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# CONSTRUCTING THE SET OF COMPLETE INTERSECTION NUMERICAL SEMIGROUPS WITH A GIVEN FROBENIUS NUMBER

A. ASSI AND P. A. GARCÍA-SÁNCHEZ

ABSTRACT. Delorme suggested that the set of all complete intersection numerical semigroups can be computed recursively. We have implemented this algorithm, and particularized it to several subfamilies of this class of numerical semigroups: free and telescopic numerical semigroups, and numerical semigroups associated to an irreducible plane curve singularity. The recursive nature of this procedure allows us to give bounds for the embedding dimension and for the minimal generators of a semigroup in any of these families.

## 1. INTRODUCTION

Let  $\mathbb{N}$  denote the set of nonnegative integers. A numerical semigroup  $\Gamma$  is a submonoid of  $\mathbb{N}$  with finite complement in  $\mathbb{N}$  (this condition is equivalent to  $\gcd(\Gamma) = 1$ ). If  $\Gamma$  is a numerical semigroup, the elements in  $\mathbb{N} \setminus \Gamma$  are the gaps of  $\Gamma$ , and its cardinality is the genus of  $\Gamma$ ,  $g(\Gamma)$ . The largest integer not in  $\Gamma$  is called the Frobenius number of  $\Gamma$ , and will be denoted by  $F(\Gamma)$ . Clearly,  $F(\Gamma) + 1 + \mathbb{N} \subseteq \Gamma$ , and this is why  $c(\Gamma) = F(\Gamma) + 1$  is known as the conductor of  $\Gamma$ .

Since for every  $x \in \Gamma$ ,  $F(\Gamma) - x$  cannot be in  $\Gamma$ , we deduce that  $g(\Gamma) \geq \frac{c(\Gamma)}{2}$ . We say that  $\Gamma$  is symmetric when the equality holds, or equivalently, for every integer  $x$ ,  $x \notin \Gamma$  implies  $F(\Gamma) - x \in \Gamma$ . In this setting,  $c(\Gamma)$  is an even integer, and thus  $F(\Gamma)$  is odd.

It can be easily proved that any numerical semigroup admits a unique minimal generating system (every element is a linear combination of elements in this set with nonnegative integer coefficients and none of its proper subsets fulfills this condition; see for instance [13, Chapter 1]). If  $A = \{r_0, \dots, r_h\}$  is the minimal generating set of  $\Gamma$ , then its elements are called minimal generators, and its cardinality is the embedding dimension of  $\Gamma$ ,  $e(\Gamma)$ .

The map

$$\mathbb{N}^{e(\Gamma)} \rightarrow \Gamma, \varphi(a_0, \dots, a_h) = a_0 r_0 + \dots + a_h r_h$$

is a monoid epimorphism. Hence  $\Gamma$  is isomorphic to  $\mathbb{N}^{e(\Gamma)} / \ker \varphi$ , where  $\ker \varphi = \{(a, b) \in \mathbb{N}^{e(\Gamma)} \times \mathbb{N}^{e(\Gamma)} \mid \varphi(a) = \varphi(b)\}$  (which is a congruence on  $\mathbb{N}^{e(\Gamma)}$ ).

A presentation for  $\Gamma$  is a set of generators of the congruence  $\varphi$ , and a minimal presentation is a set of generators minimal with respect to set inclusion (actually, in our setting also with respect to cardinality; see [13, Corollary 8.13]). It can be shown that the cardinality of any minimal presentation is greater than or equal to  $e(\Gamma) - 1$ , [13, Theorem 9.6]. A numerical semigroup is a complete intersection if this equality holds.

Given  $A$  a set positive integers, and  $A = A_1 \cup A_2$  a non trivial partition of  $A$ , we say that  $A$  is the gluing of  $A_1$  and  $A_2$  if  $\text{lcm}(d_1, d_2) \in \langle A_1 \rangle \cap \langle A_2 \rangle$ , where  $d_i = \gcd(A_i)$  and  $\langle A_i \rangle$  denotes the monoid generated by  $A_i$ ,  $i = 1, 2$ . If  $A$  is the minimal system of generators of  $\Gamma$ , and  $\Gamma_i$  is the numerical semigroup generated by  $A_i/d_i$ ,  $i = 1, 2$ , we also say that  $\Gamma$  is the gluing of  $\Gamma_1$  and  $\Gamma_2$ . It turns out that  $d_1 \in \Gamma_2$ ,  $d_2 \in \Gamma_1$ ,  $\gcd(d_1, d_2) = 1$ , and neither  $d_1$  is a minimal generator of  $\Gamma_2$  nor  $d_2$  is a minimal generator of  $\Gamma_1$  ([13, Section 8.3]). Delorme proved in [4, Proposition 9]

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that a numerical semigroup is a complete intersection if and only if it is a gluing of two complete intersection numerical semigroups (though with a different notation; the concept of gluing was introduced in [11]). The gluing of symmetric numerical semigroups is symmetric ([4, Proposition 10 (iii)]), and as a consequence of this, complete intersections are symmetric.

In [14] there is a procedure to construct the set of all numerical semigroups with a given Frobenius number. We show in this manuscript how can we use the concept of gluing to compute the set of all complete intersection numerical semigroups with a given Frobenius number (or equivalently with fixed genus). Recently there have been some experimental results that point out to the possibility that the number of numerical semigroups with a fixed genus has a Fibonacci like behaviour ([3]). However there is not even a proof that there are more numerical semigroups with genus  $g + 1$  than numerical semigroups with genus  $g$ . This is not the case for complete intersection numerical semigroups, as we see in the last section.

We also show how to calculate the set of all free (in the sense of [1]) numerical semigroups, which a special subclass of complete intersections, the set of all telescopic numerical semigroups (contained in the set of free numerical semigroups), and that of numerical semigroups associated to an irreducible plane curve singularity (these are a particular case of telescopic numerical semigroups).

The recursive nature of gluing also allows us to give some bounds for the generators and embedding dimension for these families of semigroups when we fix the Frobenius number. The deeper we go in the chain of inclusions given in the preceding paragraph, the smaller are the bounds.

