Transient elastic waves in fluid-structure multilayer systems with a probabilistic model of structural uncertainties

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TRANSIENT ELASTIC WAVES IN FLUID-STRUCTURE MULTILAYER SYSTEMS WITH A PROBABILISTIC MODEL OF STRUCTURAL UNCERTAINTIES

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ABSTRACT. This paper deals with the development of a computational model to predict transient elastic waves in fluid-structure multilayer systems for which the elasticity constants of the structure are uncertain. The fluid-structure system is a three layers system make up of an elastic solid layer sandwiched between two acoustic fluid layers and excited by an acoustic line source located in one of the two acoustic fluid layers. The mean model of the elastic solid layer is represented by a transverse isotropic material. The elasticity tensor of the solid layer is modeled by a random tensor for which the probabilistic model is constructed using the information theory. A Monte Carlo stochastic numerical solver is used in order to solve the stochastic boundary value problem. A numerical application is presented.

KEYWORDS: Uncertainties, Wave propagation, multilayer

1 INTRODUCTION

The analysis of wave phenomena [1,2] in layered elastic [3,4] and acoustic media plays a fundamental role in the fields of non-destructive testing, geophysics and seismology [2,5-10]. This paper deals with the development of a computational model to predict transient elastic waves in fluid-structure multilayer systems for which the elasticity tensor of the structure is uncertain (e.g [9]). The fluid-structure system is a three layers system constituted of a homogeneous elastic solid layer sandwiched between two acoustic fluid layers and excited by an acoustic line source located in one of the two acoustic fluid layers. The mean model of the elastic solid layer is represented by a transverse isotropic material. Due to uncertainties in the solid layer induced by heterogeneities in the material, this solid layer is modeled by a stochastic homogeneous anisotropic material for which the mean value is the mean model. A parametric probabilistic approach is used to take into account uncertainties in the dynamical system. The elasticity tensor of the solid layer is modeled by a random tensor for which the probabilistic model is constructed using the information theory. The Monte-Carlo numerical method is used to solve the stochastic boundary value problem. For each realization of the random elastic tensor, the transient elastic waves are calculated in the coupled system by using a hybrid method [11] based on a time-domain formulation associated with the space Fourier transform for the infinite dimension and using a finite element approximation [12,13] for the finite dimension.

A completed numerical application concerning the cortical bone excited with a transient acoustic line source whose central frequency is 1MHz is presented showing the propagation of uncertainties in the fluid-structure dynamical system.

2 MEAN 3D BOUNDARY VALUE PROBLEMS IN THE 3D SPACE-DOMAIN WITH A TIME-DOMAIN FORMULATION

We consider a three-dimensional multilayer medium composed of one solid layer sandwiched between two fluid layers (see Figure 1). Let $\mathbf{R} = (O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the reference Cartesian frame where O is the origin of the space and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthogonal basis for this space. Let (x_1, x_2, x_3) be the coordinates of a generic point $\mathbf{x}$ in $\mathbf{R}$. The thicknesses of the layers are denoted as h_1, h and h_2. Thus, h_1 is the thickness of the first fluid layer, h is the solid layer thickness and h_2 is the thickness of the second fluid layer. The first fluid layer occupies the unbounded domain Ω_1, the solid elastic layer occupies the domain Ω and the second acoustic fluid layer occupies the domain Ω_2. Let Γ_1, Γ_0, Γ and Γ_2 be the planes defined by

$$\begin{aligned}
\Gamma_1 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_1\} \\
\Gamma_0 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = 0\} \\
\Gamma &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z\} \\
\Gamma_2 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_2\}
\end{aligned}$$

in which $z_1 = h_1$, $z = -h$ and $z_2 = -(h + h_2)$. Then, the boundaries of domains Ω_1, Ω and Ω_2 are respectively $\partial\Omega_1 = \Gamma_1 \cup \Gamma_0$, $\partial\Omega = \Gamma_0 \cup \Gamma$, $\partial\Omega_2 = \Gamma \cup \Gamma_2$. Therefore, domains Ω_1, Ω and Ω_2 are unbounded along the transversal directions $\mathbf{e}_1$ and $\mathbf{e}_2$ whereas they are bounded along the direction $\mathbf{e}_3$.

