



HAL
open science

REDUCTION ALGORITHM FOR SIMPLICIAL COMPLEXES

Laurent Decreasefond, Philippe Martins, Anais Vergne

► **To cite this version:**

Laurent Decreasefond, Philippe Martins, Anais Vergne. REDUCTION ALGORITHM FOR SIMPLICIAL COMPLEXES. 2012. hal-00688919v1

HAL Id: hal-00688919

<https://hal.science/hal-00688919v1>

Preprint submitted on 18 Apr 2012 (v1), last revised 30 Jul 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REDUCTION ALGORITHM FOR SIMPLICIAL COMPLEXES

L. DECREUSEFOND, P. MARTINS, AND A. VERGNE

ABSTRACT. Given a simplicial complex, we present an algorithm which allows us to reduce the number of vertices in an optimized order, while keeping its homology unchanged. We show that the algorithm reaches a Nash equilibrium, moreover we find both a lower and upper bounds for the number of vertices removed, the complexity of the algorithm, and the maximal order of the resulting complex for the coverage problem. We also give some simulation results for classical cases.

1. INTRODUCTION

Wireless sensor networks attract more and more research attention due to the extent of their applications as well as the decreasing costs and sizes of the electrical circuits. Fields where wireless sensor networks can be used range from battlefield surveillance to target enumeration in agriculture and include environmental monitoring. In most applications, the topology of the network, such as its connectivity and its coverage, is a critical factor. Moreover, sensors are autonomous systems: they are not plugged in nor physically connected to each other. Battery life is thus a key problem and energy saving a crucial point in wireless sensor networks management. Sensors are often deployed in large numbers, exceeding the number of necessary sensors. A first approach to reduce energy consumption would logically be to turn off sensors randomly. However by doing so one could modify the topology of the sensor network by creating a coverage hole or breaking the connectivity. Therefore, we first have to know the network's topology before we figure out energy saving methods. To guarantee the full knowledge of the topology one solution is to deploy the sensors according to a regular pattern (hexagon, square grid, rhombus or equilateral triangle) [1]. However the target field does not always allow such a precise deployment. Furthermore, the topology may not be time-invariant: sensors could be destroyed, their batteries could die, or their communication could be disturbed by seasonal changes. Another approach is thus to consider a random deployment that creates clusters of sensors and leaves holes of coverage.

There is extensive research on the coverage problem in wireless sensor networks. Location-based [7] and ranged-based [15] methods require exact location information for the former or exact distance between sensors for the latter. However connectivity-based schemes are of great interest for us since they do not need such knowledge. In [8], Ghrist et al for the first time construct the Vietoris-Rips complex based on the connectivity graph of the network and determine the coverage by computing the homology of the complex. Random simplicial complexes allow us to represent accurately wireless sensor networks and their topology without geographical information. Many mathematical tools exist and are used for random simplicial complexes. Coverage computation via homology is used in [3], [12] and [16], while a coverage hole detection algorithm is presented in [14] [13]. Moments of various variable can be obtained in specific regimes [11], or explicitly in one dimension [4] and generally by means of Malliavin calculus [5].

In this paper, we present an algorithm which returns which sensors can be turned off without modifying the topology of the network. Given a simplicial complex, our algorithm allows us to reduce the number of vertices in an optimized order, while keeping its homology unchanged. We show that the algorithm reaches a Nash equilibrium, moreover we find both a lower and upper bounds for the number of vertices removed, the complexity of the algorithm, and the maximal order of the resulting complex for the coverage problem. We also give some simulation results for classical cases, especially coverage complexes simulating wireless sensor networks.

This is the first reduction algorithm based on simplicial complexes aimed at energy savings in wireless sensor networks. A classical approach to power management in networks is the usage of the connectivity graph, such as in the dominating graphs problem [9]. However, graphs are 2-dimensional objects. One vertex has full knowledge of its neighbours, but there is no representation of the interaction between these neighbours. Therefore there is no notion of coverage in graphs. Simplicial complexes allow us to represent higher order relations, and is a more fitted tool for the representation of wireless sensor networks. In [10] [6], the authors use reduction of chain complexes to compute homology, reducing the work domain, which make it unapplicable to a coverage problem. The authors of [2] present a game theoretic approach to power management where they define a coverage function. However this method requires precise location information.

The paper is organized as follows: in Section 2 we introduce simplicial homology concepts and variables that we will use in the next sections. Section 3 is devoted to the description of our reduction algorithm. In Section 4, we discuss its mathematical properties. Simulation results are given in Section 5. Section 6 provides possible applications for the algorithm.