## 2. THE FROBENIUS NUMBER AND MULTIPLICITY OF A COMPLETE INTERSECTION

Let  $\Gamma$  be a numerical semigroup. We know that  $\Gamma$  is a complete intersection if and only if it is the gluing of two complete intersections. Delorme (though with a different notation) highlighted in [4, Section 11] that this fact can be used to determine if a numerical semigroup is a complete intersection (this idea has already been exploited in [2]; and for affine semigroups in [12]), and also to compute the set of all complete intersections. In order to construct the set of all complete intersection numerical semigroups with given Frobenius number, we can proceed recursively by using the following formula for the Frobenius number of the gluing of two numerical semigroups, which is just a reformulation of Delorme's description of the conductor of a gluing.

**Proposition 1.** *Assume that  $\Gamma$  is a numerical semigroup minimally generated by  $A = A_1 \cup A_2$ , and that  $A$  is the gluing of  $A_1$  and  $A_2$ . Let  $d_1 = \gcd(A_1)$ ,  $d_2 = \gcd(A_2)$ , and  $d = d_1 d_2$ . Define  $\Gamma_1 = \langle A_1/d_1 \rangle$  and  $\Gamma_2 = \langle A_2/d_2 \rangle$ . Then*

$$F(\Gamma) = d_1 F(\Gamma_1) + d_2 F(\Gamma_2) + d_1 d_2.$$

*Proof.* Observe that  $\Gamma = d_1 \Gamma_1 + d_2 \Gamma_2$ . By [4, Proposition 10 (i)],

$$c(\Gamma) = d_1 c(\Gamma_1) + d_2 c(\Gamma_2) + (d_1 - 1)(d_2 - 1).$$

Having in mind the relationship between Frobenius number and conductor, the formula follows easily.  $\square$

If we repeat the process with  $\Gamma_1$  and  $\Gamma_2$  in this result, we construct a decomposition tree of  $\Gamma$ , whose leaves are copies of  $\mathbb{N}$ . Assume that  $d^{(1)}, \dots, d^{(h)}$  are the possible  $d$ 's appearing in this splitting. The Frobenius number of  $\Gamma$  is precisely  $\sum_{i=1}^h d^{(i)} - \sum_{a \in A} a$  (see [4, Section 11]).

**Example 2.** We construct a complete intersection numerical semigroup with four generators, by gluing two embedding dimension two numerical semigroups.

```
gap> s:=NumericalSemigroup(10,11);;
gap> t:=NumericalSemigroup(7,9);;
gap> g:=NumericalSemigroup(16*10,16*11,21*7,21*9);;
gap> FrobeniusNumber(g);
2747
```

gap> 16\*FrobeniusNumber(s)+21\*FrobeniusNumber(t)+16\*21;  
2747

**Proposition 3.** *If  $\Gamma$  is a complete intersection, then*

$$m(\Gamma) \geq 2^{e(\Gamma)-1}.$$

*Proof.* Let  $h = e(\Gamma) - 1$ . We use induction on  $h$ . For  $h = 1$ , the statement follows trivially. As  $\Gamma$  is a complete intersection, if  $A$  is its minimal set of generators, we can find a partition of  $A = A_1 \cup A_2$  such that  $A$  is the gluing of  $A_1$  and  $A_2$ . Set as above  $d_i = \gcd(A_i)$ , and  $\Gamma_i = \langle A_i/d_i \rangle$ . Let  $h_i = e(\Gamma_i) - 1$ . Hence  $h = h_1 + h_2 + 1$ . By induction hypothesis  $m(\Gamma_i) \geq 2^{h_i}$ . Recall that  $d_1 \in \Gamma_2$  and  $d_2 \in \Gamma_1$ , and they are not minimal generators. Thus  $d_1 \geq 2m(\Gamma_2) \geq 2^{h_2+1}$ , and analogously  $d_2 \geq 2^{h_1+1}$ . For every  $a \in A_1$ ,  $a/d_1$  is a minimal generator of  $\Gamma_1$ , whence  $a/d_1 \geq 2^{h_1}$ . Therefore  $a \geq 2^{h_1+h_2+1} = 2^h$ . The same argument shows that any element in  $A_2$  is greater than or equal to  $2^h$ .  $\square$

**Proposition 4.** *Let  $\Gamma$  be a complete intersection numerical semigroup other than  $\mathbb{N}$ , minimally generated by  $\{r_0, \dots, r_h\}$ . If  $m(\Gamma) \neq 2$ , for all  $k$ ,  $r_k < F(\Gamma)$ .*

*Proof.* The numerical semigroup  $\Gamma$  is symmetric and thus for every  $i > 0$ ,  $F(\Gamma) + r_0 - r_i \in \Gamma$ . If  $r_k > F(\Gamma)$ , for some  $k > 0$ , then  $F(\Gamma) + r_0 - r_k < r_0$ , which forces  $F(\Gamma) + r_0 = r_k$ .

If  $h > 1$ , choose  $0 < i \neq k$ . Then  $r_k - r_i = F(\Gamma) + r_0 - r_i \in \Gamma$ , contradicting that  $r_k$  is a minimal generator. This proves  $r_k < F(\Gamma)$ , whenever  $h > 1$ .

For  $h = 1$ ,  $F(\Gamma) = (r_0 - 1)(r_1 - 1) - 1$ . In this setting,  $\Gamma = \langle 2, f + 2 \rangle$  has  $F(\Gamma) = f$ . For  $m(\Gamma) > 2$ , assume without loss of generality that  $r_0 < r_1$ . Then  $F(\Gamma) = (r_0 - 1)(r_1 - 1) - 1 \geq 2(r_1 - 1) - 1 = (r_1 - 1) + (r_1 - 2) \geq r_1$ .  $\square$

**Corollary 5.** *If  $\Gamma$  is a complete intersection numerical semigroup other than  $\mathbb{N}$  and  $\langle 2, 3 \rangle$ , then*

$$e(\Gamma) \leq \log_2(c(\Gamma)) + 1.$$

*Proof.* By Proposition 3,  $2^{e(\Gamma)-1} \leq m(\Gamma)$ . If  $m(\Gamma) > 2$ , from the proof of Proposition 4, we deduce that  $m(\Gamma) \leq c(\Gamma)$ . Then  $e(\Gamma) - 1 \leq \log_2(c(\Gamma))$ .

