Generic validity of the multifractal formalism

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Abstract

The multifractal formalism is a conjecture which gives the spectrum of singularities of a signal using numerically computable quantities. We prove its generic validity by showing that almost every function in a given function space is multifractal and satisfies the multifractal formalism.

1 Introduction

One motivation of multifractal analysis was the study of fully developed turbulent flows. Indeed, some experimental results obtained in wind-tunnels showed that the regularity of the velocity of a turbulent fluid changes wildly from point to point. This quantity is therefore hardly computable. Hence, rather than measure the exponent at some point one rather estimates the fractal dimension of sets where it takes a given value H.

The *spectrum of singularities* $d(H)$ is the function which gives the Hausdorff dimension of those sets. From its definition, it is also almost impossible to obtain numerically the spectrum of singularities.

In [10], two physicists U. Frisch and G. Parisi proposed an algorithm in order to derive the spectrum of singularities from quantities that are effectively computable on a signal. They proposed to use the L^p modulus of continuity of the velocity, used in the theory

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of turbulent flows since Kolmogorov, [18]. This average quantity is called the *scaling* function, or scaling exponent and is denoted ξ_f. It is defined by $\int |f(x+l) - f(x)|^p dx \sim |l|^{\xi_f(p)}$, where $\sim$ means that $\int |f(x+l) - f(x)|^p dx$ is of the order of magnitude of $|l|^{\xi_f(p)}$ when l tends to 0 (assuming that the limit exists). Numerical estimations and further results about the scaling function and its wavelet decomposition can be found in [1, 2].

Frisch and Parisi proposed that the spectrum of singularities of a function can be obtained as follows:

$$d(H) = \inf_{p \in \mathbb{R}} (pH - \xi_f(p) + d), \tag{1}$$

see [10] for the derivation of this formula.

First, we state the mathematical framework of multifractal analysis. The main notion we need to define is the Hölder exponent.

Definition 1. *Let $\alpha \geq 0$; a function $f : \mathbb{R}^d \to \mathbb{R}$ is $C^\alpha(x_0)$ if for all $x \in \mathbb{R}^d$ such that $|x - x_0| \leq 1$ there exists a polynomial P of degree less than $[\alpha]$ and a constant C such that,*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{2}$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

It is proved in [14] that for $p \geq 1$, the scaling function $\xi_f(p)$ is closely related with Sobolev or Besov smoothness. It is thus natural for us to replace the scaling function as follows.

$$\text{If } p > 0 \quad \eta_f(p) = \sup\{s : \ f \in B_p^{s/p,\infty}\}. \tag{3}$$

So (1) applied to η_f can at most give the increasing part of the spectrum.

Defining, as in [16], an auxiliary function $s(1/p) = \eta(p)/p$, the Besov domain of a function f is the set of (q,t) such that $f \in B_{1/q}^{t,1/q}$. The boundary of the Besov domain of f is then given by the graph of s. And by Sobolev embeddings, the Besov domain of a function is a convex set. Thus, functions η satisfying (3) are increasing and concave functions. Furthermore the auxiliary function s is such that $0 \leq s'(q) \leq d$. Those conditions lead us to the following definition.

Definition 2. *A function η is admissible if $s(q) = q\eta(1/q)$ is concave and satisfies $0 \leq s'(q) \leq d$. It is strongly admissible if furthermore $s(0) > 0$.*

The following important result from [16] allows us to define a metric space using admissible functions.

Proposition 1. *Any concave function s satisfying $0 \leq s'(q) \leq d$ defines the Besov domain of a distribution f.*

Thanks to Proposition 1, to each admissible function η, a metric space V can be associated by taking

$$V = \bigcap_{\varepsilon>0, 0<p<\infty} B_{p,loc}^{(\eta(p)-\varepsilon)/p,p}.$$

To be as complete as possible, we also recall the definition of Legendre transform.

Definition 3. *Let f be a lower semi-continuous function defined in a normed vector space E. Then the Legendre transform of f is*

$$f^*(x) = \sup_{y \in E}(f(y) - xy). \tag{4}$$

This function is convex and lower semi-continuous.

In the present paper, we propose to study the validity of (1) for $\eta_f(p)$. An equivalent form of this heuristic formula is satisfied by a large class of invariant measures, see [4, 6, 20]. In the context of signal analysis, this conjecture is often satisfied if we add particular assumptions on f, such as self-similarity. On the other hand, there exist counterexamples to the general validity of this formula. If it does not hold for every function, what is its range of validity? Our purpose here is to show that the validity of formula (1) is not an exceptional phenomenon but it is satisfied for a large class of functions, without any additional assumption. More precisely, we study the validity of this formula for "almost every" functions, i.e. in a measure-theoretic sense.

In a finite dimensional space, the notion of "almost every" means "for the Lebesgue measure". The particular role played by this measure is justified by the fact that this is the only one which is σ-finite and invariant under translation. In a metric infinite dimensional space no measure enjoys this properties. The following definition, see [5, 7, 13] can thus replace the notion of vanishing Haar measure.

Definition 4. *Let V be a complete metric vector space. A Borel set B in V is called Haar null if there exists a probability measure μ with compact support such that*

$$\mu(B + v) = 0 \quad \forall v \in V. \tag{5}$$

In this case the measure μ is said transverse to B.
A subset of V is called Haar null if it is contained in a Borel Haar null set.
The complement of a Haar null set is called a prevalent set.

With a slight abuse of language we will say that a property is satisfied almost everywhere when it holds on a prevalent set.

Let us recall some properties of Haar null sets, see [7, 13].

