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Abstract

The mathematical description of laboratory fusion plasmas produced in Tokamaks is still challenging. Complete models for electrons and ions, as Vlasov-Maxwell systems, are computationally too expensive because they take into account all details and scales of magneto-hydrodynamics. In particular, for most of the relevant studies, the mass electron is negligible and the velocity of material waves is much smaller than the speed of light. Therefore it is useful to understand simplified models. Here we propose and study one of those which keeps both the complexity of the Vlasov equation for ions and the Hall effect in Maxwell's equation. Based on energy dissipation, a fundamental physical property, we show that the model is nonlinear stable and consequently prove existence.

Key words: Vlasov equations; Plasma physics; Kinetic averaging lemma; Maxwell's equations.

Mathematics Subject Classification: 35B35, 35L60, 82D10

1 Introduction

To describe the behavior of ions population in hot plasmas, it is very classical to address, at least at theoretical level, a Vlasov equation coupled to a non-linear Poisson equation which defines the electrostatic field. When the magnetic field $\mathbf{B}$ is an external datum, for the ion distribution function $f(t,x,\mathbf{v})$ (at position x and velocity $\mathbf{v}$), one addresses the following system

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \frac{\partial}{\partial \mathbf{v}}.[(-T_{\rm e}\nabla \ln n_e + \mathbf{v}\wedge\mathbf{B})\, f] &= 0, \\ -\lambda^2 \Delta \ln n_e &= \int f(\mathbf{v}) d\mathbf{v} - n_e, \end{aligned}$$

where $n_e = n_e(t,x)$ is the electron density, $T_{\rm e}$ is the mean electron temperature assumed to be constant and λ is the Debye length (a characteristic constant of the plasma). This system is a classical one, indeed the electrostatic field may be approximated by $-T_{\rm e}\nabla \ln n_e$ (at least when the electron temperature is constant) and the Poisson equation is nothing but the Gauss relation applied to this field (cf. [7] for example)

The mathematical understanding of this kind of kinetic system has made important progresses with the proof of existence of global weak solutions in the large of the Vlasov-Maxwell system in [11] and of the Vlasov-Poisson system [19, 25]. However, the mathematical description of laboratory fusion plasmas (such as those produced in Tokamaks) with this kind of ion kinetic models is still a major challenge, in particular to deal with a time scale compatible with the evolution equation for the magnetic field. As a matter of fact, complete systems of Vlasov-Maxwell type which account for all scales of electro-dynamics are not relevant since the velocity of ion waves is much smaller than the speed of light: it is well known that at the time scale of the ion population it is convenient to neglect the current of displacement in the Maxwell equations (as in magneto-hydrodynamics models cf. [20, 21]), that is to say the electric current $\mathbf{J}$ is assumed to satisfy

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$$\mathbf{J} = \nabla \wedge \mathbf{B}.$$

The aim of this work is precisely to propose, justify and study a model which couples a kinetic equation for the ions and an evolution equation for the magnetic field (as those used in magneto-hydrodynamics). The unknowns are the ion particle density $f(t,x,\mathbf{v})$, the magnetic field $\mathbf{B}(t,x)$ and the electron density $n_e(t,x)$ and they satisfy

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_I - n_e, & (a) \\ \dfrac{\partial \mathbf{B}}{\partial t} - \nabla \wedge \left(\dfrac{1}{n_e} n_I \mathbf{u}_I \wedge \mathbf{B} \right) + \nabla \wedge \left(\dfrac{1}{n_e} \mathbf{J} \wedge \mathbf{B} \right) + \nabla \wedge (\eta \nabla \wedge \mathbf{B}) = 0, & (b) \\ \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \dfrac{\partial}{\partial \mathbf{v}} \cdot \left[\left(-\dfrac{T_{\mathrm{e}}}{n_e} \nabla n_e + \dfrac{\mathbf{J} - n_{_I} \mathbf{u}_I}{n_e} \wedge \ \mathbf{B} + \mathbf{v} \wedge \mathbf{B} \right) f \right] = 0, & (c) \\ \nabla \cdot \mathbf{B} = 0, & (d) \end{cases} \tag{1}$$

where the following notations are used

$$n_I(t,x) = \int_{\mathbb{R}^3} f(t,x,\mathbf{v}) d\mathbf{v} \qquad \text{(the number density in ions)}, \tag{2}$$

$$n_I(t,x)\mathbf{u}_I(t,x) = \int_{\mathbb{R}^3} f(t,x,\mathbf{v}) \mathbf{v} d\mathbf{v} \qquad \text{(the macroscopic velocity of ions)}. \tag{3}$$

and η denotes a strictly positive bounded function corresponding to the plasma resistivity.

Notice firstly that the electric field $\mathbf{E}$ is given by the relation

$$n_e \mathbf{E} = -T_{\mathrm{e}} \nabla n_e - n_I \mathbf{u}_I \wedge \mathbf{B} + \mathbf{J} \wedge \mathbf{B} + n_e \, \eta \nabla \wedge \mathbf{B} \tag{4}$$

which is one of the classical forms of the generalized Ohm law [7, 3, 2, 14] (it includes the term $\mathbf{J} \wedge \mathbf{B}$ related to the Hall effect); so, (1)-b) is exactly the Faraday equation $\frac{\partial \mathbf{B}}{\partial t} + \nabla \wedge \mathbf{E} = 0$.

We have chosen here a rescaling, such that the ion mass is set to 1 and the electron and ion charge are also set to 1. Moreover the scaling of the ion and electron density is such that the characteristic value of n_I is equal to 1. So with this scaling, the Debye length λ is defined by $\lambda^2 = T_{\mathrm{e}} \varepsilon_0$ where ε_0 is vacuum dielectric constant and equation (1)-a) reads also as

$$-\varepsilon_0 T_{\mathrm{e}} \Delta \ln n_e = n_I - n_e$$

which is Gauss relation applied to $-T_e \nabla \ln n_e$, the dominant term in the electric field. In the rest of the paper, we use the following notations

$$\mathbf{E}^0(t,x) = -T_{\mathrm{e}} \frac{\nabla n_e}{n_e} + \frac{1}{n_e} (\mathbf{J} - n_{_I} \mathbf{u}_I) \wedge \mathbf{B} \qquad \text{and} \qquad \mathbf{F}(t,x,\mathbf{v}) = \mathbf{E}^0 + \mathbf{v} \wedge \mathbf{B}. \tag{5}$$

Here $\mathbf{F}$ is the kinetic force field which appears in equation (1)-c).

Remark 1. It is fundamental to notice that the Debye length is a small quantity in many situations; it may be proved that in the limit $\lambda \to 0$ the solution n_e to the non-linear Poisson equation satisfies $n_e \to n_I$ and many studies have analyzed this singular limit in different contexts [4, 8, 18]. Then, if we make the approximation $n_e \simeq n_I$, the Ohm law (4) leads to the classical formula

$$\mathbf{E} + \mathbf{u}_I \wedge \mathbf{B} = -T_{\mathrm{e}} \nabla \ln n_I + \frac{1}{n_I} (\nabla \wedge \mathbf{B}) \wedge \mathbf{B} + \eta \nabla \wedge \mathbf{B}$$

and from (1), we recover formerly the following system

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} - \nabla \wedge (\mathbf{u}_I \wedge \mathbf{B}) + \nabla \wedge \left(\frac{1}{n_I} (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}) \right) + \nabla \wedge (\eta \nabla \wedge \mathbf{B}) &= 0, \\ \frac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \frac{\partial}{\partial \mathbf{v}} \left[(\mathbf{E}^{\lim} + \mathbf{v} \wedge \mathbf{B}) f \right] &= 0, \\ n_I (\mathbf{E}^{\lim} + \mathbf{u}_I \wedge \mathbf{B}) &= -T_{\mathrm{e}} \nabla n_I + (\nabla \wedge \mathbf{B}) \wedge \mathbf{B}. \end{aligned}$$

Despite its apparently simpler form than the system (1), it is not used in physical literature, up to our knowledge. As a matter of fact even if the magnetic field is neglected, the equation

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \frac{\partial}{\partial \mathbf{v}} \left[(\mathbf{E}^{\lim} + \mathbf{v} \wedge \mathbf{B}) f \right] = 0, \qquad n_I \mathbf{E}^{\lim} = -T_{\mathrm{e}} \nabla n_I$$

is probably ill-posed from a mathematical point of view (nevertheless [18] has noticed that using some arguments of the paper [22], one may find a weak solution on a small time interval). Moreover it would be hard to give a meaning of the quantity $\mathbf{u}_I = \int_{\mathbb{R}^3} f(\mathbf{v})\mathbf{v}d\mathbf{v} / \int_{\mathbb{R}^3} f(\mathbf{v})d\mathbf{v}$ which appears in the magnetic equation (it is hardly possible to derive a bound of n_I away from below). Notice that this kind of model has been addressed in [16] and [1] but the framework is a little different since a velocity diffusion term (of Fokker-Planck type) is accountered for in the kinetic equation.

Back to (1), from a mathematical point of view, one can show that n_e is naturally bounded away from zero due to the nonlinear elliptic equation (1)-a and the quantity $\frac{n_I}{n_e}\mathbf{u}_I = n_e^{-1}\int_{\mathbb{R}^3} f(\mathbf{v})\mathbf{v}d\mathbf{v}$ may be defined in some Lebesgue space.

Our study is aiming at the analysis of model (1) in a bounded domain; therefore we have to specify the boundary and initial conditions.

