) - \frac{\kappa^4\mu^2 k^2}{64 a^4}. \quad (27)$$

The value of κ^2 may be expressed near the IR limit in terms of cosmological constants (12): $\kappa^2 = 32\pi Gc$. After the suitable time reparameterization we can set $8\pi Gc = 1$. Therefore the Friedmann equations take the following form near the bounce:

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2 + \frac{\mu^2 k^2}{2a^4}, \quad (28)$$

$$H^2 = \frac{1}{6}(\dot{\varphi}^2 + \varphi^2) - \frac{\mu^2 k^2}{4a^4}. \quad (29)$$

Additionally, completing dynamics of the system, there is the equation of motion for the scalar field and the definition of the Hubble parameter:

$$\ddot{\varphi} = -\varphi - 3\dot{\varphi}H, \quad (30)$$

$$\dot{a} = aH. \quad (31)$$

The value of the parameter μ may be kept arbitrary. This parameter does not alter solutions of the system (28-31), but it specifies values of a on an obtained trajectory.

4. Phase portrait

4.1. Phase space

The local geometry of the phase portrait is characterized by the nature and position of its critical points. These points are locations where the derivatives of all the dynamic variables, i.e. the r.h.s. of (43-45), vanish. Moreover, they are the only points where phase trajectories may start, end, or intersect. They can also begin or end in infinity, and then – after a suitable coordinate transformation projecting the complete phase space onto a compact region (the so called Poincaré projection) – there may be well defined *infinite* critical points. The set of finite and infinite critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, provides a qualitative description of the given dynamical system.

Dynamics of our system is described by the following set of first order ODE's:

$$u = \dot{\varphi}, \quad (32)$$

$$\dot{u} = -\varphi - 3uH, \quad (33)$$

$$\dot{a} = aH, \quad (34)$$

$$\dot{H} = -\frac{1}{2}u^2 + \frac{\mu^2 k^2}{2a^4}, \quad (35)$$

plus the constraint equation:

$$H^2 = \frac{1}{6}(u^2 + \varphi^2) - \frac{\mu^2 k^2}{4a^4}. \quad (36)$$

If spatial curvature $k = 0$ one may consider a 2-dimensional subsystem:

$$u = \dot{\varphi}, \quad (37)$$

$$\dot{u} = -\varphi - 3uH, \quad (38)$$

$$\dot{H} = -\frac{1}{2}u^2, \quad (39)$$

$$(40)$$

with a constraint equation

$$H^2 = \frac{1}{6}(u^2 + \varphi^2). \quad (41)$$

If $k \neq 0$ one may also consider a subsystem on variables (φ, u, H) , obtained via reduction of the original system with respect to constraint (36). Namely, substituting

$$\frac{\mu^2 k^2}{4a^4} = \frac{1}{6}(u^2 + \varphi^2) - H^2 \quad (42)$$

into the equation for \dot{H} and omitting equation on dynamics of a leads the following set of equations:

$$u = \dot{\varphi}, \quad (43)$$

$$\dot{u} = -\varphi - 3uH, \quad (44)$$

$$\dot{H} = \frac{1}{3}\left(\varphi^2 - \frac{u^2}{2}\right) - 2H^2. \quad (45)$$

This is a reduced 3-dimensional subset of (32-36) on variables (φ, u, H) . If one wants to obtain also dynamics of a , he needs to add to this system equation $\dot{a} = aH$ and also the constraint equation (36).

In subsequent considerations we shall focus on a case $k \neq 0$ (when HL corrections play a significant role) and a phase portrait of solution of the system (43-45) in space of (φ, u, H) , following similar a similar procedure as that described in ([60, 61]). Reducing dimensionality of phase space enables 3D phase portrait visualizations. Moreover, we will discuss shortly also a case $k = 0$, which play a role of a limiting case for $k \neq 0$ dynamics.

