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Qualitative properties of certain piecewise deterministic Markov processes

Michel Benaïm, Stephane Leborgne, Florent Malrieu, Pierre-André Zitt

April 18, 2012

Abstract

We study a class of Piecewise Deterministic Markov Processes with state space $\mathbb{R}^m \times E$ where E is a finite set. The continuous component evolves according to a smooth vector field that is switched at the jump times of the discrete coordinate. The jump rates may depend on the whole position of the process. Working under the general assumption that the process stays in a compact set, we detail a possible construction of the process and characterize its support, in terms of the solutions set of a differential inclusion. We establish results on the long time behaviour of the process, in relation to a certain set of accessible points, which is shown to be strongly linked to the support of invariant measures. Under Hörmander-type bracket conditions, we prove that there exists a unique invariant measure and that the process converges to equilibrium in total variation. Finally we give examples where the bracket condition does not hold, and where there may be one or many invariant measures, depending on the jump rates between the flows.

1 Introduction and main results

Piecewise deterministic Markov processes (PDMPs in short) are intensively used in many applied areas (molecular biology [22], storage modelling [7], Internet traffic [13, 15, 16], neuronal activity [8, 20],...). Roughly speaking, a Markov process is a PDMP if its randomness is only given by the jump mechanism: in particular, it admits no diffusive dynamics. This huge class of processes has been introduced by Davis (see [9, 10]) in a general framework.

In the present paper, we deal with an interesting subclass of the PDMPs that plays a role in molecular biology (see [22, 8]) and elsewhere [23]. We consider a PDMP evolving on $\mathbb{R}^m \times E$, where E is a finite set, as follows: the first coordinate moves continuously on \mathbb{R}^m according to a smooth vector field that depends on the second coordinate whereas the second coordinate jumps with a rate depending on the first one. We are interested in the long time qualitative behaviour of these processes. A recent paper by Bakhtin and Hurth [2] consider the particular situation where the jump rates are constants and prove the beautiful result that, under an Hörmander type condition, if there exists an invariant measure for the process, then it is unique and absolutely continuous with respect to the "Lebesgue" measure on $\mathbb{R}^m \times E$. Here we consider a more general situation and focus on different questions. However certain of our results heavily rely on Bakhtin and Hurth's paper. This class of Markov processes is reminiscent of the so-called iterated random functions in the discrete time setting (see [11] for a good review of this topic).

Let us define our process more precisely. Let E be a finite set, and for any $i \in E$, $F^i : \mathbb{R}^m \mapsto \mathbb{R}^m$ be a smooth vector field. We assume throughout that each F^i is bounded

(i.e. $\sup_x \|F_i(x)\| \leq L < \infty$) and we let $\Phi^i = \{\Phi_t^i\}$ denote the flow induced by F^i . Recall that

$$t \mapsto \Phi_t^i(x) = \Phi^i(t, x)$$

is the solution to the Cauchy problem $\dot{x} = F^i(x)$ with initial condition $x(0) = x$. We furthermore assume that there exists a compact set $M \subset \mathbb{R}^m$ positively invariant under each Φ^i . That is

$$\forall t \geq 0, \quad \Phi_t^i(M) \subset M. \quad (1)$$

We consider here a continuous time Markov process $(Z_t = (X_t, Y_t))$ living on $M \times E$ whose infinitesimal generator acts on functions

$$\begin{aligned} g : M \times E &\rightarrow \mathbb{R}, \\ (x, i) &\mapsto g(x, i) = g^i(x), \end{aligned}$$

smooth¹ in x , according to the formula

$$Lg(x, i) = \langle F^i(x), \nabla g^i(x) \rangle + \sum_{j \in E} \lambda(x, i, j)(g^j(x) - g^i(x)) \quad (2)$$

where

- (i) $x \mapsto \lambda(x, i, j)$ is continuous;
- (ii) $\lambda(x, i, j) \geq 0$ for $i \neq j$ and $\lambda(x, i, i) = 0$;
- (iii) For each $x \in M$, the matrix $(\lambda(x, i, j))_{ij}$ is irreducible.

The process is explicitly constructed in Section 2 and some of its basic properties are established. In Section 3 we describe (Theorem 3.2) the support of the law of the process in term of the solutions set of a differential inclusion induced by the collection $\{F^i : i \in E\}$. Section 4 introduces the *accessible set* which is a natural candidate to support invariant probabilities. We show (Proposition 4.8) that this set is compact, connected, strongly positively invariant and invariant under the differential inclusion induced by $\{F^i : i \in E\}$. In Section 5 basic properties of invariant and empirical occupation probabilities of the processes are established and their support is characterized in terms of the accessible set. In Section 6 we prove the regularity results of [2] in a slightly stronger form that is needed for the convergence estimates. We use these results in Section 7 to show that, under some Hörmander bracket conditions, the law of jump chain and the process converge exponentially in total variation toward the unique invariant probability of the process. Section 8 gives elementary examples, and we show in Section 9 that when the Hörmander condition is violated, the uniqueness of the invariant measure may depend on the jump mechanism between flows, and not only on the flows themselves.

Let us note here that we address related questions in two companion papers: [6] gives under more stringent assumptions an explicit rate of convergence in Wasserstein distance, via a coupling argument; [5] studies in detail an example where the invariance assumption (1) does not hold, and where the behaviour of the process depends strongly on the jump rates.

¹meaning that g^i is the restriction to M of a smooth function on \mathbb{R}^m

2 Construction and Basic properties

Since M is compact and the maps $\lambda(\cdot, i, j)$ are continuous, there exists $\lambda \in \mathbb{R}_+$ such that

$$\max_{x \in M, i \in E} \sum_{j \in E, j \neq i} \lambda(x, i, j) < \lambda.$$

Let us fix such a λ , and let

$$Q_{ij}(x) = \frac{\lambda(x, i, j)}{\lambda} \quad \text{for } i \neq j \quad \text{and} \quad Q_{ii}(x) = 1 - \sum_{j \neq i} Q_{ij}(x).$$

Note that $Q(x)$ is an irreducible aperiodic Markov transition matrix and that (2) can be rewritten as

$$Lg = Ag + \lambda(Qg - g)$$

where

$$Ag(x, i) = \langle F^i(x), \nabla g^i(x) \rangle$$

and

$$Qg(x, i) = \sum_{j \in E} Q_{ij}(x) g^j(x)$$

The process (Z_t) can be constructed as follows. Let U_1, U_2, \dots be a sequence of independent identically distributed random variables having an exponential distribution with parameter λ : for any $t > 0$,

$$\mathbb{P}(U_i < t) = 1 - e^{-\lambda t};$$

and let $Z_0 \in M \times E$ be a random variable independent of $(U_i)_{i \geq 1}$. Define a discrete time Markov chain $(\tilde{Z}_n = (\tilde{X}_n, \tilde{Y}_n))$ on $M \times E$ recursively by $\tilde{Z}_0 = Z_0$,

$$\tilde{X}_{n+1} = \Phi^{\tilde{Y}_n}(U_{n+1}, \tilde{X}_n),$$

and

$$\mathbb{P}(\tilde{Y}_{n+1} = j | \tilde{X}_{n+1}, \tilde{Y}_n = i) = Q_{ij}(\tilde{X}_{n+1}).$$

Now, set $T_0 = 0$ and $T_{n+1} = T_n + U_{n+1}$ for any $n \geq 0$ and define $(Z_t)_{t \geq 0}$ by

$$Z_{T_n+s} = (\Phi^{\tilde{Y}_n}(s, \tilde{X}_n), \tilde{Y}_n) \tag{3}$$

for all $0 \leq s < U_{n+1}$. The memoryless property of exponential random variables make $(Z_t)_{t \geq 0}$ a continuous time càd-làg Markov process.

Let us prove that the generator of this process is indeed given by (2).

We let $P = (P_t)_{t \geq 0}$ denote the semigroup induced by $(Z_t)_{t \geq 0}$. For each $t \geq 0$, P_t acts on bounded measurable maps $g : M \times E \rightarrow \mathbb{R}$ according to the formula

$$P_t g(x, i) = \mathbb{E}(g(Z_t) | Z_0 = (x, i)).$$

Let K_t and Q_t be the operators defined by

$$K_t g(x, i) = g(\Phi_t^i(x), i)$$

and

$$Q_t = K_t Q.$$

Set $Q^0 = Id$ and let Q^n be the random operator $Q_{U_1} \circ \dots \circ Q_{U_n}$. It follows from (3) that

$$P_t g = \sum_{n \geq 0} \mathbb{E} \left(\mathbb{1}_{\{N_t = n\}} Q^n K_{t-T_n} g \right) \tag{4}$$

where $N_t = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}}$ is the Poisson process induced by (T_n) .

Remark 2.1. Note that P_t can be rewritten as

$$P_t = \sum_{n \geq 0} \lambda^n e^{-\lambda t} \int_{\{u \in \mathbb{R}_+^n : \sum_{i=1}^n u_i < t\}} (Q_{u_1} \circ \dots \circ Q_{u_n} K_{t-\sum u_i}) du_1 \dots du_n. \quad (5)$$

Let $C_0 = C_0(M \times E)$ denote the set of real valued functions $f : M \times E \rightarrow \mathbb{R}$ continuous in the first variable equipped with the uniform norm $\|\cdot\|$. We also let $C_1 \subset C_0$ denote the set of C^0 functions which are C^1 in the x variable. We say that P is *Feller* provided that, for any $t \geq 0$, P_t maps C_0 into itself and $\lim_{t \rightarrow 0} \|P_t f - f\| = 0$.

Proposition 2.2. *The semigroup $P = (P_t)$ is Feller. Furthermore, for all $g \in C_1$,*

$$\lim_{t \rightarrow 0} \left\| \frac{P_t g - g}{t} - Lg \right\| = 0.$$

Proof. By Lebesgue continuity theorem and (4), $P_t g \in C_0$ whenever $g \in C_0$. Moreover, setting apart the first two terms in (4) leads to

$$P_t g = e^{-\lambda t} K_t g + \lambda e^{-\lambda t} \int_0^t Q_u K_{t-u} g du + R(g, t) \quad (6)$$

where $|R(g, t)| \leq |g| \mathbb{P}(N_t > 1) = |g|(1 - e^{-\lambda t}(1 + \lambda t))$. Therefore $\lim_{t \rightarrow 0} \|P_t g - g\| = 0$.

Let us call A the infinitesimal generator of (K_t) . If g is C^1 in x , it is in the domain of A and

$$\lim_{t \rightarrow 0} \frac{K_t g - g}{t} = Ag : (x, i) \mapsto \langle \nabla g^i(x), F^i(x) \rangle.$$

Therefore, by (6), $\frac{P_t g - g}{t} \xrightarrow{t \rightarrow 0} Ag - \lambda g + \lambda Q_0 g$, and the result follows. \square

Similarly, we let \tilde{P} denote the discrete time semigroup of \tilde{Z} . It is defined by the formula

$$\tilde{P}g(x, i) = \mathbb{E}(g(\tilde{Z}_1) | Z_0 = (x, i)) = \int_0^\infty Q_t g(x, i) \lambda e^{-\lambda t} dt. \quad (7)$$

Note that

$$\tilde{P} = \tilde{K}Q$$

where

$$\tilde{K}f = \int_0^\infty \lambda e^{-\lambda t} K_t f dt.$$

It follows from (7) that \tilde{P} is Feller, meaning that it maps C^0 into itself.

Notation. Throughout the paper we may write $\mathbb{P}_{x,i}(\cdot)$ for $\mathbb{P}(\cdot | Z_0 = (x, i))$ and $\mathbb{E}_{x,i}(\cdot)$ for $\mathbb{E}(\cdot | Z_0 = (x, i))$.

3 Support of the law of paths

In this section we shall describe the support of the law of $\{X_t : t \geq 0\}$ in terms of a certain differential inclusion induced by $\{F^i\}$.

For each $x \in \mathbb{R}^m$, let $co(F)(x) \subset \mathbb{R}^m$ be the compact convex set defined as

$$co(F)(x) = \left\{ \sum_{i \in E} \alpha_i F^i(x) : \alpha_i \geq 0, \sum_{i \in E} \alpha_i = 1 \right\}.$$

Let $C(\mathbb{R}, \mathbb{R}^m)$ denote the set of continuous paths $\eta : \mathbb{R} \mapsto \mathbb{R}^m$ equipped with the topology of uniform convergence on compact intervals. A solution to the differential inclusion

$$\dot{\eta} \in \text{co}(F)(\eta) \tag{8}$$

is an absolutely continuous function $\eta \in C(\mathbb{R}, \mathbb{R}^m)$ such that $\dot{\eta}(t) \in \text{co}(F)(\eta(t))$ for almost all $t \in \mathbb{R}$. We let $S^x \subset C(\mathbb{R}, \mathbb{R}^m)$ denote the set of solutions to (8) with initial condition x .

Lemma 3.1. *The set S^x is a nonempty compact connected set.*

Proof. Follows from standard results on differential inclusion, since the set-valued map $\text{co}(F)$ is upper-semi continuous, bounded with nonempty compact convex images; see [1] for details. \square

For $I = [0, T]$ with $T > 0$, or $I = \mathbb{R}^+$ we let $\pi_I : C(\mathbb{R}, \mathbb{R}^m) \mapsto C(I, \mathbb{R}^m)$ denote the canonical projection defined by $\pi_I(\eta)(s) = \eta(s)$; and we set $S_I^x = \pi_I(S^x)$.

Theorem 3.2. *Assume $X_0 = x$. Then the support of the law of $\{X_t : t \geq 0\}$ equals $S_{\mathbb{R}^+}^x$.*

Obviously, any path of X is a solution of the differential inclusion (8). The proof of Theorem 3.2 is made of several steps that consist in different approximations of a solution of (8).

Let $R^x \subset S_{\mathbb{R}^+}^x$ denote the set of paths η such that

$$\dot{\eta}(t) \in \bigcup_{i \in E} \{F^i(\eta(t))\}$$

for almost all $t \in \mathbb{R}^+$.

