1D-Space finite element approximation with 2D-space Fourier transform and with time-domain formulation for 3D-transient elastic waves in multilayer semi-infinite media

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1D-SPACE FINITE ELEMENT APPROXIMATION WITH 2D-SPACE FOURIER TRANSFORM AND WITH TIME-DOMAIN FORMULATION FOR 3D-TRANSIENT ELACTIC WAVES IN MULTILAYER SEMI-INFINITE MEDIA

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Abstract
The purpose of this communication is to present a hybrid method to simulate the transient elastic waves over a short time in multilayer semi-infinite media subjected to given transient loads. The medium is constituted of a finite number of unbounded layers with finite thicknesses. We present a method avoiding usual numerical difficulties for such a problem. The proposed method is based on a time domain formulation associated with a 2D-space Fourier transform for the two infinite dimensions and using a finite element discretisation for the third finite dimension. An example is presented for a three layers system constituted of an elastic solid layer sandwiched between two acoustic fluid layers and excited by an acoustic line source located in one of the two acoustic fluid layers.

INTRODUCTION

The purpose of this communication is to present a hybrid method to simulate the transient elastic waves over a short time in multilayer semi-infinite media subjected to given transient loads. The medium is constituted of a three unbounded layers with finite thicknesses. This boundary value problem can usually be solved in the frequency domain (Fourier transform with respect to the time domain) and in the spectral domain (Fourier transform with respect to the space domain) using the Green functions (see for instance [1, 2, 4, 5, 7, 9]).

Another usual approach (see for instance [3, 6, 8]) consists in solving the problem in the frequency domain and in the 2D-spectral domain (corresponding to the two infinite dimensions) and solving the boundary value problem in the 1D-space domain corresponding to the third finite dimension (direction transversal to the layers). Such a method can induce numerical difficulties which can be avoided by using an adapted algebraic formulation which can be stricky to be implemented (see for instance [3]).

In this paper, we propose an alternative approach. Since we are interested in the calculation of the transient response of the system over a relatively short time, the numerical cost is smaller in solving directly the problem in the time domain. It should be noted that the use of a Fourier transform to go in the frequency domain would require the calculation on a broad frequency band increasing the numerical cost.

Therefore, we present a method avoiding these difficulties. It is based on a time domain formulation associated with a 2D-space Fourier transform for the two infinite dimensions and using a finite element discretisation for the third finite dimension.

First, the boundary value problem is written in 1D-space and 2D-spectral domains with a time domain formulation. The weak formulation of the 1D-boundary problem is introduced. Then the finite element approximation for the 1D-space is constructed. An implicit time integration scheme is used for solving the differential equation in time. The 3D-space solution in time is then obtained by an inverse 2D-space Fourier transform. An example is presented for a three layers system constituted of an elastic solid layer sandwiched between two acoustic fluid layers and excited by an acoustic line source located in one of the two acoustic fluid layers.

3D AND 1D BOUNDARY VALUE PROBLEMS

3D Boundary value problem in the 3D-space with a time-domain formulation

We consider a three-dimensional multilayer system composed of one elastic solid layer sandwiched between two acoustic fluid layers (see Fig. 1). Let $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the Cartesian frame of reference and (x_1, x_2, x_3) be the coordinates of the generic point $\mathbf{x}$ in $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The thicknesses of the layers are denoted by h_1, h and h_2. The first acoustic fluid layer occupies the open unbounded domain Ω_1 , the second acoustic fluid layer occupies the open unbounded domain Ω_2 and the elastic solid layer occupies the open unbounded domain Ω. Let $\partial\Omega_1 = \Gamma_1 \cup \Gamma_0$, $\partial\Omega = \Gamma_0 \cup \Gamma$ and $\partial\Omega_2 = \Gamma \cup \Gamma_2$ (see Fig. 1) be respectively the boundaries of Ω_1, Ω and Ω_2 where

$$
\begin{aligned}
\Gamma_1 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_1\} \\
\Gamma_0 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = 0\} \\
\Gamma &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z\} \\
\Gamma_2 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_2\}
\end{aligned}
$$

in which $z_1 = h_1$, $z = -h$ and $z_2 = -(h + h_2)$. Therefore, the domains Ω_1, Ω and Ω_2 are unbounded along the transversal directions $\mathbf{e}_1$ and $\mathbf{e}_2$ whereas they are bounded along the vertical direction $\mathbf{e}_3$.