We start by rewriting equations (43-45) in terms of the variables

$$x \equiv \varphi; y \equiv \dot{\varphi}; z \equiv \frac{\dot{a}}{a}, \quad (46)$$

which gives three “evolution” equations

$$\dot{x} = y, \quad (47)$$

$$\dot{y} = -x - 3yz, \quad (48)$$

$$\dot{z} = \frac{1}{3}\left(x^2 - \frac{y^2}{2}\right) - 2z^2. \quad (49)$$

The space of solution of the above dynamical system is a 3D region of the phase space (x, y, z) . This region is bounded by a 2D surface defined by a constraint equation (41) – space of trajectories of a flat universe ($k = 0$). This limiting surface is a double cone $z^2 = \frac{1}{6}(x^2 + y^2)$, with the upper branch corresponding to expansion and lower one to contraction. Those two branches connect at a point: $(0,0,0)$, which is a critical point (see below). Hence there are no trajectories passing from one branch of the cone to the other. For $k = \pm 1$ all trajectories lie between the branches of this cone. Dynamical equations (32-36) contain only k^2 , their solutions are the same for either non-zero value of k : $k = -1$ or $k = 1$. This cone is also a limiting surface for trajectories with large a . The further a trajectory lies from this cone, the smaller are the values of a along it.

The bounce happens when a phase trajectory passes from the region $z < 0$ to the region $z > 0$, intersecting the plane $z = 0$. At the crossing point $\dot{z} > 0$ must hold. Equation (49) implies that this happens if the crossing point lays between the lines $y = \sqrt{2}x$ and $y = -\sqrt{2}x$ on the plane $z = 0$. Those lines are the $z = 0$ section of an elliptic cone $\frac{1}{6}\left(x^2 - \frac{y^2}{2}\right) - z^2 = 0$, whose interior consists of trajectories with $\dot{z} > 0$ (eq. (49)). The area outside this cone is filled with trajectories along which $\dot{z} < 0$.

To find the finite critical points we set all right-hand-sides of equations (47-49) to zero. This gives rise to the conditions

$$x = y = z = 0. \quad (50)$$

Stability properties of this point are determined by the eigenvalues of the Jacobian of the system (47-49). More precisely, one has to linearize transformed equations (47-49) at each point. Inserting $\vec{x} = \vec{x}_0 + \delta\vec{x}$, where $\vec{x} = (x, y, z)$, and keeping terms up to 1st order in $\delta\vec{x}$ leads to an evolution equation of the form $\delta\dot{\vec{x}} = A\delta\vec{x}$. Eigenvalues of A describe stability properties at the given point.

At the finite critical point $O = (0, 0, 0)$, the matrix A has 2 purely imaginary eigenvalues, which implies there are closed orbits in the xy -plane encircling the z -axis, i.e. point O lays on a center line surrounded by closed orbits.

To find critical points that occur at infinite values of the parameters we rescale the infinite space (x, y, z) into a finite Poincaré sphere by means of the coordinate change:

$$x = \frac{X}{1-r}, \quad (51)$$

$$y = \frac{Y}{1-r}, \quad (52)$$

$$z = \frac{Z}{1-r}, \quad (53)$$

where

$$X = r \sin \theta \cos \varphi, \quad (54)$$

$$Y = r \sin \theta \sin \varphi, \quad (55)$$

$$Z = r \cos \theta, \quad (56)$$

$$r^2 = X^2 + Y^2 + Z^2. \quad (57)$$

We shall use both Cartesian coordinates (X, Y, Z) and spherical ones: (r, θ, φ) . We also rescale the time parameter t by defining the new time parameter T such that: $dT = dt/(1-r)$. In these coordinates our phase space is contained within a sphere of radius one – infinity corresponds to $r = 1$.

This is a conformal transformation, hence the limiting cone for phase trajectories is $Z^2 = \frac{1}{6}(X^2 + Y^2)$; all physical trajectories are contained within this cone. Bounce points are located on the plane $Z = 0$ within the region bounded by the lines $Y = \sqrt{2}X$ and $Y = -\sqrt{2}X$. The region containing trajectories with $\dot{Z} > 0$ (i.e. with H increasing) is bounded by the elliptic cone $\frac{1}{6}\left(X^2 - \frac{Y^2}{2}\right) - Z^2 = 0$.

After Poincaré transformation, equations (47-49) take the following form, written in the spherical coordinates (r, θ, φ) :

$$r' = \frac{(r-1)r^2}{48} \cos \theta [82 + 14 \cos 2\theta - 42 \cos 2\phi + 21 \cos 2(\theta - \phi) + 21 \cos 2(\theta + \phi)], \quad (58)$$