2. SIMPLICIAL HOMOLOGY

Graphs can be generalized to more generic topological objects known as simplicial complexes. While graphs model binary relations, simplicial complexes represent higher order relations. A simplicial complex is a topological space made up of vertices, edges, triangles, tetrahedra, and their n -dimensional counterparts. Given a set of vertices V and an integer k , a k -simplex is an unordered subset of $k + 1$ vertices $\{v_0, v_1, \dots, v_k\}$ where $v_i \in V$ and $v_i \neq v_j$ for all $i \neq j$. As it can be seen in Figure 1, a 0-simplex is a vertex, a 1-simplex an edge, a 2-simplex a triangle, a 3-simplex a tetrahedron, etc.

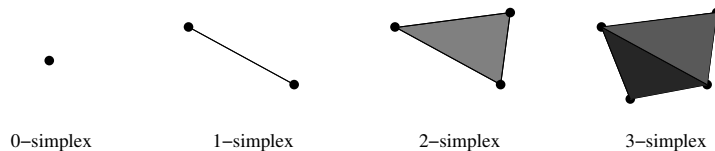


FIGURE 1. Example of simplices

Any subset of vertices included in the set of the $k + 1$ vertices of a k -simplex is a face of this k -simplex. Thus, a k -simplex has exactly $k + 1$ $(k - 1)$ -faces, which are $(k - 1)$ -simplices. For example, a tetrahedron has 4 3-faces which are triangles. A simplicial complex is a collection of simplices which is closed with respect to the inclusion of faces, i.e. all faces of a simplex are in the set of simplices,

and whenever two simplices intersect, they do so on a common face. An abstract simplicial complex is a purely combinatorial description of the geometric simplicial complex and therefore does not need the property of intersection of faces.

One can define an orientation for an abstract simplicial complex, where a change in the orientation corresponds to a change in the sign of the coefficient:

$$[v_0, \dots, v_i, \dots, v_j, \dots, v_k] = -[v_0, \dots, v_j, \dots, v_i, \dots, v_k].$$

Definition 1. Given an abstract simplicial complex X , for each integer k , $C_k(X)$ is the vector space spanned by the set of oriented k -simplices of X .

Definition 2. The boundary map ∂_k is defined to be the linear transformation $\partial_k : C_k \rightarrow C_{k-1}$ which acts on basis elements $[v_0, \dots, v_k]$ of C_k via

$$\partial_k[v_0, \dots, v_k] = \sum_{i=0}^k (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k].$$

This map gives rise to a chain complex: a sequence of vector spaces and linear transformations

$$\dots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0.$$

Definition 3. The k -th boundary group of X is $B_k(X) = \text{im } \partial_{k+1}$.

Definition 4. The k -th cycle group of X is $Z_k(X) = \text{ker } \partial_k$.

A standard result asserts that for any integer k ,

$$\partial_k \circ \partial_{k+1} = 0.$$

It follows that $B_k \subset Z_k$.

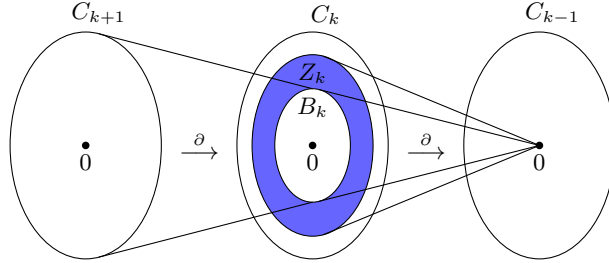


FIGURE 2. A chain complex showing the sets C_k , Z_k and B_k

Definition 5. The k -th homology group of X is the quotient vector space:

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}.$$

Definition 6. The k -th Betti number of X is the dimension:

$$\beta_k = \dim H_k = \dim Z_k - \dim B_k.$$

The Betti numbers are used to count the number of k -dimensional holes. For example, β_0 counts the number of 0-dimensional holes, that is the number of connected components. And β_1 counts the number of holes in the plane. If we are in dimension d , there is no sense in considering the k -th Betti number for $k > d$.

There are several famous types of abstract simplicial complexes, here we focus on three particular abstract simplicial complexes defined on a metric space. To characterize the randomness of the system, we consider that the set of vertices is represented by a Poisson point process ω with intensity λ on a Borel set X :

- i) For any $A \in \mathcal{B}(X)$, the number of vertices in A , $\omega(A)$, is a random variable following a Poisson law of parameter $\lambda S(A)$:

$$\mathbf{P}(\omega(A) = k) = e^{-\lambda S(A)} \frac{(\lambda S(A))^k}{k!}.$$

- ii) For any disjoint $A, A' \in \mathcal{B}(X)$, the random variables $\omega(A)$ and $\omega(A')$ are independent.