Lemma 3.3. *The set R^x is dense in $S_{\mathbb{R}^+}^x$.*

Proof. Clearly $\overline{R^x} \subset S_{\mathbb{R}^+}^x$. Conversely let $\eta \in S_{\mathbb{R}^+}^x$, and $\epsilon > 0$. Set

$$G_t(x) = \left\{ v \in \bigcup_{i \in E} F^i(x) : \langle v - \dot{\eta}(t), x - \eta(t) \rangle < \epsilon \right\}.$$

Since $\dot{\eta}(t) \in \text{co}(F)(\eta(t))$ almost surely, $G_t(x)$ is non empty. Furthermore, $(t, x) \mapsto G_t(x)$ is uniformly bounded, lower semicontinuous in x , and measurable in t . Hence, using a result by Papageorgiou [21], there exists $\xi : \mathbb{R} \mapsto \mathbb{R}^m$ absolutely continuous such that $\xi(0) = x$ and $\dot{\xi}(t) \in G_t(\xi(t))$ almost surely. In particular,

$$\frac{d}{dt} \|\xi(t) - \eta(t)\|^2 = 2\langle \dot{\xi}(t) - \dot{\eta}(t), \xi(t) - \eta(t) \rangle < 2\epsilon$$

so that

$$\sup_{0 \leq t \leq T} \|\xi(t) - \eta(t)\|^2 \leq 2\epsilon T. \tag{9}$$

Let $Q^x \subset S_{\mathbb{R}^+}^x$ denote the set of paths that are obtained by successive integrations of the F^i over positive intervals. More precisely, $\eta \in Q^x$ if there exist points $i_0, i_1, \dots \in E$ and numbers $0 = \sigma_0 \leq \sigma_1 \leq \dots \leq \sigma_n \uparrow \infty$ such that

$$\begin{cases} \eta(0) = x, \\ \eta(\sigma_k + s) = \Phi_s^{i_k}(\eta(\sigma_k)) \quad \text{for all } s \in [0, \sigma_{k+1} - \sigma_k]. \end{cases} \tag{9}$$

Lemma 3.4. *The set Q^x is dense in $S_{\mathbb{R}^+}^x$.*

Proof. By Lemma 3.3, it suffices to show that for all $\xi \in R^x, T > 0$ and $\varepsilon > 0$ there exists $\eta \in Q^x$ such that $\sup_{0 \leq t \leq T} \|\eta(t) - \xi(t)\| \leq \varepsilon$. Set $E = \{1, \dots, d\}$, and define by induction

$$\begin{aligned} \Omega_1 &= \left\{ t \in [0, T] : \dot{\xi}(t) = F^1(\xi(t)) \right\} \\ \forall i = 1, \dots, d-1, \quad \Omega_{i+1} &= \left\{ t \in [0, T] \setminus (\Omega_1 \cup \dots \cup \Omega_i) : \dot{\xi}(t) = F^{i+1}(\xi(t)) \right\}. \end{aligned}$$

Let \mathcal{C} be the algebra consisting of finite union of intervals in $[0, T]$. Since the Borel σ -field over $[0, T]$ is generated by \mathcal{C} there exists, for all $i = 1, \dots, d$, $I_i \in \mathcal{C}$ such that

$$\text{Leb}(\Omega_i \Delta I_i) \leq \varepsilon$$

where Leb stands for the normalized Lebesgue measure over $[0, T]$ and $A \Delta B$ is the symmetric difference of A and B : $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Now, set

$$\begin{aligned} J_1 &= I_1, \\ J_{i+1} &= I_{i+1} \setminus (J_1 \cup \dots \cup J_i) \end{aligned}$$

for $i = 1, \dots, d-1$ and

$$J_{d+1} = [0, T] \setminus (J_1 \cup \dots \cup J_d).$$

The family $\{J_i : i = 1, \dots, d+1\}$ forms a partition of $[0, T]$. Hence, there exists numbers $0 = \sigma_0 < \sigma_1 < \dots < \sigma_{N+1} = T$ and a map $i : \{0, \dots, N\} \mapsto \{1, \dots, d+1\}, k \mapsto i_k$ such that $]\sigma_k, \sigma_{k+1}[\subset J_{i_k}$. Set $F^{d+1} = F^1$ and define $\eta \in Q^x$ by Formula (9) for $k = 0, \dots, N$. For all $\sigma_k \leq t \leq \sigma_{k+1}$,

$$\eta(t) - \xi(t) = \eta(\sigma_k) - \xi(\sigma_k) + \int_{\sigma_k}^t \left(F^{i_k}(\eta(s)) - F^{i_k}(\xi(s)) \right) ds + \int_{\sigma_k}^t \left(F^{i_k}(\xi(s)) - \dot{\xi}(s) \right) ds.$$

Hence, by Gronwall's lemma, we get that

$$\|\eta(t) - \xi(t)\| \leq e^{K(\sigma_{k+1} - \sigma_k)} (v_k + m_k)$$

where K is a Lipchitz constant for all the vector fields (F^i) , $v_k = \|\eta(\sigma_k) - \xi(\sigma_k)\|$ and

$$m_k = L \text{Leb}([\sigma_k, \sigma_{k+1}] \setminus \Omega_{i_k}),$$

with $\|F^i\| \leq L$ and the convention $\Omega_{d+1} = \emptyset$. It then follows that, for all $k = 0, \dots, N$ and $\sigma_k \leq t \leq \sigma_{k+1}$,

$$\|\eta(t) - \xi(t)\| \leq \sum_{l=0}^k e^{K(\sigma_{k+1} - \sigma_l)} m_l \leq e^{KT} \sum_{l=0}^N m_l \leq e^{KT} \sum_{i=1}^{d+1} \text{Leb}(J_i \setminus \Omega_i). \quad (10)$$

The lemma will follow once we prove the following claim:

Claim. $\text{Leb}(\Omega_i \Delta J_i) \leq C\varepsilon$ for some constant C depending only on d .

For $i \leq d$, $J_i \setminus \Omega_i \subset I_i \subset \Omega_i$ so that $\text{Leb}(J_i \setminus \Omega_i) \leq \varepsilon$. Now,

$$\Omega_i \setminus J_i = \Omega_i \cap [I_i^c \cup J_1 \cup \dots \cup J_{i-1}] = \Omega_i \setminus \left(I_i \cup \left(\bigcup_{l=1}^{i-1} \Omega_l \cap J_l \right) \right).$$

For $l \neq i$, $\Omega_i \cap J_l \subset \Omega_i \cap I_l = \Omega_i \cap (I_l \setminus \Omega_l) \subset I_l \setminus \Omega_l$. Hence

$$\text{Leb}(\Omega_i \setminus J_i) \leq i\varepsilon \leq d\varepsilon.$$

We then have shown that $\text{Leb}(\Omega_i \Delta J_i) \leq (d+1)\varepsilon$ for $i = 1, \dots, d$. It then follows that $\text{Leb}(J_{d+1}) \leq d(d+1)\varepsilon$. This proves the claim and the lemma. \square

Let us complete the

Proof of Theorem 3.2. In view of Lemma 3.4 it suffices to show that for all $\varepsilon > 0, T > 0$ and $\eta \in Q^x$ the event $\sup_{0 \leq t \leq T} \|X_t - \eta(t)\| \leq \varepsilon$ has positive probability. The map η is determined by sequences $(i_n), (\sigma_n)$ and the formula (9). Set $t_k = \sigma_k - \sigma_{k-1}$. Since we allow t_k to be zero we may assume without loss of generality that $Q(\eta(\sigma_{k+1}))(i_k, i_{k+1}) > 0$ for all k .

Let n be such that $\sigma_n < T \leq \sigma_{n+1}$ and

$$\Sigma(n, T) = \left\{ (s_1, \dots, s_n) \in \mathbb{R}_+^n : \sum_i s_i \leq T \right\}.$$

For $\mathbf{s} \in \Sigma(n, T)$ set $\sigma_k(\mathbf{s}) = s_1 + \dots + s_k$ for $k = 1, \dots, n$, $\sigma_0(\mathbf{s}) = 0$ and $\sigma_{n+1}(\mathbf{s}) = T$. Now let $\eta_{\mathbf{s}} \in C([0, T], \mathbb{R}^m)$ be the function defined by (9) for $k = 0, \dots, n$ and $\sigma_k = \sigma_k(\mathbf{s})$. The mapping $\mathbf{s} \mapsto \eta_{\mathbf{s}}$ is easily seen to be continuous. Hence, there exist $\delta_1, \delta_2 > 0$ such that

$$\|\eta_{\mathbf{s}} - \eta\|_{[0, T]} \leq \varepsilon \tag{11}$$

and

$$Q(\eta(\sigma_{k+1}(\mathbf{s}))(i_k, i_{k+1})) \geq \delta_2 \tag{12}$$

whenever

$$\max_{i=1, \dots, n} |s_i - t_i| \leq \delta_1.$$

Let E_k be the event defined for $k = 1 \dots n$ by

$$E_k = \left\{ \max_{i=1, \dots, k} \|U_i - t_i\| \leq \delta_1, \tilde{Y}_1 = i_1, \dots, \tilde{Y}_n = i_n \right\}.$$

It follows from (11) that $\|X_t - \eta\|_{[0, T]} \leq \varepsilon$ on the event $E_n \cap \{\tau_{n+1} > T\}$. Now, there exists $r > 0$ such that

$$\mathbb{P}(U_{n+1} > T) \geq r \text{ and } \mathbb{P}(|U_k - t_k| \leq \delta_1) \geq r.$$

Therefore,

$$\begin{aligned} \mathbb{P}(E_n \cap \{\tau_{n+1} > T\}) &\geq r \mathbb{P}(E_n) \\ &= r \mathbb{E}(E_{n-1}; \mathbb{E}(Q(\tilde{X}_n)(i_{n-1}, i_n) | \mathcal{F}_{n-1})) \mathbb{P}(|U_n - t_n| \leq \delta_1) \\ &\geq \delta_2 r^2 \mathbb{P}(E_{n-1}). \end{aligned}$$

By induction we get

$$\mathbb{P}(E_n) \geq r(r\delta_2)^n > 0.$$

This concludes the proof. \square

4 The accessible set

In order to describe the long term behavior of (X_t) it is natural to define the *accessible set* of the process (X_t) as the set of points that can be "reached from everywhere" by (X_t) . The purpose of this section is to make such a definition precise and to investigate some of its basic properties.

For all $n \in \mathbb{N}_*$ let $\mathbb{T}_n = E^{n+1} \times \mathbb{R}_+^n$. Given

$$(\mathbf{i}, \mathbf{t}) = ((i_0, \dots, i_n); (t_1, \dots, t_n)) \in \mathbb{T}_n$$

and $x \in M$ we let

$$\Phi_{\mathbf{t}}^{\mathbf{i}}(x) = \Phi_{t_n}^{i_{n-1}} \circ \dots \circ \Phi_{t_1}^{i_0}(x). \quad (13)$$

The *positive* trajectory of x is the set

$$\gamma^+(x) = \{\Phi_{\mathbf{t}}^{\mathbf{i}}(x) : (\mathbf{i}, \mathbf{t}) \in \bigcup_{n \in \mathbb{N}_*} \mathbb{T}_n\}.$$

The *accessible set* of (X_t) is the (possibly empty) compact set $\Gamma \subset M$ defined as

$$\Gamma = \bigcap_{x \in M} \overline{\gamma^+(x)}.$$

Remark 4.1. The accessible set Γ is called the set of D -approachable points and is denoted by L in [2].

4.1 The accessible set and recurrence properties

The following lemmas relate the accessible set to the behavior of the processes \tilde{Z} and Z .

Lemma 4.2. *Assume that $\Gamma \neq \emptyset$. Let $p \in \Gamma$ and U be a neighborhood of p . There exists $m \in \mathbb{N}$ and $\delta > 0$ such that for all $i, j \in E$ and $x \in M$*

$$\mathbb{P}_{x,i}(\tilde{Z}_m \in U \times \{j\}) \geq \delta.$$

In particular,

$$\mathbb{P}_{x,i}(\exists m \in \mathbb{N}, \tilde{Z}_m \in U \times \{j\}) = 1.$$

Similarly we have

Lemma 4.3. *Assume that $\Gamma \neq \emptyset$. Let $p \in \Gamma$ and U be a neighborhood of p . There exists $m \in \mathbb{N}, t_1, \dots, t_m > 0$ and $\delta > 0$ such that for all $i, j \in E$ and $x \in M$*

$$\sum_{i=1}^m \mathbb{P}_{x,i}(Z_{t_i} \in U \times \{j\}) \geq \delta.$$

Since Γ is positively invariant under each flow Φ^i we deduce from Lemma 4.3 the following result.

Corollary 4.4. *Assume Γ has nonempty interior. Then*

$$\mathbb{P}_{x,i}(\exists t_0 \geq 0, \forall t \geq t_0, Z_t \in \Gamma \times E) = 1.$$

In order to provide the proof of Lemmas 4.2 and 4.3 we firstly establish two technical lemmas. For all $x \in M, (\mathbf{i}, \mathbf{t}) \in \mathbb{T}_n$ let

$$p(x, \mathbf{i}, \mathbf{t}) = Q(x_1)(i_0, i_1) \times Q(x_2)(i_1, i_2) \times \dots \times Q(x_n)(i_{n-1}, i_n)$$

where $x_0 = x$ and $x_k = \Phi_{t_k}^{i_{k-1}}(x_{k-1})$ for $k = 1, \dots, n$.

Definition 4.5. *The sequence $(\mathbf{i}, \mathbf{t}) \in \mathbb{T}_n$ is said adapted to $x \in M$ if $p(x, \mathbf{i}, \mathbf{t}) > 0$.*

For $i, j \in E$, let \mathbb{T}_n^{ij} denote the subset of \mathbb{T}_n consisting of sequences (\mathbf{i}, \mathbf{t}) for which $i_0 = i$ and $i_n = j$. The next lemma show that the positive trajectory of x can be obtained with adapted sequences.

Lemma 4.6. *There exists $r \in \mathbb{N}_*$ such that for all $x \in M, i, j \in E$*

$$\gamma^+(x) = \{\Phi_{\mathbf{t}}^i(x) : (\mathbf{i}, \mathbf{t}) \in \bigcup_{n \in \mathbb{N}_*} \mathbb{T}_{nr}^{ij} \text{ with } (\mathbf{i}, \mathbf{t}) \text{ adapted to } x\}.$$

Proof. By ergodicity of $Q(x)$ there exists $r \in \mathbb{N}$ such that $P^r(x)$ has positive entries. By continuity of P and compactness we can choose such an r to be independent of x . Now, given $(\mathbf{i}, \mathbf{t}) \in \mathbb{T}_n$ we can construct a new sequence in $\mathbb{T}_{r(n+1)}^{ij}$ as follows. Insert before i_0 a sequence of the form (i, j^1, \dots, j^{r-1}) with $Q(x)(i, j^1) \times \dots \times Q(x)(j^{r-2}, j^r) \times Q(x)(j^{r-1}, i_0) > 0$ and insert before t_1 a sequence $(0, \dots, 0)$ of length r . Replace i_n by j and for all $0 \leq k \leq n-1$ insert after i_k a sequence $(j_k^1, \dots, j_k^{r-1})$ with $Q(x_k)(i_k, j_k^1) \times \dots \times Q(x_k)(j_k^{r-2}, j_k^{r-1}) \times Q(x_k)(j_k^{r-1}, i_{k+1}) > 0$ and insert after t_k a sequence $(0, \dots, 0)$ of length r . Such a procedure doesn't affect the value of $\Phi_{\mathbf{t}}^i(x)$. Hence the result. \square

Lemma 4.7. *Assume $\Gamma \neq \emptyset$. Let $p \in \Gamma, U$ be a neighborhood of p and $i, j \in E$. There exist $m \in \mathbb{N}^*, \epsilon, \eta > 0$, finite sequences $(\mathbf{i}^1, \mathbf{t}^1), \dots, (\mathbf{i}^N, \mathbf{t}^N) \in \mathbb{T}_m^{ij}$ and a open covering O^1, \dots, O^N of M (i.e. $M = O^1 \cup \dots \cup O^N$) such that for all $x \in M$ and $\tau \in \mathbb{R}_+^m$:*

$$x \in O^k \text{ and } \|\tau - \mathbf{t}^k\| \leq \epsilon \Rightarrow \Phi_{\tau}^{\mathbf{i}^k}(x) \in U \text{ and } p(x, \mathbf{t}^k, \tau) \geq \eta.$$

Furthermore, m, ϵ and η are independent of $i, j \in E$.