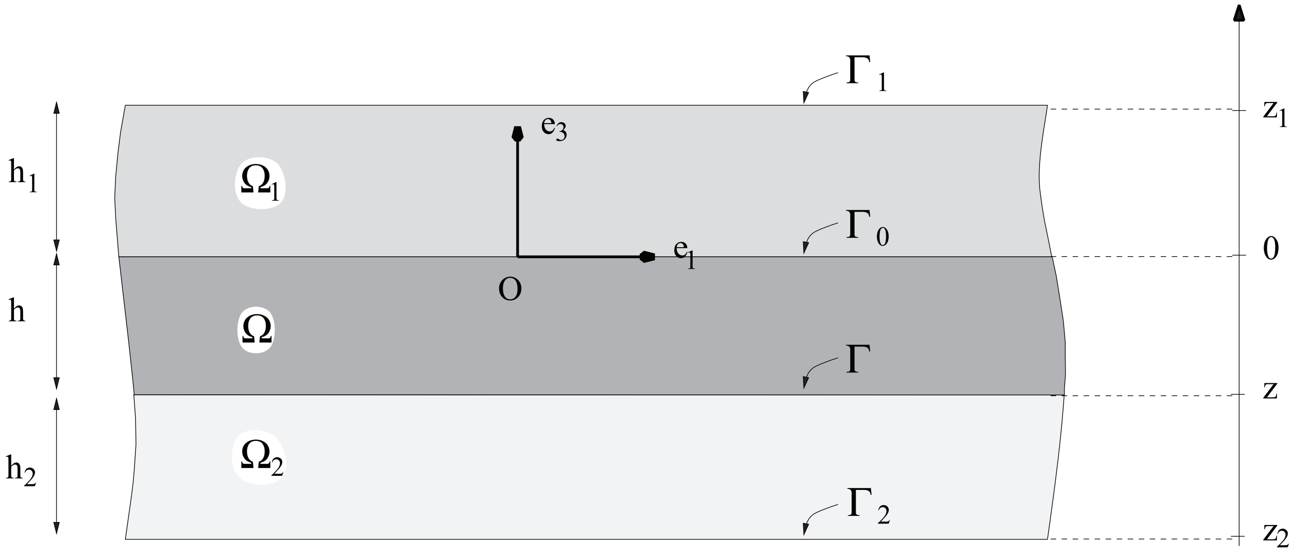

Fig 1. Geometric configuration

Let $p_1(\mathbf{x}, t)$ and $p_2(\mathbf{x}_t)$ be the disturbance of the pressure of the fluid layer at time $t > 0$ for $\mathbf{x}$ belonging to respectively Ω_1 and Ω_2. The mean boundary value problems for the two fluid layers are written as

$$\frac{1}{k_1}\frac{\partial^2 p_1}{\partial t^2} - \frac{1}{\rho_1}\Delta p_1 = \frac{1}{\rho_1}\frac{\partial Q}{\partial t} \quad , \mathbf{x} \in \Omega_1 \tag{1}$$

$$p_1 = 0 \quad , \mathbf{x} \in \Gamma_1 \tag{2}$$

$$\frac{\partial p_1}{\partial x_3} = -\rho_1 \frac{\partial^2 u_3}{\partial t^2} \quad , \mathbf{x} \in \Gamma_0 \tag{3}$$

$$\frac{1}{k_2}\frac{\partial^2 p_2}{\partial t^2} - \frac{1}{\rho_2}\Delta p_2 = 0 \quad , \mathbf{x} \in \Omega_2 \tag{4}$$

$$p_2 = 0 \quad , \mathbf{x} \in \Gamma_2 \tag{5}$$

$$\frac{\partial p_2}{\partial x_3} = -\rho_2 \frac{\partial^2 u_3}{\partial t^2} \quad , \mathbf{x} \in \Gamma \tag{6}$$

in which $k_1 = \rho_1 c_1^2$ where c_1 and ρ_1 are, respectively, the wave velocity and the mass density at equilibrium of the first fluid occupying domain Ω_1; $k_2 = \rho_2 c_2^2$ where c_2 and ρ_2 are, respectively, the wave velocity and the mass density at equilibrium of the second fluid occupying domain Ω_2; Δ is the Laplacian operator with respect to $\mathbf{x}$ and $Q(\mathbf{x}, t)$ is an acoustic source density at point $\mathbf{x} = (x_1, x_2, x_3)$

and at time $t > 0$. Acoustic source density $Q(\mathbf{x}, t)$ is such that

$$\frac{\partial Q}{\partial t}(\mathbf{x}, t) = \rho_1 \, F(t) \, \delta_0(x_1 - x_1^S) \, \delta_0(x_3 - x_3^S) \quad , \tag{7}$$

where x_3^S is a given parameter in $[0, h_1]$ and where x_1^S is a given parameter in $\mathbb{R}$. Thus, Eq. (7) describes an impulse line source parallel to $(O; \mathbf{x}_2)$, placed in the fluid Ω_1 at a given distance from the interface Γ_0.

Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement of a particle located in point $\mathbf{x}$ of Ω at time $t > 0$ and verifying the following boundary value problem,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \text{div}\sigma = \mathbf{0} \quad , \ \mathbf{x} \in \Omega \tag{8}$$

$$\sigma \, \mathbf{n} = -p_1 \, \mathbf{n} \quad , \ \mathbf{x} \in \Gamma_0 \tag{9}$$

$$\sigma \, \mathbf{n} = -p_2 \, \mathbf{n} \quad , \ \mathbf{x} \in \Gamma \tag{10}$$

in which ρ is the mass density and $\sigma(\mathbf{x}, t)$ is the Cauchy stress tensor of the elastic medium at point $\mathbf{x}$ and at time $t > 0$, $\mathbf{n}$ is the outward unit normal to domain Ω and div is the divergence operator with respect to $\mathbf{x}$. The constitutive equation of the solid elastic medium is written as

$$\sigma(\mathbf{x}, t) = \sum_{i,j,k,h=1}^{3} c_{ijkh} \varepsilon_{kh}(\mathbf{x}, t) \, \mathbf{e}_i \otimes \mathbf{e}_j \tag{11}$$

in which $\sum_{i,j,k,h=1}^{3} c_{ijkh} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_h$ is the elasticity tensor of the medium and $\varepsilon_{kh} = \frac{1}{2}(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k})$ is the linearized strain tensor. Finally, the system is at rest at time $t = 0$. Consequently, we have

$$p_1(\mathbf{x}, 0) = 0 \quad , \ \mathbf{x} \in \Omega_1 \cup \partial\Omega_1 \tag{12}$$

$$\mathbf{u}(\mathbf{x}, 0) = 0 \quad , \ \mathbf{x} \in \Omega \cup \partial\Omega \tag{13}$$

$$p_2(\mathbf{x}, 0) = 0 \quad , \ \mathbf{x} \in \Omega_2 \cup \partial\Omega_2 \tag{14}$$