Boundary conditions Let Ω be a bounded domain of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. We consider the model (1) for $\mathbf{v} \in \mathbb{R}^3$, $x \in \Omega$ and $t \geq 0$, with the following boundary conditions

$$\begin{cases} \mathbf{n}_x \cdot \nabla n_e(t,x) = 0 & x \in \partial\Omega, \quad (a) \\ \mathbf{n}_x \wedge \mathbf{B}(t,x) = 0 & x \in \partial\Omega, \quad (b) \\ f(x, \mathbf{v} - 2(\mathbf{v}\cdot\mathbf{n}_x)\mathbf{n}_x) = f(x,\mathbf{v}), \qquad \mathbf{v} \in \mathbb{R}^3, & x \in \partial\Omega. \quad (c) \end{cases} \tag{6}$$

The zero flux boundary conditions (6)-a) on electrons enforces in (1)-a) the global neutrality of the plasma

$$\int_\Omega n_e dx = \int_\Omega n_i dx.$$

The condition (b) on the magnetic field is the simplest one and a more realistic condition is discussed in section 4. Our result will be still true with more realistic boundary condition, of impedance type for example, or also a non homogeneous boundary condition such as

$$\mathbf{n}_x \wedge \mathbf{B}(t,x) = \mathbf{n}_x \wedge \mathbf{B}_{\rm imp}, \quad x \in \partial\Omega,$$

which is much more relevant in the context of confined plasmas in Tokamaks. Condition (6)-c) is the so-called specular reflection; it implies the no-slip boundary condition

$$\mathbf{u}_I(t,x)\cdot\mathbf{n}_x = 0, \quad x \in \partial\Omega. \tag{7}$$

Our main result is the following.

Theorem 1 (Nonlinear stability)**.** *Assume* (14) *and that the initial datum satisfy* (11)–(13)*. From any bounded family of solutions* $f^\varepsilon, \mathbf{B}^\varepsilon, n_e^\varepsilon$ *to* (1)*,* (6) *we can extract a subsequence that converges to a weak solution with finite energy. Moreover, we have for all* $T > 0$*,*

$$\mathbf{B}^\varepsilon \to \mathbf{B} \in \mathrm{L}^\infty\big(0,T;\mathrm{L}^2(\Omega)\big) \cap \mathrm{L}^2\big(0,T;\mathrm{L}^q(\Omega)\big) \quad \textit{strongly}, \qquad q < 6,$$

$$f^\varepsilon \to f \in \mathrm{L}^\infty\big(0,T;\mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega\times\mathbb{R}^3))\big) \quad \textit{weakly},$$

$$\mathbf{J}^\varepsilon = \nabla \wedge \mathbf{B}^\varepsilon \to \mathbf{J} \quad \text{in } \mathrm{L}^2((0,T)\times\Omega) \quad \textit{weakly},$$

$$n_e^\varepsilon \to n_e \quad \textit{in } \mathrm{L}^q((0,T)\times\Omega), \textit{ strongly and for some constants } \quad 0 < K_- \leq n_e \leq K_+,$$

$$n_I^\varepsilon \mathbf{u}_I^\varepsilon \to n_I \mathbf{u}_I \in (\mathrm{L}^q(0,T\times\Omega))^3 \quad \text{strongly},$$

for all q *such that* $1 \leq q < 5/4$*.*

The first tool towards a mathematical analysis is to state the energy balance for the full system; this is performed in section 2. It indicates that the magnetic resistivity in equation (1)-a) is useful (if not necessary) so as to control the current $\mathbf{J}$ and subsequently the term $\mathbf{J}\wedge\mathbf{B}$ describing to the Hall effect. Also the classical tools for elliptic equation with Neumann boundary conditions will lead to control n_e away from below. But more technical ingredients (namely a kinetic averaging lemma in a L^q space) are used in the proof in order to pass to the limit in the nonlinearities; this is the purpose of section 3 and of several appendices. We come back on more realistic boundary conditions in section 4 so as to include confinement. We also detail a constructive splitting procedure which can be used to design an approximate solution with the same energy law.

2 Preliminaries

The first and major ingredient in our approach to nonlinear stability is the energy balance. We state it here together with considerations on the physics sustaining the model.

2.1 The energy balance

We now turn to the study of energy dissipation for the system (1) with homogeneous boundary conditions (6). We introduce the ions kinetic energy and the magnetic energy

$$\mathcal{E}_I(t) = \frac{1}{2}\int_\Omega\int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^2 d\mathbf{v}dx, \qquad \mathcal{E}_m(t) = \frac{1}{2}\int_\Omega |\mathbf{B}(t,x)|^2 dx,$$

as well as the free energy $\displaystyle\int_\Omega (n_e(t)\ln n_e(t) - n_e(t) + 1)dx$ and the total energy which reads as

$$\mathcal{E}_{tot}(t) = \mathcal{E}_I(t) + \mathcal{E}_m(t) + \frac{\lambda^2}{2}\int_\Omega |\nabla \ln n_e(t)|^2\, dx + \int_\Omega (n_e(t)\ln n_e(t) - n_e(t) + 1)dx. \tag{8}$$

Notice that the integral $\displaystyle\frac{\lambda^2}{2}\int_\Omega |\nabla \ln n_e(t)|^2\, dx$ corresponds to the electrostatic energy. We now establish the

Proposition 1 (Energy dissipation)**.** *Classical solutions to (1),* (6) *satisfy the energy dissipation relation*

$$\frac{d}{dt}\left[\mathcal{E}_I + \mathcal{E}_m + \frac{\lambda^2}{2}\int_\Omega |\nabla(\ln n_e)|^2\, dx + \int_\Omega (n_e \ln n_e - n_e + 1)\, dx\right] = -\int_\Omega \eta\Big|\nabla\wedge\mathbf{B}\Big|^2 dx. \tag{9}$$

Proof. Notice that, as usual, the free divergence condition (1)-(d) is not needed for the energy identity. We first consider the ion kinetic energy. Using the specular reflection condition, we compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_I &= \int_\Omega\int_{\mathbb{R}^3} \mathbf{F}(t,x,\mathbf{v})\cdot\mathbf{v}d\mathbf{v} \\
&= \int_\Omega\int_{\mathbb{R}^3}\left[\left(\frac{1}{n_e}(\nabla\wedge\mathbf{B})\wedge\mathbf{B} + (\mathbf{v} - \frac{n_I}{n_e}\mathbf{u}_I)\wedge\mathbf{B} - T_e\nabla(\ln n_e)\right)f\right]\cdot\mathbf{v}d\mathbf{v}dx \\
&= \int_\Omega\left(\frac{n_I}{n_e}\mathbf{J}\wedge\mathbf{B} - n_I T_e\nabla(\ln n_e)\right)\cdot\mathbf{u}_I dx.
\end{aligned}$$

Now, we turn to the magnetic energy and first recall the definition of $\mathbf{E}$ in (4). Thanks to a classical tensorial identity, we find

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_m &= \int_\Omega \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{B}dx = -\int_\Omega \nabla\wedge\mathbf{E}\cdot\mathbf{B}dx = -\int_\Omega [\mathbf{E}\cdot\nabla\wedge\mathbf{B} + \nabla\cdot(\mathbf{E}\wedge\mathbf{B})]dx \\
&= -\int_\Omega\left[-\frac{n_I}{n_e}\mathbf{u}_I\wedge\mathbf{B} + \eta\nabla\wedge\mathbf{B}\right]\cdot(\nabla\wedge\mathbf{B})\, dx - \int_{\partial\Omega}\mathbf{n}_x\cdot(\mathbf{E}\wedge\mathbf{B})dx.
\end{aligned}$$

According to the boundary condition (6)-(b), we have $\mathbf{n}_x\cdot(\mathbf{E}\wedge\mathbf{B}) = -\mathbf{E}\cdot(\mathbf{n}_x\wedge\mathbf{B}) = 0$, then the boundary integral is zero. Now, thanks to identity $\mathbf{J}\cdot(\mathbf{u}_I\wedge\mathbf{B}) = -\mathbf{u}_I\cdot(\mathbf{J}\wedge\mathbf{B})$, we get

$$\frac{d}{dt}\mathcal{E}_m = -\int_\Omega \frac{n_I}{n_e}\mathbf{u}_I\cdot(\mathbf{J}\wedge\mathbf{B})dx - \int_{\mathbb{R}^3}\eta\Big|\nabla\wedge\mathbf{B}\Big|^2 dx.$$

At this stage, we have obtained

$$\begin{aligned}
\frac{d}{dt}[\mathcal{E}_i + \mathcal{E}_m] &= \int_\Omega [(\mathbf{J}\wedge\mathbf{B}) - n_I\nabla(\ln n_e)]\cdot\mathbf{u}_I dx - \int_\Omega \mathbf{u}_I\cdot(\mathbf{J}\wedge\mathbf{B})dx - \int_\Omega \eta\Big|\nabla\wedge B\Big|^2 dx \\
&= -\textstyle\int_\Omega n_I\nabla(\ln n_e)\cdot\mathbf{u}_I dx - \int_\Omega \eta\Big|\nabla\wedge B\Big|^2 dx.
\end{aligned} \tag{10}$$

To continue, we recall the conservation of mass on n_I

$$\frac{\partial n_I}{\partial t} + \nabla \cdot (n_I \mathbf{u}_I) = 0.$$

Then, we may use the boundary condition (7) to obtain

$$\begin{aligned} -\int_\Omega n_I \nabla (\ln n_e) \cdot \mathbf{u}_I dx &= \int_\Omega (\ln n_e) \nabla \cdot (n_I \mathbf{u}_I)\, dx \\ &= -\int_\Omega (\ln n_e) \frac{\partial n_I}{\partial t} dx \\ &= -\int_\Omega (\ln n_e) \left[\frac{\partial n_e}{\partial t} - \lambda^2 \frac{\partial}{\partial t} \Delta (\ln n_e) \right] dx \\ &= -\frac{d}{dt} \left[\int_\Omega (n_e (\ln n_e) - n_e)\, dx \right] - \lambda^2 \frac{d}{dt} \left[\int_\Omega \frac{|\nabla (\ln n_e)|^2}{2} dx \right] \end{aligned}$$

where we have used equation (1)-(a) once differentiated in time.
Altogether, we obtain the relation (9). □