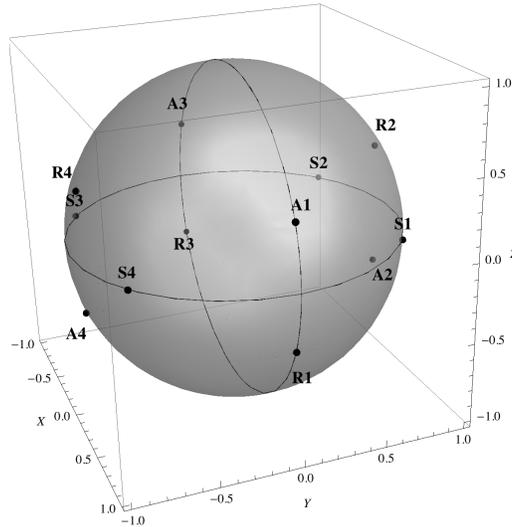
$$\theta' = \frac{1}{24} r \sin \theta (5 + 7 \cos 2\theta)(1 + 3 \cos 2\phi), \quad (59)$$

$$\phi' = r - 1 - 3r \cos \theta \cos \phi \sin \phi. \quad (60)$$

The form of the above equations is similar to the ones obtained in [60, 61]. Taking the limit $r = 1$ and putting r.h.s. of equations for θ' and ϕ' to zero, we find 12 solutions for θ, ϕ at the Poincaré sphere, shown in the Table 1. As we can see, there are 4 saddle points (more precisely saddle lines with end points at S_1, S_2, S_3, S_4). In the contracting part of the phase portrait ($z < 0$) there are two attracting nodes A_2 and A_4 and two repulsing lines starting at R_1 and R_3 . Hyperbolic areas near the nodes are bounded by repulsing lines which play role of separatrices. The expanding part is a mirror (“reversed in time”) of the contracting one.

Stability properties of infinite critical points are described in the Table 1, their position in 3D phase space, on a Poincaré sphere, is shown in Fig. 1.

Point	φ	θ	Stability
S1	$\arcsin \sqrt{2/3}$	$\pi/2$	Saddle line
S2	$\pi - \arcsin \sqrt{2/3}$	$\pi/2$	Saddle line
S3	$\pi + \arcsin \sqrt{2/3}$	$\pi/2$	Saddle line
S4	$2\pi - \arcsin \sqrt{2/3}$	$\pi/2$	Saddle line
A1	0	$\arccos \frac{\sqrt{7}}{7}$	Attracting line
R1	0	$\pi - \arccos \frac{\sqrt{7}}{7}$	Repelling line
R2	$\pi/2$	$\arccos \frac{\sqrt{7}}{7}$	Repelling node
A2	$\pi/2$	$\pi - \arccos \frac{\sqrt{7}}{7}$	Attracting node
A3	π	$\arccos \frac{\sqrt{7}}{7}$	Attracting line
R3	π	$\pi - \arccos \frac{\sqrt{7}}{7}$	Repelling line
R4	$3\pi/2$	$\arccos \frac{\sqrt{7}}{7}$	Repelling node
A4	$3\pi/2$	$\pi - \arccos \frac{\sqrt{7}}{7}$	Attracting node

Table 1. The properties of the infinite critical points.**Figure 1.** Infinite critical points located on a Poincaré sphere

4.2. Trajectories

When spatial curvature $k = 0$ then phase trajectories lay on the limiting cone $\tilde{z}^2 = \frac{1}{6}(X^2 + Y^2)$, as shown in the Figure 2. In the contracting part all trajectories start winding around z -axis, then some of them end at attracting node A_2 , some at A_4 . Those two families are separated by repelling lines with end-points at R_1 or R_3 , acting as separatrices. Expanding part is a mirror reflection with time reversed of the contracting part.

Trajectories of non-flat universes lay inside the limiting cone of the flat space. In the contracting part of the diagram ($Z < 0$), trajectories start spiraling outside from circles around the Z -axis. There are two families of such trajectories, separated by repelling lines ending at R_1 and R_3 . In each family, there are two possible scenarios for subsequent evolution. The first one, shown in Figure 3a, is to end at an attractor node (A_2 or A_4), which also lays in the contracting part of the phase diagram. On the way between O and A_2 or A_4 , a trajectory may go up through $Z = 0$ surface, undergoing a bounce there, and then recollapse, crossing the $Z = 0$ plane again, or go straight to the attractor node, without bounce. In either case, the end is the Big Crunch.

