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# Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions

M.I. Isaev and R.G. Novikov

## Abstract

We prove new global Hölder-logarithmic stability estimates for the Gel'fand inverse problem at fixed energy in dimension  $d \geq 3$ . Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Comparisons with preceding results in this direction are given.

## 1 Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \quad (1.1)$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \quad d \geq 2, \quad (1.2)$$

$$\text{with } \partial D \in C^2,$$

$$v \in \mathbb{L}^\infty(D). \quad (1.3)$$

Consider the map  $\hat{\Phi} = \hat{\Phi}(E)$  such that

$$\hat{\Phi}(E)(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}|_{\partial D} \quad (1.4)$$

for all sufficiently regular solutions  $\psi$  of (1.1) in  $\bar{D} = D \cup \partial D$ , where  $\nu$  is the outward normal to  $\partial D$ . Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \quad (1.5)$$

The map  $\hat{\Phi} = \hat{\Phi}(E)$  is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1):

**Problem 1.1.** Given  $\hat{\Phi}$  for some fixed  $E$ , find  $v$ .

This problem can be considered as the Gelfand inverse boundary value problem for the Schrödinger equation at fixed energy (see [10], [23]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [6], [23]). Problem 1.1 can be also

considered as an example of ill-posed problem: see [18], [4] for an introduction to this theory.

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness results and global reconstruction methods for Problem 1.1 were given for the first time in [23] in dimension  $d \geq 3$  and in [5] in dimension  $d = 2$ .

Global logarithmic stability estimates for Problem 1.1 were given for the first time in [1] in dimension  $d \geq 3$  and in [30] in dimension  $d = 2$ . A principal improvement of the result of [1] was given recently in [29] (for the zero energy case): stability of [29] optimally increases with increasing regularity of  $v$ .

For the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [36] for  $d \geq 3$  and in [21] for  $d = 2$ . Global logarithmic stability estimates for this problem were given for the first time in [1] for  $d \geq 3$  and [19] for  $d = 2$ . Principal increasing of global stability of [1], [19] for the regular coefficient case was found in [29] for  $d \geq 3$  and [34] for  $d = 2$ .

In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension  $d \geq 2$  were given in [7], [16]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [2] and additional studies in this direction were fulfilled in [33].

Due to [20] the logarithmic stability results of [1], [19] with their principal effectivization of [29], [34] are optimal (up to the value of the exponent). An extension of the instability estimates of [20] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [12].

On the other hand, it was found in [25], [26] (see also [28], [31]) that for inverse problems for the Schrödinger equation at fixed energy  $E$  in dimension  $d \geq 2$  (like Problem 1.1) there is a Hölder stability modulo an error term rapidly decaying as  $E \rightarrow +\infty$  (at least for the regular coefficient case). In addition, for Problem 1.1 for  $d = 3$ , global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were given in [15]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in these results of [15]. An additional study, motivated by [15], [29], was given in [22].

In the present work we give new global Hölder-logarithmic stability estimates for Problem 1.1 in dimension  $d \geq 3$  for the regular coefficient case, see Theorem 2.1 and Remark 2.6. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In particular cases, our new estimates become coherent (although less strong) with respect to results of [29], [26], see Remarks 2.2, 2.3. In general, our new estimates give some synthesis of several important preceding results.

## 2 Stability estimates

In this section we assume for simplicity that

$$v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \text{ supp } v \subset D, \quad (2.1)$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \partial^J v \in L^1(\mathbb{R}^d), |J| \leq m\}, \quad m \in \mathbb{N} \cup 0, \quad (2.2)$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^d J_i, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \quad (2.3)$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^d)}. \quad (2.4)$$

Let

$$\begin{aligned} \|A\| \text{ denote the norm of an operator} \\ A : \mathbb{L}^\infty(\partial D) \rightarrow \mathbb{L}^\infty(\partial D). \end{aligned} \quad (2.5)$$

We recall that if  $v_1, v_2$  are potentials satisfying (1.3), (1.5) for some fixed  $E$ , then

$$\hat{\Phi}_2(E) - \hat{\Phi}_1(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \quad (2.6)$$

where  $\hat{\Phi}_1, \hat{\Phi}_2$  are the DtN maps for  $v_1, v_2$ , respectively, see [23], [27]. Note also that (2.1)  $\Rightarrow$  (1.3).