Definition 7 (Čech complex). *Given (X, d) a metric space, ω a finite set of points in X , and ϵ a real positive number. The Čech complex of parameter ϵ of ω , denoted $\mathcal{C}_\epsilon(\omega)$, is the abstract simplicial complex whose k -simplices correspond to $(k + 1)$ -tuples of vertices in ω for which the intersection of the $k + 1$ balls of radius ϵ centered at the $k + 1$ vertices is non-empty.*

Thus the Čech complex characterizes the coverage of a domain as we can see in Figure 3.

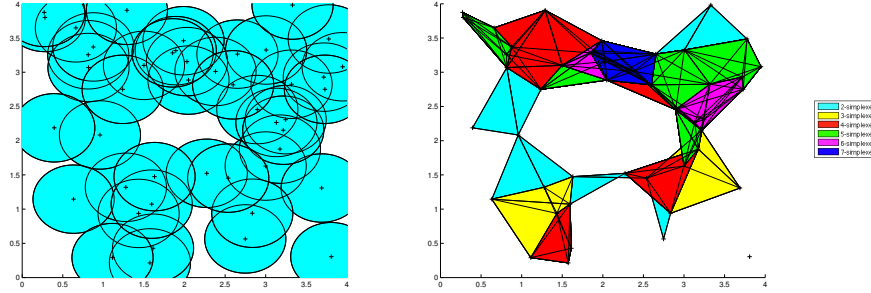


FIGURE 3. A sensor network and its associated Čech complex.

Definition 8 (Vietoris-Rips complex). *Given (X, d) a metric space, ω a finite set of points in X , and ϵ a real positive number. The Vietoris-Rips complex of parameter ϵ of ω , denoted $\mathcal{R}_\epsilon(\omega)$, is the abstract simplicial complex whose k -simplices correspond to unordered $(k + 1)$ -tuples of vertices in ω which are pairwise within distance less than ϵ of each other.*

In general, unlike the Čech one, Vietoris-Rips complexes are not topologically equivalent to the coverage of an area. However, the following gives us the relation between coverage and Vietoris-Rips complexes:

Lemma 1. *Given (X, d) a metric space, ω a finite set of points in X , and ϵ a real positive number,*

$$\mathcal{R}_{\sqrt{3}\epsilon}(\omega) \subset \mathcal{C}_\epsilon(\omega) \subset \mathcal{R}_{2\epsilon}(\omega).$$

In the Erdős-Rényi model, which is a random graph model, there is no geometric considerations, we extend the model to the homology:

Definition 9 (Erdős-Rényi complex). *Given n an integer and p a real number in $[0, 1]$, the Erdős-Rényi complex of parameters n and p , denoted $G(n, p)$, is an abstract simplicial complex with n vertices which are connected randomly. Each edge is included in the complex with probability p independent from every other edge. Then a k -simplex, for $k \geq 2$, is included in the complex if and only if all its faces already are.*

Only graph description is required to build a Vietoris-Rips or a Erdős-Rényi complex. That is why here we will give examples only on these two complexes.

3. REDUCTION ALGORITHM

In this section, we present the reduction algorithm. The algorithm takes as input a fully described abstract simplicial complex: all k -simplices must be explicit for every k integer. Then the algorithm aims at removing superfluous vertices while maintaining the homology type of the abstract simplicial complex.

There are several levels of knowledge of the homology type: you might want to maintain only the same first k_0 -th homology group and Betti numbers. For practical reasons we restrict ourselves to dimension 2, so only the first three Betti numbers might be nonzero: β_0 is the number of connected components, β_1 is the number of holes in the plane, β_2 is the number of voids and equals 1 if the simulation space is 3-dimensional, like a torus, and 0 otherwise and is therefore uninteresting. We have thus two different algorithms, for $k_0 = 1$ and $k_0 = 2$, but the main idea can be extended to greater dimensions.

The first algorithm, referred to henceforth as the connectivity algorithm, aims at maintaining β_0 . The connectivity algorithm takes as inputs the abstract simplicial complex, plus the list of connected active vertices that are to stay connected. The second algorithm, dealing with the coverage, aims at maintaining β_0 and β_1 . It takes as inputs in addition to the complex, the list of connected boundary vertices of the covered area that is to stay covered.