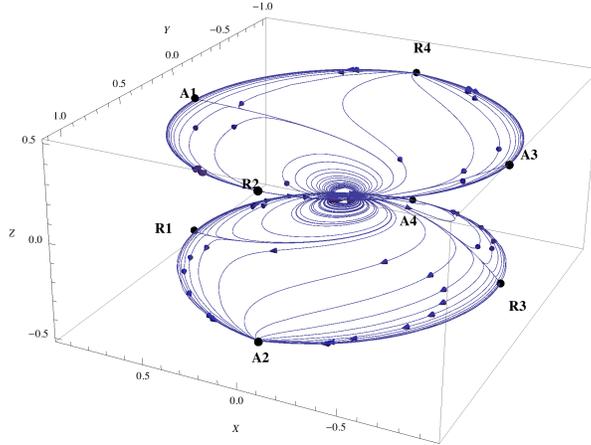


Figure 2. Phase trajectories for flat HL universe

The second scenario is shown in the Figure 3c. Here, after some oscillations and H decreasing, the trajectories reach an attractor – a repelling line (that ends either at R_1 or R_3), along which they move until $\dot{H} = 0$. Then they rapidly go up, crossing the $Z = 0$ (i.e. $H = 0$) plane, undergoing a bounce. After that, and after a period of accelerated expansion, they reach another attractor – an attracting line laying in the expanding part, with endpoint at either A_1 or A_3 . Along this line trajectories approach the Z -axis, winding around it. A subcase of this scenario is shown in the Figure 3e, where trajectories do not go through accelerated contraction and expansion, but cross the $Z = 0$ surface during oscillations around the Z -axis. This is in fact the scenario described in [37].

Trajectories may also start at repelling nodes R_2 or R_4 in the expanding part of the diagram. At those points $H = \infty$, which corresponds to the Big Bang. After that and a period of extreme, but with decreasing rate, expansion, there are again two possible scenarios. One is shown in the Figure 3b, where trajectories reach an attracting line and end up winding around the Z -axis. Before that, some of them collapse, crossing the $Z = 0$ plane, then slow down and finally stop contraction culminating in a bounce. Others show only slow expansion, without crossing $Z = 0$.

The last scenario is shown in the Figure 3d. Trajectories start at Big Bang points R_2 or R_4 , and after a period of slowing down expansion, reach a turning point and start accelerated contraction, ending at Big Crunch points A_2 or A_4 .

For better visualization we have gathered some described families of trajectories in Figure 3f.

Special attention has to be paid to circular motion around z -axis. As stability properties of the point $(0,0,0)$ and constraint equation (36) suggest, there may exist closed circular orbits laying on a $Z = 0$ plane ($H = 0$). But they cannot. Equation (49) does not allow for this, as $\dot{Z} = 0$ is fulfilled only on a class of curves laying on $X^2 - \frac{X^2}{2} - 6Z^2 = 0$, i.e. on a surface of the elliptic cone mentioned before. Yet numerical simulations exhibit oscillating solutions, such as $X^2 + Y^2 = \text{const.}$ and Z oscillated around zero, $\dot{Z} > 0$ between lines $Y = \sqrt{2}X$ and $Y = -\sqrt{2}X$, $\dot{Z} < 0$ outside this region. Such a trajectory resembles deformed circle. These solutions appear for sufficiently small X and Y , for larger values of X and Y numerical simulations show slow decreasing of the radius of this “circle” .

Finally, note that except for the special solution shown in Figure 3e – which is the bounce described by Brandenberger [37], there are also other types of bounces. One, probably the most interesting, is shown in Figure 3c. Here a big existing universe slowly starts to contract, but later on the contraction becomes exponential, until a bounce is reached and an exponential expansion begins, which finally slows down. Another type of bounce, shown in the Figure 3a, happens again when a big universe slowly starts contracting, stops and goes through an expanding phase for a while, then recollapses and ends at the Big Crunch. The last one, shown in the Figure 3b, happens