For  $m(\Gamma) = 2$  and  $\Gamma \neq \langle 2, 3 \rangle$ , we also have  $2 \leq c(\Gamma)$ .  $\square$

*Remark 6.* If we want to compute the set of all complete intersection numerical semigroups with Frobenius number  $f$ , then we can use the formula given in Proposition 1. Hence  $f = d_1 f_1 + d_2 f_2 + d_1 d_2$ , and we recursively construct all possible numerical semigroups with Frobenius number  $f_1$ , and then the set with Frobenius number  $f_2$ . We next give some useful bounds and facts to perform this task. Denote  $f + 1$  by  $c$ .

- i)  $d_1 \neq 1 \neq d_2$ . This is because  $d_1 \in \Gamma_2$  and it is not a minimal generator of  $\Gamma_2$ . The only possibility to have  $d_1 = 1 \in \Gamma_2$  would be  $\Gamma_2 = \mathbb{N} = \langle 1 \rangle$ . But then  $d_1$  would be a minimal generator. The same argument is valid for  $d_2$ .
- ii) Since  $\gcd\{d_1, d_2\} = 1$ , we can assume without loss of generality that  $2 \leq d_2 < d_1$ .
- iii) Since  $f_1, f_2 \geq -1$ ,  $f \geq -d_1 - d_2 + d_1 d_2 = (d_1 - 1)(d_2 - 1) - 1$ . Hence
  - a)  $c \geq (d_2 - 1)^2$ , and thus  $d_2 \leq \sqrt{c} + 1$ ;
  - b)  $d_2 \leq \frac{c}{d_1 - 1} + 1$ ;
 and consequently  $d_2 \leq \min\{d_1 - 1, \sqrt{c} + 1, \frac{c}{d_1 - 1} + 1\}$ .
- iv) If  $f_i = -1$ , then  $f - d_j f_j \equiv 0 \pmod{f_i}$ ,  $\{i, j\} = \{1, 2\}$ .
- v)  $d_2 < f$ , except in the case  $\Gamma = \langle 2 = d_2, f + 2 = d_1 \rangle$ .
  - a) If  $f_1 = f_2 = -1$ , then  $\Gamma_1 = \Gamma_2 = \mathbb{N}$ , and the gluing is  $\langle d_2 < d_1 \rangle$ . If  $d_2 \neq 2$ , then Proposition 4, asserts that  $d_2 < f$ .
  - b) If  $f_2 > 0$ , then  $f \geq -d_1 + d_2 + d_1 d_2 = (d_1 + 1)(d_2 - 1) + 1 \geq d_1 + 2$ . Hence  $d_1 \leq f - 2$ .
  - c) If  $f_1 > 0$ , then  $f \geq d_1 - d_2 + d_1 d_2 = (d_1 - 1)(d_2 + 1) + 1 > 3(d_1 - 1) \geq d_1$ .

vi) If  $f_1 \neq -1 \neq f_2$ , then  $f - d_1d_2 \in \langle d_1, d_2 \rangle$ . We are only interested in factorisations  $f - d_1d_2 = a_1d_1 + a_2d_2$ ,  $a_1, a_2 \in \mathbb{N}$ , with  $a_1 \equiv a_2 \equiv 1 \pmod{2}$ , since the Frobenius number of a complete intersection is an odd integer.

**Example 7.** We compute the set of all complete intersection numerical semigroups with Frobenius number 11. First note that  $\langle 2, 13 \rangle$  is in this set. The possible  $d_1$  belong to  $\{3, \dots, 10\}$ .