2.2 Physical considerations

Let first focus on the momentum balance. To find it, we multiply the kinetic equation (1)-c) by $\mathbf{v}$, integrate over $\mathbf{R}^3$ and define as usual [6] the ion pressure tensor P_I through

$$\int \nabla \cdot (\mathbf{v} \otimes \mathbf{v} f(\mathbf{v})) d\mathbf{v} = \nabla \cdot (n_I \mathbf{u}_I \otimes \mathbf{u}_I) + \nabla \cdot \mathrm{P}_I.$$

Then, we get

$$\frac{\partial (n_I \mathbf{u}_I)}{\partial t} + \nabla \cdot (n_I \mathbf{u}_I \otimes \mathbf{u}_I) + \nabla \cdot \mathrm{P}_I + T_\mathrm{e} \frac{\nabla n_e}{n_e} n_I + \frac{n_I}{n_e} n_I \mathbf{u}_I \wedge \mathbf{B} - \frac{n_I}{n_e} \mathbf{J} \wedge \mathbf{B} - n_I \mathbf{u}_I \wedge \mathbf{B} = 0,$$

that is to say

$$\frac{\partial (n_I \mathbf{u}_I)}{\partial t} + \nabla \cdot (n_I \mathbf{u}_I \otimes \mathbf{u}_I) + \nabla \cdot \mathrm{P}_I + T_\mathrm{e} \nabla n_e - \mathbf{J} \wedge \mathbf{B} = \frac{(n_e - n_I)}{n_e} \left[T_\mathrm{e} \nabla n_e + (n_I \mathbf{u}_I - \mathbf{J}) \wedge \mathbf{B} \right].$$

On the left hand side one recognizes the classical momentum equation which appears in the MHD modeling with the magnetic pressure tensor. Let us stress that with the electric field $\mathbf{E}^0$ introduced in (5) and according to (1)-a), the r.h.s. term reads as follows

$$\varepsilon_0 \left[\nabla \cdot \mathbf{E}^0 + \nabla \cdot \left(\frac{n_I \mathbf{u}_I - \mathbf{J}}{n_e} \wedge \mathbf{B} \right) \right] \mathbf{E}^0 = \varepsilon_0 \nabla \cdot \left(\frac{\overline{\overline{1}}}{2} |\mathbf{E}^0|^2 - \mathbf{E}^0 \otimes \mathbf{E}^0 \right) + \lambda^2 \mathbf{E}^0 \nabla \cdot \left(\frac{n_I \mathbf{u}_I - \mathbf{J}}{n_e T_\mathrm{e}} \wedge \mathbf{B} \right).$$

Thus we may see that this equation is in a conservative form only when the Debye length vanishes. This is due to the asymptotic expansion motivating (1); we have chosen to keep energy dissipation and smoothness for n_e to the expense of momentum balance. In order to ensure exact conservation momentum, a possible route is to use a modified Poisson equation as follows

$$-\lambda^2 \Delta \ln n_e = n_I - n_e - \lambda^2 \nabla \cdot \left(\frac{\mathbf{J} - n_I \mathbf{u}_I}{n_e T_\mathrm{e}} \wedge \mathbf{B} \right)$$

which corresponds to an approximation at the same order but which looses nice properties on n_e.

Remark 2 (Electron momentum balance). We see that $\mathbf{E}^0$ is equal to the electric field $\mathbf{E}$ up to the resistive part $\eta \nabla \wedge \mathbf{B}$. Let us stress that $(\nabla \wedge \mathbf{B}) - n_I \mathbf{u}_I = \mathbf{J} - n_I \mathbf{u}_I$ is the electron contribution to the electric current and (4) may be interpreted as the electron momentum balance equation when using the massless electron approximation.

Remark 3 (Friction). In order to account for the friction between ions and electrons in the kinetic equation, we could add a friction term to equation (1)-c) and arrive to

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \frac{\partial}{\partial \mathbf{v}}.(\mathbf{F}f) = -\frac{\partial}{\partial \mathbf{v}}(\eta \mathbf{J} f)$$

(recall that the coefficient η is related to the collision frequency of electrons against ions). Also, for the sake of compatibility, instead of equation (1)-a), the Poisson equation would read

$$\lambda^2\big(-\Delta \ln n_e + \frac{1}{T_\mathrm{e}}\nabla(\eta \mathbf{J})\big) = n_I - n_e.$$

3 Proof of the stability theorem

We give a family of initial conditions that satisfy

$$f^\varepsilon(0,x,\mathbf{v}) = f^{in,\varepsilon}(x,\mathbf{v}) \geq 0, \qquad f^{in,\varepsilon} \text{ is bounded in } \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^3), \tag{11}$$

$$\mathbf{B}^\varepsilon(0,x) = \mathbf{B}^{in,\varepsilon}(x), \qquad \nabla_x \cdot \mathbf{B}^{in,\varepsilon} = 0. \tag{12}$$

From the Lemma 1 in Appendix A and the explanations in Appendix B we see that if $\mathcal{E}_I(0)$ is uniformly bounded then the free energy at initial time $\displaystyle\int_\Omega \left|\nabla_x \left(\ln n_e^\varepsilon(0,x)\right)\right|^2 dx + \int_\Omega n_e^\varepsilon(0,x)\ln n_e^\varepsilon(0,x)dx$ is also uniformly bounded. Then we only need to assume that

$$\sup_{0<\varepsilon\leq 1} \left[\mathcal{E}_I(0) + \mathcal{E}_m(0)\right] < \infty \tag{13}$$

to have a total energy uniformly bounded for all time. Moreover we assume a control on resistivity as

$$0 < \eta_{\min} \leq \eta \in \mathrm{L}^\infty(\Omega). \tag{14}$$

Associated with such initial datum, we consider a sequence of strong solutions f^ε, $\mathbf{B}^\varepsilon$, and n_e^ε of (1) with boundary conditions (6).

3.1 A priori bounds

Before we begin the proof of Theorem 1, we recall several general a priori estimates that are used. The energy bound (13) is of course at the heart of our analysis because it allows us to deduce directly from the estimate (9) the

Proposition 2 (Bounds derived from energy)**.** *Under the assumptions* (11)–(14)*, the sequences* f^ε*,* $\mathbf{B}^\varepsilon$*, and* n_e^ε *satisfy*

$$\sup_{0<\varepsilon\leq 1}\sup_{t\in[0,\infty]}\int_{\mathbb{R}^3}\int_\Omega f_i^\varepsilon(t,x,\mathbf{v})|\mathbf{v}|^2 d\mathbf{v}dx < \infty, \tag{15}$$

$$\sup_{0<\varepsilon\leq 1}\sup_{t\in[0,\infty]}\int_\Omega |\mathbf{B}^\varepsilon(t,x)|^2 dx < \infty, \tag{16}$$

$$\sup_{0<\varepsilon\leq 1}\sup_{t\in[0,\infty]}\int_\Omega \left(\left|\nabla_x\left(\ln|n_e^\varepsilon|\right)\right|^2 + n_e^\varepsilon \ln|n_e^\varepsilon| + n_e^\varepsilon\right) dx < \infty. \tag{17}$$

Indeed, for (17), we use the fact that the map $x \in \mathbb{R} \mapsto x\ln(x) - x$ is bound from below and thus all the terms in (9) are bounded by the initial energy plus a constant.
This provides a control on the kinetic energy for the ion distribution that is essential to obtain further a priori bounds

Proposition 3 (Estimates for the kinetic densities)**.** *For strong solutions one has* $f^\varepsilon \in \mathrm{L}^\infty\big(0,\infty;\mathrm{L}^1\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)\big)$ *and*

$$\|f^\varepsilon(t,\cdot,\cdot)\|_p = \|f^{in,\varepsilon}(\cdot,\cdot)\|_p \quad \forall t \geq 0,\ \forall p \in [1,\infty],$$

$$\|n_I^\varepsilon(t,\cdot)\|_{5/3} \leq C\|f^{in,\varepsilon}(\cdot,\cdot)\|_\infty^{2/5}\left(\int_\Omega\int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^2 d\mathbf{v}dx\right)^{3/5},$$

$$\|n_I^\varepsilon \mathbf{u}_I^\varepsilon(t,\cdot)\|_{5/4} \leq C\|f^{in,\varepsilon}(\cdot,\cdot)\|_\infty^{1/5}\left(\int_\Omega\int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^2 d\mathbf{v}dx\right)^{4/5}. \tag{18}$$

The bounds of f^ε follow immediately from the observation that $\mathrm{div}_{\mathbf{v}} F = 0$. The bounds on n_I^ε and $n_I^\varepsilon \mathbf{u}_I^\varepsilon$ follow from interpolation inequalities that we recall in the Appendix A.

Next we use the $L^{5/3}$ integrability of n_I to obtain

Proposition 4 (Estimates on the electron density)**.** *The electron density satisfies for some constants $K_+ > 0$, $K_- > 0$ (depending on the bounds stated as now)*

$$0 < K_- \leq n_e(t,x) \leq K_+. \tag{19}$$

This is an easy consequence of elliptic regularity and we give a proof in the Appendix B, it is just a combination of Lemma 1 and Lemma 2.