3.1. Degree calculation. The first part of the algorithm is the calculation of the degree for every k_0 -simplex, with k_0 the number of Betti numbers to be kept unchanged.

Definition 10. For k integer, the degree of a k -simplex (v_0, v_1, \dots, v_k) is the size of the greatest simplex it is part of:

$$D(v_0, v_1, \dots, v_k) = \max\{d \mid (v_0, v_1, \dots, v_k) \subset d\text{-simplex}\}.$$

We immediately have $D(v_0, v_1, \dots, v_k) \geq k$ for any k -simplex.

For the remainder of the paper, let $s_k(X)$ be the number of k -simplices of the abstract simplicial complex X .

The computation of the s_{k_0} degrees $D_1, \dots, D_{s_{k_0}}$ is done as follows:

```

for  $i = 1 : s_{k_0}$  do
  Get  $(v_0, \dots, v_{k_0})$  the vertices of the  $i$ -th  $k_0$ -simplex;
   $k = k_0$ ;
  while  $(v_0, \dots, v_{k_0})$  are vertices of a  $(k + 1)$ -simplex do
     $k = k + 1$ ;
  end
   $D_i = k$ ;
end
return  $D_1, \dots, D_{s_{k_0}}$ 

```

3.2. Indices computation. The second part of the algorithm is the computation of the indices for each vertex.

Definition 11. The index of a vertex v is the minimum of the degrees of the k_0 -simplices it is a vertex of:

$$I(v) = \min\{D(v_0, v_1, \dots, v_{k_0}) \mid v \in (v_0, v_1, \dots, v_{k_0})\},$$

with k_0 the number of Betti numbers to be kept unchanged. If a vertex v is not a vertex of a k_0 -simplex then $I(v) = 0$.

Let v_1, v_2, \dots, v_{s_0} be the vertices of the abstract simplicial complex, the computation of the s_0 indices is done as follows:

```

for  $i = 1 : s_0$  do
   $I(v_i) = 0$ ;
  for  $j = 1 : s_{k_0}$  do
    if  $v_i$  is vertex of  $k_0$ -simplex  $j$  then
      if  $I(v_i) == 0$  then
         $I(v_i) = D_j$ ;
      end
      else
         $I(v_i) = \min\{I(v_i), D_j\}$ ;
      end
    end
  end
end
return  $I(v_1), \dots, I(v_{s_0})$ 

```

The index of a vertex is an indicator of the density of vertices around the vertex: an index of k_0 indicates that at least one k_0 -simplex of the vertex is not a face of any $(k_0 + 1)$ -simplices; whereas a high index shows that each k_0 -simplices of the vertex is part of higher simplices.

An index of zero indicates that the vertex is not a part of a any k_0 -simplex: the vertex is isolated up to the k_0 -th degree. For $k_0 = 1$, that means that the vertex is disconnected. For $k_0 = 2$, the vertex is only linked to other vertices by edges, therefore is inside a coverage hole.

3.3. Optimized order for the removal of vertices. First, we consider the particular vertices. The active or boundary vertices that are given as inputs and define the area of interest for the homology computation are critical vertices that are requested to remain in the last complex. They are flagged as unremovable by a negative index.

Then the indices give us an order for the removal of vertices: the greater the index of a vertex, the greater its probability of being superfluous for the homology of the abstract simplicial complex. So the vertices with the greatest index are candidates for removal: one is chosen randomly. The removal of a vertex leads to the degradation of all the k -simplices it was a vertex of, to $k - 1$ -simplices for every k integer.

We need to ensure that the homology is unchanged, so we compute the k_0 first Betti numbers thanks to the boundary maps once before the algorithm runs, then every time a vertex is removed. This computation is instantaneous since the complex is already build, and only adjacency matrices defining the complex are needed. If they change, the vertex is put back into the abstract simplicial complex, with a negative index to flag it as critical. If they are unchanged, the removal of the vertex is confirmed. The degrees of the k_0 -simplices and the indices of the vertices are recalculated for the new abstract simplicial complex.

Lemma 2. *When a vertex of index $I_{\max} > k_0$ is removed, only vertices sharing a I_{\max} -simplex with it, and of index I_{\max} can have their index changed to $I_{\max} - 1$.*

Proof. Let w be the removed vertex of index I_{\max} , and v a vertex of the abstract simplicial complex. If v does not share any simplex with w , none of the degrees of its k_0 -simplex will change, and neither will its index.