Let

$$s_0 = \frac{m-d}{m}, \quad s_1 = \frac{m-d}{d}, \quad s_2 = m-d. \quad (2.7)$$

**Theorem 2.1.** *Let  $D$  satisfy (1.2), where  $d \geq 3$ . Let  $v_1, v_2$  satisfy (2.1) and (1.5) for some fixed real  $E$ . Let  $\|v_j\|_{m,1} \leq N$ ,  $j = 1, 2$ , for some  $N > 0$ . Let  $\hat{\Phi}_1(E)$  and  $\hat{\Phi}_2(E)$  denote the DtN maps for  $v_1$  and  $v_2$ , respectively. Then*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C_1 (\ln(3 + \delta^{-1}))^{-s}, \quad 0 < s \leq s_1, \quad (2.8)$$

where  $C_1 = C_1(N, D, m, s, E) > 0$ ,  $\delta = \|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|$  is defined according to (2.5). In addition, for  $E \geq 0$ ,  $\tau \in (0, 1)$  and any  $s \in [0, s_1]$ ,

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C_2(1 + \sqrt{E})\delta^\tau + C_3(1 + \sqrt{E})^{s-s_1} (\ln(3 + \delta^{-1}))^{-s}, \quad (2.9)$$

where  $C_2 = C_2(N, D, m, \tau) > 0$  and  $C_3 = C_3(N, D, m, \tau) > 0$ .

**Remark 2.1.** Estimate (2.8) for  $s = s_0$  is a variation of the result of [1] (see also [29], [13]). One can see that estimate (2.8),  $s = s_1$ , of Theorem 2.1 is more strong (as much as  $s_1$  is greater than  $s_0$ ) than the aforementioned result going back to [1].

**Remark 2.2.** Estimate (2.8) for  $s = s_2$ ,  $E = 0$ ,  $d = 3$  was proved in [29]. One can see that this estimate of [29] is more strong (as much as  $s_2$  is greater than  $s_1$ ) than estimate (2.8),  $s = s_1$ , of Theorem 2.1 for  $E = 0$ ,  $d = 3$ .

**Remark 2.3.** Using results of [26] one can obtain estimate (2.9) for  $s = 0$ ,  $d = 3$ , with  $s_2$  in place of  $s_1$ , for sufficiently great  $E$  with respect to  $N$ . One can see that for this particular case the aforementioned corollary of [26] is more strong (as much as  $s_2$  is greater than  $s_1$ ) than estimate (2.9) of Theorem 2.1.

**Remark 2.4.** In a similar way with results of [13], [14], estimates (2.8), (2.9) can be extended to the case when we do not assume that condition (1.5) is fulfilled and consider an appropriate impedance boundary map instead of the Dirichlet-to-Neumann map.

**Remark 2.5.** Concerning two-dimensional analogs of results of Theorem 2.1, see [25], [31], [34], [35].

**Remark 2.6.** Actually, in the proof of Theorem 2.1 we obtain the following estimate (see formula (4.19)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq C_4 \sqrt{E + \rho^2} e^{2\rho L} \delta + C_5 (E + \rho^2)^{-s_1/2}, \quad (2.10)$$

where  $L = \max_{x \in \partial D} |x|$ ,  $C_4 = C_4(N, D, m) > 0$ ,  $C_5 = C_5(N, D, m) > 0$  and parameter  $\rho > 0$  is such that  $E + \rho^2$  is sufficiently large:  $E + \rho^2 \geq C_6(N, D, m)$ . Estimates of Theorem 2.1 follow from estimate (2.10).