Proposition 2.
1. *If S is Haar-null, then $\forall x \in V$, $x + S$ is Haar-null.*
2. *If $\dim(V) < \infty$, S is Haar-null if and only if $\mathrm{meas}(S) = 0$ (where meas denotes the Lebesgue measure).*
3. *Prevalent sets are dense.*
4. *If S is Haar null and $S' \subset S$ then S' is Haar null.*
5. *The union of a countable collection of Haar null sets is Haar null.*
6. *If $\dim(V) = \infty$, compact subsets of V are Haar-null.*

Several kinds of measures can be used as transverse measures of a Borel set. Here, we will only use the following notion.

Definition 5. *A finite dimensional space P is called a probe for the set $T \subset V$ if the Lebesgue measure on P is transverse to the complement of T.*

Those measures are not compactly supported probability measures. However one immediately checks that Definition 5 is equivalent to the same one stated with the Lebesgue measure defined on the unit ball of P. Note that in this case, the support of the measure is included in the unit ball of a finite dimensional subspace. The compactness assumption is therefore fulfilled.

The study of generic regularity for a "large" set of functions goes back to S. Banach [3], who gave differentiability properties of continuous functions, for quasi-all functions in the Baire's categories sense. Later B. Hunt [12] proved the same result in the measure-theoretic sense of prevalence.

In [16], S. Jaffard studied properties of generic functions, in the Baire's categories sense, in Sobolev spaces. He also proved that in the sense of Baire's categories quasi-all functions in V satisfy:

$$d(H) = \inf_{p \geq p_c} (pH - \eta(p) + d) \tag{6}$$

where p_c is the only critical point such that $\eta(p) = d$.

In this paper we will study the validity of the Frisch-Parisi conjecture for almost every function in the prevalence setting. The aim of this paper is to prove the following theorem.

Theorem 1. *Let η be a strongly admissible function and let V be the space defined by*

$$V = \bigcap_{\varepsilon>0, 0<p<\infty} B_{p,loc}^{(\eta(p)-\varepsilon)/p,p}; \tag{7}$$

then, in the sense of prevalence, almost every function f in V satisfies the following two conditions:

1. *For all $p > 0$,*
$$\eta_f(p) = \eta(p)$$
2. *The spectrum of singularities is defined on the interval $\left[s(0), \frac{d}{p_c}\right]$ where it is given by:*
$$d_f(H) = \inf_{p \geq p_c} (pH - \eta_f(p) + d) \tag{8}$$
where p_c is the only critical point such that $\eta(p_c) = d$.

Remark. *We have to impose that η is strongly admissible else, according to [8], almost every function in V is nowhere locally bounded.*

In section 2 we will solve a simpler problem. We will prove that almost every function in a given intersection of a Sobolev or a Besov space and an Hölder space satisfies a slight modification of the Frisch and Parisi conjecture. We will first establish their spectrum of singularities.

Theorem 2. *If $\gamma > 0$ and $s - \frac{d}{p} < 0$ the spectrum of singularities of almost every function in $B_p^{s,q} \bigcap C^\gamma$ or in $L^{p,s} \bigcap C^\gamma$ is given by:*

$$d(H) = \begin{cases} \frac{d+(\gamma-s)p}{\gamma} H & \text{if } H \in \left[\gamma, \frac{d\gamma}{d+(\gamma-s)p}\right] \\ -\infty & \text{otherwise.} \end{cases}$$

Remarks. *1. Using the Sobolev embeddings $B_p^{s,1} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s,\infty}$, the same result holds in Sobolev and in Besov spaces. As Besov spaces have a very simple wavelet expansion, we will only prove the result in those spaces. To obtain the Sobolev case, we only need to pick $q = \infty$ in the following.*

2. In Theorem 2 we only state the spectrum of singularities of functions in the case $B_p^{s,q} \cap C^\gamma$ where $s - \frac{d}{p} < 0$. Other cases are proved in [9]. To be complete, we recall the following result from [9].

Proposition 3.
- *If $s - d/p \leq 0$, then almost every function in $L^{p,s}$ or in $B_p^{s,q}$ is nowhere locally bounded, and therefore its spectrum of singularities is not defined.*
- *If $s - d/p > 0$, then the Hölder exponent of almost every function f of $L^{p,s}$, or of $B_p^{s,q}$ takes values in $[s - d/p, s]$ and*

$$\forall H \in [s - d/p, s], \qquad d_f(H) = Hp - sp + d; \qquad (9)$$

furthermore, for almost every x, $h_f(x) = s$.

Our purpose here is to expand the result of [9] in two directions. On one hand, we will work with an intersection of Besov spaces. On the other hand, we will see in the last part another stronger generic result, in the topological sense mentioned above.

The main tool that we will use in the following is the wavelet expansion of functions. First, it yields a simple characterization of functional spaces and it offer a simple condition for pointwise regularity. Let us recall some properties of wavelet expansion.

There exist $2^d - 1$ oscillating functions $(\psi^{(i)})_{i \in \{1,\dots,2^d-1\}}$ in the Schwartz class such that the functions

$$2^{dj} \psi^{(i)}(2^j x - k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^d$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$, see [19]. Wavelets are indexed by dyadic cubes $\lambda = [\frac{k}{2^j}; \frac{k+1}{2^j}[^d$. Thus, any function $f \in L^2$ can be written:

$$f(x) = \sum c_{j,k}^{(i)} \psi^{(i)}(2^j x - k)$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int f(x)\psi^{(i)}(2^j x - k)dx.$$

(Note that we use an L^∞ normalization instead of an L^2 one, which simplifies the formulas). If $p > 1$ and $s > 0$, Sobolev space have thus the following characterization, see [19]:

$$f \in L^{p,s} \Leftrightarrow \left(\sum_{\lambda \in \Lambda} |c_\lambda|^2 (1 + 4^{js}) \chi_\lambda(x) \right)^{1/2} \in L^p(\mathbb{R}^d), \qquad (10)$$

where $\chi_\lambda(x)$ denotes the characteristic function of the cube λ and Λ is the set of all dyadics cubes. Homogeneous Besov spaces, which will also be considered, are characterized (for $p, q > 0$ and $s \in \mathbb{R}$) by

$$f \in B_p^{s,q} \iff \sum_j \left(\sum_{\lambda \in \Lambda_j} |c_\lambda|^p 2^{(sp-d)j} \right)^{q/p} \leq C \qquad (11)$$

where Λ_j denotes the set of dyadics cubes at scale j, see [19]. Note that, if $p \in]0, 1[$, Besov spaces are not Banach spaces since they are not locally convex but nonetheless are separable complete metric vector spaces.