3 MEAN ELASTICITY MATRIX FOR AN ISOTROPIC TRANSVERSE MATERIAL

Let $[C]$ be the elasticity matrix whose components are the coefficients of elasticity tensor c_{ijkh} such that

$$[C] = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} & \sqrt{2}c_{1123} & \sqrt{2}c_{1131} & \sqrt{2}c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & \sqrt{2}c_{2223} & \sqrt{2}c_{2231} & \sqrt{2}c_{2213} \\ c_{3311} & c_{3322} & c_{3333} & \sqrt{2}c_{3323} & \sqrt{2}c_{3331} & \sqrt{2}c_{3312} \\ \sqrt{2}c_{2311} & \sqrt{2}c_{2322} & \sqrt{2}c_{2333} & 2c_{2323} & 2c_{2331} & 2c_{2312} \\ \sqrt{2}c_{3111} & \sqrt{2}c_{3122} & \sqrt{2}c_{3133} & 2c_{3123} & 2c_{3131} & 2c_{3112} \\ \sqrt{2}c_{1211} & \sqrt{2}c_{1222} & \sqrt{2}c_{1233} & 2c_{1223} & 2c_{1231} & 2c_{1212} \end{pmatrix} . \tag{15}$$

Let $c_{ij} = [C]_{ij}$ be the components of matrix $[C]$. If the elasticity tensor is modeling an isotropic homogeneous medium, then all the components c_{ij} are zeros except the following

$$c_{11} = \frac{e_L^2(1 - \nu_T)}{(e_L - e_L\nu_T - 2e_T\nu_L^2)}, \quad c_{22} = c_{33} = \frac{e_T(e_L - e_T\nu_L^2)}{(1 + \nu_T)(e_L - e_L\nu_T - 2e_T\nu_L^2)}$$

$$c_{12} = c_{13} = c_{21} = c_{31} = \frac{e_T e_L \nu_L}{(e_L - e_L\nu_T - 2e_T\nu_L^2)}, \quad c_{23} = c_{32} = \frac{e_T(e_L\nu_T + e_T\nu_L^2)}{(1 + \nu_T)(e_L - e_L\nu_T - 2e_T\nu_L^2)}$$

$$c_{44} = g_T = \frac{e_T}{2(1 + \nu_T)}, \quad c_{55} = c_{66} = g_L$$

in which e_L and e_T are the longitudinal and transversal Young moduli, g_L and g_T are the longitudinal and transversal shear moduli, respectively; ν_L and ν_T are the longitudinal and transversal Poisson coefficients, respectively.

4 MEAN WEAK FORMULATION IN THE 1D-SPECTRAL DOMAIN WITH A TIME-DOMAIN FORMULATION

Due to the nature of the source and to the geometrical configuration, the transverse waves polarized in the $(\mathbf{e}_1, \mathbf{e}_2)$ plane are not excited. Then, the present study can be conducted in the plane $(O; \mathbf{e}_1, \mathbf{e}_3)$ and the mean 3D boundary value problem is independent of x_2 .

For all x_3 fixed in $]z_2, z_1[$, the 1D-Fourier transform of an integrable function $x_1 \mapsto f(x_1, x_3, t)$ on $\mathbb{R}$ is defined by

$$\widehat{f}(k_1, x_3, t) = \int_{\mathbb{R}} f(x_1, x_3, t)\, e^{i k_1 x_1} dx_1 \quad .$$

Let $\widehat{p_1}$, $\widehat{\mathbf{u}}$ and $\widehat{p_2}$ be the 1D-Fourier transforms of functions p_1, $\mathbf{u}$ and p_2. Let C_1 and C_2 be the function spaces constituted of all the sufficiently differentiable complex-valued functions $x_3 \mapsto \delta p_1(x_3)$ and $x_3 \mapsto \delta p_2(x_3)$ respectively, defined on $]0, z_1[$ and $]z_2, z[$. We introduce the admissible function spaces $C_{1,0} \subset C_1$ and $C_{2,0} \subset C_2$ such that

$$C_{1,0} = \{\delta p_1 \in C_1; \quad \delta p_1(z_1) = 0\}$$
$$C_{2,0} = \{\delta p_2 \in C_2; \quad \delta p_2(z_2) = 0\}$$

Let C be the admissible function space constituted of all the sufficiently differentiable functions $x_3 \mapsto \delta\mathbf{u}(x_3)$ from $]z, 0[$ into $\mathbb{C}^2$ where $\mathbb{C}$ is the set of all the complex numbers. The weak formulation of the present problem is written as : for all k_1 fixed in $\mathbb{R}$ and for all fixed t, find $\widehat{p_1}(k_1, \cdot, t) \in C_{1,0}$, $\widehat{\mathbf{u}}(k_1, \cdot, t) \in C$ and $\widehat{p_2}(k_1, \cdot, t) \in C_{2,0}$ such that, for all $\delta p_1 \in C_{1,0}$, $\delta\mathbf{u} \in C$ and $\delta p_2 \in C_{2,0}$,

$$a_1\left(\frac{\partial^2 \widehat{p_1}}{\partial t^2}, \delta p_1\right) + k_1^2\, c_1^2\, a_1(\widehat{p_1}, \delta p_1) + b_1(\widehat{p_1}, \delta p_1) + r_1\left(\frac{\partial^2 \widehat{\mathbf{u}}}{\partial t^2}, \delta p_1\right) = f(\delta p_1; t)\,,$$
$$m\left(\frac{\partial^2 \widehat{\mathbf{u}}}{\partial t^2}, \delta\mathbf{u}\right) + s_1(\widehat{\mathbf{u}}, \delta\mathbf{u}) + k_1^2\, s_2(\widehat{\mathbf{u}}, \delta\mathbf{u}) - ik_1\, s_3(\widehat{\mathbf{u}}, \delta\mathbf{u}) + \overline{r_2(\delta\mathbf{u}, \widehat{p_2})} - \overline{r_1(\delta\mathbf{u}, \widehat{p_1})} = 0\,,$$
$$a_2\left(\frac{\partial^2 \widehat{p_2}}{\partial t^2}, \delta p_2\right) + k_1^2\, c_2^2\, a_2(\widehat{p_2}, \delta p_2) + b_2(\widehat{p_2}, \delta p_2) - r_2\left(\frac{\partial^2 \widehat{\mathbf{u}}}{\partial t^2}, \delta p_2\right) = 0\,,$$