We now come to the magnetic field. We have the following Proposition :

Proposition 5 (Estimate on the magnetic field)**.** *The family of magnetic fields satisfies for a uniform constant C*

$$\sup_{0<\varepsilon\leq 1} \int_0^\infty \int_\Omega \left| \nabla_x \wedge \mathbf{B}^\varepsilon \right|^2 dxdt \leq C,$$

$$\sup_{0<\varepsilon\leq 1} \left\| \mathbf{B}^\varepsilon \right\|_{L^2\left(0,\infty;L^6(\Omega)\right)} \leq C. \tag{20}$$

Proof. The control in $\mathrm{L}^2(0,\infty;\mathrm{H}_{\mathrm{curl}}(\Omega))$, follows from the energy dissipation in Proposition 1. Then thanks to Theorem 2, the family $\mathbf{B}^\varepsilon$ is bounded in $\mathrm{L}^2(0,\infty;\mathrm{H}^1(\Omega)^3)$, and then, thanks to the Sobolev injections, the family $\mathbf{B}^\varepsilon$ is bounded in $L^2\left(0,\infty;\mathrm{L}^6(\Omega)^3\right)$. We refer to Appendix C for precise statements and references. □

Our last task is to show that we have enough bounds to define the electric and force fields. The four terms that compose the electric field

$$\mathbf{E}^\varepsilon = -T_\mathrm{e} \nabla \ln n_e^\varepsilon - \frac{n_I^\varepsilon \mathbf{u}_I^\varepsilon}{n_e^\varepsilon} \wedge \mathbf{B}^\varepsilon + \frac{\mathbf{J}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon} + \eta \nabla \wedge \mathbf{B}^\varepsilon$$

are respectively uniformly bounded in the spaces

- $T_\mathrm{e} \nabla \ln n_e^\varepsilon \in L_t^\infty(L_x^2)$ (energy inequality),
- $\frac{n_I^\varepsilon \mathbf{u}_I^\varepsilon}{n_e^\varepsilon} \wedge \mathbf{B}^\varepsilon \in L_t^2(\mathrm{L}_x^{30/29})$; this follows from (18), (19) and (20) and the fact that, according to Hölder inequality, we get the bound $\displaystyle\int_0^T [\|AB\|_{30/29}]^2 dt \leq \int_0^T [\|A\|_{5/4} \|B\|_6]^2 dt \leq \sup_t \|A\|_{5/4}^2 \int_0^T [\|B\|_6]^2 dt$.
- $\frac{\mathbf{J}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon} \in \mathrm{L}_t^1(L_x^{3/2}) \cap \mathrm{L}_t^2(L_x^1)$ from the Hölder inequality with (19) and the a priori estimates in Proposition 5. By interpolation, we also find that

  $$\frac{\mathbf{J}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon} \in \mathrm{L}_t^r(L_x^p), \qquad 1 \leq r \leq 2, \quad \frac{1}{p} = \frac{4}{3} - \frac{2}{3r}.$$

  In particular $\frac{\mathbf{J}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon} \in \mathrm{L}_t^{20/11}(L_x^{30/29})$.
- $\eta \nabla \wedge \mathbf{B}^\varepsilon \in L_{t,x}^2$ (energy dissipation).

Therefore it is well defined and the four terms will have a weak limit in Lebesgue spaces.

For the kinetic force field,

$$\mathbf{F}^\varepsilon(t,x,v) := -T_\mathrm{e} \nabla \ln n_e^\varepsilon + \left(\mathbf{v} - \frac{n_I^\varepsilon}{n_e^\varepsilon} \mathbf{u}_I^\varepsilon \right) \wedge \mathbf{B}^\varepsilon + \frac{\mathbf{J}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon},$$

the same integrability, locally uniformly in $\mathbf{v}$, holds true because these are the same terms and thus they are bounded as for the electric field at the exception of $\mathbf{v} \wedge \mathbf{B}^\varepsilon$ which is well defined thanks to (16).

3.2 Space-time compactness

The bounds mentioned before allow us to extract subsequences that converge weakly and our task is now to prove several strong convergence results.

We first observe that, from Proposition 5 and (16), we may extract a subsequence, still denoted by $\mathbf{B}^\varepsilon$ such that

$$\mathbf{B}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{B} \quad \text{strongly in } L^p(0,T;\mathrm{L}^q(\Omega)^3),\ 1\le p<2,\ 1\le q<6, \text{ and } 1\le p<\infty,\ 1\le q<2. \tag{21}$$

This is because (see Appendix C) thanks to our a priori estimates and Theorem 2, the field $\{\mathbf{B}^\varepsilon\}$ is bounded in $\mathrm{L}^1(0,T,\mathrm{H}^1(\Omega)^3)$, and then, thanks to Rellich Theorem, $\{\mathbf{B}^\varepsilon(t,\cdot)\}$ is relatively compact in $\mathrm{L}^6(\Omega)^3$ for all $t\in]0,T[$. Compactness in time then follows from the Simon-Lions-Aubin lemma (see [28]) because, as proved at the end of section (3.1), we control $L^r_t(L^p_x)$ integrability of the electric field and

$$\frac{\partial \mathbf{B}^\varepsilon}{\partial t} = \nabla \wedge \mathbf{E}^\varepsilon.$$

According to the kinetic averaging lemma recalled in Appendix D, we know that after extraction of a subsequence, for every test function $\psi=\psi(\mathbf{v})$ compactly supported, there exists a weak limit f^* of f^ε such that

$$\langle \psi f^\varepsilon\rangle \to \langle \psi f^*\rangle \quad \text{a.e.} \quad \text{and} \quad \langle \psi \mathbf{v} f^\varepsilon\rangle \to \langle \psi \mathbf{v} f^*\rangle \quad \text{a.e.}$$

Now, since $\displaystyle\int_0^T\int_\Omega |\,\langle \psi f^\varepsilon\rangle\,|^{5/3}dxdt$ and $\displaystyle\int_0^T\int_\Omega |\,\langle \psi \mathbf{v} f^\varepsilon\rangle\,|^{5/4}dxdt$ are uniformly bounded, according to classical interpolation arguments we have for given exponents q and r (with $1<q<5/3$ and $1<r<5/4$)

$$\int_0^T\int_\Omega |\,\langle \psi f^\varepsilon\rangle - \langle \psi f^*\rangle\,|^q dxdt \to 0, \quad \text{and} \quad \int_0^T\int_\Omega |\,\langle \psi \mathbf{v} f^\varepsilon\rangle - \langle \psi \mathbf{v} f^*\rangle\,|^r dxdt \to 0.$$

Therefore we get

$$n_I^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n_I, \text{ strongly in } \mathrm{L}^q(0,T;\Omega), \tag{22}$$

$$n_I \mathbf{u}_I^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n_I \mathbf{u}_I \quad \text{strongly in } \mathrm{L}^r(0,T;\Omega)^3. \tag{23}$$

Next for the electronic density we may extract a subsequence, still denoted n_e^ε such that,

$$n_e^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n_e \quad \text{a.e. and strongly in } \mathrm{L}^q(0,T;\Omega)^3. \tag{24}$$

This is a consequence of the bounds (17) and (19) and the strong convergence (22).

3.3 Passing to the limit

We recall that a weak (distributional) solution to (1) is defined by testing against smooth test functions $\Psi(t,x)$, $\Phi(t,x,\mathbf{v})$ respectively for (1)-b) and (1)-c) and satisfying the corresponding boundary conditions as in (6).

After integration by parts, we obtain respectively for (1)-b) and (1)-c), the definitions

$$-\iint_{(0,T)\times\Omega}\Big[\frac{\partial \Psi(t,x)}{\partial t}\mathbf{B}^\varepsilon + \mathbf{E}^\varepsilon(t,x)\cdot\nabla\wedge\Psi(t,x)\Big]dtdx = \int_\Omega \mathbf{B}^{in,\varepsilon}(x,\mathbf{v})\Psi(0,x), \tag{25}$$

and

$$\begin{aligned}-\iiint_{(0,T)\times\Omega\times\mathbb{R}^3}\Big[\frac{\partial \Phi(t,x,\mathbf{v})}{\partial t} &+\mathbf{v}\cdot\nabla_x\Phi(t,x,\mathbf{v}) + \mathbf{F}^\varepsilon(t,x,\mathbf{v})\cdot\nabla_{\mathbf{v}}\Phi(t,x,\mathbf{v})\Big]f^\varepsilon(t,x,\mathbf{v})dtdxd\mathbf{v}\\ &= \iint_{\Omega\times\mathbb{R}^3} f^{in,\varepsilon}(x,\mathbf{v})\Phi(0,x,\mathbf{v}).\end{aligned} \tag{26}$$

The elliptic equation (1)-a) is more standard and does not yield difficulties, hence we do not consider it here. In order to conclude the proof of Theorem 1, our purpose is to show that the limit as $\varepsilon\to 0$ of the various unknowns $\mathbf{B}^\varepsilon$, $\mathbf{E}^\varepsilon(t,x)$, $\mathbf{F}^\varepsilon$ and f^ε still satisfy these equalities.

Additionally to the strong convergence results stated in (21)–(24), we are know other weak convergences for functions of interest. Firstly, we have

$$\mathbf{J}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{J} \quad \text{in } \mathrm{L}^2((0,T)\times\Omega)^3.$$

This is enough to pass to the limit weakly in the electric fields because the nonlinear terms are always formed of either strongly convergent terms or $\mathbf{J}^\varepsilon$ multiplied by a term that converges strongly. As a conclusion, we have

$$\mathbf{E}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{E} \quad \text{in } \mathrm{L}^{20/11}\big(0,T;\mathrm{L}^{30/29}(\Omega)\big)^3.$$

The same applies to the force field. However for later purposes, it is better to use the splitting

$$\mathbf{F}^\varepsilon = \mathbf{E}^{0,\varepsilon} + \mathbf{v}\wedge\mathbf{B}^\varepsilon,$$

and to notice that, as before,

$$\mathbf{E}^{0,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{E}^0 \quad \text{in } \mathrm{L}^{20/11}\big(0,T;\mathrm{L}^{30/29}(\Omega)\big)^3$$

while the other term converges strongly.