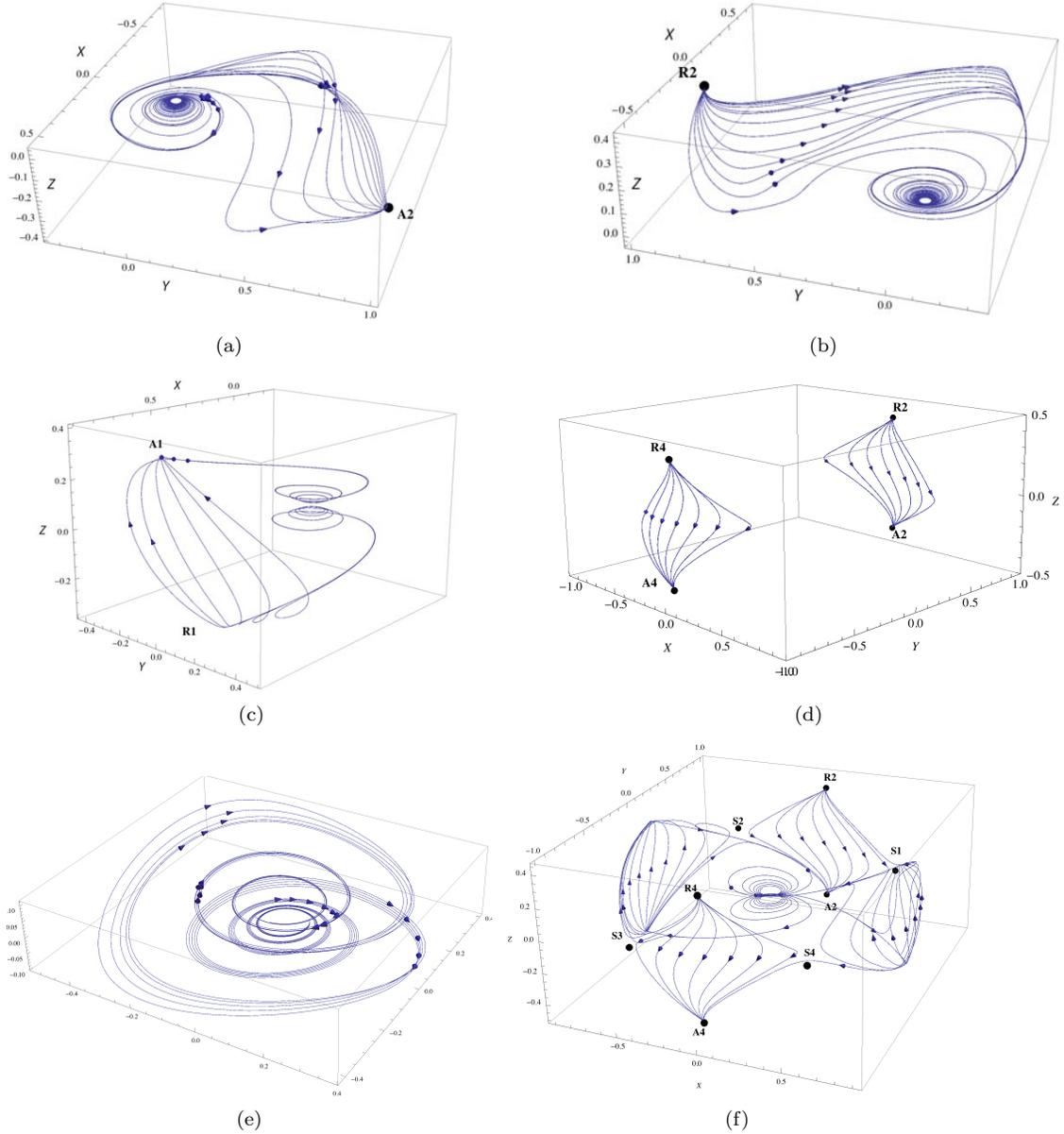


Figure 3. Different types of phase trajectories for a non-flat Hořava-Lifshitz universe

during a transition from the Big Bang to a quasi stationary final stage (with H slowly decreasing), however with a bounce on the way.

Trajectories shown in the figures discussed above are numerical solutions of the equations (58-60). To find a variety of bounce scenarios we investigated initial conditions: $\theta = \pi/2 \pm 0.01$, $\phi = i\pi/20$ ($i = 1 \dots 20$), each for $r = j/10$ ($j = 1 \dots 9$) and $r = 0, 9 + j \cdot 0.01$; time in range $[-20, 20]$. This procedure picked up the classes of trajectories discussed above. In general it may not be exhaustive in the sense that qualitatively different behavior of solutions may be possible. However it is sufficient for the purpose of understanding how bouncing scenarios emerge here due to the specific modification of general relativity which appears in Hořava's theory.

5. Discussion and conclusions

In this paper we have investigated the cosmological bounce in the Hořava-Lifshitz gravity. Using the 3D flow visualization technique we have found that phase portraits in the considered theory have a different structure than in standard cosmology. Comparing to results from the paper [61], we can see that there are additional repellers (R_1 and R_2) in the contracting part of a phase space, and mirror attractors in the expanding part. Their presence allows the existence of a bounce, because now there are possible new families of trajectories, starting at additional repellers in the contracting part, and possibly ending at new attractors in the expanding part, or surrounding the $(0, 0, 0)$ point, which is now a center, compared to saddle in standard cosmology. Those are realizations of the bounce. One of them is the solution with oscillatory behavior described in ([37]); there are however additional possibilities. The most interesting one contains a period of rapid contraction, and – after a bounce – a period of rapid expansion, what may fit inflationary scenario.