in which the positive-definite and definite sesquilinear forms a_1 and b_1 defined on $C_1 \times C_1$, the sesquilinear form r_1 defined on $C \times C_1$, the antilinear form f_1 defined on C_1, the sesquilinear forms positive-definite and positive a_2 and b_2 defined on $C_2 \times C_2$, the sesquilinear form r_2 defined on $C \times C_2$, the positive-definite sesquilinear form a defined on $C \times C$ and finally, the sesquilinear form b defined on $C \times C$ which are presented in Appendix and where

$$s_1(\widehat{\mathbf{u}}, \delta\mathbf{u}) = \int_z^0 \langle [D_1]\, \frac{\partial \widehat{\mathbf{u}}}{\partial x_3}, \overline{\frac{\partial \delta\mathbf{u}}{\partial x_3}} \rangle dx_3 \tag{16}$$

$$s_2(\widehat{\mathbf{u}}, \delta\mathbf{u}) = \int_z^0 \langle [D_2]\, \widehat{\mathbf{u}}, \overline{\delta\mathbf{u}} \rangle dx_3 \tag{17}$$

$$s_3(\widehat{\mathbf{u}}, \delta\mathbf{u}) = \int_z^0 \left(\langle [D_3]\, \widehat{\mathbf{u}}, \overline{\frac{\partial \delta\mathbf{u}}{\partial x_3}} \rangle - \langle [D_3]\, \overline{\delta\mathbf{u}}, \frac{\partial \widehat{\mathbf{u}}}{\partial x_3} \rangle \right) dx_3 \tag{18}$$

in which $\langle \cdot, \cdot \rangle$ means the usual Euclidean inner product on $\mathbb{R}^2$ extended to $\mathbb{C}^2$ and where

$$[D_1] = \begin{bmatrix} c_{55}/2 & c_{53}/\sqrt{2} \\ c_{35}/\sqrt{2} & c_{33} \end{bmatrix} \;,\; [D_2] = \begin{bmatrix} c_{11} & c_{15}/\sqrt{2} \\ c_{51}/\sqrt{2} & c_{55}/2 \end{bmatrix} \;,\; [D_3] = \begin{bmatrix} c_{51}/\sqrt{2} & c_{55}/2 \\ c_{31} & c_{35}/\sqrt{2} \end{bmatrix} \quad . \tag{19}$$

It should be noted that only $s_1(\widehat{\mathbf{u}}, \delta\mathbf{u})$, $s_2(\widehat{\mathbf{u}}, \delta\mathbf{u})$ and $s_3(\widehat{\mathbf{u}}, \delta\mathbf{u})$ depend on components of elasticity matrix $[C]$.

5 MEAN FINITE ELEMENT MODEL IN THE 1D-SPECTRAL DOMAIN WITH A TIME-DOMAIN FORMULATION

We introduce a finite element mesh of domain $[z_2, z] \cup [z, 0] \cup [0, z_1]$ which is constituted of ν_{tot} nodes. The finite elements used are Lagrangian 1D-finite element with 3 nodes. Let $\widehat{\mathbf{p}}_1(k_1, t)$, $\widehat{\mathbf{v}}(k_1, t)$ and $\widehat{\mathbf{p}}_2(k_1, t)$ be the complex vectors of the nodal values of the functions $x_3 \mapsto \widehat{p}_1(k_1, x_3, t)$, $x_3 \mapsto \widehat{\mathbf{u}}(k_1, x_3, t)$ and $x_3 \mapsto \widehat{p}_2(k_1, x_3, t)$. Let $\widehat{\mathbf{f}}(k_1, t)$ be the complex vector in $\mathbb{C}^{\nu_1}$ where ν_1 is the number of degree of freedom related to the mesh of domain $[0, z_1]$, corresponding to the finite element approximation of the antilinear form $f(\delta p_1; t)$. For all k_1 fixed in $\mathbb{R}$ and for all fixed t, the finite element approximation of the weak formulation of the 1D boundary value problem yields the following linear system of equations

$$[A_1]\ddot{\widehat{\mathbf{p}}}_1 + (k_1^2\, c_1^2[A_1] \; + [B_1])\widehat{\mathbf{p}}_1(k_1, t) + [R_1]\ddot{\widehat{\mathbf{v}}}(k_1, t) = \widehat{\mathbf{f}}(k_1, t) \tag{20}$$

$$[M]\ddot{\widehat{\mathbf{v}}}(k_1, t) + ([S_1] - ik_1\,[S_3] + k_1^2\,[S_2])\widehat{\mathbf{v}}(k_1, t) + [R_2]^T\,\widehat{\mathbf{p}}_2(k_1, t) - [R_1]^T\,\widehat{\mathbf{p}}_1(k_1, t) = 0 \tag{21}$$

$$[A_2]\ddot{\widehat{\mathbf{p}}}_2(k_1, t) + (k_1^2\, c_2^2[A_2] + [B_2])\widehat{\mathbf{p}}_2(k_1, t) - [R_2]\ddot{\widehat{\mathbf{v}}}(k_1, t) = 0 \tag{22}$$

in which the double dots means the second partial derivative with respect to t. Each of Eqs. (20) , (21) and (22) form linear systems whose the square matrices are respectively of dimensions $\nu_1 \times \nu_1$, $\nu \times \nu$ and $\nu_2 \times \nu_2$. The integer numbers ν and ν_2 are respectively the number of degree of freedom related to the meshes of domains $[z, 0]$ and $[z_2, z]$. Moreover, the components of these matrices are complex numbers. These three equations can be rewritten as