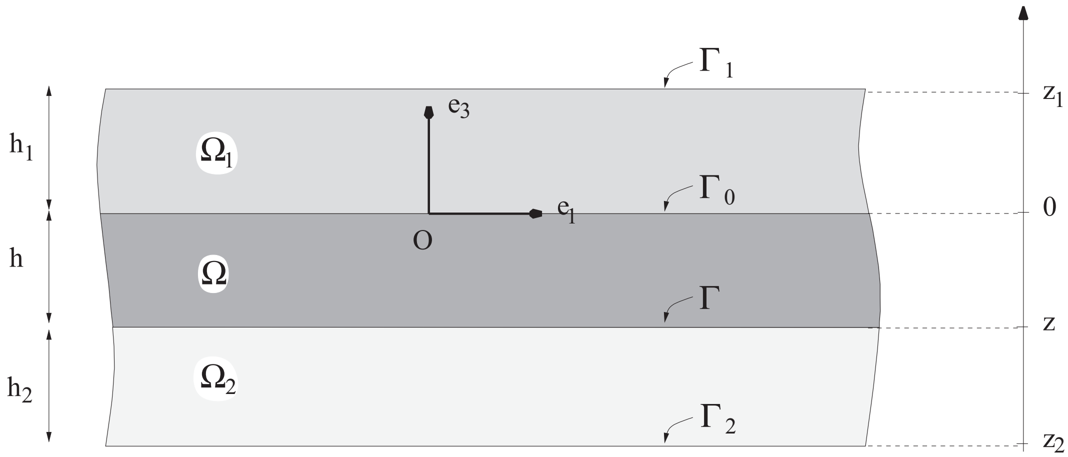

Figure 1: Geometric configuration

The displacement field of a particle located in point $\mathbf{x}$ of Ω and at time $t > 0$ is denoted by $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$. For all $\mathbf{x}$ belonging to Ω_1 and for all time $t > 0$, the disturbance of the pressure of the acoustic fluid layer occupying the domain Ω_1 is denoted by $p_1(\mathbf{x}, t)$. The boundary value problem for this acoustic fluid layer is written as

$$\frac{1}{c_1^2}\frac{\partial^2 p_1}{\partial t^2} - \Delta p_1 = \frac{\partial Q}{\partial t} \quad , \ \mathbf{x} \in \Omega_1 \tag{1}$$

$$p_1 = 0 \quad , \ \mathbf{x} \in \Gamma_1 \tag{2}$$

$$\frac{\partial p_1}{\partial x_3} = -\rho_1 \frac{\partial^2 u_3}{\partial t^2} \quad , \ \mathbf{x} \in \Gamma_0 \tag{3}$$

where c_1 and ρ_1 are the constant speed of sound and the mass density of the fluid at equilibrium, Δ is the Laplacian operator with respect to $\mathbf{x}$ and $Q(\mathbf{x}, t)$ is the acoustic source density at point $\mathbf{x} = (x_1, x_2, x_3)$ and at time $t > 0$.

The displacement field $\mathbf{u}$ of the solid elastic medium occupying the domain Ω verifies the following boundary value problem,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\boldsymbol{\sigma} = 0 \quad , \ \mathbf{x} \in \Omega \tag{4}$$

$$\boldsymbol{\sigma}\,\mathbf{n} = -p_1\,\mathbf{n} \quad , \ \mathbf{x} \in \Gamma_0 \tag{5}$$

$$\boldsymbol{\sigma}\,\mathbf{n} = -p_2\,\mathbf{n} \quad , \ \mathbf{x} \in \Gamma \tag{6}$$

in which ρ is the mass density and $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress tensor of the solid elastic medium at point $\mathbf{x}$ and at time $t > 0$, $\mathbf{n}$ is the outward unit normal to domain Ω and div is the divergence operator with respect to $\mathbf{x}$. The constitutive equation of the solid elastic medium is written as