The proof of Theorem 2.1 and estimate (2.10) is given in Section 4 and is based on results recalled in Section 3. Actually, this proof is technically very similar to the proof of estimate (2.8) for  $s = s_0$ , see [1], [29], [13]. Possibility of such a proof of estimate (2.8) for  $s = s_1$ ,  $E = 0$  was mentioned, in particular, in [32].

### 3 Faddeev functions

We consider the Faddeev functions  $G$ ,  $\psi$ ,  $h$  (see [8], [9], [11], [23]):

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (3.1)$$

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) dy, \quad (3.2)$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{C}^d$ ,  $\text{Im } k \neq 0$ ,  $d \geq 3$ ,

$$h(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \quad (3.3)$$

where

$$k, l \in \mathbb{C}^d, \quad k^2 = l^2, \quad \text{Im } k = \text{Im } l \neq 0. \quad (3.4)$$

One can consider (3.2), (3.3) assuming that

$$\begin{aligned} v \text{ is a sufficiently regular function on } \mathbb{R}^d \\ \text{with sufficient decay at infinity.} \end{aligned} \quad (3.5)$$

For example, in connection with Problem 1.1, one can consider (3.2), (3.3) assuming that

$$v \in \mathbb{L}^\infty(D), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus D. \quad (3.6)$$

We recall that (see [8], [9], [11], [23]):

- The function  $G$  satisfies the equation

$$(\Delta + k^2)G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \quad (3.7)$$

- Formula (3.2) at fixed  $k$  is considered as an equation for

$$\psi = e^{ikx} \mu(x, k), \quad (3.8)$$

where  $\mu$  is sought in  $\mathbb{L}^\infty(\mathbb{R}^d)$ ;

- As a corollary of (3.2), (3.1), (3.7),  $\psi$  satisfies (1.1) for  $E = k^2$ ;
- The Faddeev functions  $G$ ,  $\psi$ ,  $h$  are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular,  $h$  is a generalized "scattering" amplitude).

In addition,  $G$ ,  $\psi$ ,  $h$  in their zero energy restriction, that is for  $E = 0$ , were considered for the first time in [3]. The Faddeev functions  $G$ ,  $\psi$ ,  $h$  were, actually, rediscovered in [3].

Let

$$\begin{aligned} \Sigma_E &= \{k \in \mathbb{C}^d : k^2 = k_1^2 + \dots + k_d^2 = E\}, \\ \Theta_E &= \{k \in \Sigma_E, l \in \Sigma_E : \text{Im } k = \text{Im } l\}, \\ |k| &= (|\text{Re } k|^2 + |\text{Im } k|^2)^{1/2}. \end{aligned} \quad (3.9)$$

Under the assumptions of Theorem 2.1, we have that:

$$\mu(x, k) \rightarrow 1 \quad \text{as } |k| \rightarrow \infty \quad (3.10)$$

and, for any  $\sigma > 1$ ,

$$|\mu(x, k)| \leq \sigma \quad \text{for } |k| \geq r_1(N, D, m, \sigma), \quad (3.11)$$

where  $x \in \mathbb{R}^d$ ,  $k \in \Sigma_E$ ;

$$\hat{v}(p) = \lim_{\substack{(k, l) \in \Theta_E, k-l=p \\ |\text{Im } k| = |\text{Im } l| \rightarrow \infty}} h(k, l) \quad \text{for any } p \in \mathbb{R}^d, \quad (3.12)$$

$$\begin{aligned} |\hat{v}(p) - h(k, l)| &\leq \frac{c_1(D, m)N^2}{(E + \rho^2)^{1/2}} \quad \text{for } (k, l) \in \Theta_E, \quad p = k - l, \\ |\text{Im } k| = |\text{Im } l| &= \rho, \quad E + \rho^2 \geq r_2(N, D, m), \\ p^2 &\leq 4(E + \rho^2), \end{aligned} \quad (3.13)$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (3.14)$$