Proof. Let V be a neighborhood of p with closure $\bar{V} \subset U$. Define open sets

$$O(\mathbf{i}, \mathbf{t}, \eta) = \{x \in M : \Phi_{\mathbf{t}}^{\mathbf{i}}(x) \in V \text{ and } p(x, \mathbf{i}, \mathbf{t}) > \eta\},$$

$$O(\mathbf{i}, \mathbf{t}) = \bigcup_{\eta > 0} O(\mathbf{i}, \mathbf{t}, \eta)$$

$$\text{and } O_{ij}^n = \bigcup O(\mathbf{i}, \mathbf{t}),$$

where the union is taken over all $(\mathbf{i}, \mathbf{t}) \in \mathbb{T}_{nr}^{ij}$ with r like in Lemma 4.6. Now $O_{ij}^{n+1} \supset O_{ij}^n$ (the proof is similar to the proof of Lemma 4.6) and, by Lemma 4.6, $M = \bigcup_n O_{ij}^n$. It then follows, by compactness of M that $M = O_{ij}^{n(i,j)}$ for some integer $n(i, j)$ and also

$$M = O_{ij}^n$$

for $n = \max n(i, j)$. By compactness again M can be rewritten as

$$M = O_{ij}^n = \bigcup_{l=1}^N O^l$$

where $O^k = O(\mathbf{i}^k, \mathbf{t}^k, \eta^k)$ for some $\eta^k > 0$ and $(\mathbf{i}^k, \mathbf{t}^k) \in \mathbb{T}_{nr}^{ij}$. The results follows with $m = nr$ and $0 < \eta < \min_{k=1, \dots, N} \eta^k$ and ϵ sufficiently small. \square

Proof of Lemma 4.2. We use the notation of Lemma 4.7. Suppose $x \in O^k$ and to shorten notation write \mathbf{t} for \mathbf{t}^k and \mathbf{i} for \mathbf{i}^k . By Lemma 4.7, if

$$\|(U_1, \dots, U_m) - \mathbf{t}\| \leq \epsilon \text{ and } (\tilde{Y}_0, \tilde{Y}_1, \dots, \tilde{Y}_m) = \mathbf{i}$$

then $\tilde{X}_m \in U$ and

$$Q(\tilde{X}_1)(i, i_1) \times \dots \times Q(\tilde{X}_m)(i_{m-1}, j) \geq \eta. \quad (14)$$

Therefore

$$\begin{aligned} \mathbb{P}(\tilde{X}_m \in U, \tilde{Y}_m = j | \tilde{X}_0 = x, \tilde{Y}_0 = i) &\geq \mathbb{P}\left(\|(U_1, \dots, U_m) - \mathbf{t}\| \leq \epsilon, (\tilde{Y}_0, \tilde{Y}_1, \dots, \tilde{Y}_m) = \mathbf{i}\right) \\ &= \mathbb{P}\left(\bigcap_{j=1}^m \{|U_j - t_j| \leq \epsilon, \tilde{Y}_j = i_j\}\right). \end{aligned}$$

Now

$$\mathbb{P}(|U_m - t_m^k| \leq \epsilon, \tilde{Y}_m = j | \mathcal{F}_{m-1}) = Q(\tilde{X}_m)(\tilde{Y}_{m-1}, j) \mathbb{P}(|U_m - t_m| \leq \epsilon)$$

on the event

$$\bigcap_{j=1}^{m-1} \{|U_j - t_j| \leq \epsilon \text{ and } \tilde{Y}_j = i_j\}.$$

Hence, using (14),

$$\begin{aligned} \mathbb{P}\left(\bigcap_{j=1}^m \{|U_j - t_j| \leq \epsilon \text{ and } \tilde{Y}_j = i_j\}\right) &\geq \eta \mathbb{P}\left(\bigcap_{j=1}^m \{|U_j - t_j| \leq \epsilon\}\right) \\ &= \eta e^{-\sum_{j=1}^m t_j} (e^\epsilon - e^{-\epsilon}). \quad \square \end{aligned}$$

The proof of Lemma 4.3 is similar. Details are left to the reader.

4.2 Topological properties of the accessible set

The differential inclusion (8) induces a set-valued dynamical system $\Psi = \{\Psi_t\}$ defined by

$$\Psi_t(x) = \Psi(t, x) = \{\eta(t) : \eta \in S^x\}$$

enjoying the following properties

- (i) $\Psi_0(x) = \{x\}$,
- (ii) $\Psi_{t+s}(x) = \Psi_t(\Psi_s(x))$ for all $t, s \geq 0$,
- (iii) $y \in \Psi_t(x) \Rightarrow x \in \Psi_{-t}(y)$.

For subsets $I \subset \mathbb{R}$ and $A \subset \mathbb{R}^m$ we set

$$\Psi(I, A) = \bigcup_{(t,x) \in I \times A} \Psi_t(x).$$

A set $A \subset \mathbb{R}^m$ is called *strongly* positively invariant under Ψ if $\Psi_t(A) \subset A$ for $t \geq 0$. It is called *invariant* if for all $x \in A$ there exists $\eta \in S^x$ such that $\eta(\mathbb{R}) \subset A$. Given $x \in \mathbb{R}^m$, the *limit set* of x under Ψ is defined as

$$\omega_\Psi(x) = \bigcap_{t \geq 0} \overline{\Psi_{[t, \infty[}(x)}.$$

Lemma 4.8. *The set $\omega_\Psi(x)$ is compact connected invariant and strongly positively invariant.*

Proof. It is not hard to deduce the first three properties from Lemma 3.1. For the last one, let $p \in \omega_\Psi(x)$, $s > 0$ and $q \in \Psi_s(p)$. By Lemma 3.4, for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $(\mathbf{i}, \mathbf{t}) \in \mathbb{T}_n$ such that $d(\Phi_{\mathbf{t}}^{\mathbf{i}}(p), q) < \varepsilon$. Continuity of $\Phi_{\mathbf{t}}^{\mathbf{i}}$ makes the set

$$W = \{z \in M : d(\Phi_{\mathbf{t}}^{\mathbf{i}}(z), q) < \varepsilon\}$$

an open neighborhood of p . Hence $W \cap \Psi_{[t, \infty[}(x) \neq \emptyset$ for all $t > 0$. This proves that $d(q, \overline{\Psi_{[t, \infty[}(x)}) < \varepsilon$ and since ε is arbitrary $q \in \overline{\Psi_{[t, \infty[}(x)}$. \square

Remark 4.9. For a general differential inclusion with an upper semi-continuous bounded right hand side with compact convex values, the omega limit set of a point is not (in general) strongly positively invariant, see e.g. [4].

Proposition 4.10. *The set Γ enjoys the following properties*

- (i) $\Gamma = \bigcap_{x \in M} \omega_\Psi(x)$,
- (ii) $\Gamma = \omega_\Psi(p)$ for all $p \in \Gamma$,
- (iii) Γ is compact, connected, strongly positively invariant and invariant under Ψ .
- (iv) Either Γ has empty interior or its interior is dense in Γ .

Proof. (i) : Let $x \in M$ and $y \in \Psi_t(x)$. Then $\gamma^+(y) \subset \Psi_{[t, \infty[}(x)$. Hence $\Gamma \subset \overline{\Psi_{[t, \infty[}(x)}$ for all x . This proves that $\Gamma \subset \bigcap_{x \in M} \omega_\Psi(x)$. Conversely let $p \in \bigcap_{x \in M} \omega_\Psi(x)$. Then, for all $t > 0$ and $x \in M$, $p \in \overline{\Psi_{[t, \infty[}(x)} \subset \gamma^+(x)$ where the latter inclusion follows from Lemma 3.4. This proves the converse inclusion.

(ii) : By Lemma 3.4, $\omega_\Psi(p) \subset \Gamma$. The converse inequality follows from (i).

(iii) : This follows from (ii) and Lemma 4.8.

(iv) : Suppose $\text{int}(\Gamma) \neq \emptyset$. Then there exists an open set $U \subset \Gamma$ and $\bigcup_{\mathbf{t}, \mathbf{i}} \Phi_{\mathbf{t}}^{\mathbf{i}}(U)$ is an open subset of Γ dense in Γ . \square

An equilibrium p for the flow Φ^1 is called an *attracting equilibrium* if there exists a neighborhood U of p such that

$$\lim_{t \rightarrow \infty} \|\Phi_t^1(x) - p\| = 0$$

uniformly in $x \in U$. In this case, the *basin of attraction* of p is the open set

$$B(p) = \left\{ x \in \mathbb{R}^m : \lim_{t \rightarrow \infty} \|\Phi_t^1(x) - p\| = 0 \right\}.$$

Proposition 4.11. *Suppose the flow Φ^1 has an attracting equilibrium p with basin of attraction $B(p)$; and that for all $x \in M \setminus B(p)$, $\gamma^+(x) \cap B(p) \neq \emptyset$. Then*

- (i) $\Gamma = \overline{\gamma^+(p)}$,
- (ii) If furthermore $\Gamma \subset B(p)$. Then Γ is contractile. In particular, it is simply connected.

Proof. The proof of (i) is left to the reader. To prove (ii), let $h : [0, 1] \times \Gamma \rightarrow \Gamma$ be defined by

$$h(t, x) = \begin{cases} \Phi^1(-\log(1-t), x) & \text{if } t < 1, \\ p & \text{if } t = 1. \end{cases}$$

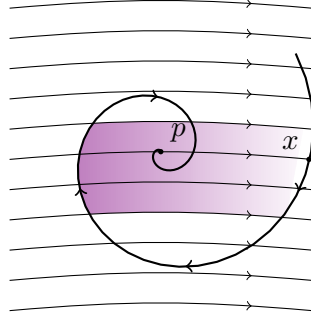
It is easily seen that h is continuous. Hence the result. \square

4.3 The two-dimensional case

We assume here that M is a subset of \mathbb{R}^2 and that one of the vector fields, say F^1 , has a globally attracting equilibrium p . Hence, by Proposition 4.11, $\Gamma \neq \emptyset$.

Proposition 4.12. *Assume that the eigenvalues of $DF^1(p)$ are non real with negative real parts and that $F^2(p) \neq 0$. Then p lies in the interior of Γ .*

Proof. The following drawing shows that this is the case: from the equilibrium p , one can follow the flow of F^2 and reach x , then follow the spiral, and switch back to F^2 to reach any point in the shaded region.



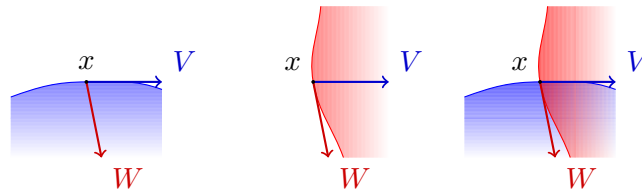
□

Proposition 4.13. *If the PDMP is driven by only two planar contractant real analytic vector fields without common trajectories, then the boundary of Γ is a finite union of pieces of trajectories.*

Proof. Let $V = F^0$ and $W = F^1$ denote the two vector fields.

First let us remark that in this analytic case the trajectories are real analytic curves. So is the curve defined by $\det(V, W) = 0$. In particular all these curves locally intersect one each other finitely many times.

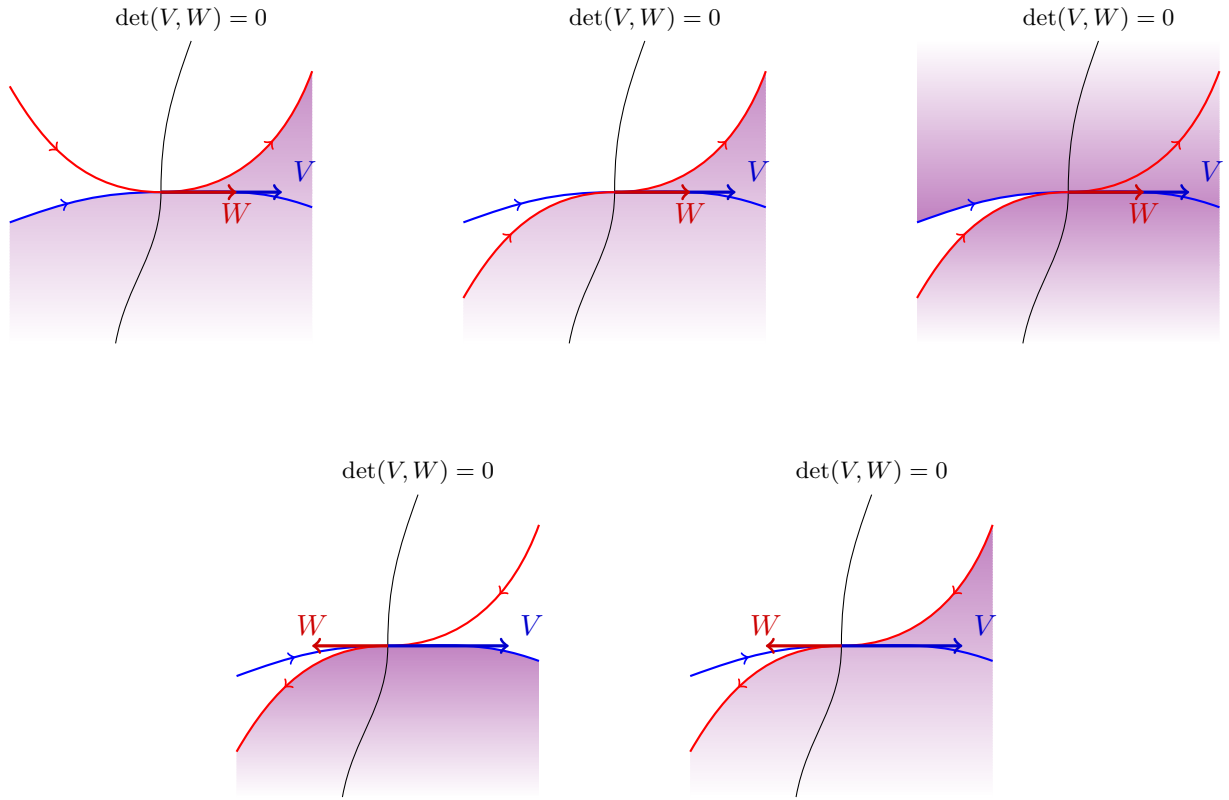
Let x be a point belonging to the boundary $\partial\Gamma$ of Γ . Suppose first that x does not belong to $\det(V, W) = 0$. Then the boundary of Γ in the neighbourhood of x has one of the three following forms.



To get this we just have to remark that at x the vector fields of our PDMP cannot point to the exterior of Γ and that it must be possible to reach x from the interior of Γ ...

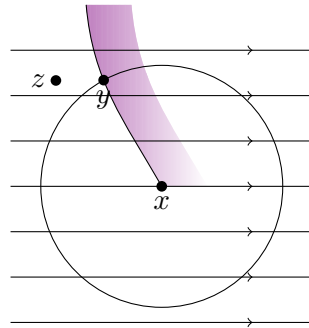
In particular for every $\varepsilon > 0$ the set $\partial\Gamma \cap |\det(V, W)| > \varepsilon$ is composed by a finite number of pieces of trajectories of both vector fields.

Let us now consider the case of a point x belonging to $\det(V, W) = 0$ but at which the two vector fields are transverse to the curve $\det(V, W) = 0$. By conjugation it is possible to suppose that V is an horizontal vector field. At such a point the different possibilities are the following. We distinguish two cases : at x , V and W are colinear, they may have the same sense or opposite sense.