$$[\mathbb{M}]\ddot{\widehat{\mathbb{v}}}(k_1, t) + ([\mathbb{K}_1] - ik_1[\mathbb{K}_2] + k_1^2[\mathbb{K}_3])\widehat{\mathbb{v}}(k_1, t) = \widehat{\mathbb{f}}(k_1, t) \tag{23}$$

in which the vectors $\widehat{\mathbb{v}}(k_1, t) = (\widehat{\mathbf{p}}_1(k_1, t), \widehat{\mathbf{v}}(k_1, t), \widehat{\mathbf{p}}_2(k_1, t))$ and $\widehat{\mathbb{f}}(k_1, t) = (\widehat{\mathbf{f}}(k_1, t), 0, 0)$ belong to $\mathbb{C}^{\nu_1+\nu+\nu_2}$ and where

$$[\mathbb{M}] = \begin{bmatrix} [A_1] & [R_1] & 0 \\ 0 & [M] & 0 \\ 0 & -[R_2] & [A_2] \end{bmatrix} \quad ,$$

$$[\mathbb{K}_1] = \begin{bmatrix} [B_1] & 0 & 0 \\ -[R_1]^T & [S_1] & [R_2]^T \\ 0 & 0 & [B_2] \end{bmatrix} \quad , \quad [\mathbb{K}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & [S_3] & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad [\mathbb{K}_3] = \begin{bmatrix} c_1^2\,[A_1] & 0 & 0 \\ 0 & [S_2] & 0 \\ 0 & 0 & c_2^2\,[A_2] \end{bmatrix}$$

where upper-script T denotes the transpose matrix. It should be noted that matrices $[S_1]$, $[S_2]$ and $[S_3]$ correspond to the finite element approximations of sesquilinear forms s_1, s_2 and s_3. Consequently, matrices $[S_1]$, $[S_2]$ and $[S_3]$ depend on components of elasticity matrix $[C]$ (see Eqs (16) to (19)) and there exist mappings $\mathbb{g}_1$, $\mathbb{g}_2$ and $\mathbb{g}_3$ such that $[S_1] = \mathbb{g}_1([C])$, $[S_2] = \mathbb{g}_2([C])$ and $[S_3] = \mathbb{g}_3([C])$.

6 PROBABILISTIC MODEL OF STRUCTURAL UNCERTAINTIES

This section is devoted to the construction of a probabilistic model of uncertainties in the solid layer. It is assumed that uncertainties are only related to the components of elasticity tensor c_{ijkh}. The stochastic finite element model is constructed by substituting matrix $[C]$ in the mean finite element model with a random matrix $[\mathbf{C}]$ whose probabilistic model is constructed using the information theory. The available information on $[\mathbf{C}]$ is defined as follows: (1) the mean value of random matrix $[\mathbf{C}]$ is the mean elasticity matrix $[C]$ of the mean model; (2) random matrix $[\mathbf{C}]$ is a second-order random variable with values in $\mathbb{M}_n^+(\mathbb{R})$ with $n = 6$ where $\mathbb{M}_n^+(\mathbb{R})$ is the set of all the $(n\times n)$ real symmetric positive-definite matrices; (3) the inverse matrix of $[\mathbf{C}]$ which exists almost surely is assumed to be a second-order random variable. Thus, random matrix $[\mathbf{C}]$ belongs to the set SE^+ (see [13]) and is written as

$$[\mathbf{C}] = [L]^T[\mathbf{G}][L] \quad , \tag{24}$$

in which the (6×6) upper triangular matrix $[L]$ corresponds to the Cholesky factorization of matrix $[C]$ and where random matrix $[\mathbf{G}]$ belongs to the set SG^+ defined in [13]. The probability density function $p_{[\mathbf{G}]}$ of random matrix $[\mathbf{G}]$ is written as

$$p_{[\mathbf{G}]}([G]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G]) \times c_n \times (\det[G])^{b_n} \times \exp\{-a_n \mathrm{tr}[G]\} \quad , \tag{25}$$

with $n = 6$ and where $a_n = (n+1)/(2\delta^2)$, $b_n = a_n(1-\delta^2)$, $\mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G])$ is equal to 1 if $[G]$ belongs to $\mathbb{M}_n^+(\mathbb{R})$ and is equal to zero if $[G]$ does not belong to $\mathbb{M}_n^+(\mathbb{R})$, $\mathrm{tr}[G]$ is the trace of a matrix $[G]$ and where positive constant c_n is such that

$$c_n = \frac{(2\pi)^{-n(n-1)/4} a_n^{n a_n}}{\prod_{j=1}^n \Gamma(\alpha_j)} \quad , \tag{26}$$

in which $\alpha_j = (n+1)/(2\delta^2) + (1-j)/2$ and Γ is the Gamma function. Parameter δ allows the dispersion of the stochastic model to be controlled. It should be noted that such a probabilistic model takes into account any anisotropic perturbation of the elasticity tensor with respect to a mean elasticity tensor of a simplified elasticity model such as, for instance, an isotropic transverse solid. Note that the components $C_{ij} = [\mathbf{C}]_{ij}$ of random matrix $[\mathbf{C}]$ are statistically dependent random variables with values in $\mathbb{R}$ and depends on dispersion parameter δ.