Nevertheless there are still initial conditions which lead to the Big Crunch, as shown in the Figures 3a and 3d, or which start at initial singularity (Fig. 3b and 3d). Hence the existence of a bounce is not generic for Hořava theory and depends on initial conditions.

Another interesting class of solutions consists of quasi stationary universes. These solutions are described in phase space by closed orbits, winding around the critical point $(0, 0, 0)$ - a center. All trajectories in the neighborhood of this point end up as closed orbits, “deformed circles”. Equations of motion do not allow closed orbits laying on $Z = \text{const.}$ plane, resulting in slight deformation of the circular orbits. The values of H oscillate around stationary stage, for sufficiently small values of φ and $\dot{\varphi}$. Values of the scale parameter a during this evolution are much bigger than the regime for which our simplifications are valid. Therefore this behavior is not a feature of the Hořava-Lifshitz theory, but of cosmologies with modified equations of motion, i.e. with the additional term $\sim 1/a^4$ in the Friedmann equations. Still, presence of this term leads to a different solution than induced by a negative potential as in [61], due to different stability properties of finite critical points there.

The visualizations presented in this paper describe the dynamics of Hořava-Lifshitz universe in the regime of small scale factor a , when standard curvature and Λ terms are not relevant. Even in such slightly limited framework they answer the question of possible scenarios realizing a bounce, and whether it is generic for the theory or not. It appears not, as we have found solutions leading to infinite collapse, or starting at the initial singularity, both staying within the regime of small a . There is also an interesting possibility of quasi stationary, oscillating universe, existence of which is clearly implied by dark radiation term in the Friedmann equations.

The above portraits of the matter bounce in HL cosmology are attributed only to a homogenous and isotropic model. Possible deviations from isotropy may become dominant in the small volume limit, as it happens in GR [62, 63, 64, 65]. Thus the next step to be taken in the research on the realistic matter bounce is to analyze the effects of anisotropies in the HL gravity, extending the standard analysis of Belinskii, Khalatnikov and Lifshitz. The addition of shearing components, due to anisotropies, may make the bounce unstable leading possibly to BKL chaotic behavior at the Big Crunch singularity. On the other hand they may prevent the Universe from collapsing to the singularity and thus avoiding the Big Crunch which is found in some solutions of the theory.

It was shown [66, 67, 68] that adding $({}^4R)^2$ (and possibly other) curvature terms to the gravitational action in GR suppressed near the singularity chaotic behavior induced by the linear curvature term. One would expect similar effect in the HL gravity, where the gravitational action potential consists of higher order curvature terms. Indeed, the recent study of the mixmaster universe in the Horava gravity [69] reveals that in certain case chaos is absent when the singularity is approached, there are also possible harmonic oscillations around the fully isotropic model. However different study [70] shows that in Hořava gravity with the zero cosmological constant the presence of the higher curvature terms in the HL action cannot suppress chaotic behaviors induced by the IR part of the action. Still there is possibility that extending the HL gravitational potential beyond the detailed balance form, i.e. adding additional higher curvature terms as in modified

HL theories [20, 22, 23, 24, 25, 26, 27] may have stabilizing effect. Clearly, further analysis is necessary, although it is beyond the scope of this paper. For this purpose the singularity theorems of General Relativity have to be revisited and examined under which conditions they remain valid in theories with anisotroping scaling.

Acknowledgements

I would like to thank Michał Spaliński for fruitful discussions. This work has been supported by the Polish Ministry of Science and Higher Education grant PBZ/MNiSW/07/2006/37.

