Modeling and numerical simulation of a nonlinear system of piano strings coupled to a soundboard.

Juliette Chabassier, Antoine Chaigne

Proceedings of 20th International Congress on Acoustics, ICA 2010

23–27 August 2010, Sydney, Australia

Modeling and numerical simulation of a nonlinear system of piano strings coupled to a soundboard.

Juliette Chabassier (1) and Antoine Chaigne (2)

(1) POems Team, INRIA, Rocquencourt, FRANCE
(2) UME, ENSTA, Palaiseau, FRANCE

PACS: 43.75.Mn ; 43.75.Zz ; 43.40.Ga

ABSTRACT

Construction of a physical model for the grand piano implies complex and multidimensional phenomena. We present a model of piano strings coupled to a soundboard, and its numerical approximation. Measurements on piano strings and bridge show phantom partials and a time precursor that both cannot be explained by the linear scalar string model. A classical model of nonlinear strings has been written by Morse & Ingard, it implies to consider the longitudinal displacement as well as the standard transversal displacement of the string, in a nonlinear coupled system. Various approximate (polynomial) models have been written from this one, by expanding the nonlinearity (a square root term) around the rest position of the string. We provide a mathematical justification of the most used model. Transmission of the string motion to the rest of the structure is essential from the acoustical point of view. We use a modal approach for the soundboard, and we write a nonstandard reciprocal coupling condition between strings and soundboard at the bridge. Numerical approximation of such a nonlinear, multidimensional and coupled problem is a difficult issue. We use an energy approach to achieve stability, which leads to an innovating implicit numerical scheme.

MODELING THE PIANO

The work presented in this paper is the ongoing subject of the first author's PhD, which consists in modeling a grand piano. This project is the third collaboration between UME-ENSTA, laboratory specialized in mechanics and musical acoustics, and POEMS-INRIA, laboratory specialized in numerical analysis. In the past, two other PhD works have considered the numerical simulation of a timpani [12] and a guitar [8]. Apart from the fluid/structure coupling, the major difficulty of the timpani modeling was to take into account the nonlinear interaction between the timpani stick and the membrane, while the major difficulty of the guitar was to model the soundboard. For such complex, coupled problems, the priority when performing numerical simulations is the stability of the numerical scheme, which is not an easy issue in a nonlinear context. The energy approach has proven to be very efficient and lead to intuitive numerical schemes, especially in order to treat the different coupling conditions.

The two issues mentioned earlier have still to be considered for the piano modeling, for the interaction between hammer and strings and the modeling of the soundboard. As explained later, we must also consider a nonlinear model for the strings. The object of this paper is to explain the modeling choices of the authors, and to construct a stable numerical scheme that represents a system of hammer, strings and soundboard. Because of all the nonlinearities and the couplings, this is not an easy task and this paper will outline the difficulties, and propose efficient solutions.

A first section will present the string models and how to approximate them, then a second section will suggest how to model the hammer/strings interaction. A description of the soundboard and the models we use is then proposed, and finally we will present the whole coupled problem and its numerical approximation.

STRINGS

A nonlinear string model has been introduced by Morse & Ingard [10], in which the string vibration problem is considered as a nonlinear coupled system referred to as "Geometrically Exact Model" (GEM). Conklin [7] has seen in his measurements on piano spectra that some partials could not be explained with the linear string vibration theory, and Bank & Sujbert [2] have shown that these so called "phantom partials" appeared at frequency values being sum or differences of harmonic partials. Several authors [2, 3] have done numerical simulations using a nonlinear string model coming from Taylor developments of the GEM. These developments are made so that the energy of the string remains positive, giving a stable numerical scheme. We present here the GEM as well as a mathematical justification of the developed models, for small transversal initial data.

The geometrically exact model

We consider an infinitely thin string, parametrized at rest with $x \in [0,L]$, where L is the length of the string in meters. We will call μ the lineic mass of the string, A the area of its section, E its Young's modulus, T_0 its tension at rest. All along this

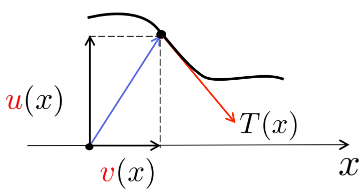

Figure 1: String unknowns and local tension

paper, the notations $\partial_t \cdot \equiv \frac{\partial \cdot}{\partial t}$ and $\partial_x \cdot \equiv \frac{\partial \cdot}{\partial x}$ will denote the partial derivative along time t and space x respectively. Vectorial unknowns will be noted in thick or underlined font unlike the scalar unknowns. The standard nonlinear geometrically exact model [10] couples the transversal displacement of the string $u(x,t)$ to the longitudinal displacement $v(x,t)$:

$$\begin{cases} \mu\,\partial_t^2 u = \partial_x \left[EA\partial_x u - (EA - T_0)\dfrac{\partial_x u}{\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2}} \right] \\ \mu\,\partial_t^2 v = \partial_x \left[EA\partial_x v - (EA - T_0)\dfrac{(1+\partial_x v)}{\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2}} \right] \end{cases} \quad (1)$$

This system is obtained by a geometrical description of the motion of a material point of the string, using the dynamic fundamental law on the elementary system subjected only to tension forces. Hooke's law is used to link the local tension of the string to the relative elongation. We can use a dimensionless system, by introducing

$$\alpha = \frac{EA - T_0}{EA} \in [0,1[, \quad T = L\sqrt{\frac{\mu}{EA}} \quad (2)$$

$$x^* = x/L,\ u^* = u/L,\ v^* = v/L,\ t^* = t/T \quad (3)$$

The new system is, forgetting the starred notations,

$$\begin{cases} \partial_t^2 u = \partial_x \left[\partial_x u - \alpha\dfrac{\partial_x u}{\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2}} \right] \\ \partial_t^2 v = \partial_x \left[\partial_x v - \alpha\dfrac{(1+\partial_x v)}{\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2}} \right] \end{cases} \quad (4)$$

If we introduce the potential energy

$$\mathscr{U}_{ex}(u,v) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - \alpha\left[\sqrt{u^2+(1+v)^2} - (1+v)\right] \quad (5)$$

then the solution of (4) satisfies the preservation of the energy:

$$\frac{d}{dt}\left\{ \frac{1}{2}\int_0^L |\partial_t u|^2 + \frac{1}{2}\int_0^L |\partial_t v|^2 + \int_0^L \mathscr{U}_{ex}(\partial_x u, \partial_x v) \right\} = 0 \quad (6)$$

Expanded models and their asymptotic justification

In the papers of [1–3], a developed model can be found, which has been established with the method presented above, but performing a Taylor development of the square root in (5) and neglecting some terms, in order to keep a positive potential energy. We wanted to give a more precise explanation of the origin of this model, by using standard asymptotic methods on the model (4). We present here the method used and the results obtained, the reader can refer to [4] for calculation details.

We solve, for a small initial amplitude ε on the transversal data,

$$\begin{cases} \dfrac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2} - \dfrac{\partial}{\partial x}\left[\nabla\mathscr{U}_{ex}\left(\dfrac{\partial \mathbf{u}^\varepsilon}{\partial x}\right)\right] = 0, \\ u^\varepsilon(t=0,x) = \varepsilon\,\overline{u}(x), \quad \partial_t u^\varepsilon(t=0,x) = \varepsilon\,\overline{u}_t, \\ v^\varepsilon(t=0,x) = 0, \quad \partial_t v^\varepsilon(t=0,x) = 0. \end{cases} \quad (7)$$

and we seek the solution $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ under the form

$$\left| \begin{aligned} u^\varepsilon &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots \\ v^\varepsilon &= \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \dots \end{aligned} \right. \quad (8)$$