In these cases also the boundary of Γ is simple. In the analytic case the preceding descriptions are valid at points x where V and W do not vanish.

Then we just have to say what happens if x is a stationary point for one of the two flows (since there are no common trajectories, a point cannot be stationary for both flows). For example let us suppose that $V(x) = 0$. Remark that the derivative of V at x does not have non real eigenvalues (because a stationary point with non real eigenvalues is in the interior of Γ). We can assume that at x , W is an horizontal flow. Take a little circle \mathcal{C} around x and a point y in $\mathcal{C} \cap \partial\Gamma$. Suppose that V is not horizontal at y . Now take z another point. If z belongs to Γ the V -trajectory starting at z stays in Γ . But if z is in Γ and sufficiently near x , the only way to reach z by one of the two flows is to follow a V -trajectory (otherwise x would not be on the boundary $\partial\Gamma$). It means that if z is not on the V -trajectory of y then z is not on the boundary $\partial\Gamma$.



This implies that the V trajectory of y is a part of $\partial\Gamma$. As a conclusion in a neighbourhood of x the boundary $\partial\Gamma$ is either a piece of a W -trajectory, the union of a piece of a W -trajectory and a piece of a V -trajectory, or the union of two pieces of V -trajectories. \square

5 Invariant Probabilities

Let $\mathcal{M}(M \times E)$ (respectively $\mathcal{M}^+(M \times E)$ and $\mathcal{P}(M \times E)$) denote the set of signed (respectively positive, and probability) measures on $M \times E$. For $\mu \in \mathcal{M}(M \times E)$ and $f \in L^1(\mu)$ we write μf for $\int f d\mu$. Recall that we let C_0 denote $C_0(M \times E)$. Given a bounded operator $K : C_0 \mapsto C_0$ and $\mu \in \mathcal{M}(M \times E)$ we let $\mu K \in \mathcal{M}(M \times E)$ denote the measure defined by duality :

$$\forall g \in C_0, \quad (\mu K)g = \mu(Kg).$$

The mappings $\mu \mapsto \mu P_t$, $\mu \tilde{P}$ preserve the set $\mathcal{M}^+(M \times E)$ and $\mathcal{P}(M \times E)$.

For $\mu \in \mathcal{M}(M \times E)$ we let μ_i denote the marginal measure on M given by $\mu_i(A) = \mu(A \times \{i\})$.

Recall that a probability measure μ is called *invariant* for (P_t) provided

$$\forall t \geq 0, \quad \mu P_t = \mu.$$

It is called invariant for \tilde{P} if $\mu \tilde{P} = \mu$.

We let \mathcal{P}_{inv} denote the set of invariant probabilities for (P_t) and $\tilde{\mathcal{P}}_{inv}$ the set of invariant probabilities for \tilde{P} .

For $n \in \mathbb{N}$ and $t > 0$ we let $\tilde{\Pi}_n$ and Π_t the (random) occupation measures defined by

$$\tilde{\Pi}_n = \frac{1}{n} \sum_{k=1}^n \delta_{\tilde{Z}_k} \quad \text{and} \quad \Pi_t = \frac{1}{t} \int_0^t \delta_{Z_t}.$$

By standard results for Feller chains on a compact space (see e.g. [12]), one has

Proposition 5.1. *The set $\tilde{\mathcal{P}}_{inv}$ is nonempty, compact (for the weak- \star topology) and convex. Furthermore, with probability one every limit point of (Π_n) lies in $\tilde{\mathcal{P}}_{inv}$.*

Proposition 5.2. *The mapping $\mu \mapsto \mu \tilde{K}$ maps $\tilde{\mathcal{P}}_{inv}$ homeomorphically onto \mathcal{P}_{inv} and extremal points of $\tilde{\mathcal{P}}_{inv}$ (i.e. ergodic probabilities for \tilde{P}) onto extremal points of \mathcal{P}_{inv} (ergodic probabilities for P).*

The inverse homeomorphism is the map $\mu \mapsto \mu Q$ restricted to \mathcal{P}_{inv} .

Proof. For all $f \in C_1$, integrating by part $\int_0^\infty \frac{dK_t f}{dt} e^{-\lambda t} dt$ and using the identities $\frac{dK_t f}{dt} = AK_t f = K_t A f$ leads to

$$\tilde{K}(\lambda I - A)f = \lambda f = (\lambda I - A)\tilde{K}f \quad (15)$$

Let $\mu \in \mathcal{P}(M \times E)$. Then, using (15) and the form of L gives

$$\mu \tilde{K} L f = \mu \tilde{K} (A - \lambda I) f + \lambda \mu \tilde{K} Q f = \lambda (-\mu f + \mu \tilde{P} f). \quad (16)$$

$$\mu L (\tilde{K} f) = \mu (A - \lambda I) \tilde{K} f + \lambda \mu Q \tilde{K} f = \lambda (-\mu f + \mu Q \tilde{K} f). \quad (17)$$

If $\mu \in \tilde{\mathcal{P}}_{inv}$, (16) implies $(\mu \tilde{K}) L f = 0$ for all $f \in C_1$ and since C_1 is dense in C_0 this proves that $\mu \tilde{K} \in \tilde{\mathcal{P}}_{inv}$. Similarly, if $\mu \in \mathcal{P}_{inv}$, (17) implies $\mu = \mu Q \tilde{K}$. Hence $(\mu Q) = (\mu Q) \tilde{K} Q = (\mu Q) \tilde{P}$ proving that $\mu Q \in \tilde{\mathcal{P}}_{inv}$. Furthermore the identity $\mu = \mu Q \tilde{K}$ for all $\mu \in \mathcal{P}_{inv}$ shows that the maps $\mu \mapsto \mu \tilde{K}$ and $\mu \mapsto \mu Q$ are inverse homeomorphisms. \square

Lemma 5.3. *Let $\mu \in \tilde{\mathcal{P}}_{inv}$. Then μ and $\mu \tilde{K}$ have the same support.*

Proof. Let $(x, i) \in \text{supp}(\mu)$ and let U be a neighborhood of x . Then for $t_0 > 0$ small enough and $0 \leq t \leq t_0$ $\Phi_{-t}^i(U)$ is also neighborhood of x . Thus

$$(\mu\tilde{K})_i(U) = \int \lambda e^{-\lambda t} \mu_i(\Phi_{-t}^i(U)) dt \geq \lambda \int_0^{t_0} e^{-\lambda t} \mu_i(\Phi_{-t}^i(U)) dt > 0.$$

This proves that $\text{supp}(\mu) \subset \text{supp}(\mu\tilde{K})$. Conversely, let $\nu = \mu\tilde{K}$ $(x, i) \in \text{supp}(\nu)$ and let U be a neighborhood of x . Then

$$\mu_i(U) = (\nu Q)_i(U) = \sum_j \int_U Q_{ji}(x) \nu_j(dx) \geq \int_U Q_{ii}(x) \nu_i(dx) > 0.$$

As a consequence, $\text{supp}(\mu) \supset \text{supp}(\mu\tilde{K})$. □

The following lemma is useful to compare empirical measures of \tilde{Z} and Z .

Lemma 5.4. *Let $f : M \times E \mapsto \mathbb{R}$ be a bounded measurable function. Then*

$$\lim_{t \rightarrow \infty} \Pi_t f - \tilde{\Pi}_{N_t} \tilde{K} f = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \Pi_{T_n} f - \tilde{\Pi}_n \tilde{K} f = 0$$

with probability one.

Proof. Write

$$\Pi_t f = \frac{N_t}{t} \left(\frac{1}{N_t} \sum_{i=0}^{N_t-1} \int_{T_i}^{T_{i+1}} f(Z_s) ds + r_t \right)$$

where $\|r_t\| \leq \|f\| \frac{U_{N_t+1}}{N_t}$.

Since $\lim_{t \rightarrow \infty} \frac{N_t}{t} = 1$ almost surely and $\mathbb{P}(U_n/n \geq \varepsilon) = e^{-\lambda n \varepsilon}$, $r_t \xrightarrow[t \rightarrow \infty]{a.s.} 0$, so

$$\Pi_t f - \frac{1}{N_t} \sum_{i=0}^{N_t-1} \int_{T_i}^{T_{i+1}} f(Z_s) ds \xrightarrow[t \rightarrow \infty]{a.s.} 0.$$

Now, note that

$$\int_{T_i}^{T_{i+1}} f(Z_s) ds = \int_0^{U_{i+1}} f(\phi_s^{\tilde{Y}^i}(\tilde{X}_i), \tilde{Y}_i) ds.$$

Therefore

$$M_n = \sum_{i=0}^{n-1} \left(\int_{T_i}^{T_{i+1}} f(Z_s) ds - \tilde{K} f(\tilde{X}_i, \tilde{Y}_i) \right)$$

is a martingale with increments bounded in L^2 : $\mathbb{E}(M_{n+1} - M_n)^2 \leq 2\|f\|^2/\lambda^2$. Therefore, by the strong law of large numbers for Martingales,

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = 0. \quad \square$$

Corollary 5.5. *The set \mathcal{P}_{inv} is nonempty compact (for the weak- \star topology) and convex. Furthermore, with probability one, every limit point of Π_t lies in \mathcal{P}_{inv} .*

Proof. This follows from Proposition 5.2 and Lemma 5.4. □

The chain \tilde{Z} (respectively (Z)) is called *stable* if it has a unique invariant probability.

Proposition 5.6. *The chain (Z_t) is stable if and only if (\tilde{Z}) is stable.*

Proof. This follows from Proposition 5.2. □

5.1 Support of invariant probabilities

Given $\mu \in \mathcal{P}(M \times E)$ we let $\text{supp}(\mu)$ denote its support. The following proposition relates Γ to $\text{supp}(\mu)$ when μ is invariant for \tilde{Z} or Z . We state and prove the result for \tilde{Z} and rely on Lemma 5.3 for Z .

Proposition 5.7. (i) If $\Gamma \neq \emptyset$ then $\Gamma \times E \subset \text{supp}(\mu)$ for all $\mu \in \tilde{\mathcal{P}}_{\text{inv}}$ (respectively \mathcal{P}_{inv}) and there exists $\mu \in \tilde{\mathcal{P}}_{\text{inv}}$ (resp. \mathcal{P}_{inv}) such that $\text{supp}(\mu) = \Gamma \times E$.

(ii) If Γ has nonempty interior, then $\Gamma \times E = \text{supp}(\mu)$ for all $\mu \in \tilde{\mathcal{P}}_{\text{inv}}$ (resp. \mathcal{P}_{inv}).

(iii) Suppose (\tilde{Z}) (resp. Z) is stable with invariant probability π . Then $\text{supp}(\pi) = \Gamma \times E$.

Proof. (i) follows from Lemma 4.2. Also, since Γ is strongly positively invariant, there are invariant measures supported by $\Gamma \times E$. (ii) follows from (i) and Corollary 4.4. To prove (iii), let $(p, i) \in \text{supp}(\pi)$. Let U, V be open neighborhoods of p with $\bar{U} \subset V$ compact. Let $0 \leq f \leq 1$ be a continuous function which is 1 on U and 0 outside V and let $\tilde{f}(x, j) = f(x)\delta_{j,i}$. Suppose $Z_0 = (x, j)$. Then with probability one

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \#\{1 \leq k \leq n : \tilde{Z}_k \in V \times \{i\}\} \\ & \geq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\tilde{Z}_k) = \int f \pi \\ & \geq \pi(U \times \{i\}) > 0. \end{aligned}$$

Hence (\tilde{Z}_n) visits infinitely often $U \times \{i\}$. In particular, $p \in \gamma^+(x)$. This proves that $\text{supp}(\pi) \subset \Gamma \times E$. The converse statement follows from (i). \square

Remark 5.8. The example given in Section 9 shows that the inclusion $\Gamma \times E \subset \text{supp}(\mu)$ may be strict when Γ has empty interior. On the other hand, the condition that Γ has non empty interior is not sufficient to ensure uniqueness of the invariant probability since there exist smooth minimal flows that are not uniquely ergodic. An example of such a flow can be constructed on a 3-manifold by taking the suspension of an analytic minimal non uniquely ergodic diffeomorphism of the torus constructed by Furstenberg in [14] (see also [19]). As shown in [2] (see also Section 6) a sufficient condition to ensure uniqueness of the invariant probability is that the vector fields verify a Hörmander bracket property at some point in Γ .

5.2 Law of pure types

Assume that M is an embedded submanifold of \mathbb{R}^m possibly with boundary. We let λ_M denote the normalized Riemannian measure on M and $\lambda_{M \times E} = \lambda_M \otimes \frac{1}{|E|} \sum_{i \in E} \delta_i$.

Proposition 5.9. Let $\mu \in \tilde{\mathcal{P}}_{\text{inv}}$ (respectively \mathcal{P}_{inv}) and let $\mu = \mu_{ac} + \mu_s$ be the Lebesgue decomposition of μ with μ_{ac} the absolutely continuous (with respect to $\lambda_{M \times E}$) measure and μ_s the singular (with respect to $\lambda_{M \times E}$) measure. Then both μ_{ac} and μ_s are in $\tilde{\mathcal{P}}_{\text{inv}}$ (respectively \mathcal{P}_{inv}). In particular, if μ is ergodic, then μ is either absolutely continuous or singular.

Proof. The key point is that \tilde{K} and Q , hence $\tilde{P} = \tilde{K}Q$, map absolutely continuous measures into absolutely continuous measures. For $\mu \in \tilde{\mathcal{P}}_{\text{inv}}$ the result now follows from the following simple Lemma 5.10 applied to \tilde{P} . \square

Lemma 5.10. *Let $(\Omega, \mathcal{A}, \lambda)$ be a measure space with λ a probability. Let \mathcal{M} (respectively $\mathcal{M}^+, \mathcal{P}, \mathcal{M}_{ac}$) denote the set of signed (positive, probability, absolutely continuous) measures on Ω . Let $K : \mathcal{M} \mapsto \mathcal{M}, \mu \mapsto \mu K$ be a linear map that maps each of the preceding sets into itself. Then if $\mu \in \mathcal{P}$ is a fixed point for K with Lebesgue decomposition $\mu = \mu_{ac} + \mu_s$ both μ_{ac} and μ_s are fixed point for K .*

Proof. Write $\mu K = \mu_{ac}K + \mu_sK = \mu_{ac}K + \nu_{ac} + \nu_s$ with $\mu_sK = \nu_{ac} + \nu_s$ the Lebesgue decomposition of μ_sK . Then, by uniqueness of the decomposition, $\mu_{ac} = \mu_{ac}K + \nu_{ac}$. Thus, $\mu_{ac} \geq \mu_{ac}K$. Now either $\mu_{ac} = 0$ and there is nothing to prove or, we can normalize by $\mu_{ac}(\Omega)$ and we get that $\mu_{ac} = \mu_{ac}K$. \square

6 Regularity of the law of the process

For simplicity we shall assume here that M is an m -dimensional submanifold of \mathbb{R}^m (typically a closed ball). However the results here carry out to the more general situation where M is any embedded submanifold of \mathbb{R}^m .

6.1 The results

Let us fix some additional notation. If u is a time, F a vector field, Φ the flow of F , and G another vector field, we can pull back G by Φ_u by defining:

$$(\Phi_u^*G)_x = (D(\Phi_u))_x^{-1} \cdot G_{\Phi_u(x)}.$$

Similarly we define the push-forward: $(\Phi_{u,*}G)_x = (D(\Phi_u))_{\Phi_{-u}(x)} \cdot G_{\Phi_{-u}(x)}$. If F is the vector field F^i , we write $\Phi_{u,*}^{i,*}$ and $\Phi_{u,*}^i$ the pull-back and push-forward.