Results of the type (3.10), (3.11) go back to [3]. For more information concerning (3.11) see estimate (4.11) of [13]. Results of the type (3.12), (3.13) (with less precise right-hand side in (3.13)) go back to [11]. Estimate (3.13) follows, for example, from formulas (3.2), (3.3) and the estimate

$$\begin{aligned} \|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d) \rightarrow \mathbb{L}^2(\mathbb{R}^d)} &= O(|k|^{-1}) \\ \text{as } |k| &\rightarrow \infty, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d, \end{aligned} \quad (3.15)$$

for  $s > 1/2$ , where  $g(k)$  denotes the integral operator with the Schwartz kernel  $g(x-y, k)$  and  $\Lambda$  denotes the multiplication operator by the function  $(1+|x|^2)^{1/2}$ .

Estimate (3.15) was formulated, first, in [17] for  $d \geq 3$ . Concerning proof of (3.15), see [37].

In addition, we have that:

$$h_2(k, l) - h_1(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} \psi_1(x, -l)(v_2(x) - v_1(x))\psi_2(x, k)dx$$

$$\text{for } (k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0,$$

$$\text{and } v_1, v_2 \text{ satisfying (3.5),} \quad (3.16)$$

$$h_2(k, l) - h_1(k, l) = (2\pi)^{-d} \int_{\partial D} \psi_1(x, -l) \left[ (\hat{\Phi}_2 - \hat{\Phi}_1) \psi_2(\cdot, k) \right] (x) dx$$

$$\text{for } (k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0,$$

$$\text{and } v_1, v_2 \text{ satisfying (1.5), (3.6),} \quad (3.17)$$

and, under assumptions of Theorem 2.1,

$$|\hat{v}_1(p) - \hat{v}_2(p) - h_1(k, l) + h_2(k, l)| \leq \frac{c_2(D, m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}}$$

$$\text{for } (k, l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho,$$

$$E + \rho^2 \geq r_3(N, D, m), \quad p^2 \leq 4(E + \rho^2), \quad (3.18)$$

where  $h_j, \psi_j$  denote  $h$  and  $\psi$  of (3.3) and (3.2) for  $v = v_j$ , and  $\hat{\Phi}_j$  denotes the Dirichlet-to-Neumann map for  $v = v_j$ , where  $j = 1, 2$ .

Formulas (3.16), (3.17) were given in [24], [27]. Estimate (3.18) follows from (3.2), (3.15), (3.16) in a similar way as estimate (3.13) follows from (3.2), (3.3), (3.15).

## 4 Proof of Theorem 2.1

Let

$$\mathbb{L}_\mu^\infty(\mathbb{R}^d) = \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\},$$

$$\|u\|_\mu = \operatorname{ess\,sup}_{p \in \mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \quad \mu > 0. \quad (4.1)$$

Note that

$$w \in \mathbb{W}^{m,1}(\mathbb{R}^d) \implies \hat{w} \in \mathbb{L}_\mu^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d),$$

$$\|\hat{w}\|_\mu \leq c_3(m, d)\|w\|_{m,1} \quad \text{for } \mu = m, \quad (4.2)$$

where  $\mathbb{W}^{m,1}, \mathbb{L}_\mu^\infty$  are the spaces of (2.2), (4.1),

$$\hat{w}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^d. \quad (4.3)$$

Using the inverse Fourier transform formula

$$w(x) = \int_{\mathbb{R}^d} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^d, \quad (4.4)$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq \sup_{x \in \bar{D}} \left| \int_{\mathbb{R}^d} e^{-ipx} (\hat{v}_2(p) - \hat{v}_1(p)) dp \right| \leq \\ &\leq I_1(r) + I_2(r) \quad \text{for any } r > 0, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} I_1(r) &= \int_{|p| \leq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\ I_2(r) &= \int_{|p| \geq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \end{aligned} \quad (4.6)$$