We wonder what would be the system of equation on $\mathbf{u}^\varepsilon$ if we neglect all terms in factor of ε^4. We write the fourth order Taylor development of (6) and inject it in (7). Then we group all the terms in factor of ε, ε^2 and ε^3 respectively, to obtain three different systems on the unknowns u_ℓ and v_ℓ, and we find by calculation that $v_1 = u_2 = v_3 = 0$. If we regroup at this point all the terms together, neglecting ε^4, we find the following system on the new unknowns $\tilde{u}^\varepsilon = \varepsilon u_1 + \varepsilon^3 u_3$ and $\tilde{v}^\varepsilon = \varepsilon^2 u_2$:

$$\begin{cases} \dfrac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} - \dfrac{\partial}{\partial x}\left[(1-\alpha)\dfrac{\partial \tilde{u}^\varepsilon}{\partial x} + \alpha\dfrac{\partial \tilde{u}^\varepsilon}{\partial x}\dfrac{\partial \tilde{v}^\varepsilon}{\partial x} + \dfrac{\alpha}{2}\left(\dfrac{\partial \tilde{u}^\varepsilon}{\partial x}\right)^3\right] = 0, \\ \dfrac{\partial^2 \tilde{v}^\varepsilon}{\partial t^2} - \dfrac{\partial}{\partial x}\left[\dfrac{\partial \tilde{v}^\varepsilon}{\partial x} + \dfrac{\alpha}{2}\left(\dfrac{\partial \tilde{u}^\varepsilon}{\partial x}\right)^2\right] = 0, \end{cases}$$

which corresponds to the Cauchy problem (7) replacing the potential energy $\mathscr{U}_{ex}$ with

$$\mathscr{U}_{BS}(u,v) = \frac{1-\alpha}{2}u^2 + \frac{1}{2}v^2 + \frac{\alpha}{2}u^2 v + \frac{\alpha}{8}u^4. \quad (9)$$

This energy is the one found empirically in [1–3]. We give here a restriction: this model is valid only if the string is excited transversally.

Remark 1 *For a more accurate development in long time, perturbation methods can be used [11]. The principle is to consider different time scales, leading to more complex expressions of the equations satisfied by the developed solution. See figure 2 for a comparison between classical method and multi-scale method.*

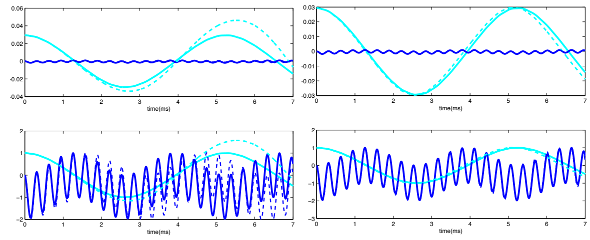

Figure 2: Comparison between the solution of developed systems (dashed line) and the numerical solution of the original system (7) (continuous line), using Taylor method (left) or perturbation method (right). The lower image is normalized, while the upper one is not.

At this point of our work, it does not represent any further difficulty to tackle the exact model rather than the developed one. The string model will, in the sequel, be represented by the potential energy $\mathscr{U}$ which can either be $\mathscr{U}_{ex}$ or $\mathscr{U}_{BS}$, or any other positive potential energy. This statement has yet to be qualified, since we have to to enlarge our class of models in order to consider stiff strings.

Stiff string

Dispersion plays a great role in the timbre of musical instruments. This is why the modeling of stiffness is a major issue when one aims at synthesizing musical instruments' sounds. We present here an unusual point of view for the modeling of string stiffness: we propose to introduce an angle φ which will be coupled with the string displacement, and use Timoshenko [14] beam theory in order to write the coupling term. This approach is quite close to considering a prestressed nonlinear beam.

The stiff string equation is the following system:

$$\begin{cases} \partial_t(\rho A\,\partial_t u) - \partial_x\Big[AGk'(\partial_x u - \varphi) + EA\,\partial_x u \\ \qquad - (EA - T_0)\dfrac{\partial_x u}{\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2}}\Big] = 0, \\ \partial_t(\rho A\,\partial_t v) - \partial_x\Big[EA\,\partial_x v \\ \qquad - (EA - T_0)\dfrac{1+\partial_x v}{\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2}}\Big] = 0, \\ \partial_t(\rho I\,\partial_t \varphi) - \partial_x\Big[EI\,\partial_x \varphi\Big] - AGk'(\partial_x u - \varphi) = 0. \end{cases} \quad (10)$$

If we write the kinetic and potential energies as:

$$\begin{cases} \mathscr{T}(\partial_t \mathbf{q}) = \frac{\rho A}{2}(\partial_t u)^2 + \frac{\rho A}{2}(\partial_t v)^2 + \frac{\rho I}{2}(\partial_t \varphi)^2 & \text{(11a)} \\ \mathscr{U}(\mathbf{q}, \partial_x \mathbf{q}) = \frac{EA}{2}(\partial_x u)^2 + \frac{EA}{2}(\partial_x v)^2 + \frac{EI}{2}(\partial_x \varphi)^2 & \\ \qquad - (EA - T_0)\sqrt{(\partial_x u)^2 + (1+\partial_x v)^2} & \text{(11b)} \\ \qquad + \frac{AGk'}{2}(\partial_x u - \varphi)^2 & \end{cases}$$

where

$$\left| \begin{array}{l} \mathbf{q} = (u, v, \varphi),\ N \text{ the size of the system (here, 3)}, \\ \rho \text{ is the volumic mass of the string: } \mu = \rho A, \\ I \text{ is the stiffness inertia coefficient of the string,} \\ G \text{ is the shear coefficient,} \\ k' \text{ is Timoshenko's parameter,} \end{array} \right. \quad (12)$$

then our stiff string can be seen as a system of the form

$$\partial_t \nabla \mathscr{T}(\partial_t \mathbf{q}) - \partial_x \nabla_{\partial_x \mathbf{q}} \mathscr{U}(\mathbf{q}, \partial_x \mathbf{q}) + \nabla_{\mathbf{q}} \mathscr{U}(\mathbf{q}, \partial_x \mathbf{q}) = 0 \quad (13)$$

which preserves the energy

$$\mathscr{E}_s(\mathbf{q}, \partial_x \mathbf{q}, \partial_t \mathbf{q}) = \int_0^L \mathscr{T}(\partial_t \mathbf{q}) + \int_0^L \mathscr{U}(\mathbf{q}, \partial_x \mathbf{q}). \quad (14)$$

We will for now on consider that

$$\boxed{\mathscr{T}(\partial_t \mathbf{q}) = \sum_k \frac{1}{2}\mathscr{T}_k(\partial_t q_k)^2.} \quad (15)$$

Remark 2 *In all what follows, we will abusively mingle notations and treat the triplet (u, v, φ) as (q_1, q_2, q_3) and equally refer to the unknowns with their number in $\mathbf{q}$ or their letter symbol in the triplet. To illustrate this fact, we shall as well, in our present case, write that $\mathscr{T}_1 = \mathscr{T}_2 = \rho A$, $\mathscr{T}_3 = \rho I$ or $\mathscr{T}_u = \mathscr{T}_v = \rho A$, $\mathscr{T}_\varphi = \rho I$.*

Numerical approximation

We have shown in [5] that in a certain class of energy preserving numerical schemes, it was impossible to construct an explicit scheme unless the original equation is linear. We have shown that the intuitive scheme, that approximate the gradient of $\mathscr{U}$ with a directional finite difference does not, in general, lead to a preserving scheme. The numerical scheme that we propose here is an implicit, second order accurate in time, unconditionally stable numerical scheme.