Recall that, if \mathbf{i} is a sequence of indices $\mathbf{i} = (i_1, \dots, i_K)$ and \mathbf{u} is a sequence of times $\mathbf{u} = (u_1, \dots, u_K)$, $\Phi_{\mathbf{u}}^{\mathbf{i}} : M \rightarrow M$ is the composite map defined by (13). We introduce a bit of additional notation.

- If \mathbf{i} and \mathbf{j} are two sequences of indices, we write $\mathbf{i} \cdot \mathbf{j}$ the concatenation of \mathbf{i} and \mathbf{j} (and use the same notation for the \mathbf{u}).
- For any sequence \mathbf{i} , we let \mathbf{i}_k be the first k elements of \mathbf{i} , and define \mathbf{u}_k in the same way.

We can push vector fields forward and pull them back by the composite flows $\Phi_{\mathbf{u}}^{\mathbf{i}}$: we denote by $\Phi_{\mathbf{u},*}^{\mathbf{i}}$ and $\Phi_{\mathbf{u}}^{\mathbf{i},*}$ these operations.

The first results are “global” condition for regularity. Similar results may be found in the proofs of [2]; we state them here separately, with an additional uniformity on the starting point.

Theorem 6.1 (Regularity at jump times — global form). *Let x_0 be a point in M , K be an integer, $\mathbf{u} = (u_1, \dots, u_K)$ be a sequence of times, and $\mathbf{i} = (i_1, \dots, i_K)$ a sequence of indices. Suppose that \mathbf{i}, \mathbf{u} is adapted to x_0 in the sense of Definition 4.5. For $k = 1, \dots, K$, let $x_k = \Phi_{\mathbf{u}_k}^{\mathbf{i}_k}(x_0)$ be the successive points in the trajectory that follows the flow of F^{i_k} for a time u_k .*

Consider the family of tangent vectors at x_0 obtained by collecting tangent vectors at each jump, and pulling them all back to x_0 along the flows:

$$\{F^{i_1}(x_0), (\Phi_{\mathbf{u}_1}^{\mathbf{i}_1,*}F^{i_2})(x_0), \dots, (\Phi_{\mathbf{u}_{K-1}}^{\mathbf{i}_{K-1},*}F^{i_K})(x_0)\}. \quad (18)$$

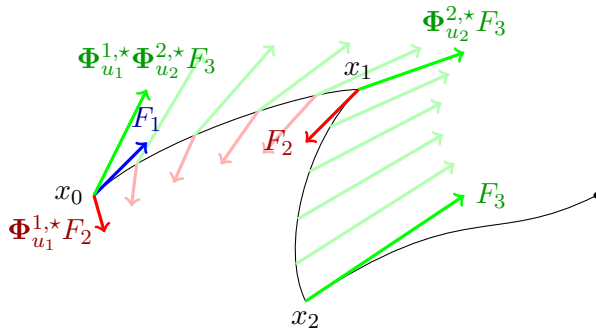


Figure 1: The global condition

The trajectory begins by following F^1 . At the first jump, it starts following F^2 ; we pull this tangent vector back to x_0 . The next tangent vector F^3 (at x_2) has to be pulled back by the two flows. If the three tangent vectors we obtain at x_0 span $T_{x_0}M$, the law of X_{T_3} , the process at its third jump, is partly regular.

If this collection spans the entire tangent plane $T_{x_0}M$, then the law of the process at its K -th jump is partly regular. More precisely, there exist \mathcal{U}_0 a neighborhood of x_0 , \mathcal{V}_0 a neighborhood of x_K , and a constant $c > 0$ such that:

$$\forall x \in \mathcal{U}_0, \quad \mathbb{P}_{x, i_1}[X_{T_K} \in \cdot] \geq c \lambda_{\mathbb{R}^m}(\cdot \cap \mathcal{V}_0). \quad (19)$$

The condition on the starting index i_1 may be relaxed: there exist \mathcal{U}'_0 and \mathcal{V}'_0 , a integer K' and a constant c' such that:

$$\forall x \in \mathcal{U}'_0, \forall i, \quad \mathbb{P}_{x, i}[X_{T_{K'}} \in \cdot] \geq c' \lambda_{\mathbb{R}^m}(\cdot \cap \mathcal{V}_0). \quad (20)$$

Theorem 6.2 (Regularity at fixed time — global form). *Let (x_k) , \mathbf{u} , \mathbf{i} be as above. Consider the following family:*

$$\begin{aligned} & \{F^{i_1}(x_0) - (\Phi_{\mathbf{u}_K}^{\mathbf{i}_K, *})F^{i_{K+1}}(x_0), \\ & (\Phi_{\mathbf{u}_1}^{\mathbf{i}_1, *})F^{i_2}(x_0) - (\Phi_{\mathbf{u}_K}^{\mathbf{i}_K, *})F^{i_{K+1}}(x_0), \\ & \quad \vdots \\ & (\Phi_{\mathbf{u}_{K-1}}^{\mathbf{i}_{K-1}, *})F^{i_K}(x_0) - (\Phi_{\mathbf{u}_K}^{\mathbf{i}_K, *})F^{i_{K+1}}(x_0)\}. \end{aligned} \quad (21)$$

If this family spans $T_{x_0}M$, then for any $t_0 > u_1 + \dots + u_K$, there exist two constants c and ϵ , and two non-empty open sets \mathcal{U}_0 and \mathcal{V}_0 such that:

$$\forall x \in \mathcal{U}_0, \forall i, \forall t \in [t_0, t_0 + \epsilon], \quad \mathbb{P}_{x, i}[X_t \in \cdot] \geq c \lambda_{\mathbb{R}^m}(\cdot \cap \mathcal{V}_0). \quad (22)$$

Remark 6.3. Since the derivative $(D\Phi_{\mathbf{i}_K, \mathbf{u}_K})_{x_0}$ is an isomorphism from $T_{x_0}M$ to $T_{x_K}M$, we can write down this condition at $T_{x_K}M$ by pushing all tangent vectors forward to the end of the trajectory rather than pulling them back to the starting point.

Remark 6.4. The condition for regularity at the K -th jump involves K vectors. Therefore, we need at least d jumps to get regularity in dimension d .

This condition for having a regular part is necessary and sufficient (if it does not hold for any n , \mathbf{u} , \mathbf{i} then the process is not regular). Unfortunately, it is not very easy to check, since one needs to “solve” the flows. However, it translates to a very nice local condition. To write down this condition, we need a bit of additional notation. Let \mathcal{F}_0 the collection of vector fields $(F^i : i \in E)$. Let $\mathcal{F}_k = \mathcal{F}_{k-1} \cup \{[F^i, V], V \in \mathcal{F}_{k-1}\}$, and $\mathcal{F}_k(x)$ the vector space (included in T_xM) spanned by $\{V(x), V \in \mathcal{F}_k\}$.

Similarly, starting from $\mathcal{G}_0 = \{F^i - F^j, i \neq j\}$, we define \mathcal{G}_k by taking Lie brackets with the F^i , and $\mathcal{G}_k(x)$ the corresponding subspace of T_xM .

Definition 6.5. We say that the weak bracket condition is satisfied at x if there exists k such that $\mathcal{F}_k(x) = T_x M$. If for some k , $\mathcal{G}_k(x) = T_x M$, we say that the strong bracket condition holds.

Since $\mathcal{G}_k(x)$ is a subspace of $\mathcal{F}_k(x)$, the strong condition implies the weak one. The converse is false, a counter-example is given below in Section 8.1.

These two conditions are called A (for the stronger) and B (for the weaker) in [2]. The following result is a version of Theorem 2 from [2], with an additional uniformity with respect to the initial point and the time t .

Theorem 6.6 (Regularity — local form, bracket condition). *If the weak bracket condition holds at x_0 , then the process is partly regular at jump times: there exist an integer K' , a constant $c > 0$ and non-empty open sets $\mathcal{U}_0, \mathcal{V}_0$ such that (20) holds.*

If the strong bracket condition holds, the process is partly regular: there is a t_0 , two constants $c > 0$ and $\epsilon > 0$, and two non-empty open sets $\mathcal{U}_0, \mathcal{V}_0$ such that (22) holds.

6.2 Three useful lemmas

The first result is a “uniform” local inversion lemma, for functions of \mathbf{t} that depend on a parameter x .

Remark 6.7. Even if x lives in some \mathbb{R}^k , we do not write it in boldface, for the sake of coherence with the rest of the paper.

Lemma 6.8. *Let m, n, k be integers, with $m \leq n$. Let f be a \mathcal{C}^1 map from $\mathbb{R}^n \times \mathbb{R}^k$ to \mathbb{R}^n ,*

$$f : (\mathbf{t}, x) \mapsto f(\mathbf{t}, x) = f_x(\mathbf{t}).$$

For any fixed x , f_x maps \mathbb{R}^n to itself; we denote its derivative at \mathbf{t} by $(Df_x)_{\mathbf{t}}$. Suppose that, for some points x_0 and \mathbf{t}_0 , $(Df_{x_0})_{\mathbf{t}_0}$ is invertible. Then we can find a neighborhood $J \subset \mathbb{R}^k$ of x_0 , an open set $I \subset \mathbb{R}^n$ and, for all $x \in J$, an open set $W_x \subset \mathbb{R}^n$, such that:

$$\tilde{f}_x : \begin{cases} W_x & \rightarrow I, \\ \mathbf{t} & \mapsto f_x(\mathbf{t}) \end{cases}$$

is a diffeomorphism. Moreover, for any $m \leq n$, and any neighborhood W of \mathbf{t}_0 , we can choose I, J and the W_x so that:

i) I is a cartesian product $I_1 \times I_2$ where $I_1 \subset \mathbb{R}^m, I_2 \subset \mathbb{R}^{n-m}$;

ii) $\forall x \in J, W_x \subset W$.

Proof. We “complete” the map f by defining:

$$H : \begin{cases} \mathbb{R}^n \times \mathbb{R}^k & \rightarrow \mathbb{R}^n \times \mathbb{R}^k \\ (\mathbf{t}, x) & \mapsto (f_x(\mathbf{t}), x). \end{cases}$$

H is \mathcal{C}^1 , and its derivative can be written in block form:

$$DH_{(\mathbf{t}, x)} = \begin{pmatrix} (Df_x)_{\mathbf{t}} & \star \\ 0 & I_k \end{pmatrix}.$$

Since $(Df_{x_0})_{\mathbf{t}_0}$ is invertible, $(DH)_{\mathbf{t}_0, x_0}$ is invertible. We apply the local inversion theorem to H : there exist open sets $\mathcal{U}_0, \mathcal{V}_0$ such that H maps \mathcal{U}_0 to \mathcal{V}_0 diffeomorphically. In order to

satisfy the properties i) and ii), we restrict H two times. First we define $\mathcal{U}_1 = \mathcal{U}_0 \cap (W \times \mathbb{R}^k)$, and $\mathcal{V}_1 = H(\mathcal{U}_1)$. Since \mathcal{V}_1 is open it contains a product set $\mathcal{V} = I_1 \times I_2 \times J$, and we let $\mathcal{U} = H^{-1}(\mathcal{V})$. For any $(y, x) \in I \times J$, define $g_x(y)$ the first component of $H^{-1}(y, x)$: composing by H , we see that $f_x(g_x(y)) = y$.

Now, define $W_x = \{\mathbf{t} \in \mathbb{R}^n; (\mathbf{t}, x) \in \mathcal{U}\}$. W_x is open, and included in W . Since f_x maps W_x to I , g_x is its inverse and both are continuous, so \tilde{f}_x is a diffeomorphism. \square

Lemma 6.9. *Let T be a continuous random variable in \mathbb{R}^n , with density f_T . Let $m \leq n$, and let ϕ be a C^1 map from $\mathbb{R}^n \times \mathbb{R}^m$ to \mathbb{R}^m :*

$$\phi : (\mathbf{t}, x) \mapsto \phi_x(\mathbf{t}).$$

Suppose that, for some x_0, \mathbf{t}_0 , $(D\phi_{x_0})_{\mathbf{t}_0} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has full rank m . Suppose additionally that f_T is bounded below by $c > 0$ on a neighborhood of \mathbf{t}_0 .

Then there exist a constant $c' > 0$, a neighborhood J of x_0 and a neighborhood I_1 of $\phi_{x_0}(\mathbf{t}_0)$ such that:

$$\forall x \in J, \quad \mathbb{P}[\phi(T, x) \in \cdot] \geq c' \lambda_{\mathbb{R}^m}(\cdot \cap I_1). \quad (23)$$

In other words, $\phi(T, x)$ has an absolutely continuous part w.r.t. the Lebesgue measure.

Proof. We know that $(D\phi_{x_0})_{\mathbf{t}_0}$ has rank m . Without loss of generality, we suppose that the first m columns are independent. In other words, writing $\mathbf{t} = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, we suppose that the derivative of $\psi_{x, \mathbf{v}} : \mathbf{u} \mapsto \phi_{x_0}(\mathbf{u}, \mathbf{v})$ is invertible in \mathbf{u}_0 for $\mathbf{v} = \mathbf{v}_0$.

Once more, we “complete” ϕ and define:

$$f_x : \begin{cases} \mathbb{R}^m \times \mathbb{R}^{n-m} & \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} \\ (\mathbf{u}, \mathbf{v}) & \mapsto (\phi_x(\mathbf{u}, \mathbf{v}), \mathbf{v}). \end{cases}$$

By Lemma 6.8, we can find $I_1 \subset \mathbb{R}^m$, $I_2 \subset \mathbb{R}^{n-m}$, $J \subset \mathbb{R}^m$ and $(W_x)_{x \in J} \subset \mathbb{R}^n$ such that f_x maps diffeomorphically W_x to $I_1 \times I_2$. Call \tilde{f}_x this diffeomorphism. By property ii) of the lemma, we can ensure that W_x is included in a given neighborhood of \mathbf{t}_0 . Since

$Df_x = \begin{pmatrix} D\psi_{x, \mathbf{v}} & \star \\ 0 & I \end{pmatrix}$, we can choose this neighborhood so that:

$$\forall x \in J, \forall \mathbf{t} \in W_x, \quad f_T(\mathbf{t}) |\det((Df_x)_{\mathbf{t}})|^{-1} \geq c'' > 0. \quad (24)$$

for some strictly positive constant c'' .