References

- [1] Khoury J, Ovrut B A, Steinhardt P J and Turok N 2001 *Phys. Rev. D* **64** 123522 [arXiv:hep-th/0103239].
- [2] Buchbinder E I, Khoury J and Ovrut B A 2007 *Phys. Rev. D* **76** 123503 (2007) [arXiv:hep-th/0702154].
- [3] Steinhardt P J and Turok N 2002 *Science* **296** no 5572 1436 [arXiv:hep-th/0111030].
- [4] Steinhardt P J and Turok N 2002 *Phys. Rev. D* **65** 126003 [arXiv:hep-th/0111098].
- [5] Steinhardt P J and Turok N 2002 *Phys. Rev. D* **66** 101302 [arXiv:astro-ph/0112537].
- [6] Tolley A J and Turok N 2002 *Phys. Rev. D* **66** 106005 [arXiv:hep-th/0204091].
- [7] Niz G and Turok N 2007 *Phys. Rev. D* **75** 026001 [arXiv:hep-th/0601007].
- [8] Copeland E J, Niz G and Turok N [arXiv:1001.5291[hep-th]].
- [9] Kallosh R, Kofman L and Linde A D 2001 *Phys. Rev. D* **64** 123523 [arXiv:hep-th/0104073].
- [10] Ashtekar A, Pawłowski T and Singh P 2006 *Phys. Rev. Lett.* **96** 14130 [arXiv:gr-qc/0602086].
- [11] Ashtekar A, Pawłowski T and Singh P 2006 *Phys. Rev. D* **73** 124038 [arXiv:gr-qc/0604013].
- [12] Ashtekar A, Pawłowski T and Singh P 2006 *Phys. Rev. D* **74** 084003 [arXiv:gr-qc/0607039].
- [13] Singh P, Vandersloot K and Vereshchagin G V 2006 *Phys. Rev. D* **74** 043510 [arXiv:gr-qc/0606032].
- [14] Corichi A and Singh P 2008 *Phys. Rev. Lett.* **100** 161302 [arXiv:0710.4543 [gr-qc]].
- [15] Cailleteau T, Cardoso A, Vandersloot K and Wands D 2008 *Phys. Rev. Lett.* **101** 251302 [arXiv:gr-qc/0808.0190].
- [16] Maartens R 2004 *Living Rev. Rel.* **7** 7 [arXiv:gr-qc/0312059].
- [17] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
- [18] Novello M and Perez S E 2008 *Phys. Rept.* **463** 127-213 [arXiv:0802.1634 [astro-ph]].
- [19] Horava P 2009 *JHEP* **0903** 020 [arXiv:0812.4287 [hep-th]].
- [20] Horava P 2009 *Phys. Rev. D* **79** 084008 [arXiv:0901.3775 [hep-th]].
- [21] Horava P 2009 *Phys. Rev. Lett.* **102** 161301 [arXiv:0902.3657 [hep-th]].
- [22] Nastase H [arXiv:0904.3604 [hep-th]]
- [23] Kehagias A and Sfetsos K 2009 *Phys. Lett. B* **678** 123 [arXiv:0905.0477 [hep-th]]
- [24] Sotiriou T P, Visser M and Weinfurtner S 2009 *Phys. Rev. Lett.* **102** 251601 [arXiv:0904.4464 [hep-th]]
- [25] Sotiriou T P, Visser M and Weinfurtner S 2009 *JHEP* **0910** 033, [arXiv:0905.2798 [hep-th]]
- [26] Blas D, Pujolas O and Sibiryakov S 2010 *Phys. Rev. Lett.* **104** 181302 [arXiv:0909.3525 [hep-th]]
- [27] Horava P and Melby-Thompson C M 2010 *Phys. Rev. D* **82** 064027 [arXiv:1007.2410 [hep-th]]
- [28] Mukohyama S 2010 *invited review for Class. Quantum Grav.* [arXiv:1007.5199[hep-th]]
- [29] Sotiriou T P 2010 [arXiv:1010.3218v1 [hep-th]].
- [30] G. Calcagni 2009 *JHEP* **0909** 112 [arXiv:0904.0829 [hep-th]].
- [31] Kiritsis E and Kofinas G. 2009 *Nucl. Phys. B* **821** 467 [arXiv:0904.1334 [hep-th]].
- [32] Saridakis E N 2010 *Eur. Phys. J. C* **67** 229 [arXiv:0905.3532 [hep-th]].
- [33] Mukohyama S, Nakayama K, Takahashi F and Yokoyama S 2009 *Phys. Lett. B* **679** 6 [arXiv:0905.0055 [hep-th]].
- [34] Minamitsuji M 2010 *Phys. Lett. B* **684** 194 [arXiv:0905.3892 [astro-ph]].
- [35] Wang A and Wu Y 2009 *JCAP* **0907** 012 [arXiv:0905.4117 [hep-th]].
- [36] Takahashi T and Soda J 2009 *Phys. Rev. Lett.* **102** 231301 [arXiv:0904.0554 [hep-th]].
- [37] Brandenberger R 2009 *Phys. Rev. D* **80** 043516 [arXiv:0904.2835 [hep-th]].
- [38] Gao X, Wang Y, Xue W and Brandenberger R 2010 *JCAP* **1002** 020 [arXiv:0911.3196 [hep-th]].
- [39] Charmousis C, Niz G, Padilla A and Saffin P M 2009 *JHEP* **0908** 070 [arXiv:0905.2579 [hep-th]].
- [40] Li M and Pang Y 2009 *JHEP* **0908** 015 [arXiv: 0905.2751[hep-th]].
- [41] Bogdanos C and Saridakis E N 2010 *Class. Quantum Grav.* **27** 075005 [arXiv:0907.1636 [hep-th]].
- [42] Tingo R, Russo J G and Serone M 2009 *JHEP* **0911** 020 [arXiv:0906.3477 [hep-th]].
- [43] Cai R G, Hu B and Zhang H B 2009 *Phys. Rev. D* **80** 041501 [arXiv:0905.0255 [hep-th]].
- [44] Blas D, Pujolas O and Sibiryakov S 2009 *JHEP* **091** 029 [arXiv:0906.3046 [hep-th]].
- [45] Blas D, Pujolas O and Sibiryakov S 2010 *Phys. Lett. B* **688** 350 [arXiv:0912.0550 [hep-th]]
- [46] Carloni S, Elizalde E and Silva P J 2010 *Class. Quantum Grav.* **27** 045004 [arXiv:0909.2219 [hep-th]].
- [47] Leon G and Saridakis E N 2009 *JCAP* **0911** 006 [arXiv:0909.3571 [hep-th]].
- [48] da Silva A M [arXiv:1009.4885 [hep-th]]
- [49] Germani C, Kehagias A and Sfetsos K 2009 *JHEP* **0909** 060 [arXiv:0906.1201[hep-th]].
- [50] Suyama T 2010 *JHEP* **1001** 093 [arXiv:0909.4833 [hep-th]].
- [51] Capasso D and Polychronakos AP 2010 *JHEP* **1002** 068 [arXiv:0909.5405 [hep-th]].
- [52] Rama S K 2009 [arXiv:0910.0411 [hep-th]].
- [53] Kiritsis E and Kofinas G 2010 *JHEP* **1001** 122 [arXiv:0910.5487 [hep-th]].