In a very general context, we had to introduce the functions

$$\begin{cases} \delta_{\partial_x,k} \mathscr{U}(\mathbf{q}; \partial_x q_k^{n+1}, \partial_x q_k^{n-1}, \partial_x q_{\ell\neq k}^{n+\sigma(\ell)}) = \\ \qquad \dfrac{\mathscr{U}(\mathbf{q}; \partial_x q_k^{n+1}, \partial_x q_{\ell\neq k}^{n+\sigma(\ell)}) - \mathscr{U}(\mathbf{q}; \partial_x q_k^{n-1}, \partial_x q_{\ell\neq k}^{n+\sigma(\ell)})}{\partial_x q_k^{n+1} - \partial_x q_k^{n-1}} \\ \delta_k \mathscr{U}(q_k^{n+1}, q_k^{n-1}, q_{\ell\neq k}^{n+\sigma(\ell)}; \partial_x \mathbf{q}) = \\ \qquad \dfrac{\mathscr{U}(q_k^{n+1}, q_{\ell\neq k}^{n+\sigma(\ell)}; \partial_x \mathbf{q}) - \mathscr{U}(q_k^{n-1}, q_{\ell\neq k}^{n+\sigma(\ell)}; \partial_x \mathbf{q})}{q_k^{n+1} - q_k^{n-1}} \end{cases}$$

where σ is a function mapping Σ_k to $\{-1, 1\}$, while Σ_k is the set of all variables except $k : \Sigma_k = [1, \ldots, N] \setminus \{k\}$. As an example, $\Sigma_u = \{v, \varphi\}$. We introduce the coefficients $\zeta(\sigma)$, and we write the numerical scheme for any test function ψ in the Lagrange $\mathbb{P}_k$ finite elements functions basis:

$$\begin{aligned} &\oint_0^L \mathscr{T}_k \frac{q_k^{n+1} - 2q_k^n + q_k^{n-1}}{\Delta t^2}\psi + \\ &\frac{1}{2}\oint_0^L \sum_{\sigma\in\Sigma_k} \zeta(\sigma)\Big[\delta_{\partial_x,k}\mathscr{U}(q_k^{n+1}, q_{l\neq k}^{n+1}; \partial_x q_k^{n+1}, \partial_x q_k^{n-1}, \partial_x q_{l\neq k}^{n+\sigma(l)}) + \\ &\qquad \delta_{\partial_x,k}\mathscr{U}(q_k^{n-1}, q_{l\neq k}^{n-1}; \partial_x q_k^{n+1}, \partial_x q_k^{n-1}, \partial_x q_{l\neq k}^{n+\sigma(l)})\Big]\partial_x \psi + \\ &\frac{1}{2}\oint_0^L \sum_{\sigma\in\Sigma_k} \zeta(\sigma)\Big[\delta_k \mathscr{U}(q_k^{n+1}, q_k^{n-1}, q_{l\neq k}^{n+\sigma(l)}; \partial_x q_k^{n+1}, \partial_x q_{l\neq k}^{n+1}) + \\ &\qquad \delta_k \mathscr{U}(q_k^{n+1}, q_k^{n-1}, q_{l\neq k}^{n+\sigma(l)}; \partial_x q_k^{n-1}, \partial_x q_{l\neq k}^{n-1})\Big]\psi = 0 \end{aligned} \quad (16)$$

preserves the energy

$$\begin{aligned} \mathscr{E}_s^{n+1/2} = \frac{1}{2}\oint_0^L \sum_{k=1}^N \mathscr{T}_k \left|\frac{q_k^{n+1} - q_k^n}{\Delta t}\right|^2 + \\ \oint_0^L \frac{\mathscr{U}(\mathbf{q}^{n+1}, \partial_x \mathbf{q}^{n+1}) + \mathscr{U}(\mathbf{q}^n, \partial_x \mathbf{q}^n)}{2} \end{aligned} \quad (17)$$

HAMMER / STRINGS INTERACTION

The interaction with the hammer is essential for timbre quality and realism in sound synthesis. Several studies have been made on the model which shall be used [13]. We will consider a contact with nonlinear interaction and hysteresis. The reality and geometry of the piano leads us to take into account the coupling of several (N_c) strings with only one hammer. We call $\mathbf{q}_i = (u_i, v_i, \varphi_i)$ the triplet of unknowns of the i^{th} string. As the strings are slightly detuned (their tension at rest T_0 is different), which makes a different potential energy per string $\mathscr{U}_i$.

The hammer is represented by a point of the space moving along a straight line. It is then a scalar unknown that we will call $\xi(t)$. The parameters defining the hammer are M^{ham}, K_i^{ham} and R_i^{ham}, which could depend on the struck string (by damage, for instance), and the function Φ which links the force of interaction to the crushing of the hammer. The contact is distributed along the string through a repartition function δ^{ham}, and we note $\langle u_i \rangle = \int_0^L \delta^{\text{ham}}(x - x^{\text{ham}}) u_i(x) dx$ the value of u_i averaged by δ^{ham}. The coupled system can now be written

$$\begin{cases} M^{\text{ham}} \dfrac{d^2\xi}{dt^2}(t) = -\sum_i F_i^{\text{ham}}(t) & \text{(18a)} \\ F_i^{\text{ham}}(t) = K_i^{\text{ham}}\Phi\big(\langle u_i\rangle(t) - \xi(t)\big) + R_i^{\text{ham}}\dfrac{d}{dt}\Phi\big(\langle u_i\rangle(t) - \xi(t)\big) & \text{(18b)} \\ \partial_t \nabla \mathscr{T}(\partial_t \mathbf{q}_i) - \partial_x \nabla_{\partial_x \mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) + \nabla_{\mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) = & \\ \qquad F_i^{\text{ham}}(t)\begin{pmatrix} \delta^{\text{ham}}(x - x^{\text{ham}}) \\ 0 \\ 0 \end{pmatrix}, \quad \forall\, i & \text{(18c)} \end{cases}$$

We introduce the function Ψ such that $\Psi' = -\Phi$ and the previous system preserves the energy as soon as R_i^{ham} vanishes:

$$\begin{aligned} \mathscr{E}_{h,s}(t) = \sum_{i=1}^{N_c}\Big[\int_0^L \mathscr{T}(\partial_t \mathbf{q}_i)\Big] + \frac{M^{\text{ham}}}{2}\left|\xi'(t)\right|^2 + \\ \sum_{i=1}^{N_c}\Big[\int_0^L \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) + K_i^{\text{ham}}\Psi\big(\langle u_i\rangle(t) - \xi(t)\big)\Big] \end{aligned} \quad (19)$$

Numerical approximation

An energy preserving numerical approximation of (18) can be obtained by using (16) for each component of each string, and adding a line for the hammer as well as the contributions coming from the interactions. Since the function Φ is nonlinear, we have to treat the hammer implicitly with the string, which is not a great over cost since it is a scalar unknown. The line for the hammer should be:

$$M^{\text{ham}}\,\frac{\xi^{n+1}-2\xi^{n}+\xi^{n-1}}{\Delta t^2}=\sum_{i=1}^{N_c}\Bigg[K_i^{\text{ham}}\,\frac{\Psi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Psi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}-R_i^{\text{ham}}\,\frac{\Phi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Phi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{2\Delta t}\Bigg] \tag{20}$$

and the following contribution should be added to the u-line of each string, for any test function ψ:

$$\Bigg[K_i^{\text{ham}}\,\frac{\Psi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Psi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}-R_i^{\text{ham}}\,\frac{\Phi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Phi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{2\Delta t}\Bigg]\langle\psi\rangle \tag{21}$$

where $U_{i,h}$ is the vector of unknowns linked to the degrees of freedom of the string for the variable u_i, $V_{i,h}$ for v_i and $\Phi_{i,h}$ for φ_i. This scheme preserves the energy

$$\mathscr{E}_{h,s}^{n+1/2}=\sum_{i=1}^{N_c}\Bigg[\frac{1}{2}\oint_0^L\sum_{k=1}^{N}\mathscr{T}_k\left|\frac{q_{i,k}^{n+1}-q_{i,k}^{n}}{\Delta t}\right|^2\Bigg]+\frac{M^{\text{ham}}}{2}\left|\frac{\xi^{n+1}-\xi^{n}}{\Delta t}\right|^2+\sum_{i=1}^{N_c}\Bigg[\oint_0^L\frac{\mathscr{U}_i(\mathbf{q}_{i,h}^{n+1},\partial_x\mathbf{q}_{i,h}^{n+1})+\mathscr{U}_i(\mathbf{q}_{i,h}^{n},\partial_x\mathbf{q}_{i,h}^{n})}{2}+K_i^{\text{ham}}\,\frac{\Psi(\langle U_{i,h}^{n+1}-\xi^{n+1}\rangle)+\Psi(\langle U_{i,h}^{n}-\xi^{n}\rangle)}{2}\Bigg] \tag{22}$$

SOUNDBOARD

Geometry

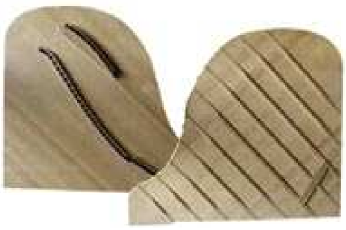

Figure 3: Soundboard with brigdes (left) and ribs (right).
Source: www.richardlipp.com.au

The piano soundboard is a wooden plate (mostly in spruce) with a variable shape depending on the piano. It is stiffened by ribs, which are glued perpendicularly to the fibers of the wood, and a bridge (see figure 3). The main purpose of the ribs is to restore a certain isotropy in a fundamentally orthotropic material: wood. This objective is achieved for the first modes (see [9]) but is irrelevant for high wavelengths. The soundboard can be modeled in different ways, from the least to the most sophisticated in terms of details, as:

- an isotropic plate,
- an orthotropic plate with variations of thickness or materials at the place of the ribs and bridge,
- an orthotropic plate with variations of thickness or materials at the place of the ribs and bridge, but only on one side each,
- a coupling between an orthotropic plate and beams modeling the ribs and the bridge.