7 STOCHASTIC FINITE ELEMENT MODEL IN THE 1D-SPECTRAL DOMAIN WITH A TIME-DOMAIN FORMULATION

The stochastic finite element model of the system is constructed substituting $[S_1]$, $[S_2]$ and $[S_3]$ in Eqs. (20) to (22) with random matrices $[\mathbf{S}_1]$, $[\mathbf{S}_2]$ and $[\mathbf{S}_3]$ by $[\mathbf{S}_1] = \mathbb{g}_1([\mathbf{C}])$, $[\mathbf{S}_2] = \mathbb{g}_2([\mathbf{C}])$ and $[\mathbf{S}_3] = \mathbb{g}_3([\mathbf{C}])$ where mappings $\mathbb{g}_1$, $\mathbb{g}_2$ and $\mathbb{g}_3$ are introduced in Section 5. Consequently, for all time t fixed in $\mathbb{R}$ and for all k_1 fixed in $\mathbb{R}$, the solution of the stochastic finite element model is a random vector $\widehat{\mathbb{V}}(k_1,t) = (\widehat{\mathbf{P}}_1(k_1,t), \widehat{\mathbf{V}}(k_1,t), \widehat{\mathbf{P}}_2(k_1,t))$ such that

$$[A_1]\ddot{\widehat{\mathbf{P}}}_1 + (k_1^2 c_1^2 [A_1] + [B_1])\widehat{\mathbf{P}}_1(k_1,t) + [R_1]\ddot{\widehat{\mathbf{V}}}(k_1,t) = \widehat{\mathbf{f}}(k_1,t) \tag{27}$$

$$[M]\ddot{\widehat{\mathbf{V}}}(k_1,t) + ([\mathbf{S}_1] - ik_1[\mathbf{S}_3] + k_1^2[\mathbf{S}_2])\widehat{\mathbf{V}}(k_1,t) + [R_2]^T\widehat{\mathbf{P}}_2(k_1,t) - [R_1]^T\widehat{\mathbf{P}}_1(k_1,t) = 0 \tag{28}$$

$$[A_2]\ddot{\widehat{\mathbf{P}}}_2(k_1,t) + (k_1^2 c_2^2 [A_2] + [B_2])\widehat{\mathbf{P}}_2(k_1,t) - [R_2]\ddot{\widehat{\mathbf{V}}}(k_1,t) = 0 \tag{29}$$

These three equations can be rewritten as

$$[\mathbb{M}]\ddot{\widehat{\mathbb{V}}}(k_1,t) + ([\mathbf{K}_1] - ik_1[\mathbf{K}_2] + k_1^2[\mathbf{K}_3])\widehat{\mathbb{V}}(k_1,t) = \widehat{\mathbb{f}}(k_1,t) \tag{30}$$

in which, matrix $[\mathbb{M}]$ and vector $\widehat{\mathbb{f}}(k_1,t)$ are defined in Section 5 and where random matrices $[\mathbf{K}_1]$, $[\mathbf{K}_2]$ and $[\mathbf{K}_3]$ are such that

$$[\mathbf{K}_1] = \begin{bmatrix} [B_1] & 0 & 0 \\ -[R_1]^T & [\mathbf{S}_1] & [R_2]^T \\ 0 & 0 & [B_2] \end{bmatrix} \quad , \quad [\mathbf{K}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & [\mathbf{S}_3] & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad [\mathbf{K}_3] = \begin{bmatrix} c_1^2[A_1] & 0 & 0 \\ 0 & [\mathbf{S}_2] & 0 \\ 0 & 0 & c_2^2[A_2] \end{bmatrix}$$

8 STOCHASTIC SOLVER

By construction, for all k_1 fixed in $\mathbb{R}$ and for all fixed time $t > 0$, random vector $\widehat{\mathbf{P}}_1(k_1,t)$ in Eqs. (27) to (29) is the finite element approximation of random field $\widehat{P}_1(k_1,\cdot,t)$ indexed by $[0,h_1]$ associated with the deterministic field $\widehat{p}_1(k_1,\cdot,t)$. The inverse 1D-Fourier transform in k_1 of $\widehat{P}_1$ is denoted as P_1 and is a random field indexed by $\mathbb{R} \times [0,h_1] \times [0,+\infty[$ modeling the random disturbance of the pressure in the first fluid layer due to uncertainties in the solid layer (see Section 6). Then, there exists a deterministic mapping $\mathbb{g}_{P_1}$ such that $P_1 = \mathbb{g}_{P_1}(\widehat{\mathbf{P}}_1)$. Let the random arrival time T be the first local maximum of stochastic field $\{P_1(x_1^R, x_3^R, t)\}_{t>0}$ in which x_1^R and x_3^R are given parameters in $\mathbb{R}$ and $[0, h_1]$, respectively. Let $\mathfrak{g}_T$ be the mapping defined as $T = \mathfrak{g}_T(P_1)$.

The stochastic solver used in order to construct statistical estimations of T and P_1 is based on the Monte-Carlo numerical simulation. For each realization $[\mathbf{C}(\theta)]$ of random matrix $[\mathbf{C}]$, realization $[\mathbf{S}_1(\theta)] = \mathfrak{g}_1([\mathbf{C}(\theta)])$,$[\mathbf{S}_2(\theta)] = \mathfrak{g}_2([\mathbf{C}(\theta)])$ and $[\mathbf{S}_3(\theta)] = \mathfrak{g}_3([\mathbf{C}(\theta)])$ of random matrices $[\mathbf{S}_1]$, $[\mathbf{S}_2]$ and $[\mathbf{S}_3]$ are constructed. Then, for all k_1 fixed in $\mathbb{R}$ and for all fixed time $t > 0$, the realization $\widehat{\mathbf{P}}_1(k_1, t, \theta)$ of random vector $\widehat{\mathbf{P}}_1(k_1, t)$ is calculated solving the deterministic equation associated with stochastic Eq. (30) using an implicit time integration scheme. Then, the realization $P_1(\theta) = \mathfrak{g}_{P_1}(\widehat{\mathbf{P}}_1(\theta))$ of random field P_1 is calculated. Finally, the realization $T(\theta) = \mathfrak{g}_T(P_1(\theta))$ of random arrival time T can be calculated.