- $d_1 = 10$ . Then  $2 \leq d_2 \leq \min\{9, \lfloor \frac{12}{9} \rfloor + 1, \lfloor \sqrt{12} \rfloor + 1\} = 2$ . Hence  $d_2$  must be 2, but then  $\gcd\{d_1, d_2\} \neq 1$ , and we have no complete intersections under these conditions.
- $d_1 = 9$ . Then  $2 \leq d_2 \leq \min\{8, \lfloor \frac{12}{8} \rfloor + 1, \lfloor \sqrt{12} \rfloor + 1\} = 2$ . This forces  $d_2 = 2$ , which in addition is coprime with 9.
  - $11 + 9 \equiv 0 \pmod{2}$ , and thus  $f_1 = -1$  ( $\Gamma_1 = \mathbb{N}$ ) is a possible choice. In this setting  $f_2 = (11 - 18 + 0)/2 = 1$ , whence  $\Gamma_2 = \langle 2, 3 \rangle$ . We obtain a new complete intersection  $\Gamma = 9\mathbb{N} + 2\langle 2, 3 \rangle = \langle 4, 6, 9 \rangle$ , because  $9 \in \langle 2, 3 \rangle$  is not a minimal generator.
  - $11 + 2 \not\equiv 0 \pmod{9}$ , so  $f_2$  cannot be  $-1$ .
  - $11 - 19 \notin \langle 2, 9 \rangle$ , so we have no more complete intersections with this data.
- For  $d_1 = 8$ , we have  $2 \leq d_2 \leq \min\{7, \lfloor \frac{12}{7} \rfloor + 1, \lfloor \sqrt{12} \rfloor + 1\} = 2$ . However  $\gcd\{d_1, d_2\} \neq 1$ .
- If  $d_1 = 7$ , then  $2 \leq d_2 \leq \min\{6, \lfloor \frac{12}{6} \rfloor + 1, \lfloor \sqrt{12} \rfloor + 1\} = 3$ .
  - $d_2 = 2$ .
    - \*  $11 + 7 \equiv 0 \pmod{2}$ , and thus  $\Gamma_1$  can be  $\mathbb{N}$ . But then  $f_2 = (11 - 14 + 7)/2 = 2$ , which is even. So this case cannot occur.
    - \*  $11 + 2 \not\equiv 0 \pmod{7}$ , and so  $\Gamma_2$  will not be  $\mathbb{N}$ .
    - \* Finally,  $11 - 14 \notin \langle 2, 7 \rangle$ , so no complete intersections can be found with properties.
  - $d_2 = 3$ .
    - \*  $11 + 7 \equiv 0 \pmod{3}$ , and thus  $\Gamma_1$  could be  $\mathbb{N}$ . In this setting  $f_2 = (11 - 21 + 7)/3 = -1$ , and so  $\Gamma_2$  is also  $\mathbb{N}$ . We get a new complete intersection  $\Gamma = 7\mathbb{N} + 3\mathbb{N} = \langle 3, 7 \rangle$  with Frobenius number 14.
    - \*  $11 - 21 \notin \langle 3, 7 \rangle$ , so no more complete intersections are obtained for this choice of  $d_1$  and  $d_2$ .
- For  $d_1 = 6$ ,  $2 \leq d_2 \leq \min\{5, \lfloor \frac{12}{5} \rfloor + 1, \lfloor \sqrt{12} \rfloor + 1\} = 3$ , but both 2 and 3 are not coprime with 6.
- $d_1 = 5$ . Then  $d_2 \in \{2, 3, 4\}$ .
  - $d_2 = 2$ .
    - \*  $11 + 5 \equiv 0 \pmod{2}$ , and so  $\Gamma_1$  can possibly be  $\mathbb{N}$ . Hence  $f_2 = (11 - 10 + 5)/2 = 3$ . The only possible complete intersection numerical semigroup with Frobenius number 3 is  $\langle 2, 5 \rangle$ . But 5 is a minimal generator of this semigroup.
    - \*  $11 + 2 \not\equiv 0 \pmod{5}$ .
    - \*  $11 - 10 \notin \langle 2, 5 \rangle$ .
  - $d_2 = 3$ . In this case  $11 + 5 \not\equiv 0 \pmod{3}$ ,  $11 + 3 \not\equiv 0 \pmod{5}$ , and  $11 - 15 \notin \langle 3, 5 \rangle$ .
  - $d_2 = 4$ .
    - \*  $11 + 5 \equiv 0 \pmod{4}$ , and  $f_2 = (11 - 20 + 5)/4 = -1$ . So  $\Gamma = 5\mathbb{N} + 4\mathbb{N} = \langle 4, 5 \rangle$  is another complete intersection with Frobenius number 11.
    - \*  $11 - 20 \notin \langle 4, 5 \rangle$ .
- $d_1 = 4$ ,  $2 \leq d_2 \leq \min\{3, \lfloor \frac{12}{3} \rfloor + 1, \lfloor \sqrt{12} \rfloor + 1\} = 3$ , and as  $\gcd\{2, 4\} \neq 1$ , we get  $d_2 = 3$ .
  - $11 + 4 \equiv 0 \pmod{3}$ . So  $\Gamma_1$  could be  $\mathbb{N}$ . If this is the case,  $f_2 = (11 - 12 + 4)/3 = 1$ , which forces  $\Gamma_2$  to be  $\langle 2, 3 \rangle$ , and  $4 \in \Gamma_2$  is not a minimal generator. So we obtain  $\Gamma = 4\mathbb{N} + 3\langle 2, 3 \rangle = \langle 4, 6, 9 \rangle$ , which was already computed before.
  - $11 + 3 \not\equiv 0 \pmod{4}$ .
  - $11 - 12 \notin \langle 3, 4 \rangle$ .
- $d_3 = 3$  and  $d_2 = 2$ .

- $11 + 3 \equiv 0 \pmod{2}$ , and  $\Gamma_1 = \mathbb{N}$  can be a possibility. Then  $f_2 = (11 - 6 + 3)/2 = 7$ . If we apply this procedure recursively for  $f = 7$ , we obtain that  $\{\langle 2, 9 \rangle, \langle 3, 5 \rangle, \langle 4, 5, 6 \rangle\}$  is the set of all possible complete intersection numerical semigroups with Frobenius number 7. However,  $3 \notin \langle 2, 9 \rangle$ , 3 is a minimal generator of  $\langle 3, 5 \rangle$ , and  $3 \notin \langle 4, 5, 6 \rangle$ .
- $11 + 2 \not\equiv 0 \pmod{3}$ .
- $11 - 6 = 5 \in \langle 2, 3 \rangle$ , and  $5 = 1 \cdot 2 + 1 \cdot 3$  is the only factorization. So the only possible choice for  $f_1$  and  $f_2$  is 1. This means that  $\Gamma_1$  and  $\Gamma_2$  must be  $\langle 2, 3 \rangle$ . Again we obtain no new semigroups, since 2 and 3 are minimal generators of  $\langle 2, 3 \rangle$ .

Thus the set of complete intersection numerical semigroups with Frobenius number 11 is

$$\{\langle 2, 13 \rangle, \langle 4, 6, 9 \rangle, \langle 3, 7 \rangle, \langle 4, 5 \rangle\}.$$

### 3. FREE NUMERICAL SEMIGROUPS

Throughout this section, let  $\Gamma$  be the numerical semigroup  $\Gamma$  minimally generated by  $\{r_0, \dots, r_h\}$ . For  $k \in \{1, \dots, h+1\}$ , set  $d_k = \gcd\{r_0, \dots, r_{k-1}\}$  ( $d_1 = r_0$ ).

Write  $\Gamma_k = \left\langle \frac{r_0}{d_{k+1}}, \dots, \frac{r_k}{d_{k+1}} \right\rangle$ , and  $c_k = c(\Gamma_k)$  for all  $k \in \{1, \dots, h\}$ . Set  $c = c_h = c(\Gamma)$ .

We say that  $\Gamma$  is free if either  $h = 0$  (and thus  $r_0 = 1$ ) or  $\Gamma$  is the gluing of the free numerical semigroup  $\Gamma_{h-1}$  and  $\mathbb{N}$ . Free numerical semigroups were introduced in [1]. For other characterizations and properties of free numerical semigroups see [13, Section 8.4].

**Example 8.** Notice that the order in which the generators are given is crucial. For instance,  $S = \langle 8, 10, 9 \rangle$  is free for the arrangement  $(8, 10, 9)$  but it is not free for  $(8, 9, 10)$ . And a numerical semigroup can be free for different arrangements, for example,  $S = \langle 4, 6, 9 \rangle$  has this property.