Remark 3 *Even if this last model will be quite difficult to consider and implement, it should represent a wide improvement in the soundboard modeling, since measurements on the bridge have shown that it has eigen vibration modes (see figure 4).*

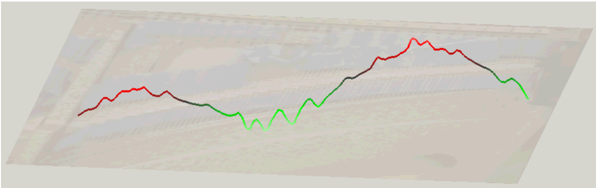

Figure 4: Motion of the upper part of the main bridge of a PLEYEL upright piano, at frequency f=2,52 kHz. The shape is very similar to the one of a bending mode for a free-free beam. As a consequence, the bridge mobility at the end of the strings varies substantially from one note to the next, which, in turn, induces significant differences in the tone duration.

Modeling

Whatever choice we make in the above cases, we must choose a set of equations which govern the soundboard motion. The soundboard's thickness is small compared to its other dimensions. It fits the frame of plate theories, reducing the great amount of unknowns of linear elasticity to only three or one unknown. The price to pay for this simplification is several hypothesis on the deformations, as for instance an hypothesis of non-coupling between transversal and longitudinal displacements. The two models that we will consider are Kirchhoff Love and Mindlin Reissner models. They both model a stiff plate, and can take into account orthotropy, variations in the thickness or the material. The Kirchhoff Love model can be derived from the Mindlin Reissner model when making an additional assumption: the straight sections remain orthogonal to the neutral fiber (excluding the ribs and the bridges).

These two plate models can be grouped into the following general frame, with A and B two selfadjoint operators. We seek a displacement field $u:\omega\times\mathbb{R}^+\rightarrow\mathbb{R}$ and an angle field $u:\omega\times\mathbb{R}^+\rightarrow\mathbb{R}^2$ such that:

$$\left\{\begin{aligned} &c_\theta\,\frac{\partial^2\underline{\theta}}{\partial t^2}+A\,\underline{\theta}+C\,u=0 && \text{(23a)}\\ &c_u\,\frac{\partial^2 u}{\partial t^2}+B\,u+C^*\,\underline{\theta}=f\,\chi_\omega(x,y) && \text{(23b)}\end{aligned}\right.$$

where χ_ω is a repartition function which distributes the force $f(t)$ over the plate.

	Mindlin Reissner	Kirchhoff Love
c_θ	$\rho\,\delta^3/12$	0
c_u	$\rho\,\delta$	$\rho\,\delta$
$A\,\underline{\theta}$	$-(\delta^3/12)\,\underline{\mathrm{Div}}(\underline{\underline{C}}\,\varepsilon(\underline{\theta}))+\delta\,G\,\underline{\theta}$	0
$B\,u$	$-\delta\,\mathrm{div}(G\,\underline{\nabla}u)$	$(\delta^3/12)\,\mathrm{div}\,\underline{\mathrm{Div}}\,\mathbf{C}\,\varepsilon(\underline{\nabla}u)$
$C\,u$	$\delta\,G\,\underline{\nabla}u$	0
$C^*\,\underline{\theta}$	$-\delta\,G\,\mathrm{div}\,\underline{\theta}$	0

Numerical approximation

We proceed to the semi discretization of the plate problem, the problem becomes to seek $(\underline{\theta}_p,u_p):[0,T]\rightarrow(\Theta_p^{disc},\mathscr{U}_p^{disc})$ such that $\forall\,\underline{\theta}^*\in\Theta_p,\ \forall\,u_p^*\in\mathscr{U}_p$

$$\left\{\begin{aligned} &\int_\omega c_\theta\,\frac{\partial^2\underline{\theta}_p}{\partial t^2}\cdot\underline{\theta}_p^*+\int_\omega A\,\underline{\theta}_p\cdot\underline{\theta}_p^*+\int_\omega(C\,u_p)\cdot\underline{\theta}_p^*=0,\\ &\int_\omega c_u\,\frac{\partial^2 u_p}{\partial t^2}\,u_p^*+\int_\omega B\,u_p u_p^*+\int_\omega(C^*\,\underline{\theta}_p)u_p^*=f\int_\omega\chi_\omega(x,y)\,u_p^*.\end{aligned}\right. \tag{24}$$

If we introduce $(\underline{\zeta}_k)_k$ a basis of Θ_p^{disc}, and $(\psi_n)_n$ a basis $\mathscr{U}_p^{disc}$, and the matrices

$$(M_h^{\theta_p})_{k,\ell} = \oint_\omega c_\theta\, \underline{\zeta}_k \cdot \underline{\zeta}_\ell, \qquad (M_h^{u_p})_{n,m} = \oint_\omega c_u\, \psi_n \psi_m, \quad (25)$$

$$(A_h)_{k,\ell} = \oint_\omega A\, \underline{\zeta}_k \cdot \underline{\zeta}_\ell, \qquad (B_h)_{n,m} = \oint_\omega B \psi_n \psi_m, \quad (26)$$

$$(C_h)_{k,n} = \oint_\omega (C\psi_n) \cdot \underline{\zeta}_k, \qquad (J_h)_n = \oint_\omega \chi_\omega \psi_n, \quad (27)$$

the problem becomes to find the vectors $U_{p,h}$ and $\Theta_{p,h}$ s.t.

$$\begin{cases} \partial_t^2 (M_h^{\theta_p}\, \Theta_{p,h}) + A_h \Theta_{p,h} + C_h U_{p,h} = 0, \\ \partial_t^2 (M_h^{u_p}\, U_{p,h}) + B_h U_{p,h} + {}^t C_h \Theta_{p,h} = f\, J_h. \end{cases} \quad (28)$$

which is equivalent to

$$\partial_t^2\, \mathbb{M}_h \Lambda_h^{EF} + \mathbb{R}_h \Lambda_h^{EF} = \begin{pmatrix} f\, J_h \\ 0 \end{pmatrix},$$

where $\Lambda_h^{EF} = \begin{pmatrix} U_{p,h} \\ \Theta_{p,h} \end{pmatrix}$, $\mathbb{M}_h = \begin{pmatrix} M_h^{u_p} & 0 \\ 0 & M_h^{\theta_p} \end{pmatrix}$, $\mathbb{R}_h = \begin{pmatrix} B_h & {}^tC \\ C & A_h \end{pmatrix}$.