Pointwise regularity can also be expressed in terms of a condition on wavelet coefficients, see [14].

Proposition 4. *Let x be in $\mathbb{R}^d$. If f is in $C^\alpha(x)$ then there exists $c > 0$ such that for all λ:*

$$|c_\lambda| \leq c 2^{-\alpha j} (1 + |2^j x - k|)^\alpha. \qquad (12)$$

2 Multifractal formalism in a given Besov space

The Frisch-Parisi conjecture gives the spectrum of singularities as the Legendre transform of the scaling function. We will determine the validity of this formula for measure theoretic generic functions in a given Besov space, in two steps. First we will prove Theorem 2, which one gives the spectrum of singularities of almost every function. Afterwards, we will give the prevalent scaling function. This allows us to merge the spectrum obtained with formula (1) applied to the scaling function.

2.1 Proof of Theorem 2

Proposition 3 states that if $s - \frac{d}{p} < 0$, almost every function in $B_p^{s,q}$ is nowhere locally bounded and the spectrum of singularities is not defined for any H. To define this spectrum, we need to assume a minimum uniform regularity. That is why, in the following, we choose $s - \frac{d}{p} < 0$ and $0 < \gamma < s$ and we study almost every function in $B_p^{s,q} \cap C^\gamma$.

Theorem 2.1 from [17], yields an upper bound of the spectrum of singularities.

Lemma 1. *Let $s - \frac{d}{p} < 0$. For all functions $f \in B_p^{s,q} \bigcap C^\gamma$, the Hausdorff dimension of the set $\{x : f \notin C^\alpha(x)\}$ is bounded by $\frac{d+(\gamma-s)p}{\gamma}\alpha$.*

We need also the following definition.

Definition 6. *Let $\alpha \in [1, \frac{d}{d+(\gamma-s)p}]$. A point x_0 belongs to J_α if there exists an infinite sequence $(j,k) \in \mathbb{N} \times \{0, ..., 2^j - 1\}^d$, $k = (k_1, ..., k_d)$ such that for each $i = 1, ..., d$ k_i can be written $l_i 2^{j-L}$ and:*

$$\frac{1}{2^j} + \left| x_0 - \frac{k}{2^j} \right| < \frac{1}{2^{\alpha L}} \tag{13}$$

where $L := \left[\frac{(d+(\gamma-s)p)j}{d}\right]$. We define the exponent of approximation of x as $\alpha'(x) = \sup\{\alpha : x \in J_\alpha\}$.

In [15], it is proved that the Hausdorff dimension of J_α is $\frac{d}{\alpha}$.

Let $\alpha \in \left[1, \frac{d}{d+(\gamma-s)p}\right]$, $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $N = 2^{dn} > \frac{d}{\varepsilon\alpha} + 1$ be fixed. We denote $H(\alpha) = \frac{d\gamma}{\alpha(d+(\gamma-s)p)}$ and $\beta(\alpha) = H(\alpha) + \varepsilon$. Each dyadic cube of size 2^{-dj} can be split into 2^{dn} subcubes $i(\lambda)$ with side $2^{-(j+n)}$. We define the probe P spanned by N functions g^r with the following wavelet coefficients d_λ^r:

$$d_\lambda^r = \begin{cases} j^{-2/q} 2^{-\gamma j} & \text{if each } k_i \text{ is a multiple of } 2^{j-L} \text{ and } r = i(\lambda) \\ 0 & \text{elsewhere} \end{cases} \tag{14}$$

where for each j we denote $L = \left[\frac{(d+(\gamma-s)p)j}{d}\right]$.

One can check that these functions g^r belong to $B_p^{s,q} \cap C^\gamma$, see [16].

Let $J_\alpha(i,l) = \frac{l}{2^i} + \left[-\frac{1}{2^{\alpha L}}\frac{1}{2^{\alpha L}}\right]^d$.

Let us first check that the set of points $S_c(\alpha)$ defined by

$$S_c(\alpha) = \{f = \sum c_\lambda \psi_\lambda \in B_p^{s,q} \cap C^\gamma : \exists x \in J_\alpha \ \forall j,k \ |c_\lambda| \leq c 2^{-\beta(\alpha)j}(1+|2^j x - k|)^{\beta(\alpha)}\}.$$

is a Haar null Borel set. Indeed this set can be included in the lim sup on i of the countable union over l of sets:

$$S_c(\alpha)^{i,l} = \{f = \sum c_\lambda \psi_\lambda \in B_p^{s,q} \cap C^\gamma : \exists x \in J_\alpha(i,l) \ \forall j,k \ |c_\lambda| \leq c 2^{-\beta(\alpha)j}(1+|2^j x - k|)^{\beta(\alpha)}\}.$$

which are closed sets.
We pick a sequence of functions f_n in $S_c(\alpha)^{i,l}$ and such that f_n converges to f in $B_p^{s,q} \cap C^\gamma$. For each n, there exists x_n in $J_\alpha(i,l)$ such that f_n satisfies condition (12) at x_n. But $J_\alpha(i,l)$ is a compact set, so there exists $x \in J_\alpha(i,l)$ and a subsequence $(x_{n(i)})_{i\in\mathbb{N}}$ such that $x_{n(i)}$ converges to x. As the mapping which gives wavelet coefficients of a function is continuous, f satisfies also (12) at x.