These observations allow us to pass to the weak limit in equations (1)-a) and (1)-b), on the electronic density and on the magnetic field.

For passing to the weak limit in the third equation (1)-c), let us introduce two smooth test functions χ and φ with compact support respectively in $[0,T)\times\Omega$ and in $\mathbf{R}^3$; then we have to deal with two terms. The first one which reads as

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3} \mathbf{B}^{0,\varepsilon}\wedge\mathbf{v}\cdot\frac{\partial\varphi}{\partial\mathbf{v}}(\mathbf{v})\chi(t,x)f^\varepsilon(t,x,\mathbf{v})dtdxd\mathbf{v},$$

can be treated as usual because the magnetic field converges strongly as stated earlier. The second one reads as follows

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3} \mathbf{E}^{0,\varepsilon}\cdot\frac{\partial\varphi}{\partial\mathbf{v}}(\mathbf{v})f^\varepsilon(t,x)\chi(t,x)dtdxd\mathbf{v} = \int_0^T\int_\Omega \chi(t,x)\mathbf{E}^{0,\varepsilon}\cdot\langle\psi f^\varepsilon\rangle(t,x)dtdx$$

with $\psi_i = \frac{\partial\varphi}{\partial\mathbf{v}_i}$; we need to use the kinetic averaging lemma which is stated in appendix D. It states that, after extraction, velocity averages converge strongly

$$\chi\langle\psi_i f^\varepsilon\rangle(t,x) \to \chi\langle\psi_i f^*\rangle \quad \text{in} \quad L^q([0,T]\times\Omega)$$

for a given value of the index q, with $1\le q<p$. Therefore they converge almost everywhere and this is enough to pass to the weak-strong limit in the term $\int_0^T\int_\Omega \chi\mathbf{E}^{0,\varepsilon}\cdot\langle\psi f^\varepsilon\rangle\,dtdx$.

This concludes the proof of the Theorem 1.

4 Non homogeneous magnetic boundary condition

In view of applications to the physics of confined plasmas, it is worthwhile considering, instead of (6)-(b), a non homogeneous boundary condition for the magnetic field

$$\mathbf{n}_x\wedge\mathbf{B}(t,x) = \mathbf{n}_x\wedge\mathbf{B}_{\mathrm{imp}}, \quad x\in\partial\Omega, \tag{27}$$

The imposed magnetic field $B_{\mathrm{imp}}(x)$ is given and can be a function of time as well eventhough we restrict our analysis to space dependency only. In order to introduce this boundary condition in the problem, we assume that $\mathbf{B}_{\mathrm{imp}}$ can be defined globally as a smooth function in the closure of Ω. It turns out that a convenient regularity assumption is

$$\|\mathbf{B}_{\mathrm{imp}}\|_{1,\infty}\le C. \tag{28}$$

We will write $\mathbf{B} = \mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}$ where $\mathbf{B}_{\rm pert}$ is the perturbation. Using these notations the system (1)-(a,b,c) is rewritten as

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_I - n_e, & (a) \\ \dfrac{\partial \mathbf{B}_{\rm pert}}{\partial t} - \nabla \wedge \left(\dfrac{1}{n_e} n_I \, \mathbf{u}_I \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \right) + \nabla \wedge \left(\dfrac{1}{n_e} (\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}) \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \right) \\ \qquad + \nabla \wedge (\eta \, (\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert})) = 0, & (b) \\ \dfrac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \dfrac{\partial}{\partial \mathbf{v}} \left[\left((-\dfrac{T_{\rm e}}{n_e} \nabla n_e + \dfrac{(\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}) - n_I \mathbf{u}_I}{n_e} \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \right) + \mathbf{v} \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \right) f \right] = 0, & (c). \end{cases} \tag{29}$$

where the total current is $\mathbf{J} = \mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}$ with

$$\mathbf{J}_{\rm imp} = \nabla \wedge \mathbf{B}_{\rm imp} \text{ and } \mathbf{J}_{\rm pert} = \nabla \wedge \mathbf{B}_{\rm pert}.$$

We now use the boundary conditions

$$\begin{cases} \mathbf{n}_x \cdot \nabla n_e(t,x) = 0 & x \in \partial\Omega, \quad (a) \\ \mathbf{n}_x \wedge \mathbf{B}(t,x)_{\rm pert} = 0 & x \in \partial\Omega, \quad (b) \\ f(x, \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n}_x)\mathbf{n}_x) = f(x,\mathbf{v}), \qquad \mathbf{v} \in \mathbb{R}^3, & x \in \partial\Omega. \quad (c) \end{cases} \tag{30}$$

Let us define the perturbed magnetic energy and the total perturbed energy

$$\mathcal{E}_m^{\rm pert}(t) = \frac{1}{2} \int_\Omega |\mathbf{B}_{\rm pert}(t,x)|^2 dx,$$

$$\mathcal{E}_{\rm tot}^{\rm pert} = \mathcal{E}_I + \mathcal{E}_m^{\rm pert} + \frac{\lambda^2}{2} \int_\Omega |\nabla_x (\ln n_e)|^2 \, dx + \int_\Omega (n_e \ln n_e - n_e + 1) \, dx \geq 0.$$

The energy balance is modified by the imposed magnetic field and we have

Proposition 6. *Classical solutions to (29)–(30) satisfy the perturbed energy dissipation relation*

$$\frac{d}{dt} \mathcal{E}_{\rm tot}^{\rm pert} = - \int_\Omega \eta \Big| \mathbf{J}_{\rm pert} \Big|^2 dx + S$$

where the source is

$$S = - \int_\Omega \eta \ \mathbf{J}_{\rm imp} \cdot \mathbf{J}_{\rm pert} dx - \int_\Omega \frac{1}{n_e} \mathbf{J}_{\rm imp} \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \cdot \mathbf{J}_{\rm pert} dx + \int_\Omega \frac{1}{n_e} \mathbf{J}_{\rm imp} \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \cdot n_I \ \mathbf{u}_I dx.$$

Proof. Performing the same manipulations as in the proof of the energy identity (9), we obtain

$$\frac{d}{dt} \mathcal{E}_I = \int_\Omega \left(\frac{n_I}{n_e} (\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}) \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) - n_I T_{\rm e} \nabla (\ln n_e) \right) \cdot \mathbf{u}_I dx.$$

Taking the scalar product of the magnetic equation against $\mathbf{B}_{\rm pert}$ and using the homogeneous boundary condition (30)-(b), we obtain

$$\frac{d}{dt} \mathcal{E}_m^{\rm pert} = - \int_\Omega \frac{n_I}{n_e} \mathbf{u}_I \cdot (\mathbf{J}_{\rm pert} \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert})) \, dx - \int_\Omega \frac{1}{n_e} (\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}) \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \cdot \mathbf{J}_{\rm pert} dx$$

$$- \int_{\mathbb{R}^3} \eta \, (\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}) \cdot \mathbf{J}_{\rm pert} dx$$

$$= - \int_\Omega \frac{n_I}{n_e} \mathbf{u}_I \cdot (\mathbf{J}_{\rm pert} \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert})) \, dx - \int_\Omega \frac{1}{n_e} \mathbf{J}_{\rm imp} + \wedge (\mathbf{B}_{\rm imp} + \mathbf{B}_{\rm pert}) \cdot \mathbf{J}_{\rm pert} dx - \int_{\mathbb{R}^3} \eta \, (\mathbf{J}_{\rm imp} + \mathbf{J}_{\rm pert}) \cdot \mathbf{J}_{\rm pert} dx$$

Therefore one gets

$$\frac{d}{dt} \left(\mathcal{E}_I + \mathcal{E}_{\rm tot}^{\rm pert} \right) = - \int_\Omega T_{\rm e} \nabla (\ln n_e) \cdot (n_I \mathbf{u}_I) dx - \int_\Omega \eta \Big| \mathbf{J}_{\rm pert} \Big|^2 dx + S$$

which is very similar to (10). The rest of the proof is unchanged. □

Proposition 7. *There exists a constant $K > 0$ depending only on the initial data and on the constant in (28) such that the perturbed energy is bounded for all time $t < T^\star = \log\left(1 + \frac{C}{\|\mathbf{J}_{\rm imp}\|_\infty}\right)$.*

Remark 4. Our analysis of weak stability can easily be extended to this non-homogeneous boundary condition for $t < T^\star$. This estimate expresses the interest of a good control on the imposed current. Indeed the smaller is $\|\mathbf{J}_{\rm imp}\|_\infty$, the greater $T^\star$ is.

Proof. (of proposition 7). We observe that the new terms in S are proportional to $\mathbf{J}_{\rm imp}$. Therefore two cases occur.

First case: $\mathbf{J}_{\rm imp} = 0$. This idealized case might be encountered in Tokamaks: for example if $\mathbf{B}_{\rm imp} = F\nabla\theta$ where θ is the toroidal angle and F is a constant. Physically it corresponds to an imposed exterior magnetic with vanishing current [10]. The magnetic lines of $\mathbf{B}_{\rm imp}$ form a ring. In this case $S = 0$, so the perturbed energy dissipation relation has the same form as (9). It turns out that the perturbed energy is bounded for all times, which indeed corresponds to the claim since $\mathbf{J}_{\rm imp} = 0$.