We diagonalize the real symmetric matrix $\mathbb{R}_h$ in a $\mathbb{M}_h$-orthogonal basis. Let Λ_h^{mod} be the basis of eigenvectors. Then, there exist a diagonal matrix $\mathbb{D}_h$ and a matrix $\mathbb{P}_h$ orthogonal for the scalar product $\langle \mathbb{M}_h \cdot, \cdot \rangle$ such that

$$\begin{cases} {}^t\mathbb{P}_h \mathbb{R}_h \mathbb{P}_h = \mathbb{D}_h \\ {}^t\mathbb{P}_h \mathbb{M}_h \mathbb{P}_h = \mathbb{I}_d \end{cases} \text{and} \begin{cases} \Lambda_h^{EF} & = \mathbb{P}_h \Lambda_h^{mod} \\ \Lambda_h^{mod} & = {}^t\mathbb{P}_h \mathbb{M}_h \Lambda_h^{EF} \end{cases} \quad (29)$$

The problem is now constituted of decoupled ODEs with second member and initial values: $\forall t \in [t^{n-1/2}, t^{n+1/2}]$

$$\begin{cases} \partial_t^2 \Lambda_h^{mod} + \mathbb{D}_h \Lambda_h^{mod} = f\, {}^t\mathbb{P}_h \begin{pmatrix} J_h \\ 0 \end{pmatrix}, \\ \Lambda_h^{mod}(t = t^{n-1/2}) = \Lambda_h^{mod,n-1/2}, \\ \partial_t \Lambda_h^{mod}(t = t^{n-1/2}) = \partial_t \Lambda_h^{mod,n-1/2}. \end{cases} \quad (30)$$

Which we can solve exactly, using the technique of [8]. We introduce the operators $\mathscr{S}_{\Delta t}(U_0, U_1)$, which gives the solution, after Δt time, to the homogeneous problem with initial values U_0 and U_1, and $\mathscr{R}_{n,\Delta t}(F)$ which gives the solution to the problem between $t^{n-1/2}$ and $t^{n+1/2}$ with second member F (see (43c)). Out of linearity, we can write that

$$\Lambda_h^{mod}(t^{n+1/2}) = \mathscr{S}_{\Delta t}(\Lambda_h^{mod,n-1/2}, \partial_t \Lambda_h^{mod,n-1/2}) + f\, \mathscr{R}_{n,\Delta t}\left({}^t\mathbb{P}_h \begin{pmatrix} J_h \\ 0 \end{pmatrix} \right). \quad (31)$$

The energy preserved by this method, if $f \equiv 0$, is

$$\begin{aligned} \mathscr{E}_p^{n+1/2} &= \frac{1}{2} \|\partial_t \Lambda_h^{mod,n+1/2}\|^2 + \frac{1}{2} \|\Lambda_h^{mod,n+1/2}\|^2_{\mathbb{D}_h} \\ &= \frac{1}{2} \|\partial_t \Lambda_h^{EF,n+1/2}\|^2_{\mathbb{M}_h} + \frac{1}{2} \|\Lambda_h^{EF,n+1/2}\|^2_{\mathbb{R}_h} \end{aligned} \quad (32)$$

Remark 4 *An interesting possibility is to solve the problem only on the first eigen modes of the soundboard, which concentrates most energy, by considering a truncated (hence rectangular) matrix $\tilde{\mathbb{P}}_h$. This approach justifies the (expensive) diagonalization of the matrix $\mathbb{M}_h$ since it entitles us to use a physically more relevant basis of approximation with less degrees of freedom.*

COMPLETE COUPLED PROBLEM

Complete model

The system that we consider is the following:

$$\begin{cases} M^{\text{ham}} \dfrac{d^2 \xi}{dt^2}(t) = -\sum_i F_i^{\text{ham}}(t) & (33a) \\ \left| \begin{aligned} F_i^{\text{ham}}(t) = & K_i^{\text{ham}} \Phi\big(\langle u_i \rangle(t) - \xi(t)\big) + \\ & R_i^{\text{ham}} \frac{d}{dt} \Phi\big(\langle u_i \rangle(t) - \xi(t)\big) \end{aligned} \right. & (33b) \\ \left| \begin{aligned} & \partial_t \nabla \mathscr{T}(\partial_t \mathbf{q}_i) - \partial_x \nabla_{\partial_x \mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) + \nabla_{\mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) = \\ & \qquad F_i^{\text{ham}}(t) \begin{pmatrix} \delta^{\text{ham}}(x - x^{\text{ham}}) \\ 0 \\ 0 \end{pmatrix}, \quad \forall\, i \end{aligned} \right. & (33c) \\ \begin{cases} c_\theta \dfrac{\partial^2 \underline{\theta}_p}{\partial t^2} + A\, \underline{\theta}_p + C\, u_p = 0 \\ c_u \dfrac{\partial^2 u_p}{\partial t^2} + B\, u_p + C^* \underline{\theta}_p = \Big[-\sum_i F_i^{\text{coupl}}(t) \Big] \chi_\omega(x,y) \end{cases} & (33d) \\ F_i^{\text{coupl}}(t) = \underline{\nu} \cdot \nabla^{u,v}_{\partial_x \mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i)(x = L, t) & (33e) \end{cases}$$

Coupling condition at the bridge are (see remark (5)):

$$\begin{cases} \begin{pmatrix} u_i(x = L, t) \\ v_i(x = L, t) \end{pmatrix} = \underline{\nu} \times \displaystyle\int_\omega u_p(x,y,t)\, \chi_\omega(x,y)\, dx\, dy & (34a) \\ \partial_x \varphi_i(x = L, t) = 0 & (34b) \end{cases}$$

All the other limit conditions are standard (string attached and plate embedded). The only non zero initial condition is the hammer's, which has an initial velocity.

The solution of this system preserves the following energy:

$$\begin{cases} \mathscr{E}(t) = \mathscr{E}^k(t) + \mathscr{E}^p(t), \qquad \text{where:} & (35a) \\ \mathscr{E}^{kin}(t) = \displaystyle\sum_{i=1}^{N_c} \Big[\int_0^L \mathscr{T}(\partial_t \mathbf{q}_i) \Big] + \frac{c_u}{2} \|\partial_t u_p\|^2_\omega + \frac{c_\theta}{2} \|\partial_t \theta_p\|^2_\omega + \\ \qquad \dfrac{M^{\text{ham}}}{2} |\xi'(t)|^2 & (35b) \\ \mathscr{E}^{pot}(t) = \displaystyle\sum_{i=1}^{N_c} \Big[\int_0^L \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) + K_i^{\text{ham}} \Psi\big(\langle u_i \rangle(t) - \xi(t)\big) \Big] \\ \qquad + \dfrac{1}{2} \langle A \underline{\theta}_p, \underline{\theta}_p \rangle + \dfrac{1}{2} \langle B u_p, u_p \rangle + \dfrac{1}{2} \|C u_p + \underline{\theta}_p\|^2_\omega & (35c) \end{cases}$$

The proof of energy preservation is obtained classically, multiplying each line with the appropriate time derivative and adding all lines. We must notice that we use (34a) after derivation in time.

We consider the variational spaces such that

$$\begin{aligned} \xi &: [0,T] \to \mathbb{R}, \\ (\mathbf{q}_i)_i = (u_i, v_i, \varphi_i)_i &: [0,T] \to \mathscr{Q} = \mathscr{U}_c \times \mathscr{V}_c \times \Phi_c, \\ (\underline{\theta}_p, u_p) &: [0,T] \to (\Theta_p, \mathscr{U}_p) \end{aligned} \quad (36)$$

Writing the variational formulation leads us to calculate the following formal equation, for each string i:

$$\begin{aligned} & \int_0^L \partial_t \nabla \mathscr{T}(\partial_t \mathbf{q}_i) \cdot \mathbf{q}_i^* + \int_0^L \nabla_{\partial_x \mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \partial_x \mathbf{q}_i^* - \\ & \quad \Big[\nabla^{u,v}_{\partial_x \mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i)(x = L, t) \cdot \begin{pmatrix} u_i^*(x = L, t) \\ v_i^*(x = L, t) \end{pmatrix} \Big] \\ & \quad + \int_0^L \nabla_{\mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \mathbf{q}_i^* = F_i^{\text{ham}}(t) \begin{pmatrix} \langle u_i^* \rangle(t) \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (37)$$

The standard choice of $\mathscr{U}_c$, $\mathscr{V}_c$, Φ_c, Θ_p, $\mathscr{U}_p$ is to set

$$\begin{cases} (\mathscr{U}_c,\mathscr{V}_c) = \Big\{ u_c \in H^1([0,L]),\ v_c \in H^1([0,L]) \ \Big| \\ \quad \begin{pmatrix} u_c(x=0) \\ v_c(x=0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_c(x=L) \\ v_c(x=L) \end{pmatrix} \cdot \underline{\tau} = 0 \Big\}, & (38a) \\ \Phi_c = H^1([0,L]), & (38b) \\ \Theta_p = H_0^1(\omega)^2,\ \mathscr{U}_p = H_0^1(\omega). & (38c) \end{cases}$$

But in this case, the expression $\underline{\nu} \cdot \nabla^{u,v}_{\partial_x \mathbf{q}} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i)(x=L,t)$ does not make sense anymore. We choose to introduce new unknowns, Lagrange multipliers, $F_i^{\text{coupl}} = [0,T] \to \mathbb{R}^{N_c}$, which will be implicitly determined by the equation (34a).