9 NUMERICAL APPLICATION

For the numerical application presented in this section, the fluid layer Ω_1 is excited by a line source located at $x_1^S = 0$ and $x_3^S = 2\times10^{-3}$m with a time-history defined with the function F in Eq. (7) such that

$$F(t) = F_1 \sin(2\pi f_c t)\, e^{-4(t f_c - 1)^2} \quad ,$$

where $f_c = 1$ MHz is the center frequency and $F_1 = 100$ m.s^{-2} is an amplitude factor. Figure 2 shows the power spectrum of F (left) and the graph of function $t \mapsto F(t)$ (right). The thicknesses of the three layers are $h_1 = 2 \times 10^{-3}$m, $h = 4 \times 10^{-3}$m and $h_2 = 10^{-2}$m. The mechanical parameters of the first fluid layer are $\rho_1 = 1000$ kg/m^3 and $c_1 = 1500$ m/s. For the second fluid layer, the mechanical parameters are $\rho_2 = 1000$ kg/m^3 and $c_2 = 1500$ m/s. Finally, for the elastic solid layer we will use the longitudinal and transversal Young moduli $e_L = 16.6$ GPa and $e_T = 9.5$ GPa, respectively; the longitudinal and transversal shear moduli $g_L = 4.7$ GPa and $g_T = 3.3$ GPa, respectively; the longitudinal and transversal Poisson coefficients $\nu_L = 0.38$ and $\nu_T = 0.44$, respectively.

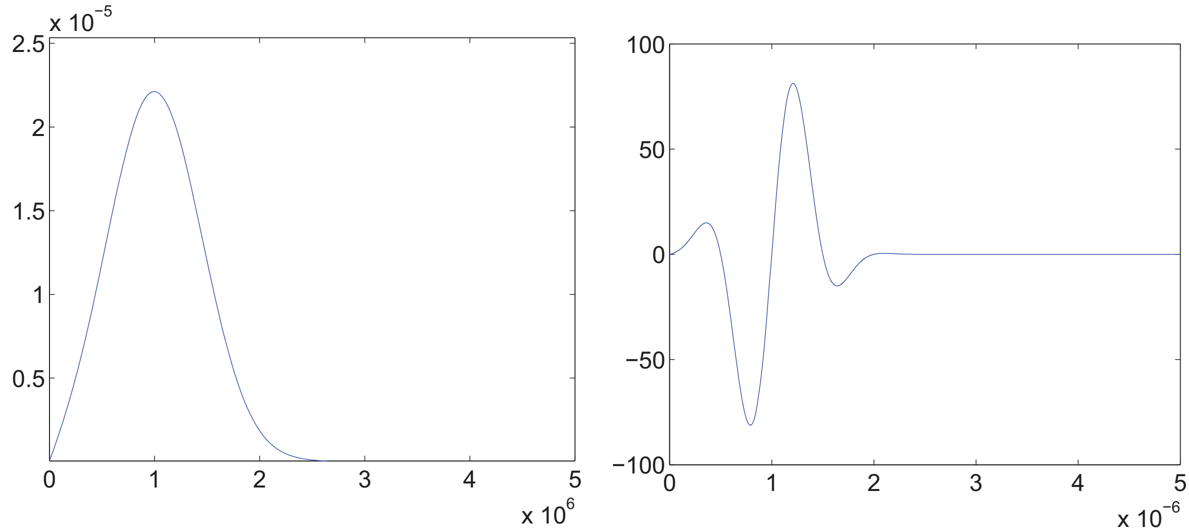

Fig 2. Definition of the function F. Graphs of the power spectrum of F (left) and function $t \mapsto F(t)$ (right). Vertical axis: power spectrum (left) and F(t) (right). Horizontal axis: frequency (left) and t (right).

Figure 3 shows the graph of the confidence region of stochastic field $P_1(x_1^R, x_3^R, \cdot)$ indexed by $[0, +\infty[$ for a probability level $P_c = 0.95$ and with a dispersion parameter $\delta = 0.2$, $x_1^R = 2 \times 10^{-3}$m and $x_3^R = 2 \times 10^{-3}$m. Figure 4 shows the graph of the density probability function of random arrival time T with $\delta = 0.2$ and $x_1^R = 2 \times 10^{-3}$m and $x_3^R = 2 \times 10^{-3}$m.