Thus let us consider that the maximum common simplex of v and w is a k -simplex with $k \leq 1$. If $k < I_{\max}$ then w have index $k < I_{\max}$, which is absurd.

Then we can assume that $k \geq I_{\max}$. Either the index I of v is strictly less than I_{\max} and thus comes from an I -simplex not shared with w , therefore it is unmodified by the removal of w . Or the index of v is I_{\max} , it either still comes from a I_{\max} -simplex not shared with w and remains unmodified, or it comes from their common I_{\max} -simplex. In the latter, after the removal of w , the index of v becomes $I_{\max} - 1$. \square

The algorithm goes on removing vertices until every remaining vertex has its index below or equal to $k_0 - 1$. We give here the general algorithm for the conservation of the k_0 first Betti numbers:

```

Data: Abstract simplicial complex  $X$ , list  $L_C$  of critical vertices (active or
        boundary)
    Computation of  $\beta_0(X), \beta_1(X), \dots, \beta_{k_0-1}(X)$ ;
    Computation of  $D_1(X), \dots, D_{s_{k_0}}(X)$ ;
    Computation of  $I(v_1(X)), \dots, I(v_{s_0}(X))$ ;
for every  $v \in L_C$  do
    |  $I(v) = -1$ ;
end
 $I_{\max} = \max\{I(v_1(X)), \dots, I(v_{s_0}(X))\}$ ;
while  $I_{\max} > k_0 - 1$  do
    | Draw  $w$  a vertex of index  $I_{\max}$ ;
    |  $X' = X \setminus \{w\}$ ;
    | Computation of  $\beta_0(X'), \beta_1(X'), \dots, \beta_{k_0-1}(X')$ ;
    | if  $\beta_i(X') \neq \beta_i(X)$  for any  $i = 0, \dots, k_0 - 1$  then
    | |  $I(w) = -1$ ;
    | end
    | else
    | | Computation of  $D_1(X'), \dots, D_{s'_{k_0}}(X')$ ;
    | | for  $i = 1 : s'_0$  do
    | | | if  $I(v_i(X')) \geq 0$  then
    | | | | Computation of  $I(v_i(X'))$ ;
    | | | end
    | | end
    | |  $I_{\max} = \max\{I(v_1(X')), \dots, I(v_{s'_0}(X'))\}$ ;
    | |  $X = X'$ ;
    | end
end
return  $X$ 

```

Lemma 3. *Let us suppose the input critical vertices define the whole input space. For the connectivity algorithm, that means that the active vertices are a dominating set of the initial abstract simplicial complex. For the coverage algorithm, that means that the area defined by the boundary vertices is the entire domain.*

Then the algorithm can stop when all vertices' indices are below or equal to k_0 instead of $k_0 - 1$.

Proof. Let us suppose we are in the input data satisfies the hypotheses of the lemma. Let v be a vertex of index k_0 , which means that at least one of the k_0 -simplex it is vertex of is not a face of any $(k_0 + 1)$ -simplex. The removal of this vertex would lead to the removal of this particular k_0 -simplex. Since we need to maintain the homology on the entire domain, this would lead to a k_0 -dimensional hole, and an incrementation of β_{k_0-1} . \square

4. PROPERTIES

Theorem 4 (Nash equilibrium). *The algorithm reaches a Nash equilibrium. Every vertex in the final abstract simplicial complex is needed to maintain its homology.*

Proof. In the final abstract simplicial complex, every vertex is of index less than or equal to $k_0 - 1$. By the definition of a computed index, it is impossible for an index to be strictly less than k_0 if nonzero or negative. We then differentiate two types of vertices: vertices of index -1 and 0 .

First, negative indices are given to vertices to flag them as critical. Either a vertex is of negative index because it is an active/boundary vertex, in which case it is required to stay in the complex. Or a vertex is of negative index if its removal leads to a change in the Betti numbers.

Then, a vertex of index 0 is an isolated vertex. If it is isolated up to the k_0 -th of degree, its removal will decrease one of the k_0 th Betti number. For example, the removal of a first degree isolated vertex, that is a disconnected one, would lead to the decrementation of β_0 . As well, the removal of a vertex inside a hole will lead to the union of 2 or more holes.