Let $f \in B_p^{s,q} \cap C^\gamma$ be fixed. Consider the affine subset $M = \{\delta \in \mathbb{R}^N; \quad f + \sum \delta^i g^i \in S_c(\alpha)\}$. Let δ_1 and δ_2 be in M. There exists $x_1 \in J_\alpha$ and $x_2 \in J_\alpha$ such that for $l = 1, 2$:

$$|c_\lambda + \sum \delta_l^i d_\lambda^i| \leq c 2^{-\beta(\alpha)j}(1 + |2^j x - k|)^{\beta(\alpha)} \leq c 2^{-\alpha\beta(\alpha)L}. \qquad (15)$$

Furthermore $H(\alpha) > \gamma$ and, if λ is such that each k is a multiple of 2^{j-L}.

$$|d_\lambda^i| > \frac{1}{j^{2/q}} 2^{-H(\alpha)L}. \qquad (16)$$

So, taking (15) and (16) we obtain:

$$\|\delta_1 - \delta_2\|_{\mathbb{R}^N} \leq 2c 2^{-\alpha\beta(\alpha)L} 2^{H(\alpha)L} j^{2/q} = 2c j^{2/q} 2^{-\alpha\varepsilon L}.$$

When j tends to infinity, the Lebesgue measure of $S_c(\alpha)$ tends to zero. Now, we take the countable union over c and $\varepsilon_n \to 0$. As Haar null set are stable under inclusion, we obtain:

$$\forall \alpha \in [1, \frac{d}{d+(\gamma-s)p}] \ \ a.e.\ in\ B_p^{s,q} \cap C^\gamma \ \ \forall x \in J_\alpha \ h_f(x) \leq H(\alpha).$$

Let (α_n) be a dense sequence in $[1, \frac{d}{d+(\gamma-s)p}]$. As a countable union of Haar null sets is still a Haar null set, for almost every function in $B_p^{s,q} \cap C^\gamma$,

$$h_f(x) \leq H(\alpha_n) \quad \forall n \, \forall x \in J_{\alpha_n}. \qquad (17)$$

Let f be a function satisfying (17). Let α be fixed, there exists a nondecreasing subsequence (α_{φ_n}) which converges to α and the intersection of the subsets $J_{\alpha_{\varphi_n}} (:= \tilde{J}_\alpha)$ contains J_α. Furthermore there exist a measure such that any set of dimension less than d/α is of measure zero. And the measure of J_α is positive. If $G_H = \{x : \ h_f(x) \leq H\}$, with Lemma 1 we have that the Hausdorff dimension of G_H is $\frac{d+(\gamma-s)p}{\gamma} H$. And the $\frac{d}{\alpha}$ Hausdorff measure of the set $\{x : \ h_f(x) < H\}$ equals zero. This way for almost every function in $B_p^{s,q} \cap C^\gamma$,

$$d(H) = \frac{d+(\gamma-s)p}{\gamma} H \quad \text{for } H \in \left[\gamma, \frac{d\gamma}{d+(\gamma-s)p}\right]$$

2.2 The scaling function

Let us now determine the scaling function of almost every function in a given Besov space. We will now show the following result.

Proposition 5. *Let s_0 and p_0 be fixed such that $s_0 - \frac{d}{p_0} > 0$. Outside a Haar null set in $B_{p_0}^{s_0,\infty}$, we have:*

$$\eta_f(p) = \begin{cases} ps_0 & p \leq p_0 \\ d + p(s_0 - \frac{d}{p_0}) & p \geq p_0. \end{cases} \tag{18}$$

Let $0 < \gamma < s$ be fixed. If $s_0 - \frac{d}{p_0} < 0$, then outside a countable union of compact set in $B_{p_0}^{s_0,p_0} \bigcap C^\gamma$:

$$\eta_f(p) = \begin{cases} ps_0 & p \leq p_0 \\ \gamma p + p_0(s_0 - \gamma) & p \geq p_0. \end{cases} \tag{19}$$

Proof: In each case, we can find in [21] the lower bound. Indeed, this bound is given by the Sobolev embedding.

To prove the upper bound, we will first consider the case $s_0 - \frac{d}{p_0} > 0$. Let $\varepsilon > 0$ be fixed and denote

$$\tilde{s}(p) = \begin{cases} ps_0 + \varepsilon & p \leq p_0 \\ d + p(s_0 - \frac{d}{p_0}) + \varepsilon & p \geq p_0. \end{cases}.$$

Let $0 < p < \infty$ be fixed. We want to show that the set of functions belonging to $B_p^{\tilde{s}(p),\infty}$ for all $0 < p < \infty$ is Haar null. This set is clearly

closed and Borel. Let $j \geq 1$ and $k \in \{0, ..., 2^j - 1\}^d$. We define $J \leq j$ and $K \in \mathbb{Z}^d$ such that

$$\frac{K}{2^J} = \frac{k}{2^j}$$

is an irreducible fraction. Let $a > \frac{3}{p_0}$. We define a probe spanned by the function F with the following wavelet coefficients:

$$d_\lambda = j^{-a} 2^{(\frac{d}{p_0} - s_0)j} 2^{-\frac{d}{p_0}J}.$$

This function belongs to $B_{p_0}^{s_0,p_0}$.

Let f be in $B_{p_0}^{s_0,p_0}$ and consider the affine subset

$$M = \{\alpha \in \mathbb{R};\ f + \alpha F \in B_p^{\tilde{s}(p),\infty}\}.$$

Suppose that there exist α_1 and α_2 in M. We have then three cases, following position of p.