Second case: $\mathbf{J}_{\rm imp} \neq 0$. This situation is more realistic in Tokamaks, since the magnetic lines form an helix or at least must be close to helicoidal geometry [10]. This is the general case.

Let $\sigma > 0$ be a given positive number. The energy identity writes also

$$\frac{d}{dt}\left(e^{-\sigma t}\mathcal{E}_{\rm tot}^{\rm pert}\right) = Re^{-\sigma t}, \quad R = -\sigma\mathcal{E}_{\rm tot}^{\rm pert} - \int_\Omega \eta\left|\mathbf{J}_{\rm pert}\right|^2 dx + S. \tag{31}$$

Our goal is to control the source term S in R as much as possible by $-\sigma\mathcal{E}_{\rm tot}^{\rm pert} - \eta\|\mathbf{J}_{\rm pert}\|_2$ and to control the remaining part with a Gronwall technique.

Using the assumption (28), the fact that η is constant and the Hölder inequality between the conjugated spaces $L^5(\Omega)$ and $L^{\frac{5}{4}}(\Omega)$, we can write

$$|S| \leq \left(\alpha_1\|\mathbf{J}_{\rm pert}\|_2 + \alpha_2\|\mathbf{J}_{\rm pert}\|_2\|\mathbf{B}_{\rm pert}\|_2 + \alpha_2\|\mathbf{B}_{\rm imp}\|_5\|n_I\ \mathbf{u}_I\|_{\frac{5}{4}} + \alpha_2\|\mathbf{B}_{\rm pert}\|_5\|n_I\ \mathbf{u}_I\|_{\frac{5}{4}}\right)\|\mathbf{J}_{\rm imp}\|_\infty$$

with $\alpha_1 = \eta$ and $\alpha_2 = \|n_e^{-1}\|_\infty$. Let $\epsilon > 0$ be an arbitrary real number. Then

$$\alpha_1\|\mathbf{J}_{\rm pert}\|_2 + \alpha_2\|\mathbf{J}_{\rm pert}\|_2\|\mathbf{B}_{\rm pert}\|_2 \leq \frac{\alpha_1}{2\epsilon} + \epsilon\frac{\alpha_1+\alpha_2}{2}\|\mathbf{J}_{\rm pert}\|_2^2 + \frac{\alpha_2}{2\epsilon}\|\mathbf{B}_{\rm pert}\|_2^2.$$

The identity (34), written as $\|n_I\ \mathbf{u}_I\|_{\frac{5}{4}} \leq \alpha_4\mathcal{E}_I^{\frac{4}{5}}$, implies successively the controls

$$\|\mathbf{B}_{\rm imp}\|_5\|n_I\ \mathbf{u}_I\|_{\frac{5}{4}} \leq \alpha_3\mathcal{E}_I^{\frac{4}{5}}$$

$$\|\mathbf{B}_{\rm pert}\|_5\|n_I\ \mathbf{u}_I\|_{\frac{5}{4}} \leq \alpha_5\left(\|\mathbf{B}_{\rm pert}\|_2 + \|\mathbf{J}_{\rm pert}\|_2\right)\mathcal{E}_I^{\frac{4}{5}} \leq \alpha_6\left(\mathcal{E}_m^{\rm pert}\right)^{\frac{1}{2}}\mathcal{E}_I^{\frac{4}{5}} + \epsilon\frac{\alpha_5}{2}\|\mathbf{J}_{\rm pert}\|_2^2 + \frac{\alpha_5}{2\epsilon}\mathcal{E}_I^{\frac{8}{10}}.$$

So the right hand side in (31) is bounded by

$$R \leq \beta_1 + \beta_2\mathcal{E}_I^{\frac{4}{5}} + \beta_3\left(\mathcal{E}_m^{\rm pert}\right)^{\frac{1}{2}}\mathcal{E}_I^{\frac{4}{5}}$$

for some constants $\beta_{1,2,3}$. Since the electronic density is bounded and $0 \leq \mathcal{E}_I + \mathcal{E}_m^{\rm pert} \leq \mathcal{E}_{\rm tot}^{\rm pert}$ by construction, we also have

$$R \leq \gamma_1 + \gamma_2\left(\mathcal{E}_{\rm tot}^{\rm pert}\right)^{\frac{4}{5}} + \gamma_3\left(\mathcal{E}_{\rm tot}^{\rm pert}\right)^{\frac{13}{10}}$$

for some constants $\gamma_{1,2,3}$ which do not depend on time. Since $y^{\frac{4}{5}} \leq 1 + y^{\frac{13}{10}}$ for positive y, we get the more compact form

$$R \leq \delta_1 + \delta_2\left(\mathcal{E}_{\rm tot}^{\rm pert}\right)^{\frac{13}{10}}$$

for some constants $\delta_{1,2}$ which do not depend on time.

We see that the right hand side is more than linear with respect to $\mathcal{E}_{\rm tot}^{\rm pert}$ due to the power $\frac{13}{10}$. As a consequence this inequality cannot prove that $e^{-\sigma t}\mathcal{E}_{\rm tot}^{\rm pert}$ or $\mathcal{E}_{\rm tot}^{\rm pert}$ is bounded for all time.

Next we wish to obtain a evaluation of the time of existence with respect to $\|\mathbf{J}_{\rm imp}\|_\infty$. We set $u(t) = e^{-\sigma t}\mathcal{E}_{\rm tot}^{\rm pert}$. One can simplify the inequality as

$$u'(t) \leq \left(\delta_1 + \delta_2 u^{\frac{13}{10}}\right)\|\mathbf{J}_{\rm imp}\|_\infty e^{\frac{3\sigma}{10}t}.$$

Rescaling of the time variable as $d\tau = \|\mathbf{J}_{\rm imp}\|_\infty e^{\frac{3\sigma}{10}t}dt$, that is

$$\tau = \frac{10}{3}\|\mathbf{J}_{\rm imp}\|_\infty \left(e^{\frac{3\sigma}{10}t} - 1\right),$$

yields the inequality $\frac{d}{d\tau}u \leq \delta_1 + \delta_2 u^{\frac{13}{10}}$. It is finally convenient to define $v = \delta_1 + \delta_2 u^{\frac{13}{10}}$ so that

$$v'(t) = \delta_2 \frac{13}{10} u^{\frac{3}{10}} u'(t) \text{ which yields } v'(t) \leq \delta_3 v^{\frac{3}{13}} v = \delta_3 v^{\frac{16}{13}}.$$

Therefore $-\frac{d}{d\tau}v^{-\frac{3}{13}} \leq \delta_4 = \frac{3}{13}\delta_3$. which implies $v^{-\frac{3}{13}}(0) - v^{-\frac{3}{13}}(\tau) \leq \delta_4 \tau$. It yields

$$v^{\frac{3}{13}}(\tau) \leq \frac{1}{v^{\frac{3}{13}}(0) - \delta_4 \tau}$$

which is valid for $\tau < \tau^\star = \frac{v^{\frac{3}{13}}(0)}{\delta_4}$. Going back to the time variable t, the solution is defined for $t < T^\star$ where

$$\frac{10}{3}\|\mathbf{J}_{\rm imp}\|_\infty \left(e^{\frac{3\sigma}{10}T^\star} - 1\right) = \tau^\star.$$

The proof is complete.

□

5 Construction of an approximate solution

In order to complete our theory, we now detail how to use the so-called splitting strategy, which is a constructive method, for the design of an approximate solution to the system

$$\left\{\begin{array}{ll} -\lambda^2 \Delta \ln n_e = n_I - n_e, & (a) \\ \dfrac{\partial \mathbf{B}}{\partial t} - \nabla \wedge \left(\frac{1}{n_e} n_I \ \mathbf{u}_I \wedge \mathbf{B}\right) + \nabla \wedge (\frac{1}{n_e}\mathbf{J} \wedge \mathbf{B}) + \nabla \wedge (\eta \nabla \wedge \mathbf{B}) = 0, & (b) \\ \dfrac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \dfrac{\partial}{\partial \mathbf{v}}\left[\left((-\dfrac{T_{\rm e}}{n_e}\nabla n_e + \dfrac{\mathbf{J} - n_{_I}\mathbf{u}_I}{n_e} \wedge \ \mathbf{B}) + \mathbf{v} \wedge \mathbf{B}\right) f\right] = 0, & (c). \end{array}\right. \tag{32}$$

The idea is clearly inspired from numerical methods. It consists of a convenient splitting strategy *à la* Strang, together with the linearization and freezing of certain coefficients *à la* Temam. The main point is to decompose the total system in simpler parts which are conceptually easier to solve or easier to analyse, preserving at the same time the decay of the energy identity

$$\mathcal{E}_{tot} = \frac{1}{2}\int_\Omega \int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^2 d\mathbf{v}dx + \frac{1}{2}\int_\Omega |\mathbf{B}(t,x)|^2 dx + \frac{\lambda^2}{2}\int_\Omega |\nabla_x (\ln n_e)|^2 dx + \int_\Omega (n_e \ln n_e - n_e + 1)\, dx.$$

Let $\Delta t > 0$ be a time step which is ultimately destinated to tend to zero. We consider that

$$f(t_k) \text{ and } \mathbf{B}(t_k)$$

are known at the beginning of the time step $t_k = k\Delta t$. We restrict the presentation to the core of the method. This constructive method also provides additional insights into the mathematical structure of the model.