If we are in the case of lemma 3, its proof shows that the vertices of indices k_0 are needed to maintain the homology. \square

Theorem 5 (Upper and lower bounds). *Let E_k be the set of vertices that have indices k . The number of removed vertices M is bounded by:*

$$\sum_{k=k_0}^{I_{\max}} \mathbf{1}_{[E_k \neq \emptyset]} \leq M \leq \sum_{k=k_0}^{I_{\max}} |E_k|.$$

Proof. Let us begin with the upper bound, the algorithm runs until all indices are less than or equal to $k_0 - 1$. So the maximum vertices the algorithm can remove is the number of vertices that initially have their index strictly greater than $k_0 - 1$. This is an optimal upper bound since this number of removed vertices is reached in the following case:

Let a k -simplex, with $k > k_0$, be the initial abstract simplicial complex, and n_C of its vertices be the initial critical vertices, necessarily $n_C \leq k$. The n_C critical vertices will have negative indices, the $k + 1 - n_C$ other vertices will have an index of k , and they will all be removed.

For the lower bound, we have seen in lemma 2 that the removal of a vertex of index I_{\max} can only decrease the index of vertices of index I_{\max} . In the worst case, it decreases all indices I_{\max} and the value of I_{\max} changes to next one, which is not necessarily $I_{\max} - 1$ depending on the critical vertices. Thus we can see, that at least one vertex per index value can be removed, hence the result.

The lower bound is reached in the previous case if $n_C = k$. \square

Theorem 6 (Complexity). *The complexity of the algorithm is upper bounded by:*

$$(N - n_C + 1)s_{k_0} \left(N + \sum_{k=k_0+1}^{N-1} s_k \right),$$

with n_C being the number of initial critical vertices, N the total number of vertices, and s_k the number of k -simplices in the abstract simplicial complex.

Proof. For the computation of the degrees of every k_0 -simplex, the algorithm traverses all the k -simplices for $k_0 < k \leq N - 1$ to see if the k_0 -simplex is included in it. Since there is at most s_k k -simplices, the computation of the degrees is of complexity at most $s_{k_0} \sum_{k=k_0+1}^{N-1} s_k$.

Then for the computation of the indices, the algorithm traverses, for every one of the N vertices, the k_0 -simplices it is vertex of, which is at most all the k_0 -simplices. The complexity of the computation of all the indices is therefore $s_{k_0}N$.

These computations are done once for the initial complex, then each time a vertex is removed, which is at most the total number of vertices minus the number of initial critical vertices, hence the result. \square

Corollary 7. *The complexity of the algorithm is $O(N4^N)$ when N goes to infinity.*

Proof. This is a direct consequence of Theorem 6. \square

Corollary 8. *The complexity of the algorithm for the Erdős-Rényi complex $G(N, p)$ if $p = O(1)$ is polynomial.*

Proof. In the Erdős-Rényi complex, the average number of k -simplices is given by $p^{\binom{k+1}{2}}$. The rest of the proof is similar to the one of Theorem 6. \square

Theorem 9 (Highest order simplex). *In the case of the algorithm applied to a Vietoris-Rips complex or a Čech complex in d dimensions, the highest order simplex of the final complex is at most a $(2d - 1)$ -simplex, excluding critical vertices.*

Proof. Let k be an integer strictly greater than $2d - 1$, and let us suppose that the $k + 1$ non-critical vertices v_0, \dots, v_k form a k -simplex. Since we are in d dimensions we can consider the vertex the furthest on each of the $2d$ directions. Note that two of these vertices can be the same vertex. Let us reorder the vertices such that v_0, \dots, v_{i-1} with $i \leq 2d$ are the extremity vertices defined above. Then v_i, \dots, v_k must fall in the covered area of v_0, \dots, v_{i-1} . Such vertices can be removed without changing the homology of the complex, therefore they can not be in the final complex which is absurd. \square

Corollary 10. *In the case of the algorithm applied to a Vietoris-Rips complex or a Čech complex on the entire definition area in 2 dimensions, to make the final simplicial complex planar, it is necessary and sufficient to remove edges that are second diagonals of a square.*

Proof. This is a direct consequence of Theorem 9. \square

Theorem 11 (Dominating set). *For the coverage algorithm applied to a Vietoris-Rips complex or a Čech complex on the entire definition area, the set of remaining vertices is a dominating set of the set of vertices of the initial abstract simplicial complex inside the coverage boundary.*

Proof. Let ϵ be the parameter of the Vietoris-Rips complex. The area inside the coverage boundary is covered by the final abstract simplicial complex: every point in it is at distance lower than ϵ from three vertices of the final complex. This is true for every vertex of the initial complex. The proof is the same in the case of the Čech complex. \square

5. SIMULATION RESULTS

We can see in Figure 4 the realisation of the connectivity algorithm on a Vietoris-Rips complex of parameter $\epsilon = 1$ based on a Poisson point process of intensity $\lambda = 4$ on a square of side length 4, with random active vertices. A vertex is active with probability $p = 0.5$ independantly from every other vertices. We can see active vertices circled in red, and non-active vertices which are kept to maintain the connectivity between active vertices starred in green. The removed vertices are black plusses.