- If $p = p_0$, then $\tilde{s}(p) = p_0 + \varepsilon$ and

$$\begin{aligned}
\|f + \alpha_1 F - (f + \alpha_2 F)\|_{B_p^{\tilde{s}(p),\infty}} &= \sup_j \sum_{k \in \{0,...,2^j-1\}^d} |\frac{\alpha_1 - \alpha_2}{j^a} 2^{(\tilde{s} - \frac{d}{p_0})j} 2^{(\frac{d}{p_0} - s_0)j} 2^{-\frac{d}{p_0}J}|^{p_0} \\
&= \sup_j |\frac{\alpha_1 - \alpha_2}{j^a}| 2^{p_0 \varepsilon j} \sum_{J=0}^{j} \sum_{K \in \{0,...,2^J-1\}^d} 2^{-dJ} \\
&= \sup_j \frac{|\alpha_1 - \alpha_2|}{j^a} 2^{p_0 \varepsilon j}.
\end{aligned}$$

But if α_1 and α_2 belong to M, this implies that

$$f + \alpha_1 F - (f + \alpha_2 F) \text{ belong to } B_p^{\tilde{s}(p),\infty}.$$

This is possible only if $\alpha_1 = \alpha_2$.

- If $p > p_0$, then $\tilde{s}(p) = d + p(s_0 - \frac{d}{p_0}) + \varepsilon$. In this case,

$$f + \alpha_1 F - (f + \alpha_2 F) \in B_p^{\tilde{s}(p),\infty}$$

implies that there exist $c > 0$ such that:

$$\|f + \alpha_1 F - (f + \alpha_2 F)\|_{B_p^{\tilde{s}(p),\infty}} \leq \|f + \alpha_1 F\|_{B_p^{\tilde{s}(p),\infty}} + \|f + \alpha_2 F\|_{B_p^{\tilde{s}(p),\infty}} \leq c.$$

We have then the following inequalities:

$$\forall j>0 \quad \sum_{k\in\{0,\dots,2^j-1\}^d} |\frac{\alpha_1-\alpha_2}{j^a}2^{(\tilde{s}-\frac{d}{p})j}2^{(\frac{d}{p_0}-s_0)j}2^{-\frac{d}{p_0}J}|^p \leq c$$

$$\forall j>0 \quad |\frac{\alpha_1-\alpha_2}{j^a}|^p 2^{(\tilde{s}-\frac{d}{p})pj}2^{(\frac{d}{p_0}-s_0)pj}\sum_{J=0}^{j}\sum_{K\in\{0,\dots,2^J-1\}^d} 2^{-\frac{dp}{p_0}J} \leq c$$

By definition of J. And

$$\begin{aligned} \forall j>0 \quad \left|\frac{\alpha_1-\alpha_2}{j^a}\right|^p \sum_{J=0}^{j} 2^{(d-\frac{dp}{p_0})J} &\leq c2^{(-\tilde{s}+\frac{d}{p}-\frac{d}{p_0}+s_0)pj} \\ \forall j>0 \quad \left|\frac{\alpha_1-\alpha_2}{j^a}\right| &\leq c2^{(-\tilde{s}+\frac{d}{p}-\frac{d}{p_0}+s_0)j}\left|\frac{1}{1-2^{j(d-d\frac{p}{p_0})}}\right|^{\frac{1}{p}} \end{aligned} \tag{20}$$

As $p>p_0$, $1-2^{j(d-d\frac{p}{p_0})}$ is equivalent to 1 for large j and (2.2) implies

$$|\alpha_1-\alpha_2| \leq cj^a 2^{-\varepsilon j}$$

which tends to zero when j tends to infinity.

- If $p<p_0$, then $\tilde{s}(p)=s_0+\varepsilon$ and $1-2^{j(d-d\frac{p}{p_0})}$ is equivalent to $2^{j(d-d\frac{p}{p_0})}$ when j tends to infinity. Thus in (2.2), we obtain again

$$|\alpha_1-\alpha_2| \leq cj^a 2^{-\varepsilon j}.$$

In each case we have obtained that M is of Lebesgue measure zero. Taking countable union over $\varepsilon \to 0$, and over p, we obtain the desired scaling exponent.

The second case, for $s_0-\frac{d}{p_0}<0$ can be treated the same way for $p\leq p_0$. The case $p>p_0$ is obtained taking the function which coefficients are given by (14) instead of F.

□

From Theorem 2 and Proposition 5, we obtain the following Legendre transform of the scaling function of almost every function in a given Besov space.

Proposition 6. *Let $s_0>0$ and $0\leq p_0<\infty$.*

- *If* $s_0 - \frac{d}{p_0} > 0$*, then for almost every function in* $B_p^{s,q}$*:*

$$\forall H \in \left[s_0 - \frac{d}{p_0}, s_0 \right] \qquad \inf_{p>0}(d - \eta(p) + Hp) = d - p_0 s_0 + H p_0. \tag{21}$$

- *If* $s_0 - \frac{d}{p_0} < 0$*, then for almost every function in* $B_p^{s,q} \bigcap C^\gamma$ *we have:*

$$\forall H \in [\gamma, s_0] \qquad \inf_{p>0}(d - \eta(p) + Hp) = d - p_0 s_0 + H p_0. \tag{22}$$

This proposition shows that for $s_0 - \frac{d}{p_0} > 0$, the increasing part of the spectrum given by Frisch-Parisi conjecture is valid for almost every function. But for $s_0 - \frac{d}{p_0} < 0$, this Legendre transform does not correspond to the spectrum of singularities given by Theorem 2.