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \sum_{i,j,k,h=1}^{3} a_{ijkh}(\mathbf{x})\varepsilon_{kh}(\mathbf{x}, t)\,\mathbf{e}_i \otimes \mathbf{e}_j \tag{7}$$

in which $\sum_{i,j,k,h=1}^{3} a_{ijkh}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_h$ is the elasticity tensor of the medium and $\varepsilon_{kh} = \frac{1}{2}(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k})$ is the linearized strain tensor. It is assumed that the components $a_{ijkh}(\mathbf{x})$ depend only on x_3. We then have $a_{ijkh}(\mathbf{x}) = a_{ijkh}(x_3)$.

For all $\mathbf{x}$ belonging to Ω_2 and for all time $t > 0$, the disturbance $p_2(\mathbf{x}, t)$ of the pressure of the acoustic fluid occupying the domain Ω_2 is such that

$$\frac{1}{c_2^2}\frac{\partial^2 p_2}{\partial t^2} - \Delta p_2 = 0 \quad , \mathbf{x} \in \Omega_2 \tag{8}$$

$$p_2 = 0 \quad , \mathbf{x} \in \Gamma_2 \tag{9}$$

$$\frac{\partial p_2}{\partial x_3} = -\rho_2 \frac{\partial^2 u_3}{\partial t^2} \quad , \mathbf{x} \in \Gamma \tag{10}$$

where c_2 and ρ_2 are the constant speed of sound and the mass density of the fluid at equilibrium.

Futhermore, the system is at rest at time $t = 0$. Consequently, we have

$$p_1(\mathbf{x}, 0) = 0 \quad , \mathbf{x} \in \Omega_1 \cup \partial\Omega_1 \tag{11}$$

$$\mathbf{u}(\mathbf{x}, 0) = 0 \quad , \mathbf{x} \in \Omega \cup \partial\Omega \tag{12}$$

$$p_2(\mathbf{x}, 0) = 0 \quad , \mathbf{x} \in \Omega_2 \cup \partial\Omega_2 \tag{13}$$

1D Boundary value problem in the 2D-spectral domain with a time-domain formulation

For all x_3 fixed in $]z_2, z_1[$, the 2D-Fourier transform of an integrable function $(x_1, x_2) \mapsto f(x_1, x_2, x_3)$ on $\mathbb{R}^2$ is defined by

$$\widehat{f}(\mathbf{k}, x_3, t) = \int_{\mathbb{R}^2} f(x_1, x_2, x_3, t)\, e^{i(k_1\, x_1 + k_2\, x_2)} dx_1\, dx_2$$

in which $\mathbf{k} = (k_1, k_2)$ belongs to $\mathbb{R}^2$. Applying the 2D-Fourier transform to Eqs. (1) to (13) yields the 1D boundary value problem of the system in the 1D space domain with a 2D-spectral and time domains formulation. Such a boundary value problem is written with respect to the functions $\widehat{p}_1$, $\widehat{\mathbf{u}}$ and $\widehat{p}_2$ which are respectively the 2D-Fourier transforms of functions p_1, $\mathbf{u}$ and p_2.

WEAK FORMULATION AND FINITE ELEMENT MODEL

Weak formulation of the 1D boundary value problem

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the function spaces constituted of all the sufficiently differentiable complex-valued functions $x_3 \mapsto \delta p_1(x_3)$ and $x_3 \mapsto \delta p_2(x_3)$ respectively, defined on $]0, z_1[,]z_2, z[$. We introduce the admissible function spaces $\mathcal{C}_{1,0} \subset \mathcal{C}_1$ and $\mathcal{C}_{2,0} \subset \mathcal{C}_2$ such that

$$\mathcal{C}_{1,0} = \{\delta p_1 \in \mathcal{C}_1; \quad \delta p_1(z_1) = 0\} \tag{14}$$

$$\mathcal{C}_{2,0} = \{\delta p_2 \in \mathcal{C}_2; \quad \delta p_2(z_2) = 0\} \tag{15}$$

Let $\mathcal{C}$ be the admissible function space constituted of all the sufficiently differentiable functions $x_3 \mapsto \delta\mathbf{u}(x_3)$ from $]z, 0[$ into $\mathbb{C}^3$.