If we take  $c_0, \dots, c_h$  pairwise coprime integers greater than one, and  $r_i = \prod_{j=0, i \neq j}^h c_j$ ,  $j = 0, \dots, h$ , then the numerical semigroup generated by  $\{r_0, \dots, r_h\}$  is free for any arrangement of its minimal generating set (see [8]).

According to Proposition 1, with  $A_2 = \{r_h\}$ , we obtain the following consequence.

**Corollary 9.** *If  $\Gamma$  is free, then*

$$F(\Gamma) = d_h F(\Gamma_{h-1}) + r_h(d_h - 1).$$

In this way we retrieve Johnson's formula ([9]). Notice also that  $\Gamma'$  is again free, so if we expand recursively this formula we obtain the formula given by Bertin and Carbonne for free numerical semigroups (see [1]; these authors named these semigroups in this way).

This equation can be reformulated in terms of the conductor as

$$(1) \quad c(\Gamma) = c_h = d_h c_{h-1} + (d_h - 1)(r_h - 1).$$

**Lemma 10.** *If  $\Gamma$  is free, then*

- (1)  $\gcd\{d_h, r_h\} = 1$ ;
- (2)  $d_h \mid F(\Gamma) + r_h$  (consequently  $d_h \nmid F(\Gamma)$ );
- (3) if we define  $e_k = \frac{d_k}{d_{k+1}}$ ,  $k = 1, \dots, h$ , then  $e_k r_k \in \langle r_0, \dots, r_{k-1} \rangle$ , for all  $k = 1, \dots, h$ ; in particular,  $e_k \geq 2$ ;
- (4)  $d_1 > d_2 > \dots > d_{h+1} = 1$ ;
- (5)  $d_h \leq \frac{c(\Gamma)}{r_h - 1} + 1$ ;
- (6) for  $h \geq 1$ ,  $(d_h - 1)(r_h - 1) \geq 2^h$ .

*Proof.* (1) This follows from the fact that  $\Gamma$  is a numerical semigroup, and thus  $\gcd\{d_h, r_h\} = d_{h+1} = 1$ .

$$(2) \quad F(\Gamma_{h-1}) = (F(\Gamma) + r_h(1 - d_h))/d_h = (F(\Gamma) + r_h)/d_h - 1.$$

- (3) As  $\Gamma_k$  is the gluing of  $\Gamma_{k-1}$  and  $\mathbb{N}$ , we have that  $\frac{r_k}{d_{k+1}} \in \Gamma_{k-1}$ . Hence  $\frac{d_k}{d_{k-1}}r_k \in \langle r_0, \dots, r_{k-1} \rangle$ . If  $e_k = 1$ , then  $r_k \in \langle r_0, \dots, r_{k-1} \rangle$ , contradicting that  $r_k$  is a minimal generator.
- (4) By definition,  $d_k \geq d_{k+1}$ . As  $e_k = \frac{d_k}{d_{k+1}} \geq 2$ , we get  $d_k > d_{k-1}$ .
- (5) Notice that  $F(\Gamma) \geq (r_h - 1)(d_h - 1) - 1$ , since  $F(\Gamma_{h-1}) \geq -1$ .
- (6) If  $d_h = 2$ , then we show that  $r_h > m(\Gamma)$ . Assume to the contrary that  $r_h = m(\Gamma)$ . Then we already proved above that  $e_h r_h \in \langle r_0, \dots, r_{h-1} \rangle$ . Since  $e_h = d_h$  and  $r_i$  is a minimal generator of  $\Gamma$  for all  $i$ , we deduce that  $2r_h = \sum_{i=0}^{h-1} a_i r_i$ , with  $\sum_{i=0}^{h-1} a_i \geq 2$ . As  $r_i > r_h$  for every  $i = 0, \dots, h-1$ , we get  $2r_h > r_h \sum_{i=0}^{h-1} a_i$ , and thus  $\sum_{i=0}^{h-1} a_i < 2$ , a contradiction. Thus in view of Proposition 3, we have that  $r_h \geq 2^h$ , and if  $d_h = 2$ , then  $r_h \geq 2^h + 1$ . Hence for  $d_h = 2$  the proof follows easily, and for  $d_h > 2$  we get  $(d_h - 1)(r_h - 1) \geq 2(r_h - 1) \geq 2(2^h - 1) \geq 2^h$  (we are assuming  $h \geq 1$ ).  $\square$

The following result gives a bound for the embedding dimension of a free numerical semigroup.

**Proposition 11.** *Let  $\Gamma$  be a free numerical semigroup. If  $\Gamma \neq \mathbb{N}$ , then*

$$(e(\Gamma) - 1)2^{e(\Gamma)-1} \leq c(\Gamma).$$

*Proof.* Assume that  $\Gamma$  is minimally generated by  $\{r_0, \dots, r_h\}$ , and write  $c = c(\Gamma)$ . We use induction on  $h$ . For  $h = 1$ ,  $c = (r_0 - 1)(r_1 - 1)$ ,  $r_0 \geq 2$ , and  $r_1 > r_0$  (otherwise  $\Gamma = \mathbb{N}$ , and  $h = 0$ ). Hence  $\max(r_0, r_1) \geq 3$ , and our assertion follows.