Semi discretization in space

We choose, to approximate the string spaces $(\mathscr{U}_c, \mathscr{V}_c, \Phi_c)$ (which are very close to $H^1([0,L])$), Lagrange finite elements $\mathbb{P}_k$, with, associated to the degree of freedom j, the basis function ϕ_j. We introduce

$$(M_h^c)_{i,j} = \oint_0^L \phi_i \phi_j, \text{ and } \alpha_j = \phi_j(L).$$

As previously, we call $U_{i,h}$ the vector of coordinates of u_i in the basis $(\phi_j)_j$, $V_{i,h}$ the coordinates of v_i, and $\Phi_{i,h}$ the coordinates of φ_i.

The semi discrete problem is to find

$$\begin{aligned} \xi &: [0,T] \to \mathbb{R}, \\ (U_{i,h}, V_{i,h}, \Phi_{i,h})_i &: [0,T] \to \mathscr{U}_c^{disc} \times \mathscr{V}_c^{disc} \times \Phi_c^{disc}, \\ (\Theta_{p,h}, U_{p,h}) &: [0,T] \to (\Theta_p^{disc}, \mathscr{U}_p^{disc}), \\ F_i^{\text{coupl}} &: [0,T] \to \mathbb{R}^{N_c}, \end{aligned} \tag{39}$$

such that

$$\begin{cases} M^{\text{ham}} \dfrac{d^2\xi}{dt^2}(t) = -\sum_i F_i^{\text{ham}}(t) & (40a) \\[2ex] F_i^{\text{ham}}(t) = K_i^{\text{ham}} \Phi(\langle \sum_p U_{i,h,p}\phi_p \rangle(t) - \xi(t)) + \\ \qquad R_i^{\text{ham}} \dfrac{d}{dt} \Phi(\langle \sum_p U_{i,h,p}\phi_p \rangle(t) - \xi(t)) & (40b) \\[2ex] \begin{cases} \mathscr{T}_u \partial_t^2 (M_h^c U_{i,h})_j + \oint_0^L \partial_{\partial_x u} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \partial_x \phi_j + \\ \quad \oint_0^L \partial_u \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \phi_j = F_i^{\text{coupl}}(t)\, \alpha_j\, \nu_u + F_i^{\text{ham}}(t) \langle \phi_j \rangle, \\ \mathscr{T}_v \partial_t^2 (M_h^c V_{i,h})_j + \oint_0^L \partial_{\partial_x v} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \partial_x \phi_j + \\ \quad \oint_0^L \partial_v \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \phi_j = F_i^{\text{coupl}}(t)\, \alpha_j\, \nu_v, \\ \mathscr{T}_\varphi \partial_t^2 (M_h^c \Phi_{i,h})_j + \oint_0^L \partial_{\partial_x \varphi} \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \partial_x \phi_j + \\ \quad \oint_0^L \partial_\varphi \mathscr{U}_i(\mathbf{q}_i, \partial_x \mathbf{q}_i) \cdot \phi_j = 0, \quad \forall\, i, \forall\, j. \end{cases} & (40c) \\[2ex] \begin{cases} \partial_t^2 (M_h^{\theta_p} \Theta_{p,h}) + A_h \Theta_{p,h} + C_h U_{p,h} = 0, \\ \partial_t^2 (M_h^{u_p} U_{p,h}) + B_h U_{p,h} + {}^t C_h \Theta_{p,h} = \Big[-\sum_i F_i^{\text{coupl}}(t) \Big] J_h \end{cases} & (40d) \\[2ex] \begin{pmatrix} U_{i,h} \cdot \alpha \\ V_{i,h} \cdot \alpha \end{pmatrix} \cdot \underline{\nu} = U_{p,h} \cdot J_h & (40e) \end{cases}$$

Remark 5 *The coupling condition at the bridge is a delicate issue that needs to be treated with caution. As drawn in figure 5, the bridge (directed by the vector ν) is not orthogonal to the position of the string at rest, hence the variable tension of the string acts on the bridge also through its longitudinal part. For this reason, longitudinal waves are transmitted to the bridge, giving to the piano its "striking" sound, and resulting in high frequency precursor at the bridge (see [6] for a further study and measurements regarding this issue). In a first approach, we decided to model the coupling condition as a contact condition between the last point of the string and a distribution zone on the soundboard, before coming to a more realistic model of the whole bridge.*

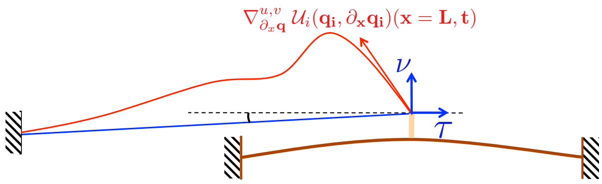

Figure 5: Schematic of the bridge condition.

Time discretization

The introduction of the Lagrange multiplier is a standard tool often used in linear problems where arithmetic combinations of linear sub-problems can lead to eliminate all unknowns but the Lagrange multiplier, leading to an equation on this multiplier only. It is then possible to determine it, and to treat all sub-problems independently. Here, the nonlinearity of the string problem forbids us to do the same. We will still be able to decouple the plate problem from the hammer/strings/Lagrange multiplier problem, which is rather good since the Lagrange multiplier (N_c scalar unknowns) does not represent a major cost compared to the string. The idea is to notice that we can write the solution of (40d) in the eigenvectors basis as the superposition of elementary solutions (see (31)). The coupling constraint that we have to deal with is written to obtain an energy preservation, using a discrete equivalent of the time derivative of (34a):

$$\begin{pmatrix} \dfrac{U_{i,h}^{n+1} - U_{i,h}^{n-1}}{2\Delta t} \cdot \alpha \\ \dfrac{V_{i,h}^{n+1} - V_{i,h}^{n-1}}{2\Delta t} \cdot \alpha \end{pmatrix} \cdot \underline{\nu} = \frac{\Lambda_h^{mod,n+1/2} - \Lambda_h^{mod,n+1/2}}{\Delta t} \cdot {}^t\mathbb{P}_h \begin{pmatrix} J_h \\ 0 \end{pmatrix} \tag{41}$$

which can be seen as the following equation, linear in all the F_i^{coupl} and the $U_{i,h}$ and $V_{i,h}$:

$$\begin{pmatrix} \dfrac{U_{i,h}^{n+1} - U_{i,h}^{n-1}}{2\Delta t} \cdot \alpha \\ \dfrac{V_{i,h}^{n+1} - V_{i,h}^{n-1}}{2\Delta t} \cdot \alpha \end{pmatrix} \cdot \underline{\nu} = \frac{\Big[\mathscr{S}_{\Delta t}(\Lambda_h^{mod,n-1/2}, \partial_t \Lambda_h^{mod,n-1/2}) + \big[-\sum_i (F_i^{\text{coupl}})^n\big] \mathscr{R}_{n,\Delta t}\Big({}^t\mathbb{P}_h \begin{pmatrix} J_h \\ 0 \end{pmatrix}\Big) \Big] - \Lambda_h^{mod,n-1/2}}{\Delta t} \cdot {}^t\mathbb{P}_h \begin{pmatrix} J_h \\ 0 \end{pmatrix} \tag{42}$$

In this equation, we can evaluate beforehand the quantities $\mathscr{S}_{\Delta t}(\Lambda_h^{mod,n-1/2}, \partial_t \Lambda_h^{mod,n-1/2})$ and $\mathscr{R}_{n,\Delta t}\Big({}^t\mathbb{P}_h \begin{pmatrix} J_h \\ 0 \end{pmatrix}\Big)$ without knowing the values of the hammer, strings and Lagrange multiplier unknowns. Then, the much smaller sub-problem {hammer, strings, coupling equation} can be solved using a fully implicit scheme. The strings and hammer are treated as in the previous dedicated section, and the coupling is treated as mentioned above in (42).