Write the random variable T as a couple (U, V) , and let A be a Borel set included in I_1 .

$$\begin{aligned} \mathbb{P}[\phi(T, x) \in A] &\geq \mathbb{P}[\phi(T, x) \in A, V \in I_2] \\ &= \mathbb{P}[f_x(U, V) \in A \times I_2] \\ &\geq \mathbb{P}[(U, V) \in \tilde{f}_x^{-1}(A \times I_2)] \\ &= \int_{\tilde{f}_x^{-1}(A \times I_2)} f_T(\mathbf{u}, \mathbf{v}) d\mathbf{u} d\mathbf{v} \\ &= \int_{\tilde{f}_x^{-1}(A \times I_2)} f_T(\mathbf{u}, \mathbf{v}) \left| \det((D\tilde{f}_x)_{\mathbf{u}, \mathbf{v}}) \right|^{-1} \cdot \left| \det((D\tilde{f}_x)_{\mathbf{u}, \mathbf{v}}) \right| d\mathbf{u} d\mathbf{v}. \end{aligned}$$

Since $\tilde{f}_x^{-1}(A \times I_2) \subset W_x$, we may use the bound (24). Then we can change variables by defining $(\mathbf{s}, \mathbf{v}) = \tilde{f}_x(\mathbf{u}, \mathbf{v})$. We obtain:

$$\begin{aligned} \mathbb{P}[\phi(T, x) \in A] &\geq c'' \int_{\tilde{f}_x^{-1}(A \times I_2)} \left| \det((D\tilde{f}_x))_{\mathbf{u}, \mathbf{v}} \right| d\mathbf{u} d\mathbf{v} \\ &= c'' \int_{A \times I_2} d\mathbf{s} d\mathbf{v} \\ &\geq c'' \lambda_{\mathbb{R}^m}(A) \lambda_{\mathbb{R}^{n-m}}(I_2). \end{aligned}$$

Therefore (23) holds with $c = c'' \lambda_{\mathbb{R}^{n-m}}(I_2)$. \square

Finally we need the following result.

Lemma 6.10. *Let U be a non-empty open set. There exist $0 < \epsilon_1 < \epsilon_2$, $k \in \mathbb{N}$ and an open set $U' \subset U$ such that:*

$$\begin{aligned} \forall x \in U', \forall i, j, & \quad \mathbb{P}_{x,i}[\forall t \in [\epsilon_1, \epsilon_2], X_t \in U, Y_t = j] \geq c, \\ \forall x \in U', \forall i, j & \quad \mathbb{P}_{x,i}[X_{T_k} \in U, Y_{T_k} = j] \geq c. \end{aligned}$$

Proof. The vector fields F^i are continuous and the space is compact, so the speed of the process is bounded: given the open set U , we can find $U' \subset U$ and ϵ_2 such that, starting from U' , the process cannot exit U before time ϵ_2 . Let $\epsilon_1 = \epsilon_2/2$: there is a positive probability that between $t = 0$ and $t = \epsilon_1$, the index jumps exactly one time, from i to j , and does not jump again before time $t = \epsilon_2$. The second result is similar; if all jump rates are bounded below, we can even choose $k = 1$. \square

6.3 Proofs for the global criteria

Since the flows are, by essence, deterministic, the randomness can only come from one source: the jump times. The proofs of (19) and (22) are very similar. We prove the former in a simple case, to illustrate the ideas, and only give a detailed proof of the latter.

Let us begin by the regularity of the law after the second jump, when the dimension is $d = 2$, and there are only two flows governed by F^1 and F^2 . Let the process start from x_0 along F^1 , and let X_{T_2} be its position at the moment of its second jump.

Let E_1, E_2 be independent exponential random variables. Using the flows, we can write:

$$X_{T_2} = \Phi^2(E_2, \Phi^1(E_1, x_0)) = \phi(E_1, E_2),$$

where ϕ is defined by:

$$\begin{aligned} \phi : \mathbb{R}^2 &\rightarrow M, \\ (u_1, u_2) &\mapsto \Phi_{(u_1, u_2)}^{(1,2)}(x_0). \end{aligned}$$

The derivative of ϕ is given by:

$$\begin{aligned} \frac{\partial \phi}{\partial u_1} &= D\phi \cdot \partial_1 = (D\Phi^2(u_2, \cdot))_{x_1} \cdot F^1(x_1) = (\Phi_{u_2, \star}^2 F^1)_{x_2}, \\ \frac{\partial \phi}{\partial u_2} &= D\phi \cdot \partial_2 = F^2(x_2). \end{aligned}$$

This family of two vectors in $T_{x_2}M$ has the same rank as its image by $(D\Phi_{(u_1, u_2)}^{(1,2)})^{-1}$:

$$(\Phi_{u_1, \star}^1 F^1, \Phi_{(u_1, u_2)}^{(1,2), \star} F^2)_x = (F^1, \Phi_{u_1, \star}^1 F^2)_x.$$

The hypothesis precisely states that this family spans $T_{x_0}M$, so ϕ has full rank at (u_1, u_2) . The product law $\mathcal{E}(1) \otimes \mathcal{E}(1)$ has a continuous density, which is strictly positive near (u_1, u_2) , so Lemma 6.9 applies and (19) holds, with \mathcal{U}_0 and \mathcal{V}_0 given by J and I_1 of Lemma 6.9. The more general (20) follows by an application of the second item in Lemma 6.10.

Let us now turn to the proof of (22), *i.e.* the regularity of the law at a fixed time under the global hypothesis. The idea is very similar. To avoid additional complications we suppose that the jump rates $\lambda(i, j, x)$ are strictly positive, so that all sequences are adapted. Let $\mathbf{i} = (i_1, \dots, i_K)$, $\mathbf{u} = (u_1, \dots, u_K)$ be sequences of indices and times, and suppose t_0 satisfies $t_0 > \sum_l u_l$. Let $\tilde{\mathbf{i}} = \mathbf{i} \cdot (i_{K+1})$, $\tilde{\mathbf{u}} = \mathbf{u} \cdot (t_0 - \sum u_k)$. The final position of the process is given by $X_{t_0} = \Phi_{\tilde{\mathbf{u}}}^{\tilde{\mathbf{i}}}(x_0)$. Fixing $\tilde{\mathbf{i}}$, this position can be seen as $\psi(\mathbf{u})$, where the function ψ (which depends on $\tilde{\mathbf{i}}$ and t_0) maps \mathbb{R}^K to M . Once more, we differentiate with respect to u_k . Since u_k appears twice in $\tilde{\mathbf{u}}$, we get:

$$\frac{\partial \psi}{\partial u_k}(\mathbf{u}) = (D\psi) \cdot \frac{\partial}{\partial u_k} = -F^{i_{K+1}}(\psi(\mathbf{u})) + \left(\Phi_{u_{k+1}, \dots, u_K, t_0 - \sum u_j, \star}^{i_{k+1}, \dots, i_{K+1}} F^{i_k} \right) (\psi(\mathbf{u})),$$

which is a tangent vector at $\psi(\mathbf{u})$. Mapping it back to $T_{x_0}M$, together with the vectors obtained for the other choices of k , we obtain the family of vectors described in the hypothesis. Therefore ψ has full rank as soon as the vectors defined in (21) span $T_{x_0}M$.

Define two events

$A =$ “the process makes k jumps before time t ”

$B =$ “the first k jumps are real and the indices followed are given by \mathbf{i} ”.

and let \mathcal{F} be the sigma field generated by the waiting times E_k . The construction of the process via independent exponential random variables (Section 2) shows that:

$$\mathbb{P}[B|\mathcal{F}] = \prod_l \frac{\lambda(i_{l-1}, i_l, \Phi_{(E_1, \dots, E_l)}^{i_l}(x_0))}{\lambda} \geq c.$$

Therefore

$$\begin{aligned} \mathbb{P}[X_t \in \cdot] &\geq \mathbb{P}[A \cap B \cap (X_t \in \cdot)] \\ &= \mathbb{P}[A \cap B \cap (\psi(E_1, \dots, E_k) \in \cdot)] \\ &= \mathbb{E} \left[\mathbb{P}[B|\mathcal{F}] \mathbf{1}_A \mathbf{1}_{\psi(E_1, \dots, E_k) \in \cdot} \right] \\ &\geq c \mathbb{P}[A \cap \psi(E_1, \dots, E_k) \in \cdot] \\ &\geq ce^{-\lambda t_0} \mathbb{P}[\psi(E_1, \dots, E_k) \in \cdot], \end{aligned}$$

since $A \supset \{E_{k+1} \geq t_0\}$. Thanks to Lemma 6.9, this implies (22), but only with $i = i_1$ and $\epsilon = 0$.

To prove the general form of (22) with the additional freedom in the choice of i and t , we need to restrict the starting point. By Lemma 6.10, we can find a neighborhood \mathcal{U}'_0 of x_0 , and three constants $0 < \epsilon_1 < \epsilon_2$ and $c > 0$ such that:

$$\forall x \in \mathcal{U}'_0, \quad \mathbb{P}_{x,i}[\forall t \in [\epsilon_1, \epsilon_2], X_t \in \mathcal{U}_0, I_t = i_1] \geq c.$$

Let $t'_0 = t_0 - \epsilon_1$ and $\epsilon = \epsilon_2 - \epsilon_1$, so that $[t'_0, t'_0 + \epsilon] = [t_0 + \epsilon_1, t_0 + \epsilon_2]$. Then, for any $x \in \mathcal{U}'_0$, and any $t \in [t'_0, t'_0 + \epsilon]$,

$$\begin{aligned} \mathbb{P}_{x,i}[X_t \in \cdot] &\geq \mathbb{E}_{x,i} \left[\mathbf{1}_{X_{t-t_0} \in \mathcal{U}_0, I_{t-t_0} = i_1} \mathbb{P}_{X_{t-t_0}, I_{t-t_0}} [X_{t_0} \in \cdot] \right] \\ &\geq c' \lambda_{\mathbb{R}^m}(\cdot \cap \mathcal{V}_0). \end{aligned}$$

This concludes the proof of Theorem 6.2.

6.4 Proof of the local criteria

Suppose that the weak bracket condition is satisfied at some point x . Then, by Theorem 5 of [2], the global condition (18) holds for some integer K . Similarly, under the strong bracket condition, Theorem 4 of [2] shows that (21) holds. This concludes the proof of Theorem 6.6.

7 Ergodicity

7.1 The embedded chain

Theorem 7.1. *Suppose there exists $p \in \Gamma$ at which the weak bracket condition holds. Then the chain \tilde{Z} admits a unique invariant probability $\tilde{\pi}$, absolutely continuous with respect to λ and*

$$\|\mathbb{P}(\tilde{Z}_n \in \cdot) - \tilde{\pi}\| \leq C\rho^n$$

for some constants $C, 0 \leq \rho < 1$ where $\|\cdot\|$ stands for the total variation norm.

Proof. By Theorem 6.6, (20) holds for some K' . It is easy to see that, up to a restriction of \mathcal{U}'_0 and a change of c , (20) also holds for $K'' = K' + 1$. By Proposition 4.2, the chain comes back to \mathcal{U}'_0 . Therefore, by usual regeneration/coupling arguments for Harris chains, detailed e.g. in [18], Sections III.8-10, $\tilde{\pi}$ is unique; the fact that (20) holds for K'' guarantees that the embedded renewal process defined by the times of return to \mathcal{U}'_0 is aperiodic. Once more by usual arguments, two copies of \tilde{Z} may be coupled in a time T that has geometric tails; this implies the exponential convergence (see e.g. [18], Section I.3 for details).

To see that $\tilde{\pi} \ll \lambda$ observe that by Proposition 4.2 and Theorem 6.6 $\tilde{\pi} \geq \delta c \psi$, where ψ , the Lebesgue measure on $\mathcal{V}_0 \times E$, is absolutely continuous with respect to λ . Therefore $\tilde{\pi}_{ac}$ (the absolutely continuous part of $\tilde{\pi}$ with respect to λ) is non zero and the result follows from Proposition 5.10. \square

Corollary 7.2. *Under the assumptions of Theorem 7.1, with probability one*

$$\lim_{n \rightarrow \infty} \tilde{\Pi}_n = \tilde{\pi}$$

and

$$\lim_{t \rightarrow \infty} \Pi_t = \tilde{\pi} \tilde{K}.$$

Proof. Follows from Theorem 7.1 and Propositions 5.2, 5.1 and 5.5. \square

7.2 The continuous time process

Lemma 7.3. *If the strong bracket condition holds at $x_0 \in \Gamma$, then for any neighborhood U of x_0 , there exist t_0, t_1 , two non-empty open sets $\mathcal{U}_0 \subset U$ and \mathcal{V}_1 , and three constants α_0, α_1 and ϵ such that:*

$$\forall x \in \mathcal{U}_0, \forall t \in [t_0, t_0 + \epsilon], \forall i \quad \mathbb{P}_{x,i}[X_t \in \cdot \cap \mathcal{V}_1] \geq \alpha_0 \lambda(\cdot \cap \mathcal{V}_1), \quad (25)$$

$$\forall y \in \mathcal{V}_1, \forall i, \quad \mathbb{P}_{y,i}[X_{t_1} \in \mathcal{U}_0] \geq \alpha_1. \quad (26)$$

Proof. The regularity result of Theorem 6.6 gives us $\mathcal{U}_0, \mathcal{V}_0, t_0$ and ϵ such that:

$$\forall x \in \mathcal{U}_0, \forall t \in [t_0, t_0 + \epsilon], \forall i, \quad \mathbb{P}_{x,i}[X_t \in \cdot] \geq c\lambda(\cdot \cap \mathcal{V}_0). \quad (27)$$

At this stage we can replace \mathcal{U}_0 by $\mathcal{U}_0 \cap U$ to ensure $\mathcal{U}_0 \subset U$. By Lemma 4.3 we cover M by a finite number of open sets O_1, \dots, O_k , such that, for some (u_1, \dots, u_k) ,

$$\forall y \in O_k, \forall i, \quad \mathbb{P}_{y,i}[X_{u_k} \in \mathcal{U}_0] \geq \alpha_2. \quad (28)$$

One of those open sets (say O_l) has a non-empty intersection with \mathcal{V}_0 : call \mathcal{V}_1 this intersection. Since $\mathcal{V}_1 \subset \mathcal{V}_0$, (27) implies (25). Since $\mathcal{V}_1 \subset O_l$, (28) implies (26), with $t_1 = u_l$. \square

Lemma 7.4. *If the strong bracket condition holds at $x_0 \in \Gamma$, then for any neighborhood U of x_0 , there exists a t and an $\alpha > 0$ such that for all x and i ,*

$$\mathbb{P}_{x,i}[X_t \in U] \geq \alpha. \quad (29)$$

Proof. We reuse Lemma 4.3 to define open sets O_1, \dots, O_k , such that, for some (u_1, \dots, u_k) , Equation (28) holds. Now we apply Lemma 7.3 to find t_0, t_1 such that (25) and (26) hold.