3 The Frisch-Parisi conjecture

We will now prove Theorem 1. Instead of $B_{p_0}^{s_0,q_0}$ we will now work with:

$$V = \bigcap_{\varepsilon>0, 0<p<\infty} B_{p,loc}^{(\eta(p)-\varepsilon)/p,p}.$$

This set V can also be written as a countable intersection over $B_{p_n,loc}^{(\eta(p_n)-\varepsilon_n)/p_n,p_n}$

Note that V is a topological vector space. For $p < 1$ Besov spaces are only quasi-Banach spaces, as the triangle inequality is only satisfied up to a constant, V is not a Banach space but a complete metric space. Indeed, if $p \geq 1$ we take for distance between two functions f and g in $B_p^{s,q}$:

$$d(f,g) = \sum_{j\geq 0} \left(\sum_{k\in\{0,\ldots,2^j-1\}^d} \left| (c_{j,k} - d_{j,k}) 2^{(s-\frac{d}{p})j} \right|^p \right)^{\frac{q}{p}}$$

where $c_{j,k}$ are the wavelet coefficients of f and $d_{j,k}$ are those of g.
If $p < 1$ Besov spaces are not Banach spaces, but complete metric space with the following distance:

$$d(f,g) = \left(\sum_{j\geq 0} \left(\sum_{k\in\{0,\ldots,2^j-1\}^d} |(c_{j,k} - d_{j,k}) 2^{(s-\frac{d}{p})j}|^p \right)^{\frac{q}{p}} \right)^{\frac{\min(p,q)}{q}}.$$

Thus, we obtain a distance in V taking:

$$\forall f, g \in V \quad d(f,g) = \sum_n 2^{-n} \frac{d_n(f,g)}{1 + d_n(f,g)}$$

where d_n denotes the distance in $B_{p_n,loc}^{(\eta(p_n)-\varepsilon_n)/p_n,p_n}$. With this distance V is clearly a complete space. Note that the measure used is the Lebesgue measure in the unit ball of a probe, so this is a probability measure with compact support.

In the following subsection we prove that the spectrum of singularities of almost every function in V satisfies:

$$d(H) = \inf_{p \geq p_c} (pH - \eta(p) + d).$$

3.1 Proof of Theorem 1

Let us now study the spectrum of singularities on a prevalent set of functions in V.

Proposition 7. *For almost every function $f \in V$, the spectrum of singularities satisfies:*

$$\forall H \in [s(0), \frac{d}{p_c}] \quad d(H) = \inf_{p \geq p_c} (Hp - \eta(p) + d). \tag{23}$$

Proof:

We will first construct the probe. Denote:

$$a(j,k) = \inf_p \left(\frac{d(j-J) - \eta(p)j}{p} \right)$$

and define g via its wavelet coefficients:

$$d_\lambda = \frac{1}{j^a} 2^{a(j,k)} \tag{24}$$

where we denote $a = a_j = \log j$ and $J \leq j$ is such that there exists $K \in \mathbb{Z}^d$ and $\frac{k}{2^j} = \frac{K}{2^J}$ is an irreducible form.
First, we check that g belongs to V. Let $p > 0$ be fixed. Thus we have to show that $g \in B_p^{\eta(p)/p,\infty}$. Let $s = \frac{\eta(p)}{p}$. Since $a(j,k) \leq \frac{d(j-J)}{p} - sj$, $pa(j,k) + (\eta(p) - p)j = -Jd$ and $g \in B_p^{\eta(p)/p,\infty}$. For further details upon this function g, we refer to [16].

Definition 7. *Let α be fixed. We denote*

$$F_\alpha = \left\{ x : \exists \text{ a sequence } ((k_n, j_n))_{n\in\mathbb{N}} \ \left| x - \frac{k_n}{2^{j_n}} \right| \leq \frac{1}{2^{\alpha j_n}} \right\}. \qquad (25)$$

The dyadic exponent of x is defined by $\alpha(x_0) = \sup\{\alpha : x_0$ is α-approximable by dyadics$\}$

As it is stated in [16], the Hausdorff dimension of the set F_α is at least $\frac{d}{\alpha}$.

First, let $\alpha \in (1, \infty)$ be fixed and let F_α be the set given by Definition 7. Let $\varepsilon > 0$ be fixed, and let

$$H(\alpha) = \frac{1}{\alpha} \sup_{\omega \geq \alpha} \left(\omega \sup_{q>0} (s(q) - d(1 - \frac{1}{\omega})q) \right)$$

and $\gamma = \gamma(\alpha) = H(\alpha) + \varepsilon$.
Let $n \in \mathbb{N}$ be such that $N = 2^{dn} > \frac{d}{\varepsilon} + 1$ be fixed. The probe P is spanned by N functions g_i which are deduced from g by taking its wavelets coefficients only over some sub-cubes $i(\lambda)$ with size $2^{-d(j+n)}$. The aim of this part is to prove that the set of functions f such that there exist a point in F_α where f is C^γ is a Haar null set. This set is included in the countable union of:

$$S_c(\alpha) = \{ f = \sum c_\lambda \psi_\lambda : \ \exists x \in F_\alpha \ \forall j, k \ |c_\lambda| \leq c 2^{-\gamma(\alpha) j} (1 + |2^j x - k|)^{\gamma(\alpha)} \}.$$

We can find a subsequence (j, k) such that $J \leq \alpha j$ and:

$$H(\alpha) = \frac{1}{\alpha} \sup_{\omega \geq \alpha} \left(\omega \sup_{q>0} (s(q) - d(1 - \frac{1}{\omega})q) \right) \geq -a(j, k).$$

If $x \in F_\alpha$ is fixed and λ is such that $|x - \lambda| \leq A$ for $A > 2N$, the wavelet coefficients of g_i satisfy:

$$|d^i_\lambda| \geq \frac{c(A)}{j^a} 2^{-H(\alpha) j}. \qquad (26)$$

We will now prove that the set $S_c(\alpha)$ is a Borel Haar null set. First, this set is included in the countable union over λ of:

$$S_c(\alpha)^{j,k} = \{ f = \sum c_\lambda \psi_\lambda : \ \exists x \in F^{j,k}_\alpha \ \forall j, k \ |c_\lambda| \leq c 2^{-\gamma(\alpha) j} (1 + |2^j x - k|)^{\gamma(\alpha)} \}.$$