- [54] Sindoni L [arXiv:0910.1329 [gr-qc]].
- [55] Kouretsis A P, Stathakopoulos M and Stavrinou P C [arXiv:1003.5640 [gr-qc]].
- [56] Carloni S, Elizalde E and Silva P J 2010 [arXiv:1009.5319v1 [hep-th]].
- [57] Binétruy P, Deffayet C, Ellwanger U and Langlois D 2000 *Phys. Lett. B* **477** 285 [hep-th/9910219].
- [58] Bogdanos C and Saridakis E N 2010 *Class. Quant. Grav.* **27** 075005 [arXiv:0907.1636].
- [59] Lu H, Mei H and Pope C N 2009 *Phys. Rev. Lett.* **103** 091301 [arXiv:0904.1595 [hep-th]].
- [60] Felder G N, Frolov A and Kofman L 2002 *Class. Quantum Grav.* **19** 2983 [arXiv:hep-th/0112165].
- [61] Felder G N, Frolov A, Kofman L and Linde A 2002 *Phys. Rev. D* **66** 023507 [arXiv:hep-th/0202017].
- [62] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 *Adv. Phys.* **19** 525.
- [63] Belinskii V A, Khalatnikov I M and Lifshitz E M 1972 *Sov. Phys. JETP* **35** 838.
- [64] Belinskii V A, Khalatnikov I M and Lifshitz E M 1982 *Adv. Phys.* **31** 639
- [65] Damour T, Henneaux M, Nicolai H 2003 *Class. Quantum Grav.* **20** R145 [arXiv:hep-th/0212256].
- [66] Barrow J D and Sirousse-Zia H 1989 *Phys. Rev. D* **39** 2187.
- [67] Barrow J D and Cotsakis S *Phys. Lett. B* **232** 172.
- [68] Cotsakis S, Demaret J, de Rop Y and Querella L 1993 *Phys. Rev. D* **48** 4595.
- [69] Bakas I, Bourliot F, Lust D and Petropoulos M 2010 *Class. Quantum Grav.* **27** 045013 [arXiv:0911.2665 [hep-th]].
- [70] Myung Y S, Kim Y W, Son W S and Park Y J 2010 *JHEP* **1003** 085 [arXiv:1001.3921 [gr-qc]].