5.1 Vlasov-Poisson

One first solves during the time step Δt

$$\left\{\begin{array}{ll} -\lambda^2 \Delta \ln n_e = n_I - n_e, & (a) \\ \dfrac{\partial \mathbf{B}}{\partial t} = 0, & (b) \\ \dfrac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \dfrac{\partial}{\partial \mathbf{v}}\left[-\dfrac{T_{\rm e}}{n_e}\nabla n_e f\right] = 0. & (c) \end{array}\right.$$

This is a non linear Vlasov-Poisson equation which can be considered as standard even if we know very little mathematical literature about it. It is easy to show that regular solutions preserve the energy. This procedure defines a new solution

$$f^\star(t_k + \Delta t) \text{ and } \mathbf{B}^\star(t_k + \Delta t) = \mathbf{B}(t_k).$$

5.2 Magnetic part, first stage

For convenience we split the magnetic part of the equations in two stages, the first one which is fundamental, and the second which deals with less involved terms. The first stage writes

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} - \nabla\wedge\left(\frac{1}{n_e} n_I\ \mathbf{u}_I \wedge \mathbf{B}_{\text{frozen}}\right) + \nabla \wedge (\frac{1}{n_e}\mathbf{J} \wedge \mathbf{B}_{\text{frozen}}) + \nabla \wedge (\eta \nabla \wedge \mathbf{B}) = 0, & (b) \\ \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \mathbf{v}}\left[\left(\dfrac{\mathbf{J}}{n_e} \wedge\ \mathbf{B}_{\text{frozen}})\right) f\right] = 0, & (c). \end{cases}$$

The initial data is provided by the previous step of the algorithm

$$f^{\square}(t_k) = f^{\star}(t_k + \Delta t) \text{ et } \mathbf{B}^{\square}(t_k) = \mathbf{B}^{\star}(t_k + \Delta t).$$

The frozen magnetic field is

$$\mathbf{B}_{\text{frozen}} = \mathbf{B}^{\star}(t_k + \Delta t).$$

This frozen field is constant in time during the whole time step. This trick was first introduced by Temam in the context of trilinear forms and Navier-Stokes equations for magnetic equations [26, 27], see also [15].

One notices that equation b) is now a linear one, even if equation c) is still formerly non linear because it has a n_e dependence. However an explicit procedure allows to compute the solution. Indeed solutions of equation c) are such that

$$\partial_t n_I = 0.$$

It means that n_I and n_e are frozen quantities

$$n_I = {n_I}^{\text{frozen}} \text{ et } n_e = {n_e}^{\text{frozen}}.$$

One has

$$\partial_t n_I\ \mathbf{u}_I = \mathbf{J} \wedge \mathbf{d}, \quad \mathbf{d} = \frac{{n_I}^{\text{frozen}}\ \mathbf{B}_{\text{frozen}}}{{n_e}^{\text{frozen}}}$$

which yields

$$n_I\ \mathbf{u}_I = n_I\ \mathbf{u}_I(t_k) + \int_{t_k}^{t} \nabla \wedge \mathbf{B}(s) ds \wedge \mathbf{d}.$$

It shows that $n_I\ \mathbf{u}_I$ integro-differential and linear with respect to $\mathbf{B}$. Plugging this form of $n_I\ \mathbf{u}_I$ in equation b), we end up with a linear equation for $\mathbf{B}$. This linear integro-differential equation is well posed under general assumptions.

Once the magnetic field is computed, we can report the current $\mathbf{J}$ in equation c) which is now easily solved with the method of characteristics. It is immediate that, due to the resistive operator, the energy decreases during this step. Since n_i is constant, the electronic energy is constant.

The solution at the end of this stage is referred to as

$$f^{\square}(t_k + \Delta t) \text{ and } \mathbf{B}^{\square}(t_k + \Delta t) = \mathbf{B}(t_k).$$

5.3 Magnetic part, second stage

It remains to solve

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} = 0, & (b) \\ \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \mathbf{v}}\left[\left(\dfrac{-n_I \mathbf{u}_I}{n_e} \wedge\ \mathbf{B}) + \mathbf{v} \wedge \mathbf{B}\right) f\right] = 0, & (c), \end{cases}$$

with prescribed initial data

$$f^{\bullet}(t_k) = f^{\square}(t_k + \Delta t) \text{ and } \mathbf{B}^{\bullet}(t_k) = \mathbf{B}^{\square}(t_k + \Delta t).$$

Since the magnetic field is frozen

$$\mathbf{B} = \mathbf{B}_{\text{frozen}},$$

equation c) is greatly simplified. Once again the ionic density n_I and the electronic density n_e are frozen

$$n_I = n_I{}^{\text{frozen}} \text{ and } n_e = n_e{}^{\text{frozen}}.$$

A consequence is

$$\partial_t n_I \mathbf{u}_I = n_{_I} \mathbf{u}_I \wedge \mathbf{d}, \quad \mathbf{d} = \left(-\frac{n_I{}^{\text{frozen}}}{n_e{}^{\text{frozen}}} + 1 \right) \mathbf{B}_{\text{frozen}}.$$

The solution of this linear equation is immediate. Therefore $n_I \mathbf{u}_I$ is known. And finally the method of characteristics can be used to solve c). The energy is preserved during this second magnetic stage.

5.4 Iterations

The previous procedure allows us to design an approximate solution

$$f(t_k + \Delta t) = f^\bullet(t_k + \Delta t) \text{ and } \mathbf{B}(t_k + \Delta t) = \mathbf{B}^\bullet(t_k + \Delta t)$$

one time step after the other. With this procedure the total energy decreases and we can apply our stability analysis for proving existence.

A Control on moments of f_I

Several type of controls on velocity moments of f_I are available, see [23]. Here we recall one of the most fundamental control in L^p spaces based on the kinetic energy.

Lemma 1. *Let $f \in L^\infty_{t,x,\mathbf{v}}((0,T) \times \Omega \times \mathbb{R}^3) \cap L^\infty_t(0,T; L^1_{x,\mathbf{v}}(\Omega \times \mathbb{R}^3, |\mathbf{v}|^2 dx d\mathbf{v})$. Define n_I by (2) and $n_I \mathbf{u}_i$ by (3). Then $n_I \in L^\infty_t\big(0,T; L^{5/3}_x(\Omega)\big)$, $n_I \mathbf{u}_I \in L^\infty_t\big(0,T; L^{5/4}_x(\Omega)\big)$ and we have for all $t \in [0,T]$,*

$$\|n_I(t,\cdot)\|_{5/3} \leq C \|f(t,\cdot,\cdot)\|_\infty{}^{2/5} \left(\int_\Omega \int_{\mathbb{R}^3} f(t,x,\mathbf{v}) |\mathbf{v}|^2 d\mathbf{v} dx \right)^{3/5}, \tag{33}$$

$$\|n_I(t,\cdot)\mathbf{u}_I\|_{5/4} \leq C' \|f(t,\cdot,\cdot)\|_\infty{}^{1/5} \left(\int_\Omega \int_{\mathbb{R}^3} f(t,x,\mathbf{v}) |\mathbf{v}|^2 d\mathbf{v} dx \right)^{4/5}. \tag{34}$$

Proof. We only recall the proof of the result on n_I. Let $R > 0$. We have

$$\begin{aligned} n_I(t,x) &= \int_{|\mathbf{v}| \leq R} f(t,x,\mathbf{v}) d\mathbf{v} + \int_{|\mathbf{v}| \geq R} f(t,x,\mathbf{v}) d\mathbf{v} \\ &\leq C R^3 \|f(t,\cdot,\cdot)\|_\infty + \frac{1}{R^2} \int_{|\mathbf{v}| \leq R} f(t,x,\mathbf{v}) |\mathbf{v}|^2 d\mathbf{v}. \end{aligned}$$

Then by minimization over R we get

$$n_I(t,x) \leq C \, \|f(t,\cdot,\cdot)\|_\infty^{2/5} \left(\int_{\mathbb{R}^3} f(t,x,\mathbf{v}) |\mathbf{v}|^2 d\mathbf{v} \right)^{3/5}$$

and after integration we obtain the claim. □

B Uniform lower bound on n_e

The purpose of this section is to prove several properties that we have used throughout the paper for the elliptic equation

$$\begin{cases} -\lambda^2 \Delta \ln n_e + n_e = n_I, & x \in \Omega, \\ \dfrac{\partial n_e}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

with a right hand side data satisfying $n_I \geq 0$, $\int n_I = M_{initial} > 0$ and $n_I \in L^{5/3}$.

We are going to prove the estimate

Lemma 2. *Let $n_I \in L^\infty(0,T;L^{5/3}\cap L^1(\Omega))$ and n_e a strong solution to the above equation, then we have the two-sided control*

$$0 < K_-(\|n_I\|_{5/3}) \le n_e \le K_+(\|n_I\|_{5/3}), \tag{35}$$

for some continuous positive functions $K_\pm(\cdot)$ with $K_+ > 1$ increasing, K_- decreasing.