Using (4.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq 2c_3(m, d)N(1 + |p|)^{-m}, \quad p \in \mathbb{R}^d. \quad (4.7)$$

Due to (3.18), we have that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| &\leq |h_2(k, l) - h_1(k, l)| + \frac{c_2(D, m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}}, \\ &\text{for } (k, l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ &E + \rho^2 \geq r_3(N, D, m), \quad p^2 \leq 4(E + \rho^2). \end{aligned} \quad (4.8)$$

Let

$$\begin{aligned} c_4 &= (2\pi)^{-d} \int_{\partial D} dx, \quad L = \max_{x \in \partial D} |x|, \\ \delta &= \|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|, \end{aligned} \quad (4.9)$$

where  $\|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|$  is defined according to (2.5).

Due to (3.17), we have that

$$\begin{aligned} |h_2(k, l) - h_1(k, l)| &\leq c_4 \|\psi_1(\cdot, -l)\|_{\mathbb{L}^\infty(\partial D)} \delta \|\psi_2(\cdot, k)\|_{\mathbb{L}^\infty(\partial D)}, \\ &(k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0. \end{aligned} \quad (4.10)$$

Using (3.11), we find that

$$\begin{aligned} \|\psi(\cdot, k)\|_{\mathbb{L}^\infty(\partial D)} &\leq \sigma \exp\left(|\operatorname{Im} k|L\right), \\ k &\in \Sigma_E, \quad |k| \geq r_1(N, D, m, \sigma). \end{aligned} \quad (4.11)$$

Here and bellow in this section the constant  $\sigma$  is the same that in (3.11).

Combining (4.10) and (4.11), we obtain that

$$\begin{aligned} |h_2(k, l) - h_1(k, l)| &\leq c_4 \sigma^2 e^{2\rho L} \delta, \quad \text{for } (k, l) \in \Theta_E, \\ &\rho = |\operatorname{Im} k| = |\operatorname{Im} l|, \\ &E + \rho^2 \geq r_1^2(N, D, m, \sigma). \end{aligned} \quad (4.12)$$

Using (4.8), (4.12), we get that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| &\leq c_4 \sigma^2 e^{2\rho L} \delta + \frac{c_2(D, m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}}, \\ &p \in \mathbb{R}^d, \quad p^2 \leq 4(E + \rho^2), \quad E + \rho^2 \geq \max\{r_1^2, r_3\}. \end{aligned} \quad (4.13)$$



Let

$$\varepsilon = \left( \frac{1}{2c_2(D, m)Nc_5} \right)^{1/d}, \quad c_5 = \int_{p \in \mathbb{R}^d, |p| \leq 1} dp, \quad (4.14)$$

and  $r_4(N, D, m, \sigma) > 0$  be such that

$$E + \rho^2 \geq r_4(N, D, m, \sigma) \implies \begin{cases} E + \rho^2 \geq r_1^2(N, D, m, \sigma), \\ E + \rho^2 \geq r_3(N, D, m), \\ \left( \varepsilon(E + \rho^2)^{\frac{1}{2d}} \right)^2 \leq 4(E + \rho^2). \end{cases} \quad (4.15)$$

Let

$$c_6 = \int_{p \in \mathbb{R}^d, |p|=1} dp. \quad (4.16)$$

Using (4.6), (4.13), we get that

$$\begin{aligned} I_1(r) &\leq c_5 r^d \left( c_4 \sigma^2 e^{2\rho L} \delta + \frac{c_2(D, m)N \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}} \right), \\ &\quad r > 0, \quad r^2 \leq 4(E + \rho^2), \\ &\quad E + \rho^2 \geq r_4(N, D, m, \sigma). \end{aligned} \quad (4.17)$$

Using (4.6), (4.7), we find that, for any  $r > 0$ ,

$$I_2(r) \leq 2c_3(m, d)Nc_6 \int_r^{+\infty} \frac{dt}{t^{m-d+1}} \leq \frac{2c_3(m, D)Nc_6}{m-d} \frac{1}{r^{m-d}}. \quad (4.18)$$