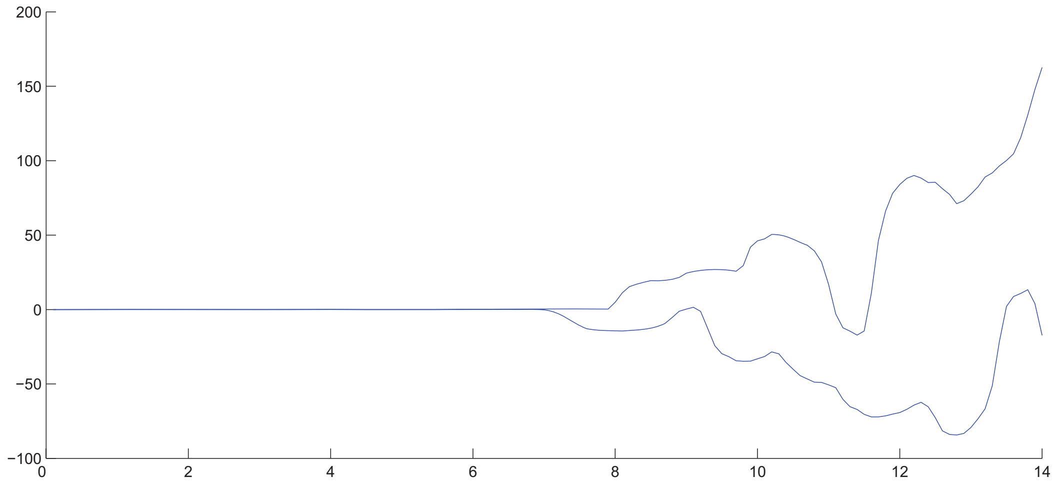

Fig 3. Confidence region of the stochastic process $\{P_1(x_1^R, x_3^R, t)\}_{t>0}$ *with a probability level* $P_c = 0.95$ $\delta = 0.2$, $x_1^R = 2 \times 10^{-3}$m *and* $x_3^R = 2 \times 10^{-3}$m . *Vertical axis: disturbance of the pressure in the first fluid layer. Horizontal axis: time t* (10^{-6}*s*).

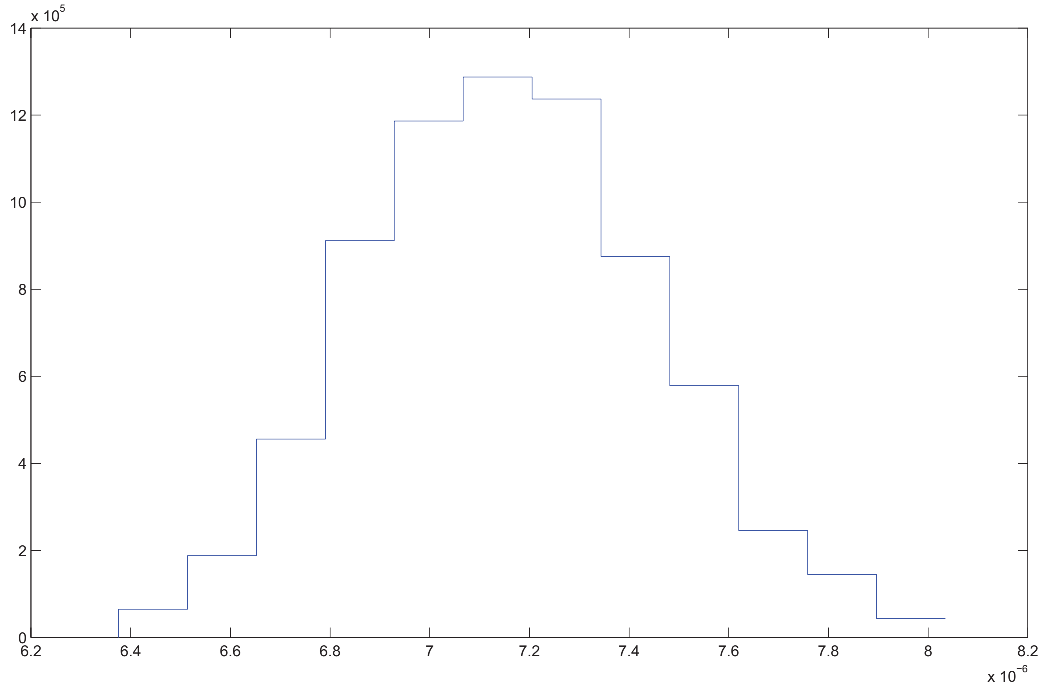

Fig 4. Probability density function of random arrival time T with $\delta = 0.2$, $x_1^R = 2 \times 10^{-3}$m *and* $x_3^R = 2 \times 10^{-3}$m. *Vertical axis: probability density. Horizontal axis: arrival time (s).*

10 CONCLUSION

We have presented a probabilistic model to predict the transient elastic wave propagation in a multilayer unbounded media with uncertainties in the solid layer. Uncertainties are taken into account with a probabilistic model. Thanks to the introduction of an efficient numerical solver, the Monte-Carlo numerical method can be used as solver of the stochastic equations. The numerical application devoted to the cortical bone shows the interest of such an approach.

11 APPENDIX

The different quantities introduced in Section 4 are defined below

$$a_1(\widehat{p}_1, \delta p_1) = \frac{1}{K_1} \int_0^{z_1} \widehat{p}_1 \, \overline{\delta p_1} \, dx_3 \tag{31}$$

$$b_1(\widehat{p}_1, \delta p_1) = \frac{1}{\rho_1} \int_0^{z_1} \frac{\partial \widehat{p}_1}{\partial x_3} \overline{\frac{\partial \delta p_1}{\partial x_3}} \, dx_3 \tag{32}$$

$$r_1(\widehat{\mathbf{u}}, \delta p_1) = \widehat{u}_3(0) \overline{\delta p_1(0)} \tag{33}$$

$$f(\delta p_1; t) = F(t) \, e^{ik_1 x_1^S} \, \overline{\delta p_1(x_3^S)} \tag{34}$$

$$m(\widehat{\mathbf{u}}, \delta \mathbf{u}) = \int_z^0 \rho \langle \widehat{\mathbf{u}}, \overline{\delta \mathbf{u}} \rangle dx_3 \tag{35}$$

$$a_2(\widehat{p}_2, \delta p_2) = \frac{1}{K_2} \int_{z_2}^{z} \widehat{p}_2 \, \overline{\delta p_2} \, dx_3 \tag{36}$$

$$b_2(\widehat{p}_2, \delta p_2) = \frac{1}{\rho_2} \int_{z_2}^{z} \frac{\partial \widehat{p}_2}{\partial x_3} \overline{\frac{\partial \delta p_2}{\partial x_3}} \, dx_3 \tag{37}$$

$$r_2(\widehat{\mathbf{u}}, \delta p_2) = \widehat{u}_3(z) \overline{\delta p_2(z)} \tag{38}$$

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