The fully discrete problem can be written, centered around time t^n:

$$
M^{\text{ham}}\ \frac{\xi^{n+1}-2\xi^{n}+\xi^{n-1}}{\Delta t^2}-\sum_{i=1}^{N_c}\Big[K_i\frac{\Psi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Psi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}-
R_i\frac{\Phi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Phi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{2\Delta t}\Big]=0 \tag{43a}
$$

$$
\begin{cases}
\oint_\Omega \mathscr{T}_u\Big(M_h^c\,\dfrac{U_{i,h}^{n+1}-2U_{i,h}^{n}+U_{i,h}^{n-1}}{\Delta t^2}\Big)_j+\\
\dfrac12\oint_\Omega\sum_{\sigma\in\Sigma_k}\zeta(\sigma)\Big[\delta_{\partial_x u}\mathscr{U}_i(U_{i,h}^{n+1},V_{i,h}^{n+1},\Phi_{i,h}^{n+1};\partial_x U_{i,h}^{n+1},\partial_x U_{i,h}^{n-1},\partial_x V_{i,h}^{n+\sigma(v)},\partial_x\Phi_{i,h}^{n+\sigma(\varphi)})+\\
\qquad\delta_{\partial_x u}\mathscr{U}_i(U_{i,h}^{n-1},V_{i,h}^{n-1},\Phi_{i,h}^{n-1};\partial_x U_{i,h}^{n+1},\partial_x U_{i,h}^{n-1},\partial_x V_{i,h}^{n+\sigma(v)},\partial_x\Phi_{i,h}^{n+\sigma(\varphi)})\Big]\partial_x\phi_j+\\
\dfrac12\oint_\Omega\sum_{\sigma\in\Sigma_k}\zeta(\sigma)\Big[\delta_u\mathscr{U}_i(U_{i,h}^{n+1},U_{i,h}^{n-1},V_{i,h}^{n+\sigma(v)},\Phi_{i,h}^{n+\sigma(\varphi)};\partial_x U_{i,h}^{n+1},\partial_x V_{i,h}^{n+1},\partial_x\Phi_{i,h}^{n+1})+\\
\qquad\delta_u\mathscr{U}_i(U_{i,h}^{n+1},U_{i,h}^{n-1},V_{i,h}^{n+\sigma(v)},\Phi_{i,h}^{n+\sigma(\varphi)};\partial_x U_{i,h}^{n-1},\partial_x V_{i,h}^{n-1},\partial_x\Phi_{i,h}^{n-1})\Big]\phi_j\\
\qquad+\Big[K_i\dfrac{\Psi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Psi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}-\\
\qquad R_i\dfrac{\Phi\big(\langle U_{i,h}^{n+1}\rangle-\xi^{n+1}\big)-\Phi\big(\langle U_{i,h}^{n-1}\rangle-\xi^{n-1}\big)}{2\Delta t}\Big]\langle\phi_j\rangle=(F_i^{\text{coupl}})^n\alpha_j\nu_u\\
\oint_\Omega \mathscr{T}_v\Big(M_h^c\,\dfrac{V_{i,h}^{n+1}-2V_{i,h}^{n}+V_{i,h}^{n-1}}{\Delta t^2}\Big)_j+\\
\dfrac12\oint_\Omega\sum_{\sigma\in\Sigma_k}\zeta(\sigma)\Big[\delta_{\partial_x v}\mathscr{U}_i(U_{i,h}^{n+1},V_{i,h}^{n+1},\Phi_{i,h}^{n+1};\partial_x V_{i,h}^{n+1},\partial_x V_{i,h}^{n-1},\partial_x U_{i,h}^{n+\sigma(u)},\partial_x\Phi_{i,h}^{n+\sigma(\varphi)})+\\
\qquad\delta_{\partial_x v}\mathscr{U}_i(U_{i,h}^{n-1},V_{i,h}^{n-1},\Phi_{i,h}^{n-1};\partial_x V_{i,h}^{n+1},\partial_x V_{i,h}^{n-1},\partial_x U_{i,h}^{n+\sigma(u)},\partial_x\Phi_{i,h}^{n+\sigma(\varphi)})\Big]\partial_x\phi_j+\\
\dfrac12\oint_\Omega\sum_{\sigma\in\Sigma_k}\zeta(\sigma)\Big[\delta_v\mathscr{U}_i(V_{i,h}^{n+1},V_{i,h}^{n-1},U_{i,h}^{n+\sigma(u)},\Phi_{i,h}^{n+\sigma(\varphi)};\partial_x U_{i,h}^{n+1},\partial_x V_{i,h}^{n+1},\partial_x\Phi_{i,h}^{n+1})+\\
\qquad\delta_v\mathscr{U}_i(V_{i,h}^{n+1},V_{i,h}^{n-1},U_{i,h}^{n+\sigma(u)},\Phi_{i,h}^{n+\sigma(\varphi)};\partial_x U_{i,h}^{n-1},\partial_x V_{i,h}^{n-1},\partial_x\Phi_{i,h}^{n-1})\Big]\phi_j=(F_i^{\text{coupl}})^n\alpha_j\nu_v\\
\oint_\Omega \mathscr{T}_\varphi\Big(M_h^c\,\dfrac{\Phi_{i,h}^{n+1}-2\Phi_{i,h}^{n}+\Phi_{i,h}^{n-1}}{\Delta t^2}\Big)_j+\\
\dfrac12\oint_\Omega\sum_{\sigma\in\Sigma_k}\zeta(\sigma)\Big[\delta_{\partial_x\varphi}\mathscr{U}_i(U_{i,h}^{n+1},V_{i,h}^{n+1},\Phi_{i,h}^{n+1};\partial_x\Phi_{i,h}^{n+1},\partial_x\Phi_{i,h}^{n-1},\partial_x U_{i,h}^{n+\sigma(u)},\partial_x V_{i,h}^{n+\sigma(v)})+\\
\qquad\delta_{\partial_x\varphi}\mathscr{U}_i(U_{i,h}^{n-1},V_{i,h}^{n-1},\Phi_{i,h}^{n-1};\partial_x\Phi_{i,h}^{n+1},\partial_x\Phi_{i,h}^{n-1},\partial_x U_{i,h}^{n+\sigma(u)},\partial_x V_{i,h}^{n+\sigma(v)})\Big]\partial_x\phi_j+\\
\dfrac12\oint_\Omega\sum_{\sigma\in\Sigma_k}\zeta(\sigma)\Big[\delta_\varphi\mathscr{U}_i(\Phi_{i,h}^{n+1},\Phi_{i,h}^{n-1},U_{i,h}^{n+\sigma(u)},V_{i,h}^{n+\sigma(v)};\partial_x U_{i,h}^{n+1},\partial_x V_{i,h}^{n+1},\partial_x\Phi_{i,h}^{n+1})+\\
\qquad\delta_\varphi\mathscr{U}_i(\Phi_{i,h}^{n+1},\Phi_{i,h}^{n-1},U_{i,h}^{n+\sigma(u)},V_{i,h}^{n+\sigma(v)};\partial_x U_{i,h}^{n-1},\partial_x V_{i,h}^{n-1},\partial_x\Phi_{i,h}^{n-1})\Big]\phi_j=0
\end{cases} \tag{43b}
$$