Combining (4.5), (4.17), (4.18) for  $r = \varepsilon(E + \rho^2)^{\frac{1}{2d}}$  and (4.15), we get that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq c_7(N, D, m, \sigma) \sqrt{E + \rho^2} e^{2\rho L} \delta + \\ &\quad + c_8(N, D, m) (E + \rho^2)^{-\frac{m-d}{2d}} + \frac{1}{2} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}, \\ &\quad E + \rho^2 \geq r_4(N, D, m, \sigma). \end{aligned} \quad (4.19)$$

Let  $\tau' \in (0, 1)$  and

$$\beta = \frac{1 - \tau'}{2L}, \quad \rho = \beta \ln(3 + \delta^{-1}), \quad (4.20)$$

where  $\delta$  is so small that  $E + \rho^2 \geq r_4(N, D, m, \sigma)$ . Then due to (4.19), we have that

$$\begin{aligned} \frac{1}{2} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq \\ &\leq c_7(N, D, m, \sigma) \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{1/2} (3 + \delta^{-1})^{2\beta L} \delta + \\ &\quad + c_8(N, D, m) \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{-\frac{m-d}{2d}} = \\ &= c_7(N, D, m, \sigma) \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{1/2} (1 + 3\delta)^{1-\tau'} \delta^{\tau'} + \\ &\quad + c_8(N, D, m) \left( E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{-\frac{m-d}{2d}}, \end{aligned} \quad (4.21)$$

where  $\tau'$ ,  $\beta$  and  $\delta$  are the same as in (4.20).

Using (4.21), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq c_9(N, D, E, m, \sigma, \tau') (\ln(3 + \delta^{-1}))^{-\frac{m-d}{d}} \quad (4.22)$$

for  $\delta = \|\hat{\Phi}_2 - \hat{\Phi}_1\| \leq \delta_1(N, D, E, m, \sigma, \tau')$ , where  $\delta_1$  is a sufficiently small positive constant. Estimate (4.22) in the general case (with modified  $c_9$ ) follows from (4.22) for  $\delta \leq \delta_1(N, D, E, m, \sigma, \tau')$  and the property that

$$\|v_j\|_{\mathbb{L}^\infty(D)} \leq c_{10}(D, m)N. \quad (4.23)$$

This completes the proof of (2.8).

If  $E \geq 0$  then there is a constant  $\delta_2 = \delta_2(N, D, m, \sigma, \tau') > 0$  such that

$$\delta \in (0, \delta_2) \implies \begin{cases} E + (\beta \ln(3 + \delta^{-1}))^2 \geq r_4(N, D, m, \sigma), \\ E + (\beta \ln(3 + \delta^{-1}))^2 \leq \left( (1 + \sqrt{E})\beta \ln(3 + \delta^{-1}) \right)^2, \\ \beta \ln(3 + \delta^{-1}) \geq 1, \end{cases} \quad (4.24)$$

where  $\beta$  is the same as in (4.20). Combining (4.21), (4.24), we obtain that for  $s \in [0, (m-d)/d]$ ,  $\tau \in (0, \tau')$  and  $\delta \in (0, \delta_2)$  the following estimate holds:

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_{11}(1 + \sqrt{E})\delta^\tau + c_{12}(1 + \sqrt{E})^{s-\frac{m-d}{d}} (\ln(3 + \delta^{-1}))^{-s}, \quad (4.25)$$

where constants  $c_{11}, c_{12} > 0$  depend only on  $N, D, m, \sigma, \tau'$  and  $\tau$ .

Estimate (4.25) in the general case (with modified  $c_{11}$  and  $c_{12}$ ) follows from (4.25) for  $\delta \leq \delta_2(N, D, m, \sigma, \tau')$  and (4.23).

This completes the proof of (2.9)

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