Suppose without loss of generality that $\min(u_i) = u_1 \leq u_k = \max(u_i)$. Let $s_k = t_0 + t_1$. For any n , define $t = u_k + ns_k$. For $j < k$, define s_j by:

$$s_j = \frac{t - u_j}{n} = s_k \frac{t - u_j}{t - u_k}.$$

We have the following bounds:

$$s_k \leq s_j \leq \frac{t - u_1}{t - u_k} s_k.$$

Choose n large enough to ensure that $\frac{t - u_1}{t - u_k} \leq (1 + \epsilon/s_k)$. Then each s_j is in the interval $[t_0 + t_1, t_0 + \epsilon + t_1]$. Therefore we have found t, n and (s_j) such that:

$$\begin{aligned} \forall j, \quad s_j &\in [t_0 + t_1, t_0 + \epsilon + t_1], \\ \forall j, \quad t &= u_j + ns_j. \end{aligned}$$

For $x \in O_j$, we have:

$$\begin{aligned} \mathbb{P}_{x,i}[X_t \in \mathcal{U}_0] &= \mathbb{P}_{x,i}[X_{u_j} \in \mathcal{U}_0, X_{u_j+s_j} \in \mathcal{U}_0, \dots, X_t \in \mathcal{U}_0] \\ &\geq \mathbb{E}_{x,i} \left[\prod_{l=1}^n \mathbb{1}_{X_{u_j+ls_j} \in \mathcal{U}_0} \right] \\ &= \mathbb{E}_{x,i} \left[\prod_{l=1}^{n-1} \mathbb{1}_{X_{u_j+ls_j} \in \mathcal{U}_0} \cdot \mathbb{P}_{X_{u_j+(n-1)s_j}, I_{u_j+(n-1)s_j}} [X_{s_j} \in \mathcal{U}_0] \right] \end{aligned}$$

by the Markov property. Now, for any $x \in \mathcal{U}_0$ and any i ,

$$\begin{aligned} \mathbb{P}_{x,i}[X_{s_j} \in \mathcal{U}_0] &\geq \mathbb{P}_{x,i}[X_{s_j-t_1} \in \mathcal{V}_1, X_{s_j} \in \mathcal{U}_0] \\ &\geq \mathbb{E}_{x,i} \left[\mathbb{1}_{X_{s_j-t_1} \in \mathcal{V}_1} \mathbb{P}_{X_{s_j-t_1}, I_{s_j-t_1}} [X_{t_1} \in \mathcal{U}_0] \right] \\ &\geq \alpha_1 \mathbb{P}_{x,i}[X_{s_j-t_1} \in \mathcal{V}_1] && \text{by (26)} \\ &\geq \alpha_0 \alpha_1 \lambda(\mathcal{V}_1) && \text{by (25),} \end{aligned}$$

since $s_j - t_1 \in [t_0, t_0 + \epsilon]$. Therefore

$$\mathbb{P}_{x,i}[X_t \in \mathcal{U}_0] \geq \alpha_0 \alpha_1 \lambda(\mathcal{V}_1) \mathbb{E}_{x,i} \left[\prod_l \mathbb{1}_{X_{u_j+ls_j} \in \mathcal{U}_0} \right].$$

By induction and thanks to (28), we get:

$$\mathbb{P}_{x,i}[X_t \in U] \geq \alpha_2(\alpha_0\alpha_1\lambda(\mathcal{V}_1))^n.$$

Therefore (29) holds with $\alpha = \alpha_2(\alpha_0\alpha_1\lambda(\mathcal{V}_1))^n$. \square

Theorem 7.5. *Suppose that there is a point $x \in \Gamma$ at which the strong bracket condition is satisfied. Let $\tilde{\pi}$ be the unique invariant probability of \tilde{Z} and let $\pi = \pi\tilde{K}$. Then*

$$\exists \alpha > 0, C > 0, \quad \|\mathbb{P}(Z_t \in \cdot) - \pi\| \leq C \exp(-\alpha t). \quad (30)$$

Remark 7.6. Theorem 1 of [2] states that, if the weak bracket condition is satisfied at $x \in \Gamma$, and if (Z_t) has an invariant measure, then it is unique and absolutely continuous with respect to the Lebesgue measure on $M \times E$. Under the strong bracket assumption, we have shown that the distribution of Z_t itself converges, and not only its empirical measure.

Proof of Theorem 7.5. Lemma 7.4 shows that two processes starting from anywhere can be coupled in some time t with positive probability. This implies (30) by the usual coupling argument. \square

8 Elementary examples

We give here a few examples of systems given by (2).

8.1 On the torus

Consider the system defined on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by the constant vector fields $F^i = e^i$, where (e_1, \dots, e_n) is the standard basis on \mathbb{R}^n . Then, as argued in [2], the weak bracket condition holds everywhere, and the strong condition does not hold. Therefore the chain \tilde{Z} is ergodic and converges exponentially fast, the empirical means of \tilde{Z}_n and Z_t converge, but the law of Z_t is singular with respect to the invariant measure.

8.2 Two planar linear flows

Let A be a 2×2 real matrix whose eigenvalues η_1, η_2 have negative real parts. Set $E = \{0, 1\}$ and consider the process defined on $\mathbb{R}^2 \times E$ by

$$F^0(x) = Ax \quad \text{and} \quad F^1(x) = A(x - a)$$

for some $a \in \mathbb{R}^2$. The associated flows are $\Phi_t^0(x) = e^{tA}x$ and $\Phi_t^1(x) = e^{tA}(x - a) + a$.

First note that, by using the Jordan decomposition of A , it is possible to find a scalar product $\langle \cdot \rangle$ on \mathbb{R}^2 (depending on A) and some number $0 < \alpha \leq \min(-\operatorname{Re}(\eta_1), -\operatorname{Re}(\eta_2))$ such that $\langle Ax, x \rangle \leq -\alpha \langle x, x \rangle$. Therefore

$$\langle A(x - a), x \rangle \leq -\alpha \langle x, x \rangle - \langle Aa, x \rangle \leq \|x\|(-\alpha\|x\| + \|Aa\|).$$

This shows that, for $R > \|Aa\|/\alpha$, the ball $M = \{x \in \mathbb{R}^2, \|x\| \leq R\}$ is positively invariant by Φ^0 and Φ^1 . Moreover every solution to the differential inclusion induced by $\{F^0, F^1\}$ eventually enters M . In particular $M \times E$ is an absorbing set for the process (Z_t) .

Another remark that will prove useful in our analysis is that

$$\det(F^0(x), F^1(x)) = \det(A) \det(a, x),$$

so that

$$\det(F^0(x), F^1(x)) > 0 \quad (\text{resp.} = 0) \Leftrightarrow \det(a, x) > 0 \quad (\text{resp.} = 0). \quad (31)$$

Case 1: a is an eigenvector

If a is an eigenvector of A the line $\mathbb{R}a$ is invariant by both flows, so that

$$\Gamma = \overline{\gamma^+(0)} = [0, a]$$

and there is one unique invariant probability π (whose support has to be Γ by Proposition 5.7.) Indeed, it is easily seen that Γ is an attractor for the set-valued dynamics induced by F^0 and F^1 . Therefore the support of every invariant measure equals Γ . If we consider the process restricted to Γ , it becomes one-dimensional and the strong bracket condition holds, proving uniqueness.

Remark 8.1. If $X(0) \notin \mathbb{R}a$, X will never reach Γ . As a consequence, the law of X_t and π are singular. In particular, their total variation distance is constant, equal to 1. Note also that the strong bracket condition being satisfied everywhere except on $\mathbb{R}a$, the law of X_t at finite times has a regular part.

Remark 8.2. Consider the following example: $A = -Id$, $a = (1, 0)$ and $\mathbb{R}a$ is identified to \mathbb{R} . If the jump rates are constant and equal to λ , it is easy to check (see [17, 22]) that the invariant measure μ on $[0, 1] \times \{0, 1\}$ is given by:

$$\mu = \frac{1}{2} (\mu_0 \times \delta_0 + \mu_1 \times \delta_1),$$

where μ_0 and μ_1 are Beta laws on $[0, 1]$,

$$\begin{aligned} \mu_0(dx) &= C_\lambda x^{\lambda-1} (1-x)^\lambda, \\ \mu_1(dx) &= C_\lambda x^\lambda (1-x)^{\lambda-1}. \end{aligned}$$

In particular, this example shows that the density of the invariant measure (with respect to the Lebesgue measure) may be unbounded: when the jump rate λ is smaller than 1, the densities blow up at 0 and 1.

Case 2: Eigenvalues are reals and a is not an eigenvector

Suppose $\eta_1, \eta_2 < 0$ and that a is not an eigenvector.

Let $\gamma_0 = \{\Phi_t^0(a), t \geq 0\}$, $\gamma_1 = \{\Phi_t^1(0), t \geq 0\}$. Note that γ_1 and γ_0 are image of each other by the transformation $T(x) = a - x$. The curve γ_0 (respectively γ_1) crosses the line $\mathbb{R}a$ only at point a (respectively 0). For, otherwise, the trajectory $t \mapsto \Phi_t^0(a)$ would have to cross the line $\text{Ker}(A - \lambda_1 I)$ which is invariant. This makes the curve $\gamma = \gamma_0 \cup \gamma_1$ a simple closed curve in \mathbb{R}^2 crossing $\mathbb{R}a$ at 0 and a . By Jordan curve Theorem, $\mathbb{R}^2 \setminus \gamma = B \cup U$ where B is a bounded component and U an unbounded one. We claim that

$$\Gamma = \overline{B}.$$

To prove this claim, observe that thanks to (31), F^0 and F^1 both point inward B at every point of γ . This makes \overline{B} positively invariant by Φ^0 and Φ^1 . Thus $\Gamma \subset \overline{B}$. Conversely, $\gamma \subset \Gamma$ (because 0 and a are accessibles from everywhere). If $x \in B$ there exists $s > 0$ such that $\Phi_{-s}^0(x) \in \gamma$ (because $\lim_{t \rightarrow -\infty} \Phi_t^0(a) = -\infty$) and necessarily $\Phi_{-s}^0(x) \in \gamma_1$. This proves that $x \in \gamma^+(0)$. Finally note that the strong bracket condition is verified in $\Gamma \setminus \mathbb{R}a$ proving uniqueness and absolute continuity of the invariant probability.

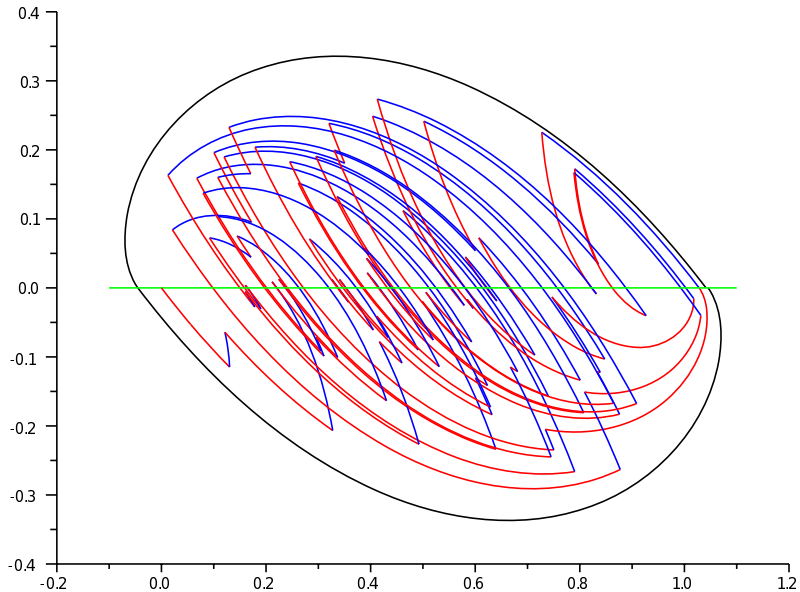


Figure 2: Double rotation.

Case 3: Eigenvalues are complex conjugates

Suppose now that the eigenvalues have a nonzero imaginary part. By Jordan decomposition, it is easily seen that trajectories of Φ^i converge in spiralling, so that the mappings $\tau^i(x) = \inf\{t > 0 : \Phi_t^i(x) \in \mathbb{R}a\}$ and $h^i(x) = \Phi_{\tau^i(x)}^i$ are well defined and continuous. Let $H : \mathbb{R}a \mapsto \mathbb{R}a$ be the map $h^0 \circ h^1$ restricted to $\mathbb{R}a$. Since two different trajectories of the same flow have empty intersection, the sequence $x_n = H^n(0)$ is decreasing (for the ordering on $\mathbb{R}a$ inherited from \mathbb{R} .) Being bounded (recall that M is compact and positively invariant), it converges to $x^* \in \mathbb{R}a$ such that $x^* = H(x^*)$. Let now $\gamma^0 = \{\Phi_t^1(x^*), 0 \leq t \leq \tau^1(x^*)\}$, $\gamma^1 = \{\Phi_t^0(h^1(x^*)), 0 \leq t \leq \tau^0(h^1(x^*))\}$ and $\gamma = \gamma^0 \cup \gamma^1$. Reasoning as previously shows that Γ is the bounded component of $\mathbb{R}^2 \setminus \gamma$ and that there is a unique invariant and absolutely continuous invariant probability.

We illustrate this situation in Figure 2, with

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark 8.3. Note that if the jump rates are small, the situation is similar to the one described in Remark 8.2, the process spends most of its time near the attractive points, and the density is unbounded at these points. Since they are in the interior of Γ , the density is not even continuous in the interior of Γ .

9 Knowing the flows is not enough

In this section we study in detail a PDMP on \mathbb{R}^2 , where the strong bracket condition holds everywhere except on Γ , and where there may be one or more invariant measures, depending on the dynamics of the discrete part of the process.

This model has been suggested by O. Radulescu. The continuous part of the process takes its values on \mathbb{R}^2 whereas its discrete part belongs to $\{0, 1\}$. For sake of simplicity we will denote (in a different way than in the beginning of the paper) by $(X_t, Y_t) \in \mathbb{R}^2$

the continuous component. The discrete component $(I_t)_{t \geq 0}$ is a continuous time Markov chain on $E = \{0, 1\}$ with jump rates $(\lambda_i)_{i \in E}$. Let $\alpha > 0$. The two vector fields F^0 and F^1 are given by

$$F^0(x, y) = \begin{pmatrix} -x + \alpha \\ -y + \alpha \end{pmatrix} \quad \text{and} \quad F^1(x, y) = \begin{pmatrix} -x + \frac{\alpha}{1 + y^2} \\ -y + \frac{\alpha}{1 + x^2} \end{pmatrix}$$

with $(x, y) \in \mathbb{R}^2$. Notice that the quarter plane $(0, +\infty)^2$ is invariant under the action of the vector fields F^0 and F^1 . If the support of the initial law of (X, Y) is included in the quarter plane (which is assumed from now on), then it is still the case for the law of (X_t, Y_t) at any time.

9.1 General properties of the two vector fields

Obviously, the vector fields F^0 has a unique stable point (α, α) , whereas F^1 may admits one or three critical points, according to the value of α .

Lemma 9.1. *Let us define*

$$a = \frac{\alpha + \sqrt{|\alpha^2 - 4|}}{2} \quad \text{and} \quad b = \left(\frac{\sqrt{4/27 + \alpha^2} + \alpha}{2} \right)^{1/3} - \left(\frac{\sqrt{4/27 + \alpha^2} - \alpha}{2} \right)^{1/3}.$$

Notice that b is positive and is the unique real solution of $b^3 + b = \alpha$. Then

- *if $\alpha \leq 2$, then F^1 admits a unique critical point (b, b) and it is stable,*
- *if $\alpha > 2$, then F^1 admits three critical points: (b, b) is unstable whereas (a, a^{-1}) and (a^{-1}, a) are stable.*

Proof. If (x, y) is a critical point of F^1 then (x, y) is solution of

$$\begin{cases} x(1 + y^2) = \alpha \\ y(1 + x^2) = \alpha. \end{cases}$$

As a consequence, x is solution of

$$0 = x^5 - \alpha x^4 + 2x^3 - 2\alpha x^2 + (1 + \alpha^2)x - \alpha = (x^2 - \alpha x + 1)(x^3 + x - \alpha).$$

The equation $x^3 + x - \alpha$ admits a unique real solution b . It belongs to $(0, \alpha)$. Obviously, if $\alpha \leq 2$, (b, b) is the unique critical point of F^1 whereas, if $\alpha > 2$ and a and a^{-1} are the roots of $x^2 - \alpha x + 1 = 0$, then F^1 admits the three critical points: (b, b) , (a, a^{-1}) and (a^{-1}, a) . Let us have a look to the stability of (b, b) . The Jacobian matrix of F^1 at (x, y) is given by

$$\text{Jac}(F^1)(x, y) = \begin{pmatrix} -1 & -\frac{2\alpha y}{(1 + y^2)^2} \\ -\frac{2\alpha x}{(1 + x^2)^2} & -1 \end{pmatrix}.$$

Since $1 + b^2 = \alpha/b$ one gets that

$$\text{Jac}(F^1)(b, b) = \begin{pmatrix} -1 & -2 + \frac{2b}{\alpha} \\ -2 + \frac{2b}{\alpha} & -1 \end{pmatrix}$$

and its eigenvalues are given by

$$\eta_1 = -3 + \frac{2b}{\alpha} = -1 - 2\frac{\alpha - b}{\alpha} \quad \text{and} \quad \eta_2 = 1 - \frac{2b}{\alpha} = \frac{b^3 - b}{\alpha}$$

and are respectively associated to the eigenvectors $(1, 1)$ and $(1, -1)$. Since $b < \alpha$, η_1 is smaller than -1 . Moreover, η_2 has the same sign than $b - 1$ *i.e.* the same sign than $\alpha - 2$. As a conclusion, (b, b) is stable (resp. unstable) if $\alpha < 2$ (resp. $\alpha > 2$).