$$
\begin{cases}
\begin{cases}
\mathscr{S}_{\Delta t}(\Lambda_h^{mod,n-1/2},\partial_t\Lambda_h^{mod,n-1/2})=\mathring{\Lambda}_h^{mod}(t=t^{n+1/2})\ ,\ \mathscr{R}_{n,\Delta t}\Big({}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}\Big)=\tilde{\Lambda}_h^{mod}(t=t^{n+1/2})\\
\dot{\mathscr{S}}_{\Delta t}(\Lambda_h^{mod,n-1/2},\partial_t\Lambda_h^{mod,n-1/2})=\partial_t\mathring{\Lambda}_h^{mod}(t=t^{n+1/2})\ ,\ \dot{\mathscr{R}}_{n,\Delta t}\Big({}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}\Big)=\partial_t\tilde{\Lambda}_h^{mod}(t=t^{n+1/2})
\end{cases}\text{, where}\\
\forall t\in[t^{n-1/2},t^{n+1/2}],\quad
\begin{cases}
\partial_t^2\mathring{\Lambda}_h^{mod}+\mathbb{D}_h\mathring{\Lambda}_h^{mod}=0,\\
\mathring{\Lambda}_h^{mod}(t=t^{n-1/2})=\ \Lambda_h^{mod,n-1/2},\\
\partial_t\mathring{\Lambda}_h^{mod}(t=t^{n-1/2})=\partial_t\Lambda_h^{mod,n-1/2}
\end{cases}
\quad\text{and}\quad
\begin{cases}
\partial_t^2\tilde{\Lambda}_h^{mod}+\mathbb{D}_h\tilde{\Lambda}_h^{mod}={}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix},\\
\tilde{\Lambda}_h^{mod}(t=t^{n-1/2})=0,\\
\partial_t\tilde{\Lambda}_h^{mod}(t=t^{n-1/2})=0
\end{cases}
\end{cases} \tag{43c}
$$

$$
\begin{pmatrix}\dfrac{U_{i,h}^{n+1}-U_{i,h}^{n-1}}{2\Delta t}\cdot\alpha\\[2mm]\dfrac{V_{i,h}^{n+1}-V_{i,h}^{n-1}}{2\Delta t}\cdot\alpha\end{pmatrix}\cdot\underline{\nu}=\frac{\Big[\mathscr{S}_{\Delta t}(\Lambda_h^{mod,n-1/2},\partial_t\Lambda_h^{mod,n-1/2})+\big[-\sum_i(F_i^{\text{coupl}})^n\big]\mathscr{R}_{n,\Delta t}\Big({}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}\Big)\Big]-\Lambda_h^{mod,n-1/2}}{\Delta t}\cdot{}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix} \tag{43d}
$$

$$
\begin{cases}
\Lambda_h^{mod,n+1/2}=\Big[\mathscr{S}_{\Delta t}(\Lambda_h^{mod,n-1/2},\partial_t\Lambda_h^{mod,n-1/2})+\big[-\sum_i(F_i^{\text{coupl}})^n\big]\mathscr{R}_{n,\Delta t}\Big({}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}\Big)\Big]\cdot{}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}\\
\partial_t\Lambda_h^{mod,n+1/2}=\Big[\dot{\mathscr{S}}_{\Delta t}(\Lambda_h^{mod,n-1/2},\partial_t\Lambda_h^{mod,n-1/2})+\big[-\sum_i(F_i^{\text{coupl}})^n\big]\dot{\mathscr{R}}_{n,\Delta t}\Big({}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}\Big)\Big]\cdot{}^t\mathbb{P}_h\begin{pmatrix}J_h\\0\end{pmatrix}
\end{cases} \tag{43e}
$$

At each time step, this system can be solved in three steps:

- Exact resolution of (43c) (needing only the knowledge of $\Lambda_h^{mod,n-1/2}$ and $\partial_t \Lambda_h^{mod,n-1/2}$),
- Implicit resolution of the system {(43a), (43b), (43d)} : Newton iterations on the strings, hammer and Lagrange multipliers unknowns,
- Adjustment of the soundboard unknowns with (43e).

The soundboard unknowns can finally be calculated in the finite element basis thanks to (29).

The energy preserved by the numerical solution of the previous system is

$$\begin{aligned} \mathscr{E}_{h,s,p}^{n+1/2} = & \frac{M^{\mathrm{ham}}}{2} \left| \frac{\xi^{n+1} - \xi^n}{\Delta t} \right|^2 + \sum_{i=1}^{N_c} \Big[\frac{1}{2} \oint_0^L \mathscr{T}_u \left| \frac{U_{i,h}^{n+1} - U_{i,h}^n}{\Delta t} \right|^2 + \\ & \mathscr{T}_v \left| \frac{V_{i,h}^{n+1} - V_{i,h}^n}{\Delta t} \right|^2 + \mathscr{T}_\varphi \left| \frac{\Phi_{i,h}^{n+1} - \Phi_{i,h}^n}{\Delta t} \right|^2 \Big] + \\ & \sum_{i=1}^{N_c} \Big[\oint_0^L \frac{\mathscr{U}_i(\mathbf{q}_{i,h}^{n+1}, \partial_x \mathbf{q}_{i,h}^{n+1}) + \mathscr{U}_i(\mathbf{q}_{i,h}^{n}, \partial_x \mathbf{q}_{i,h}^{n})}{2} + \\ & K_i^{\mathrm{ham}} \frac{\Psi(\langle U_{i,h}^{n+1} - \xi^{n+1} \rangle) + \Psi(\langle U_{i,h}^{n} - \xi^{n} \rangle)}{2} \Big] + \\ & \frac{1}{2} \|\partial_t \Lambda_h^{EF,n+1/2}\|_{\mathbb{M}_h}^2 + \frac{1}{2} \|\Lambda_h^{EF,n+1/2}\|_{\mathbb{R}_h}^2 \end{aligned} \tag{44}$$

PERSPECTIVES

This paper has presented an ongoing work on the modeling of the grand piano. Innovating models, coupling energy preserving conditions and numerical schemes have been presented, as well as the precise method to solve the proposed equations. The paper is divided into pedagogical parts, treating the discretization of each sub system respectively, coming finally to the model and energy preserving numerical approximation of a system {hammer, strings, soundboard}. Damping can be considered, in return for some modifications in the numerical approximation. The total discrete energy must decrease in accordance with the continuous energy decay. Numerical results and comparison with real measurements will be presented in the oral session. The first perspective of this work is to extend the model to the radiation of sound. Accurate modeling of bridges and ribs is another goal that should help in a better understanding of the function of the soundboard.

ACKNOWLEDGEMENTS

We would like to thank Aliénor Burel for her work and patience on perturbation methods. A great thanks also goes to Patrick Joly for all his time, remarks and help concerning this work.

REFERENCES

[1] GV Anand. Large-amplitude damped free vibration of a stretched string. Journal of the Acoustical Society of America, 45(5):1089–1096, 1969.

[2] B Bank and L Sujbert. Generation of longitudinal vibrations in piano strings: From physics to sound synthesis. Journal of the Acoustical Society of America, 117:2268–2278, 2005.

[3] S Bilbao. Conservative numerical methods for nonlinear strings. Journal of the Acoustical Society of America, 118(5):3316–3327, 2005.

[4] Juliette Chabassier, Antoine Chaigne, and Patrick Joly. Transitoires de piano et non-linéarités des cordes : mesures et simulations. Proceedings of the 10^{th} French Congress on Acoustics.

[5] Juliette Chabassier and Patrick Joly. Energy preserving schemes for nonlinear hamiltonian systems of wave equations. application to the vibrating piano string. Computer Methods in Applied Mechanics and Engineering, In Press, Accepted Manuscript IS - SP - EP - PY - T2.

[6] Antoine Chaigne. Mesures d'admitance aux chevalets d'un piano droit. Proceedings of the 10^{th} French Congress on Acoustics.

[7] H A Conklin. Generation of partials due to nonlinear mixing in a stringed instrument. Acoustical Society of America Journal, 105:536–545, 1999.

[8] Grégoire Derveaux. Modélisation numérique de la guitare acoustique. PhD thesis, Sep 2002.

[9] Kerem Ege. La table d'harmonie du piano - Études modales en basses et moyennes fréquences. PhD thesis, Mar 2010.

[10] PM Morse and KU Ingard. Theoretical Acoustics. 1968.

[11] Ali Nayfeh. Perturbation Methods. 1873.

[12] Leila Rhaouti. Domaines fictifs pour la modélisation d'un problème d'interaction fluide-structure : Simulation de la timbale. PhD thesis, Juin 1999.

[13] A Stulov. Experimental and theoretical studies of piano hammer. Proceedings of the Stockholm Music Acoustics Conference, Jan 2003.

[14] S Timoshenko, F Campus, and A de Riva-Berni. Théorie des vibrations à l'usage des ingénieurs. Jan 1939.