We also recall that integration of the equation gives the electric neutrality relation

$$\|n_e\|_1 = \|n_I\|_1.$$

We can now explain why the total energy (8) is well defined for weak solutions and also at initial time. From the lower and upper bound in (35), we conclude that $n_e \ln n_e \in L^1(\Omega)$. Also, multiplying the equation by $\ln n_e$, we find

$$\lambda \int_\Omega |\nabla_x \ln n_e|^2 \, dx = \int_\Omega (n_I - n_e) \ln n_e \le 2 \ln K_+ \|n_I\|_1.$$

Proof. Then, we argue in two steps. Firstly, we multiply the equation by $n_e^{2/3}$ and integrate by parts. The Hölder inequality gives

$$\frac{2\lambda^2}{3} \int \frac{|\nabla n_e|^2}{n_e^{1/3}} + \int n_e^{5/3} = \int n_I n_e^{2/3} \le \|n_I\|_{5/3} \, \|n_e\|_{5/3}^{2/3},$$

from which we conclude the bound

$$\|n_e\|_{5/3} \le \|n_I\|_{5/3}.$$

Secondly we use the elliptic regularity theory to conclude that $\ln n_e - \langle \ln n_e \rangle_\Omega \in W^{2,p}$, $1 \le p \le 5/3$ and thus

$$\ln n_e - \langle \ln n_e \rangle_\Omega \in L^q, \qquad \forall q > 1, \qquad \frac{1}{q} = \frac{1}{p} - \frac{2}{3},$$

where $\langle \phi \rangle_\Omega$ denotes the average of the $L^1(\Omega)$ function ϕ over Ω. Finally, because $\frac{2}{3} > \frac{3}{5}$ we conclude from the Morrey estimates [13] that

$$\| \ln n_e - \langle \ln n_e \rangle_\Omega \|_\infty \le C(\|n_I\|_{5/3}).$$

The result follows immediately thanks to the control of $\langle \ln n_e \rangle_\Omega$ through (35). □

C Compactness of the magnetic field

We have also used the Sobolev injection for Maxwell equations and we recall it in this appendix. We introduce the following spaces

$$\begin{aligned} \mathrm{H}_{\mathrm{curl}}(\Omega) &= \{\mathbf{b} \in \mathrm{L}^2(\Omega)^3 / \, \nabla \wedge \mathbf{b} \in \mathrm{L}^2(\Omega)^3\}, \\ \mathrm{H}_{\mathrm{div}}(\Omega) &= \{\mathbf{b} \in \mathrm{L}^2(\Omega)^3 / \, \nabla \cdot \mathbf{b} \in \mathrm{L}^2(\Omega)^3\}, \\ X_N(\Omega) &= \{\mathbf{b} \in \mathrm{H}_{\mathrm{curl}}(\Omega) \cap \mathrm{H}_{\mathrm{div}}(\Omega) \, / \quad \mathbf{b} \wedge \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\}. \end{aligned} \tag{36}$$

We recall the following result (see [9])

Theorem 2. *Assume that the domain Ω is of class $\mathcal{C}^{1,1}$. Then the space $X_N(\Omega)$ is continuously imbedded in $\mathrm{H}^1(\Omega)^3$.*

D Kinetic averaging lemma

We recall here one result of the theory of averaging lemmas for kinetic equations. When $f(t,x,\mathbf{v})$ is solution of a kinetic equation, it cannot be more regular that the initial data or the right hand-side. However, averages in velocity gain regularity. Recall that the macroscopic quantity $\langle f\psi \rangle$ is defined as

$$\langle f\psi \rangle(t,x) = \int_{\mathbb{R}^d} f(t,x,\mathbf{v}) \psi(\mathbf{v}) d\mathbf{v}$$

where ψ is a given function in $\mathcal{C}_c^\infty(\mathbb{R}^d)$ (i.e. smooth with compact support), the averaging lemmas aim at proving compactness properties on $\langle f\psi \rangle$. The first version of these averaging Lemma has been established by Golse, Lions

Perthame and Sentis [17] for $f \in L^2_{t,x,v}$ solution of the equation $\partial_t f + \mathbf{v}\cdot\nabla_x f = S$ with $S \in L^2_{t,x,\mathbf{v}}$: it was proved that locally in time $\langle f\psi\rangle \in H^{1/2}_{t,x}$. This version has been then be extended (by complex interpolation) for the L^p framework, $1 < p < \infty$ in [12]. Moreover, more complex versions have been proved by Di Perna, Lions and Meyer [12], who treat the case where S is the k-th derivative in velocity with a fractional derivative in x strictly less than one.

We use here a version proved by Perthame and Sougadinis [24] (the equality is the exponent is due to Bouchut [5]) which expresses an optimal gain of regularity (a full derivative)

Theorem 3. *Let $1 < q < \infty$ and f, $\mathbf{g} = (g_1, ..., g_d)$ belong to $\mathrm{L}^q_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$ and satisfy*

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_x f = \frac{\partial}{\partial \mathbf{v}}.\big((\mathbb{I} - \Delta_{t,x,\mathbf{v}})^{1/2}\mathbf{g}\big). \tag{37}$$

Then $\langle f\psi\rangle \in \mathrm{L}^q_{t,x}(\mathbf{R}^{1+3})$ and there exists $C(p,\psi)$ such that

$$\|\langle f\psi\rangle\|_q \leq C(q,\psi)\|f\|_q^{1-\alpha}\|\mathbf{g}\|_q^{\alpha}$$

for a positive exponent $\alpha \leq \frac{1}{2}\min(\frac{1}{q}, 1-\frac{1}{q})$.

In fact this theorem also proves strong compactness, which is the way we use it in our context. Consider a truncation function $\chi = \chi(t,x)$ with $\mathrm{supp}\chi \subset (0,T]\times\Omega$ which is fixed in this paragraph. Define the sequence of functions $\mathbf{w}^\varepsilon$ as $\mathbf{w}^\varepsilon = \chi\mathbf{F}^\varepsilon f^\varepsilon$ and $z^\varepsilon = Zf^\varepsilon$, with $Z = \dfrac{\partial\chi}{\partial t} + \mathbf{v}.\nabla_x\chi$. They satisfy in $\mathbf{R}^{1+3+3}$

$$\frac{\partial}{\partial t}(\chi f^\varepsilon) + \mathbf{v}\cdot\nabla_x(\chi f^\varepsilon) = \frac{\partial}{\partial\mathbf{v}}.\mathbf{w}^\varepsilon + z^\varepsilon. \tag{38}$$

We know that $\mathbf{w}^\varepsilon$ is bounded in $\mathrm{L}^1 \cap \mathrm{L}^p_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$ for some $p > 1$ (here $p = 30/29$) and z^ε and χf^ε are bounded in $\mathrm{L}^1 \cap \mathrm{L}^\infty_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$.

We are going to prove the following result for an exponent q (with $1 < q < p$).

Lemma 3. *Consider, after extraction, the weak limit f^* in $\mathrm{L}^p_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$ of the sequence f^ε weakly then*

$$\rho^\varepsilon_\psi = \chi\langle f^\varepsilon\psi\rangle \to \rho^*_\psi = \chi\langle f^*\psi\rangle, \quad \text{strongly in} \quad \mathrm{L}^q_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3}). \tag{39}$$

Proof. We first define

$$\mathbf{g}^\varepsilon = (\mathbb{I} - \Delta_{x,t,v})^{-1/2}\mathbf{w}.^\varepsilon$$

By regularizing effects the family $\mathbf{g}^\varepsilon$ is compact in $\mathrm{L}^1_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$: indeed $\mathbf{g}^\varepsilon$ is given by a convolution product between $\mathbf{w}^\varepsilon$ and the fundamental solution of $(\mathbb{I} - \Delta_{x,t,v})^{-1/2}$. Moreover $\mathbf{g}^\varepsilon$ is bounded in $\mathrm{L}^p_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$ and then, by interpolation argument, it is compact in $\mathrm{L}^q_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$ for some q (with $1 < q < p$). Since the family $\{\chi f^\varepsilon\}$ is uniformly bounded in $\mathrm{L}^q_{t,x,\mathbf{v}}$ as well as z^ε, the theorem recalled above (the term z^ε can be written as a $\mathbf{v}$ -derivative without loss of generality) applied to χf^ε allows to claim that the following bound holds

$$\|\rho^\varepsilon_\psi\|_q \leq C(q,\psi)C\|\mathbf{g}^\varepsilon\|_q^\alpha.$$

Then, there exists a subsequence $f^\varepsilon, \mathbf{w}^\varepsilon$ and functions $f^*, \rho^*_\psi, \mathbf{g}^*, \mathbf{w}^*$ such that

$$\begin{cases} f^\varepsilon \rightharpoonup f^*, \qquad \mathbf{w}^\varepsilon \rightharpoonup \mathbf{w}^* \quad \text{weakly in} \quad \mathrm{L}^p_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3}), \\ \rho^\varepsilon_\psi \rightharpoonup \rho^*_\psi \quad \text{weakly in} \quad \mathrm{L}^q_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3}), \\ \mathbf{g}^\varepsilon \to \mathbf{g}^* \quad \text{strongly in} \quad \mathrm{L}^q_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3}). \end{cases}$$

and $\mathbf{g}^* = (\mathbb{I} - \Delta_{x,t,v})^{-1/2}\mathbf{w}^*$. Of course we have also, passing to the weak limit,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_x)(\chi f^*\right) &= \frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{w}^* + Zf^* \\ &= \frac{\partial}{\partial\mathbf{v}}\cdot((\mathbb{I} - \Delta_{x,t,v})^{1/2}\mathbf{g}^*) + Zf^*. \end{aligned}$$

We combine this with (38) and we get

$$(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_x)[\chi f^\varepsilon - \chi f^*] = \frac{\partial}{\partial \mathbf{v}}\cdot((\mathbb{I} - \Delta_{x,t,v})^{1/2}(\mathbf{g}^\varepsilon - \mathbf{g}^*)) + Z(f^\varepsilon - f^*).$$

And according to the previous theorem (the term $Z(f^\varepsilon - f^*)$ can be written as a $\mathbf{v}$-derivative also) once again we see that

$$\|\chi\langle f^\varepsilon\psi\rangle - \chi\langle f^*\psi\rangle\|_q \leq C(q,\psi)\cdot\|\mathbf{g}^\varepsilon - \mathbf{g}^*\|_q^\alpha \, \|f^\varepsilon - f^*\|_q^{(1-\alpha)}.$$

That is to say, the property (39) holds. □

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