The weak formulation of the 1D boundary value problem is written as : for all $\mathbf{k}$ fixed in $\mathbb{R}^2$ and for all fixed t, find $\widehat{p}_1(\mathbf{k},\cdot,t) \in \mathcal{C}_{1,0}$, $\widehat{\mathbf{u}}(\mathbf{k},\cdot,t) \in \mathcal{C}$ and $\widehat{p}_2(\mathbf{k},\cdot,t) \in \mathcal{C}_{2,0}$ such that, for all $\delta p_1 \in \mathcal{C}_{1,0}$, $\delta\mathbf{u} \in \mathcal{C}$ and $\delta p_2 \in \mathcal{C}_{2,0}$,

$$a_1\left(\frac{\partial^2\widehat{p}_1}{\partial t^2}, \delta p_1\right) + b_1(\mathbf{k}, \widehat{p}_1, \delta p_1) + r_1\left(\frac{\partial^2\widehat{\mathbf{u}}}{\partial t^2}, \delta p_1\right) = f_1(\delta p_1, t)\,, \tag{16}$$

$$a\left(\frac{\partial^2\widehat{\mathbf{u}}}{\partial t^2}, \delta\mathbf{u}\right) + b(\mathbf{k}, \widehat{\mathbf{u}}, \delta\mathbf{u}) + \overline{r_2(\delta\mathbf{u}, \widehat{p}_2)} - \overline{r_1(\delta\mathbf{u}, \widehat{p}_1)} = 0\,, \tag{17}$$

$$a_2\left(\frac{\partial^2\widehat{p}_2}{\partial t^2}, \delta p_2\right) + b_2(\mathbf{k}, \widehat{p}_2, \delta p_2) - r_2\left(\frac{\partial^2\widehat{\mathbf{u}}}{\partial t^2}, \delta p_2\right) = 0\,, \tag{18}$$

in which the overline denotes the complex conjugate, where a_1 and b_1 are positive-definite and positive sesquilinear forms on $\mathcal{C}_1 \times \mathcal{C}_1$, the sesquilinear form r_1 is defined on $\mathcal{C} \times \mathcal{C}_1$, the antilinear form f_1 is defined on $\mathcal{C}_1$, the sesquilinear forms a_2 and b_2 are positive-definite and positive on $\mathcal{C}_2 \times \mathcal{C}_2$, the sesquilinear form r_2 is defined on $\mathcal{C} \times \mathcal{C}_2$, the sequilinear form a is positive-definite on $\mathcal{C} \times \mathcal{C}$ and finally, the sesquilinear form b is defined on $\mathcal{C} \times \mathcal{C}$.

Finite element discretisation of the 1D boundary rvalue problem

Let $\widehat{\mathbf{p}}_1(\mathbf{k},t)$, $\widehat{\mathbf{v}}(\mathbf{k},t)$ and $\widehat{\mathbf{p}}_2(\mathbf{k},t)$ be the complex vectors of the nodal values of the functions $x_3 \mapsto \widehat{p}_1(\mathbf{k},x_3,t)$, $x_3 \mapsto \widehat{\mathbf{u}}(\mathbf{k},x_3,t)$ and $x_3 \mapsto \widehat{p}_2(\mathbf{k},x_3,t)$ related to the finite element mesh of the domain $[z_2, z_1]$ which is constituted of ν_{tot} nodes. The finite elements used are Lagrangian 1D-finite element with 3 nodes. For all $\mathbf{k}$ fixed in $\mathbb{R}^2$ and for all fixed t, the finite element discretisation of the weak formulation of the 1D boundary value problem yields the following linear system of equations