Suppose that  $h \geq 2$ , and that our inequality holds for  $h-1$ . By (1),  $c = d_h c_{h-1} + (d_h - 1)(r_h - 1)$ . By induction hypothesis,  $c_{h-1} \geq (h-1)2^{h-1}$ , and as  $d_h \geq 2$ , and  $(d_h - 1)(r_h - 1) \geq 2^h$  (Lemma 10), we deduce  $c \geq 2(h-1)2^{h-1} + 2^h = h2^h$ .  $\square$

For all  $h \geq 2$ ,  $c_{h-1} = \frac{c - (d_h - 1)(r_h - 1)}{d_h}$  is an even integer, and  $c \geq h2^h$ . In particular,

$$-c_{h-1}d_h \leq -(h-1)2^{h-1}d_h.$$

Hence

$$(r_h - 1)(d_h - 1) \leq c - (h-1)2^{h-1}d_h$$

This gives us the following upper bound for  $r_h$ .

$$r_h \leq \frac{c}{d_h - 1} - (h-1)2^{h-1} \frac{d_h}{d_h - 1} + 1.$$

**Corollary 12.** *For all  $h \geq 2$ ,*

$$2^h + 1 \leq r_h \leq \frac{c}{d_h - 1} - (h-1)2^{h-1} \frac{d_h}{d_h - 1} + 1 \leq c - (h-1)2^{h-1} + 1.$$

*Remark 13.* In order to compute the set of all free numerical semigroups with a given Frobenius number, we make use of the formula given in Corollary 9, by taking into account the restrictions given in this section for  $d_h$  and  $r_h$ .

#### 4. TELESCOPIC NUMERICAL SEMIGROUPS

We keep using the same notation as in the preceding section. We say that the numerical semigroup  $\Gamma$  minimally generated by  $\{r_0 < \dots < r_h\}$  is telescopic if it is free for the arrangement of the generators  $\{r_0, \dots, r_h\}$  (see for instance [10]).

Notice that in addition to the properties we had for free numerical semigroups, if  $\Gamma$  is telescopic, then

- (1)  $d_h < r_h$ , because  $d_h \mid r_{h-1} < r_h$ ;
- (2)  $F(\Gamma) \geq (r_h - 1)(d_h - 1) - 1 > (d_h - 1)^2 - 1$ , whence  $d_h \leq \min \left\{ r_h - 1, \frac{c(\Gamma)}{r_h - 1} + 1, \sqrt{c(\Gamma)} + 1 \right\}$ .

**Proposition 14.** *Let  $\Gamma$  be a telescopic numerical semigroup minimally generated by  $\{r_0 < \dots < r_h\}$ . If  $h \geq 2$ , then  $r_h \geq 2^{h+1} - 1$ .*

*Proof.* Let  $h = 2$ , and let  $\Gamma_1 = \langle \frac{r_0}{d_2}, \frac{r_1}{d_2} \rangle$ . Since  $\frac{r_1}{d_2} \geq 3$  and  $d_2 \geq 2$ , we have  $r_1 \geq 6$ . Besides,  $r_2 > r_1$ , whence  $r_2 \geq 7$ . Note that this bound is attained for  $\Gamma_2 = \langle 4, 6, 7 \rangle$ .

Assume that  $h \geq 3$ , and that the formula is true for  $h - 1$ . We have  $r_h \geq r_{h-1} + 1$  and  $r_h \in \langle \frac{r_0}{d_h}, \dots, \frac{r_{h-1}}{d_h} \rangle$ . By induction hypothesis, we have  $\frac{r_{h-1}}{d_h} \geq 2^h - 1$ . Hence  $r_h \geq 2(2^h - 1) + 1 = 2^{h+1} - 1$ . Note that this bound is reached by  $\Gamma_h = \langle 2^h, 3 \cdot 2^{h-1}, 7 \cdot 2^{h-2}, \dots, (2^{k+1} - 1) \cdot 2^{h-k}, \dots, 2^{h+1} - 1 \rangle$ .  $\square$

As in the free case, we can describe a bound for the embedding dimension of a telescopic numerical semigroup.

**Proposition 15.** *Let  $\Gamma$  be a telescopic numerical semigroup other than  $\mathbb{N}$ . Then*

$$(e(\Gamma) - 2)2^{e(\Gamma)} + 2 \leq c(\Gamma).$$

*Proof.* Assume that  $\Gamma$  is minimally generated by  $\{r_0 < \dots < r_h\}$ . Denote as usual  $c(\Gamma)$  by  $c$ . We use once more induction on  $h$ .

Embedding dimension two telescopic numerical semigroups are free, and thus the case  $h = 1$  follows from Proposition 11.

Suppose that  $h \geq 2$ , and that our inequality is true for  $h - 1$ . By (1), we have  $c = d_h c_{h-1} + (d_h - 1)(r_h - 1)$ . By induction hypothesis,  $c_{h-1} \geq (h - 2)2^h + 2$ , and as  $d_h \geq 2$ , and  $r_h \geq 2^{h+1} - 1$ , we get  $c \geq (h - 2)2^{h+1} + 4 + 2^{h+1} - 2 = (h - 1)2^{h+1} + 2$ .  $\square$

Note that for all  $h \geq 2$ ,  $c_{h-1} = \frac{c - (d_h - 1)(r_h - 1)}{d_h}$  is an even integer, and that  $c \geq (h - 1)2^{h+1} + 2$ . In particular

$$-c_{h-1}d_h \leq -((h - 2)2^h + 2)d_h.$$

Hence

$$(r_h - 1)(d_h - 1) \leq c - ((h - 2)2^h + 2)d_h.$$

This gives us the following upper bound for  $r_h$ :

$$r_h \leq \frac{c}{d_h - 1} - ((h - 2)2^h + 2)\frac{d_h}{d_h - 1} + 1 \leq c - (h - 2)2^h - 1.$$

**Corollary 16.** *For all  $h \geq 2$ , we have*

$$2^{h+1} - 1 \leq r_h \leq \frac{c}{d_h - 1} - ((h - 2)2^h + 2)\frac{d_h}{d_h - 1} + 1 \leq c - (h - 2)2^h - 1.$$

*Remark 17.* For computing the set of all telescopic numerical semigroups with fixed Frobenius number, we proceed as in the free case, ensuring that  $r_h$  is larger than the largest generator of  $\Gamma_1$  multiplied by  $d_h$ . Notice that  $d_h$  must now be smaller than  $r_h$ .

## 5. PLANE CURVE SINGULARITIES

Let  $\Gamma$  be the numerical semigroup minimally generated by  $\{r_0 < r_1 < \dots < r_h\}$ . Let  $d_k, \Gamma_k, c_k$ , and  $e_k$  be as in the preceding section. The numerical semigroup  $\Gamma$  is the numerical semigroup associated to an irreducible plane curve singularity if  $\Gamma$  is telescopic and  $e_k r_k < r_{k+1}$  for all  $k = 1, \dots, h - 1$  (see [16]).