Where $F^{j,k}_\alpha = \{ x : \ |x - \frac{k}{2^j}| \leq \frac{1}{2^{\alpha j}} \}$. This set $S_c(\alpha)^{j,k}$ is a closed set and $S_c(\alpha)$ is a Borel set. Let f be in V and β_1 and β_2 be such that

the functions $f+\sum \beta_1^i g^i$ and $f+\sum \beta_2^i g^i$ are in $S_c(\alpha)$. There exist two points x_1 and x_2 in F_α such that in the cone of influence above x_1 and x_2:

$$|c_\lambda + \sum \beta_1^i d_\lambda^i - (c_\lambda + \sum \beta_2^i d_\lambda^i)| \leq 2c2^{-\gamma(\alpha)j}.$$

Or,

$$|c_\lambda + \sum \beta_1^i d_\lambda^i - (c_\lambda + \sum \beta_2^i d_\lambda^i)| = |\sum \beta_1^i d_\lambda^i - \beta_2^i d_\lambda^i|$$

but, using (26),

$$|\sum \beta_1^i d_\lambda^i - \beta_2^i d_\lambda^i| \geq |\sum \beta_1^i - \beta_2^i| \frac{c(A)}{j^a} 2^{-H(\alpha)j}.$$

Thus,

$$\|\beta_1 - \beta_2\|_{\mathbb{R}^N} \leq \tilde{c} j^a 2^{-\varepsilon j}.$$

So the Lebesgue measure in $\mathbb{R}^N$ of the set $\{\beta : \ f+\beta g \in S_c(\alpha)^{j,k}\}$ is bounded by $(\tilde{c}j^a)^N 2^{-N\varepsilon j}$.
The Lebesgue measure of the set of β such that $f+\sum \beta^i g^i$ belongs to $S_c(\alpha)$ vanishes. Therefore $S_c(\alpha)$ is Haar null.
Taking a countable union over $c_n > 0$ of sets $S_c(\alpha)$,the set of functions in V with a pointwise Hölder exponent greater than $\gamma(\alpha)$ at a point of F_α is also Haar null. If $\varepsilon_n \to 0$, taking the union over ε_n it follows that for all $\alpha \geq 1$ the set of functions in V with an Hölder exponent greater than $H(\alpha)$ at some point of F_α is Haar null.
Let α_n be a dense sequence in $(1,\infty)$. By countable intersection:

$$M = \{f \in V : \forall n \ \ \forall x \in F_{\alpha_n} \ \ h_f(x) \leq H(\alpha)\} \tag{27}$$

is prevalent. Let $f \in M$ and let $\alpha \geq 1$. There exist a subsequence $\alpha_{\phi(n)}$ which is nondecreasing and tends to α. If we denote $\tilde{F}_\alpha$ the intersection of sets F_{α_n}, it follows that $\tilde{F}_\alpha$ contains F_α. Furthermore, the Hausdorff dimension of $\tilde{F}_\alpha$ is greater than $\frac{d}{\alpha}$ and for all $x \in \tilde{F}_\alpha$, $h_f(x) \leq H(\alpha)$. Finally we obtain

To conclude the second point of Theorem 1, we rewrite $H(\alpha)$ in the following form

$$H(\alpha) = \frac{1}{\alpha} \inf_{a \geq \alpha} G(a)$$

where $G(a) = \sup_q(a(-qd+s(q))+qd) = a \sup_q(qd(-1+\frac{1}{a})+s(q)) = a\ s^*\left(d\left(1-\frac{1}{a}\right)\right)$. Here s^* is the Legendre transform of s. By definition of the Legendre transform, this is a convex function. Furthermore it satisfies

$$\begin{cases} s^*(h) = +\infty & \text{if } h < s'(+\infty) \\ s^*(h) = s(0) & \text{if } h > s'(0) \end{cases}. \tag{28}$$

And if s^* is twice differentiable (we refer to [11] for a general case), G is also twice differentiable and its derivative is

$$G'(a) = s^*\left(d\left(1-\frac{1}{a}\right)\right) + \frac{d}{a}(s^*)'\left(d\left(1-\frac{1}{a}\right)\right)$$

and:

$$G''(a) \geq 0.$$

Thus G is also convex and there exists a_0 such that $G(a_0) = \inf_{a\geq 0} G(a)$, a_0 being such that $G'(a_0) = 0$. We also deduce from (28) that:

$$\begin{cases} G(a) = +\infty & \text{si } a < \frac{d}{d-s'(+\infty)} \\ G(a) = as(0) & \text{si } a > \frac{d}{d-s'(0)} \end{cases}. \tag{29}$$

By definition of s and with the hypothesis that η is an admissible function we have $0 \leq s'(q) \leq d$, for all $q > 0$. It follows from (29) that a_0 belongs to the interval $(\frac{d}{d-s'(+\infty)}, \frac{d}{d-s'(0)})$ which is included in $[0, \infty)$.

Another way to treat G is to write $G(a) = \sup_q(\tilde{s}(q))$ where $\tilde{s}(q) = a(s(q) - qd) + qd$. And if s is also twice differentiable, $\tilde{s}'(q) = -da + as'(q) + d$ and $\tilde{s}''(q) = as''(q) < 0$. Thus $\tilde{s}$ is a concave function, and there exists an upper bound q_0 which satisfies $\tilde{s}'(q_0) = -da + as'(q_0) + d = 0$, and $s'(q_0) = \frac{da-d}{a}$. The value of q_0 also depends of a so we write now $q_0 = q(a)$.