$$[A_1]\,\ddot{\widehat{\mathbf{p}}}_1 + [B_1(\mathbf{k})]\,\widehat{\mathbf{p}}_1 + [R_1]\,\ddot{\widehat{\mathbf{v}}} = \mathbf{f}_1(t) \tag{19}$$

$$[A]\,\ddot{\widehat{\mathbf{v}}} + [B(\mathbf{k})]\,\widehat{\mathbf{v}} + [R_2]^T\,\widehat{\mathbf{p}}_2 - [R_1]^T\,\widehat{\mathbf{p}}_1 = 0 \tag{20}$$

$$[A_2]\,\ddot{\widehat{\mathbf{p}}}_2 + [B_2(\mathbf{k})]\,\widehat{\mathbf{p}}_2 - [R_2]\,\ddot{\widehat{\mathbf{v}}} = 0 \tag{21}$$

in which the double dots mean the second derivative with respect to t and where Eqs. (19), (20) and (21) are on $\mathbb{C}^{\nu_1}$, $\mathbb{C}^{\nu}$ and $\mathbb{C}^{\nu_2}$ respectively. These three equations can be rewritten as

$$[\mathbb{M}]\,\ddot{\widehat{\mathbb{v}}} + [\mathbb{K}(\mathbf{k})]\,\widehat{\mathbb{v}} = \mathbb{f}(t) \tag{22}$$

in which $\widehat{\mathbb{v}}(\mathbf{k},t) = (\widehat{\mathbf{p}}_1(\mathbf{k},t), \widehat{\mathbf{v}}(\mathbf{k},t), \widehat{\mathbf{p}}_2(\mathbf{k},t))$ belongs to $\mathbb{C}^{\nu_1+\nu+\nu_2}$.

Finite element solution in the 3D-space domain with a time-domain formulation

Let $\widehat{\mathbb{v}}^{n,m,\ell} = (\widehat{\mathbf{p}}_1^{n,m,\ell}, \widehat{\mathbf{v}}^{n,m,\ell}, \widehat{\mathbf{p}}_2^{n,m,\ell})$ be the solution of Eq. (22) at time $t = n\Delta t$, with $k_1 = m\,\Delta k_1$ and $k_2 = \ell\,\Delta k_2$ for $n = 0,\ldots,N$, for $m = -M,\ldots,M-1$ and $\ell = -L,\ldots,L-1$. The implicit inconditionaly stable Newmark scheme is used in order to solve the differential equation (22) in time. Let $\mathbb{v}^{n,m,\ell} = (\mathbf{p}_1^{n,m,\ell}, \mathbf{v}^{n,m,\ell}, \mathbf{p}_2^{n,m,\ell})$ be the vector of the nodal values of

$x_3 \mapsto p_1(x_1, x_2, x_3, t)$, $x_3 \mapsto \mathbf{u}(x_1, x_2, x_3, t)$ and $x_3 \mapsto p_2(x_1, x_2, x_3, t)$ related to the finite element mesh of the domain $[z_2, z_1]$ at time $t = n\Delta t$ with $x_1 = m\Delta x_1$ and $x_2 = \ell\Delta x_2$ for $n = 0, \ldots, N$, for $m = -M, \ldots, M-1$ and for $\ell = -L, \ldots, L-1$. We then have

$$\mathbb{v}^{n,m,\ell} = \frac{\Delta k_1 \Delta k_2}{4\pi^2} \sum_{p=-M}^{M-1} \sum_{q=-L}^{L-1} \widehat{\mathbb{v}}^{n,p,q} e^{-i\,(p\,m\,\Delta k_1 \Delta x_1 + q\,\ell\,\Delta k_2 \Delta x_2)} \tag{23}$$

which can be evaluated by using a 2D Fast Fourier Transform.