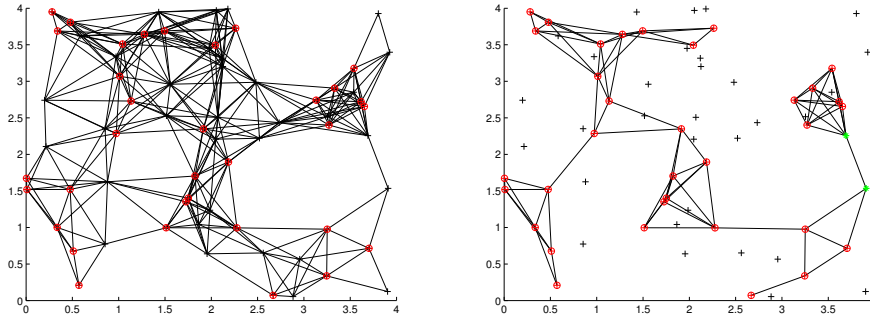


FIGURE 4. A Vietoris-Rips complex before and after the connectivity reduction algorithm.

For this configuration, on average on 200 runs, the algorithm removed 96.33% of the non-active vertices, and computed in 399.73 seconds.

We can see in Figure 5 the realisation of the coverage algorithm on a Vietoris-Rips complex of parameter $\epsilon = 1$ based on a Poisson point process of intensity $\lambda = 4.2$ on a square of side length 2, with a fixed boundary of vertices on the square perimeter. The boundary vertices are circled in red.

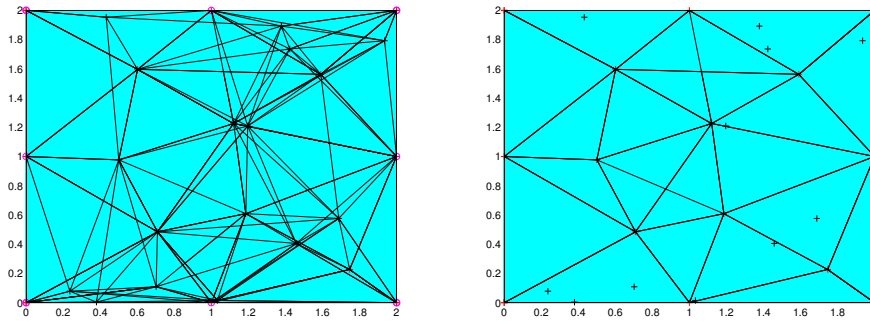


FIGURE 5. A Vietoris-Rips complex before and after the coverage reduction algorithm.

For this configuration, on average on 200 runs, the algorithm removed 69.22% of the non-boundary vertices, and computed in 206.01 seconds.

We can see in Figure 6 the realisation of the connectivity algorithm on a Erdős-Rényi complex of parameter $n = 15$ and $p = 0.3$, with random active vertices. We chose a small number of vertices for the figure to be readable. A vertex is active with probability $p_a = 0.5$ independantly from every other vertices. The graph key is the same as before.

For a configuration with $n = 60$ vertices and $p = 0.2$, on average on 200 runs, the algorithm removed 99.70% of the non-active vertices, and computed in 11.52 seconds.

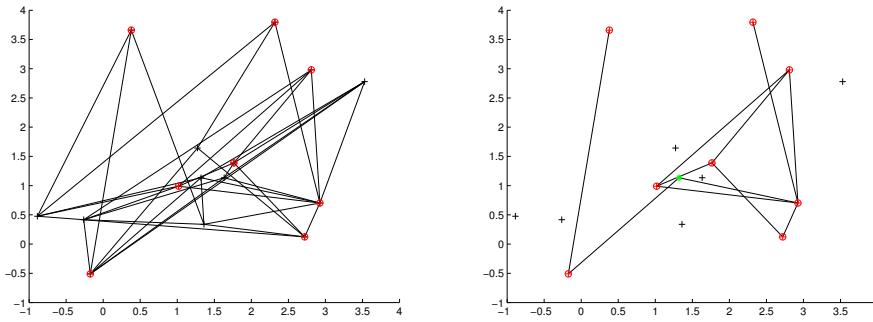


FIGURE 6. A Erdős-Rényi complex before and after the connectivity reduction algorithm.