**Proposition 18.** *Let  $\Gamma$  be the semigroup associated to an irreducible a plane curve singularity minimally generated by  $\{r_0 < \dots < r_h\}$ , with  $h \geq 2$ . Then  $r_h \geq 12 \cdot 4^{h-2} + \sum_{i=0}^{h-2} 4^i = \frac{1}{3}(5 \cdot 2^{2h-1} - 1)$ .*



*Proof.* For  $h = 2$ , as  $\Gamma_1 = \left\langle \frac{r_0}{d_2} < \frac{r_1}{d_2} \right\rangle$ , we obtain  $\frac{r_1}{2} \geq 3$ . Since  $d_2 \geq 2$ , we deduce that  $r_1 \geq 6$ . The plane singularity condition implies  $r_2 > e_1 r_1 \geq 12$ , because we know that  $e_1 \geq 2$  (Lemma 10). Hence  $r_2 \geq 13$ .

Assume that  $h \geq 3$ , and that the formula is true for  $h - 1$ . The plane singularity condition for  $k = h - 1$  implies that  $r_h \geq \frac{r_{h-1}}{d_h} d_{h-1} + 1$ . The quotient  $\frac{r_{h-1}}{d_h}$  is the largest generator of  $\Gamma_{h-1}$ . The induction hypothesis then asserts that  $\frac{r_{h-1}}{d_h} \geq 12 \cdot 4^{h-3} + \sum_{i=0}^{h-3} 4^i$ . By using that  $e_k \geq 2$  for all  $k$  (Lemma 10), we deduce that  $d_{h-1} \geq 4$ . By putting all this together, we get  $r_h \geq 4(12 \cdot 4^{h-3} + \sum_{i=0}^{h-3} 4^i) + 1 = 12 \cdot 4^{h-2} + \sum_{i=0}^{h-2} 4^i$ .  $\square$

**Proposition 19.** *Let  $\Gamma \neq \Gamma$  be the semigroup associated to an irreducible a plane curve singularity minimally generated by  $\{r_0 < \dots < a_h\}$  and with conductor  $c$ . Then*

$$c \geq 2 + 2 \sum_{i=0}^{h-2} 4^i + 12 \sum_{i=h-2}^{2h-4} 2^i = \frac{5}{3} 2^{2h} - 3 \cdot 2^h + \frac{4}{3}.$$

*Proof.* Since for  $h = 1$  the plane singularity condition adds nothing to the telescopic condition, the embedding dimension two case follows from Proposition 11.

Assume that  $h \geq 2$  and that our inequality holds for  $h - 1$ . We have:  $c = d_h c_{h-1} + (d_h - 1)(r_h - 1)$ . By induction hypothesis  $c_{h-1} \geq 2 + 2 \sum_{i=0}^{h-3} 4^i + 12 \sum_{i=h-3}^{2h-6} 2^i$ . Notice that  $d_h \geq 2$ . Thus our assertion follows from Proposition 18.  $\square$

We proceed now as we did in the telescopic case to obtain also an upper bound for  $r_h$ . Note that for all  $h \geq 2$ ,  $c_{h-1} = \frac{c - (d_h - 1)(r_h - 1)}{d_h}$  is an even integer, and that  $c \geq \frac{5}{3} 2^{2h-2} - 3 \cdot 2^{h-1} + \frac{4}{3}$ . Thus

$$-c_{h-1} d_h \leq - \left( \frac{5}{3} 2^{2h-2} - 3 \cdot 2^{h-1} + \frac{4}{3} \right) d_h.$$

Hence

$$(r_h - 1)(d_h - 1) \leq c - \left( \frac{5}{3} 2^{2h-2} - 3 \cdot 2^{h-1} + \frac{4}{3} \right) d_h.$$

This gives us the following upper bound for  $r_h$ .

$$r_h \leq \frac{c}{d_h - 1} - \left( \frac{5}{3} 2^{2h-2} - 3 \cdot 2^{h-1} + \frac{4}{3} \right) \frac{d_h}{d_h - 1} + 1.$$

**Corollary 20.** *For all  $h \geq 2$ , we have*

$$\frac{5}{3} 2^{2h-1} - \frac{1}{3} \leq r_h \leq \frac{c}{d_h - 1} - \left( \frac{5}{3} 2^{2h-2} - 3 \cdot 2^{h-1} + \frac{4}{3} \right) \frac{d_h}{d_h - 1} + 1 \leq c - \frac{5}{3} 2^{2h-2} - 3 \cdot 2^{h-1} + \frac{7}{3}.$$

A bound for the embedding dimension also follows from the above proposition.

**Corollary 21.** *If  $h \geq 2$ , then*

$$h \leq \log_2 \left( \frac{\sqrt{60c+1} + 9}{10} \right).$$

*Proof.* From Proposition 19,  $\frac{5}{3} 2^{2h} - 3 \cdot 2^h + \frac{4}{3} \leq c$ . Write  $x = 2^h$ , we get  $5/3x^2 - 3x + \frac{4}{3} \leq c$ . By solving  $5/3x^2 - 3x + \frac{4}{3} - c = 0$ , we get  $x \in \left\{ -\frac{\sqrt{60c+1}+9}{10}, \frac{\sqrt{60c+1}+9}{10} \right\}$ . As the minimum of  $5/3x^2 - 3x + \frac{4}{3} - c$  is reached in  $x = 9/10$ , and in our setting  $x = 2^h > 1$ , we have that the maximum possible  $x > 0$  such that  $\frac{5}{3} 2^{2h} - 3 \cdot 2^h + \frac{4}{3} \leq c$  is  $x = \frac{\sqrt{60c+1}+9}{10}$ .  $\square$

*Remark 22.* The set of all numerical semiogrups with fixed Frobenius number associated to an irreducible planar curve singularity is calculated as in the free case, by imposing the condition  $e_k r_k < r_{k+1}$ .