NUMERICAL EXAMPLE

In order to show the efficiency of the method, we present a numerical example for which the first acoustic fluid layer is excited by an acoustic line source located at positions $x_1 = x_1^S$, $x_3 = x_3^S$ and $x_2 \in \mathbb{R}$ where x_1^S and x_3^S are given parameters of the problem (see Table 1). Such an acoustic line source is modelled by an acoustic source density Q (see Eq. (1)) such that

$$\frac{\partial Q}{\partial t}(\mathbf{x}, t) = \sin(2\pi f_c t) e^{-4(t\, f_c - 1)^2} \delta_0(x_1 - x_1^S)\delta_0(x_3 - x_3^S) \quad .$$

For such an acoustic line source, p_1, $\mathbf{u}$ and p_2 do not depend on variable x_2. Consequently, the 3D formulation presented above is restricted to the 2D case.

It is assumed that the domain Ω related to the solid layer is constituted of an elastic transverse isotropic medium for which the longitudinal Young modulus is denoted by E_L, the transversal Young modulus by E_T, the longitudinal shear modulus by G_L, the transversal shear modulus by G_T, the longitudinal Poisson coefficient by ν_L and the transversal Poisson coefficient is denoted by ν_T. The numerical values of these mechanical parameters are given in Table 1. The parameters for the numerical method are presented in Table 2.

h_1	10^{-2}m	h	4×10^{-3}m	h_2	10^{-2}m
ρ_1	1000 Kg.m^{-3}	ρ	1722 Kg.m^{-3}	ρ_2	1000 Kg.m^{-3}
c_1	1500 m.s^{-1}	E_L	16.6 GPa	c_2	1500 m.s^{-1}
f_c	1 MHz	E_T	9.5 GPa		
x_1^S	0	G_L	4.7 GPa		
x_3^S	2×10^{-3}m	G_T	3.3 GPa		

Table 1. Values of the mechanical parameters

Let $\mathrm{vm}(\mathbf{x}, t)$ be the von Mises stress at point $\mathbf{x} \in \Omega$ and at time t. Figure 2 shows the graph of functions $(x_1, x_3) \mapsto p_1(x_1, x_2, x_3, t)$, $(x_1, x_3) \mapsto \mathrm{vm}(x_1, x_2, x_3, t)$ and $(x_1, x_3) \mapsto p_2(x_1, x_2, x_3, t)$ for any x_2 fixed in $\mathbb{R}$ and at $t = 0.6\,\mu$s (Fig. A), $t = 1.075\,\mu$s (Fig. B), $t = 1.475\,\mu$s (Fig. C), $t = 3.15\,\mu$s (Fig. D), $t = 4.775\,\mu$s (Fig. E), $t = 7.3\,\mu$s (Fig. F), $t = 7.875\,\mu$s (Fig. G), $t = 10.725\,\mu$s (Fig. H). For this simulation, the total CPU time is 140 s using a 3.8 MHz Xeon processor. Such a CPU time represents a very low computational cost with respect to a full finite element computation.