Assume now that $\alpha > 2$. Then

$$\text{Jac}(F^1)(a, a^{-1}) = \begin{pmatrix} -1 & -\frac{2a}{\alpha} \\ -\frac{2}{\alpha a} & -1 \end{pmatrix}$$

and its two eigenvalues $-1 \pm 2\alpha^{-1}$ are negative. The critical points (a, a^{-1}) and (a^{-1}, a) are stable. \square

In the sequel, we assume that $\alpha > 2$. The sets

$$\begin{aligned} D &= \{(x, x) : x > 0\}, \\ L &= \{(x, y) : x > 0 \text{ and } 0 < y < x\}, \\ U &= \{(x, y) : y > 0 \text{ and } 0 < x < y\} \end{aligned}$$

are invariant under the action of the flows F^0 and F^1 . Moreover, the set D (and in particular the unique stable point (α, α) of F^0) is included in the stable manifold of the unstable equilibrium (b, b) of F^1 .

What happens if (X, Y) starts at a point $(x, y) \in L$? The answer may depend on the parameters $\lambda_0, \lambda_1, \alpha$.

9.2 Transience

Lemma 9.2. *Assume that $(X_0, Y_0) \in L$. Then, for any $t > 0$,*

$$0 \leq X_t - Y_t \leq (X_0 - Y_0) \exp\left(-\int_0^t \alpha(I_s) ds\right),$$

with $\alpha(0) = 1$ and $\alpha(1) = 1 - c\alpha < 0$ with $c = (3/8)\sqrt{3}$.

Proof. If $I_t = 0$ then

$$\frac{d}{dt}(X_t - Y_t) = -(X_t - Y_t).$$

On the other hand, if $I_t = 1$ then

$$\begin{aligned} \frac{d}{dt}(X_t - Y_t) &= -(X_t - Y_t) + \alpha \frac{X_t^2 - Y_t^2}{(1 + X_t^2)(1 + Y_t^2)} \\ &= -(1 - \alpha h(X_t, Y_t))(X_t - Y_t) \end{aligned}$$

where the function h is defined on $[0, \infty)^2$ by

$$h(x, y) = \frac{x + y}{(1 + x^2)(1 + y^2)}.$$

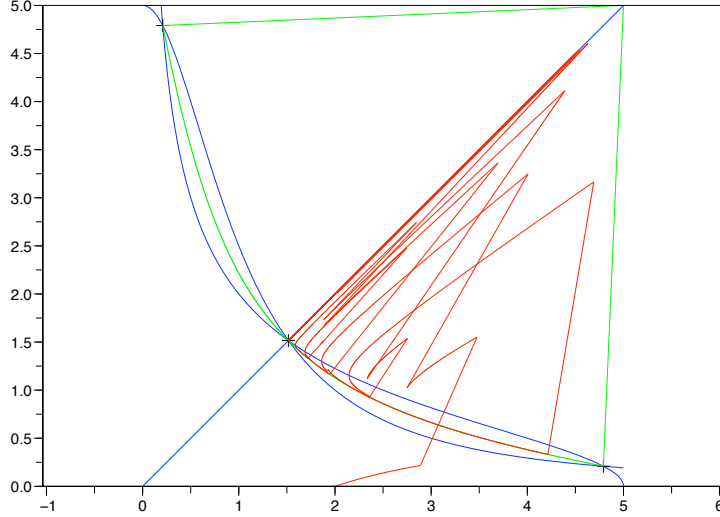


Figure 3: Trajectory of (X, Y) (red line) in the plane with $\lambda_0 = 1$, $\lambda_1 = 0.6$ and $\alpha = 5$.

The unique critical point of h on $[0, \infty)^2$ is $(1/\sqrt{3}, 1/\sqrt{3})$ and h reaches its maximum at this point:

$$c := \sup_{x, y > 0} h(x, y) = \frac{3\sqrt{3}}{8}.$$

As a consequence, for any $t \geq 0$,

$$\frac{d}{dt}(X_t - Y_t) \leq -\alpha(I_t)(X_t - Y_t) \quad \text{where} \quad \begin{cases} \alpha(0) = 1, \\ \alpha(1) = 1 - c\alpha. \end{cases}$$

Integrating this relation concludes the proof. \square

Corollary 9.3. *Assume that $(X_0, Y_0) \in L$. If $\lambda_1 > \lambda_0(c\alpha - 1)$ then (X_t, Y_t) converges exponentially fast to D almost surely. More precisely,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(X_t - Y_t) \leq -\frac{\lambda_1 - (c\alpha - 1)\lambda_0}{\lambda_0 + \lambda_1} < 0 \quad a.s. \quad (32)$$

In particular, the process (X, Y, I) admits a unique invariant measure μ which support is the set

$$S = \{(x, x) : x \in [b, \alpha]\}.$$

Proof. The ergodic theorem for the Markov process $(I_t)_{t \geq 0}$ ensures that

$$\frac{1}{t} \int_0^t \alpha(I_s) ds \xrightarrow[t \rightarrow \infty]{a.s.} \int \alpha(i) d\nu(i)$$

where the invariant measure ν of the process $(I_t)_{t \geq 0}$ is the Bernoulli measure with parameter $\lambda_0/(\lambda_0 + \lambda_1)$. The upper bound (32) is a straightforward consequence of Lemma 9.2. This ensures that the sets L and U are transient. At last, it is quite obvious that the set of recurrent points in D is exactly S . \square

One can also get an estimate for the p^{th} moment of $X_t - Y_t$.

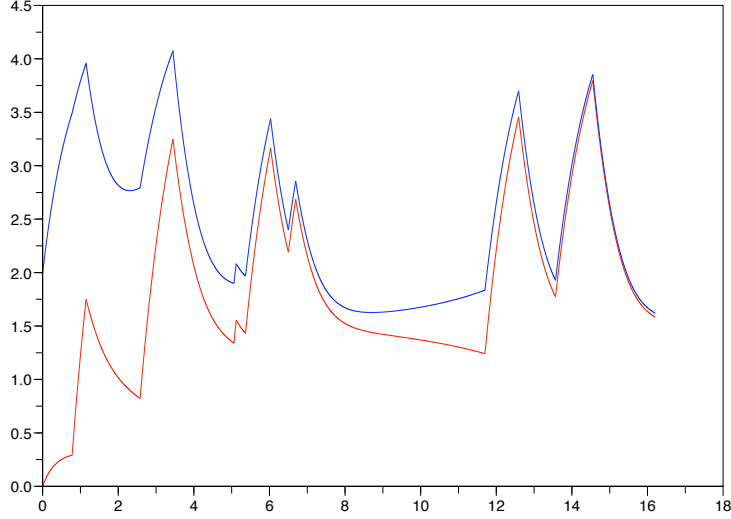


Figure 4: Trajectories of X (blue line) and Y (red line) with $\lambda_0 = 1$, $\lambda_1 = 0.6$ and $\alpha = 5$.

Corollary 9.4. *Assume that $(X_0, Y_0) \in L$. Let $p > 0$ such that*

$$\lambda_1 > (\lambda_0 + p)(c\alpha - 1). \quad (33)$$

Then there exists two positive constants c_p, μ_p such that

$$\mathbb{E}(|X_t - Y_t|^p) \leq c_p \mathbb{E}(|X_0 - Y_0|^p) e^{-\mu_p t}.$$

Proof. Once again, Lemma 9.2 ensures that

$$0 \leq \mathbb{E}(|X_t - Y_t|^p) \leq \mathbb{E}(|X_0 - Y_0|^p) \mathbb{E} \left[\exp \left(- \int_0^t p\alpha(I_s) ds \right) \right].$$

According to [3, Prop. 4.1], there exists $c_p \geq 1$ such that, for any $t \geq 0$,

$$\frac{1}{c_p} e^{-\mu_p t} \leq \mathbb{E} \left[\exp \left(- \int_0^t p\alpha(I_s) ds \right) \right] \leq c_p e^{-\mu_p t}$$

where $\mu_p = -\max \{ \operatorname{Re} \eta : \eta \in \operatorname{Spec}(M_p) \}$ and

$$M_p = \begin{pmatrix} -\lambda_0 - p & \lambda_0 \\ \lambda_1 & -\lambda_1 + p(c\alpha - 1) \end{pmatrix}.$$

The real parts of the eigenvalues of M_p are negative if and only if their sum S is negative and their product P is positive with

$$\begin{aligned} -S &= \lambda_0 + \lambda_1 + p(2 - c\alpha), \\ P &= p(\lambda_1 - (c\alpha - 1)(\lambda_0 + p)). \end{aligned}$$

The sum S is always negative and the positivity of P is given by (33). \square

9.3 Recurrence

In this section, we aim to show that (X, Y, I) may admit several invariant measures if the jump rate λ_0 is large enough. Let us define

$$U_t = \frac{X_t + Y_t}{2} \quad \text{and} \quad V_t = \frac{X_t - Y_t}{2}.$$

Of course (U, V, I) is still a PDMP. If

$$\frac{d}{dt} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = F^1(X_t, Y_t) \quad \text{then} \quad \frac{d}{dt} \begin{pmatrix} U_t \\ V_t \end{pmatrix} = G^1(U_t, V_t),$$

with

$$G^1(u, v) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} F^1(u+v, u-v) = \begin{pmatrix} -u + \frac{\alpha(1+u^2+v^2)}{(1+(u+v)^2)(1+(u-v)^2)} \\ -v + \frac{2\alpha uv}{(1+(u+v)^2)(1+(u-v)^2)} \end{pmatrix}.$$

Corollary 9.3 ensures that, if λ_1/λ_0 is large enough, then V_t goes to 0 exponentially fast.

Let us show that this is no longer true if λ_1/λ_0 is small enough. Let $\varepsilon > 0$. Assume that, with positive probability, $V_t \in (0, \varepsilon)$ for any $t \geq 0$. Then, for any time $t \geq 0$, $(U_t, V_t) \in [b, \alpha] \times [0, \varepsilon]$. Indeed, one can show that the set $X_t + Y_t \geq 2b$ for any $t \geq 0$ as soon as it is true at the initial time.

Lemma 9.5. *Assume that $(u, v) \in [b, \alpha] \times [0, \varepsilon]$. Then there exists $u_c \in (b, \alpha)$ and $K, \delta, \gamma, \tilde{\gamma} > 0$ (that do not depend on ε) such that $b_\varepsilon = b + K\varepsilon^2$ and*

$$G_1^1(u, v) \leq H_1^1(u, v) \quad \text{with} \quad H_1^1(u, v) = -\delta(u - b_\varepsilon).$$

and

$$G_2^1(u, v) \geq H_2^1(u, v) \quad \text{with} \quad H_2^1(u, v) = ((\gamma + \tilde{\gamma})\mathbb{1}_{\{u \leq u_c\}} - \tilde{\gamma})v.$$

Proof. Notice firstly that

$$\left| (1 + (u+v)^2)(1 + (u-v)^2) - (1 + u^2)^2 \right| \leq K\varepsilon^2. \quad (34)$$

Thus, using that $u^3 + u - \alpha = (u-b)(u^2 + bu + \alpha/b)$ we get that

$$\begin{aligned} G_1^1(u, v) &\leq -u + \frac{\alpha}{1+u^2} + K\varepsilon^2 \\ &\leq -(u-b) \frac{u^2 + bu + \alpha/b}{1+u^2} + K\varepsilon^2 \\ &\leq -(u-b) \frac{2b^2 + \alpha/b}{1+\alpha^2} + K\varepsilon^2. \end{aligned}$$

We get the desired upper bound for G_1^1 with

$$\delta = \frac{2b^2 + \alpha/b}{1 + \alpha^2} \quad \text{and} \quad b_\varepsilon = b + (K/\delta)\varepsilon^2.$$

Similarly, Equation (34) ensures that

$$G_2^1(u, v) \geq vk(u) \quad \text{with} \quad k(u) = \frac{2\alpha u}{(1+u^2)^2} - 1 - K\varepsilon^2.$$

Obviously, if ε is small enough, $k(b) > 0$, $k(\alpha) < 0$ and k is decreasing. Thus, if \tilde{u} is the unique zero of k on (b, α) , then one can choose

$$u_c = \frac{\tilde{u} + b}{2}, \quad \gamma = k(u_c) \quad \text{and} \quad \tilde{\gamma} = k(\alpha).$$

To get a simpler bound in the sequel we can even set $\tilde{\gamma} = k(\alpha) \vee 1$. \square

Finally, define $H_1^0(u, v) = G_1^0(u, v) = -(u - \alpha)$ and $H_2^0(u, v) = G_2^0(u, v) = -v$ and introduce the PDMP $(\tilde{U}, \tilde{V}, \tilde{I})$ where $\tilde{I} = I$ is the switching process of (U, V, I) and (\tilde{U}, \tilde{V}) is driven by H^0 and H^1 instead of G^0 and G^1 . From Lemma 9.5, we get that

$$U_t \leq \tilde{U}_t \quad \text{and} \quad \tilde{V}_t \leq V_t \quad (t \geq 0)$$

assuming that $(\tilde{U}_0, \tilde{V}_0, \tilde{I}_0) = (U_0, V_0, I_0)$. The last step is to study briefly the process $(\tilde{U}, \tilde{V}, \tilde{I})$. Let us firstly notice that if λ_1/λ_0 is small enough, then (I_s, \tilde{U}_s) spends an arbitrary large amount of time near $(1, b_\varepsilon)$ (and b_ε can be assumed smaller than u_c if ε is small enough). Thus

$$\frac{1}{t} \log \frac{\tilde{V}_t}{\tilde{V}_0} \geq \frac{1}{t} \int_0^t ((\gamma + \tilde{\gamma}) \mathbb{1}_{\{I_s=1, \tilde{U}_s < u_c\}} - \tilde{\gamma}) ds$$

since $\tilde{\gamma} \geq 1$. The right hand side converges almost surely to a positive limit as soon as λ_1/λ_0 is small enough. This implies that V cannot be bounded by ε forever.

Corollary 9.6. *If λ_1/λ_0 is small enough, the process (X, Y, I) admits three ergodic measures.*

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