We can finally write $G(a) = a(-q(a)d + s(q(a))) + q(a)d$. This function is twice differentiable and its derivative satisfies:

$$G'(a) = s(q(a)) - dq(a).$$

If $a = a_0$ is the lower bound of G, we obtain $G'(a_0) = s(q(a_0)) - dq(a_0) = 0 \Rightarrow s(q(a_0)) = dq(a_0)$, that is $q(a_0) = q_c = 1/p_c$. Furthermore, G is decreasing for $a \leq a_0$ and increasing for $a \geq a_0$. The following cases are now possible:

- If $\alpha \geq \frac{d}{d-s'(0)}$, then for all $a \geq \alpha$, $G(a) = as(0)$ and $H(\alpha) = s(0)$. So,
$$\dim_H(\{x :\ h_f(x) \leq s(0)\}) = d + s'(0).$$
- If $1 \leq \alpha \leq a_0$ then
$$\inf_{a\geq\alpha} G(a) = G(a_0) = (a_0(-q_c d + s(q_c)) + q_c d) = dq_c.$$

and the corresponding value of H is

$$H(\alpha) = \frac{1}{\alpha} \inf_{a \geq \alpha} G(a) = \frac{dq_c}{\alpha}$$

Thus, the spectrum of singularities is defined on the interval $[\frac{dq_c}{a_0}, dq_c]$ and for almost every function, and for all $H \in \left[\frac{dq_c}{a_0}, dq_c\right]$

$$\dim_H(\{x : \ h_f(x) \leq H\}) = \frac{H}{q_c}.$$

Furthermore, we have already seen that :

$$s'(q(a)) = \frac{da - d}{a}$$

which is an increasing function. As s' is decreasing, the application $a \mapsto q(a)$ is itself decreasing. So, for $\alpha \leq a_0$, $q \geq q_c = \frac{1}{p_c}$ and :

$$\dim_H(\{x : \ h_f(x) \leq H\}) \leq \inf_{p \geq p_c} (pH - \eta(p) + d).$$

- If $a_0 \leq \alpha \leq \frac{d}{d - s'(0)}$, which is equivalent to

$$\inf_{a \geq \alpha} G(a) = G(\alpha) = \alpha \sup_q \left(-qd + s(q) + \frac{qd}{\alpha} \right).$$

We obtain :

$$H(\alpha) = \sup_q \left(-qd + s(q) + \frac{qd}{\alpha} \right).$$

So for almost every function, for all $H \in [s(0), \frac{dq_c}{a_0}]$,:

$$\dim_H(\{x : \ h_f(x) \leq H\}) = \inf_{p \geq p_c} (pH - \eta(p) + d) \leq \inf_{p \geq p_c} (pH - \eta(p) + d).$$

Furthermore, see [16], the spectrum of singularities of all functions of V satisfies:

$$d(H) \leq \inf_{p \geq p_c} (pH - \eta(p) + d). \tag{30}$$

This implies that the Hausdorff dimension of the set $\{x : h_f(x) < H\}$ is strictly less than $\frac{d}{\alpha}$. As proved in [14], there exists a measure m_α

such that $m_\alpha(\{x : h_f(x) \leq H\}) > 0$. But by definition of the Hausdorff dimension, $m_\alpha(\{x : h_f(x) < H\}) = 0$. Then, $m_\alpha(\{x : h_f(x) = H) > 0$ and

$$d(H) \geq \inf_{p \geq p_c} (pH - \eta(p) + d).$$

□

Proposition 8. *For almost every function f in V, the scaling function of f satisfy:*

$$\eta_f(p) = \eta(p), \quad \forall 0 < p < \infty.$$

Proof: As we have $V = \bigcap_{\varepsilon>0, 0<p<\infty} B_{p,loc}^{(\eta(p)-\varepsilon)/p,p}$, for any $f \in V$ the scaling function is greater than $\eta(p)$ for all p. Let $\tau > 0$ and $p > 0$ be fixed. We denote $\tau(p) = \frac{\eta(p)}{p} + \tau$. We first prove that the set

$$M(p) = \{f \in V;\ f \in B_p^{\tau(p),\infty}\}$$

is a Haar null set. Let g be the function defined by wavelet coefficients given by (24) and P be the probe spanned by g. First, we check that g does not belong to $B_p^{\tau(p),\infty}$. We write $\beta_j = d(1 - \frac{j}{J})$, where J is defined as in (24). This term β_j takes discrete values, spaced by $\frac{d}{j}$ and between 0 and d. As the function s is a concave function and $0 \leq s'(q) \leq d$ for all q, there exists, for j large enough, a β_j near from $s'(q)$ such that the line given by $\tau(p) + \beta_j \left(\frac{1}{\tilde{p}} - \frac{1}{p}\right)$ is always above the graph of s. Thus

$$\forall \tilde{p} > 0 \quad \tau(p) + \beta_j \left(\frac{1}{\tilde{p}} - \frac{1}{p}\right) > s(1/p).$$

But $a(j,k) = j \inf_p(\frac{\beta_j}{p} - s(1/p))$, and this infimum is attained for a $p_0 \in (0, \infty)$. Therefore,

$$a(j,k) \geq j(\frac{\beta_j}{p} - \tau(p)),$$

and

$$\|g\|_{B_p^{\tau(p),\infty}} \geq \sup_j j^{-a_j p} 2^{\tau j p}.$$

Thus $g \notin B_p^{\tau(p),\infty}$.

Let f be in V. Suppose that there exist α_1 and α_2 such that the functions $f_1 = f + \alpha_1 g$ and $f_2 = f + \alpha_2 g$ belong to $B_p^{\tau(p),\infty}$. Then $f_1 - f_2$ also belongs to $B_p^{\tau(p),\infty}$. But

$$f_1 - f_2 = (\alpha_1 - \alpha_2)g.$$

As g does not belong to $B_p^{\tau(p),\infty}$, we have then $\alpha_1 = \alpha_2$. Thus $M(p)$ is Haar null.

Taking countable union over $\tau \to 0$ and $p > 0$, we obtain that for almost every f in V, $\eta_f(p) \leq \eta(p)$ for all $p > 0$.

□

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