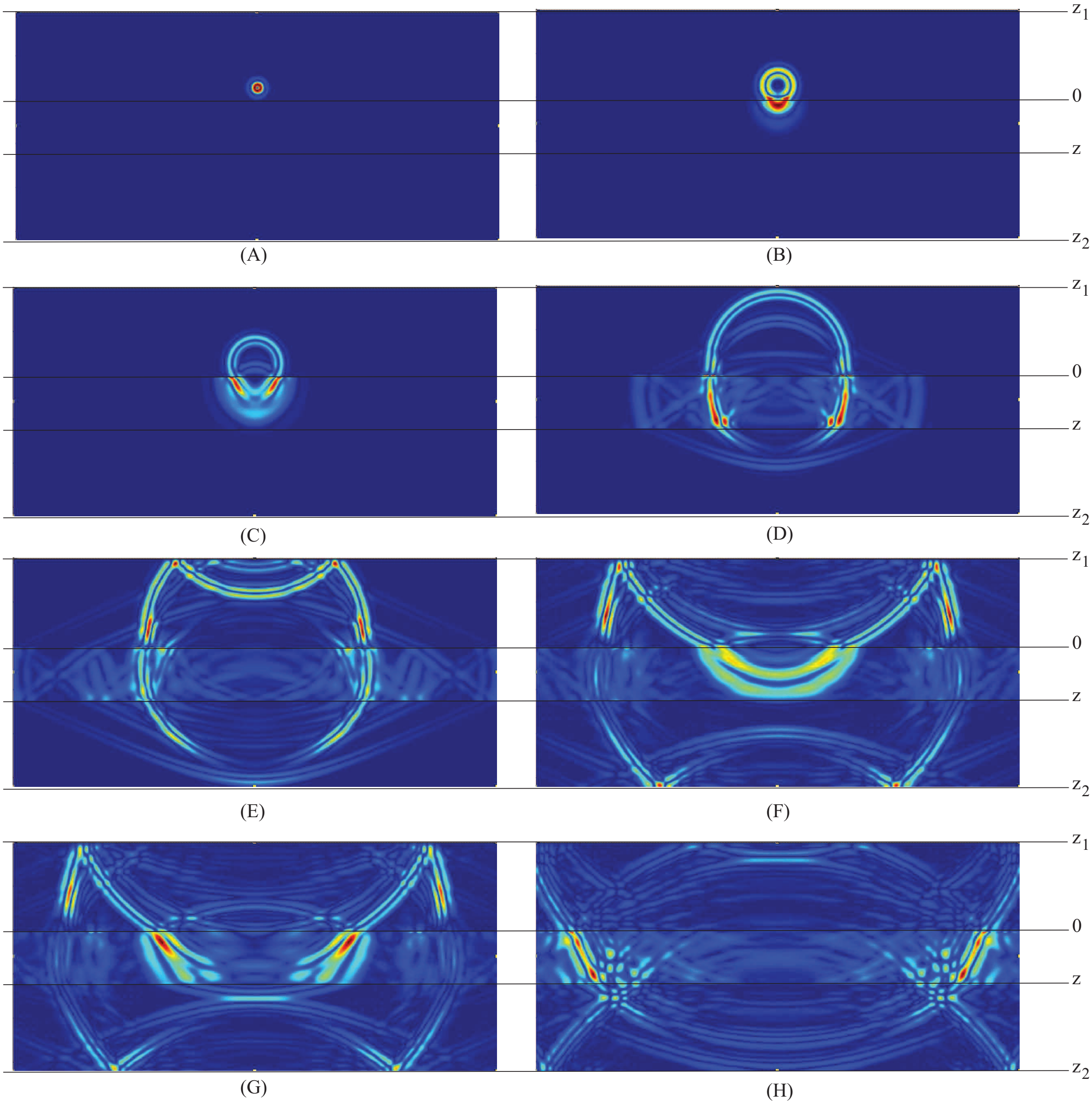

Figure 2: Wave propagation in the three layers (pressure field in the fluid layers and von Mises stress field in the elastic layer) at $t = 0.6\,\mu$s (Fig. A), $t = 1.075\,\mu$s (Fig. B), $t = 1.475\,\mu$s (Fig. C), $t = 3.15\,\mu$s (Fig. D), $t = 4.775\,\mu$s (Fig. E), $t = 7.3\,\mu$s (Fig. F), $t = 7.875\,\mu$s (Fig. G), $t = 10.725\,\mu$s (Fig. H).

Δk_1	Δx_1	M	Δt	N	ν_{h_1}	ν_h	ν_{h_2}
4.8828	0.3×10^{-3}	2048	2.5×10^{-8}	432	101	82	101

Table 2. Values of the numerical method

CONCLUSION

We have presented a method to simulate the transient elastic waves over a short time in multilayer semi-infinite media subjected to given transient loads. First, the boundary value problem is rewritten in the 1D-space domains with 2D-spectral and a time domains formulation by applying a 2D-space Fourier transform to the usual 3D boundary value problem. The weak formulation of such a 1D boundary value problem and the corresponding finite element discretisation have been constructed. The differential system of equation is solve in time by using the implicit inconditionaly stable Newmark scheme. A numerical example has been presented. Even if this numerical example is restricted to the 2D case of the method, it can be seen that the proposed method is very efficient and well adapted to the numerical simulation of transient elastic waves in multilayer systems. This method can easily be extended to the study of viscoelastic transient waves in multilayer media.

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