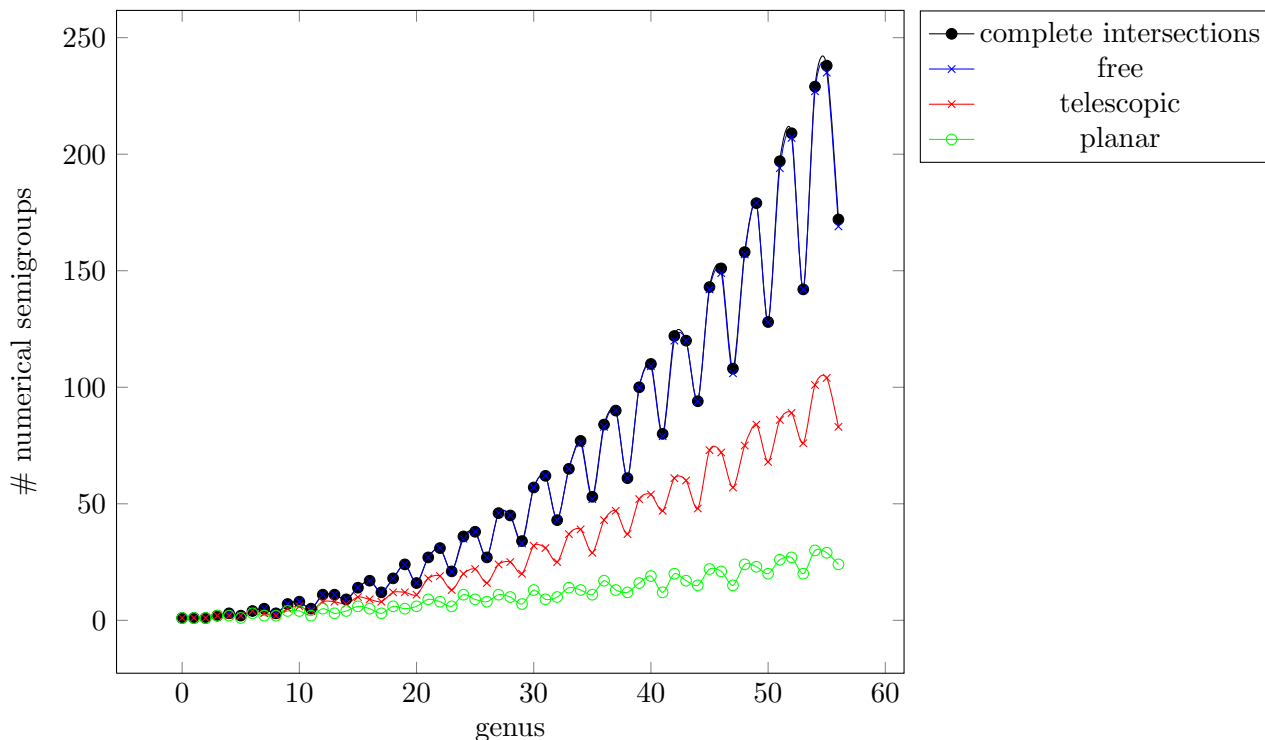
## 6. EXPERIMENTAL RESULTS

With the ideas given in the preceding sections, we implemented in `GAP` ([7]), with the help of the `numericalsgps` package ([6]), functions to compute the set of all complete intersection, free and telescopic numerical semigroups, as well as the set of all numerical semigroups associated to irreducible planar curve singularities with fixed Frobenius number (these functions will be included in the next release of this package).

The following table was computed in 6932 milliseconds, and it shows, for fixed genus  $g$ , the number of complete intersections ( $ci(g)$ ), free ( $fr(g)$ ), telescopic ( $tl(g)$ ), associated to an irreducible planar curve singularity ( $pc(g)$ ) numerical semigroups, respectively. Observe that almost all complete intersections in this table are free. This is due to the fact that the embedding dimension of all free semigroups appearing here is at most 4, because of the bound given in Proposition 11 ( $4 \cdot 2^4 = 64 > 55$ ). For embedding dimension three or less, the concepts of free and complete intersections coincide.

$g$	$ci(g)$	$fr(g)$	$tl(g)$	$pc(g)$	$g$	$ci(g)$	$fr(g)$	$tl(g)$	$pc(g)$	$g$	$ci(g)$	$fr(g)$	$tl(g)$	$pc(g)$
0	1	1	1	1	19	24	24	12	5	38	61	61	37	12
1	1	1	1	1	20	16	16	11	6	39	100	100	52	16
2	1	1	1	1	21	27	27	18	9	40	110	109	54	19
3	2	2	2	2	22	31	31	19	8	41	80	79	47	12
4	3	3	2	2	23	21	21	13	6	42	122	120	61	20
5	2	2	2	1	24	36	35	20	11	43	120	120	60	17
6	4	4	4	3	25	38	38	22	9	44	94	94	48	15
7	5	5	3	2	26	27	27	16	8	45	143	142	73	22
8	3	3	2	2	27	46	46	24	11	46	151	149	72	21
9	7	7	5	4	28	45	45	25	10	47	108	106	57	15
10	8	8	6	4	29	34	33	20	7	48	158	157	75	24
11	5	5	4	2	30	57	57	32	13	49	179	179	84	23
12	11	11	8	5	31	62	62	31	9	50	128	128	68	20
13	11	11	8	3	32	43	43	25	10	51	197	194	86	26
14	9	9	7	4	33	65	65	37	14	52	209	207	89	27
15	14	14	10	6	34	77	76	39	13	53	142	142	76	20
16	17	17	9	5	35	53	52	29	11	54	229	227	101	30
17	12	12	8	3	36	84	83	43	17	55	238	235	104	29
18	18	18	12	6	37	90	90	47	13	56	172	169	83	24

The largest genus, for which the set of numerical semigroups with this genus is known, is 55, and the number of numerical semigroups with genus 55 is 1142140736859 ([5]), while there are just 2496 symmetric numerical semigroup with genus 55 (this last amount can be computed by using the `IrreducibleNumericalSemigroupsWithFrobeniusNumber` command of the `numericalsgps` package). The average of complete intersections among symmetric numerical semigroups is small, and tiny compared with the whole set of numerical semigroups.



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