6. APPLICATIONS

This algorithm can be used in wireless sensor networks. It can reduce the number of sensors awake in a surveillance area, or in a communication network where only few of them are active. We can see an example of execution of the algorithm in Figure 7 for a sensors' network simulated by a Poisson point process of intensity $\lambda = 4.2$ and its associated Čech complex of parameter $\epsilon = 0.5$ on a square of side length 2 with a fixed boundary on the square perimeter.

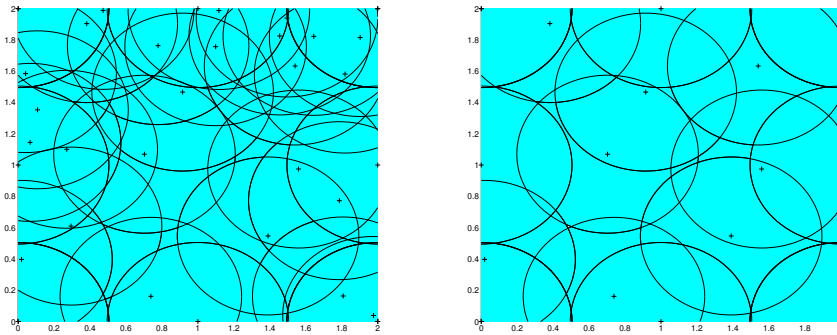


FIGURE 7. A sensor network before and after the coverage reduction algorithm.

REFERENCES

1. X. Bai, S. Kumar, D. Xuan, Z. Yun, and T. H. Lai, *Deploying wireless sensors to achieve both coverage and connectivity*, Proceedings of the 7th ACM international symposium on Mobile ad hoc networking and computing (New York, NY, USA), MobiHoc '06, ACM, 2006, pp. 131–142.
2. E. Campos-Nañez, A. Garcia, and C. Li, *A game-theoretic approach to efficient power management in sensor networks*, Oper. Res. **56** (2008), no. 3, 552–561. MR 2436853
3. V. de Silva and R. Ghrist, *Coordinate-free coverage in sensor networks with controlled boundaries via homology*, International Journal of Robotics Research **25** (2006).
4. L. Decreusefond and E. Ferraz, *On the one dimensional Poisson random geometric graph*, J. Probab. Stat. (2011), Art. ID 350382, 21. MR 2842939
5. L. Decreusefond, E. Ferraz, H. Randriambololona, and A. Vergne, *Simplicial homology of random configurations*, 2011.
6. P. Dłotko, R. Ghrist, M. Juda, and M. Mrozek, *Distributed computation of coverage in sensor networks by homological methods*, Applicable Algebra in Engineering, Communication and Computing, 1–30, 10.1007/s00200-012-0167-7.
7. Q. Fang, J. Gao, and L.J. Guibas, *Locating and bypassing routing holes in sensor networks*, INFOCOM 2004. Twenty-third Annual Joint Conference of the IEEE Computer and Communications Societies, vol. 4, march 2004, pp. 2458 – 2468 vol.4.
8. R. Ghrist and A. Muhammad, *Coverage and hole-detection in sensor networks via homology*, Proceedings of the 4th international symposium on Information processing in sensor networks (Piscataway, NJ, USA), IPSN '05, IEEE Press, 2005.
9. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 208, Marcel Dekker Inc., New York, 1998. MR 1605684 (2001a:05112)
10. T. Kaczyński, M. Mrozek, and M. Ślusarek, *Homology computation by reduction of chain complexes*, Comput. Math. Appl. **35** (1998), no. 4, 59–70. MR 1604791 (99a:18001)
11. M. Kahle, *Random geometric complexes*, Discrete & Computational Geometry **45** (2011), 553–573, 10.1007/s00454-010-9319-3.
12. A. Muhammad and A. Jadbabaie, *Decentralized computation of homology groups in networks by gossip*, American Control Conference, 2007. ACC '07, july 2007, pp. 3438 –3443.
13. F. Yan, P. Martins, and L. Decreusefond, *Accuracy of homology based approaches for coverage hole detection in wireless sensor networks*, dec. 2011.
14. ———, *Connectivity-based distributed coverage hole detection in wireless sensor networks*, Global Telecommunications Conference (GLOBECOM 2011), 2011 IEEE, dec. 2011, pp. 1 –6.
15. C. Zhang, Y. Zhang, and Y. Fang, *Detecting coverage boundary nodes in wireless sensor networks*, Networking, Sensing and Control, 2006. ICNSC '06. Proceedings of the 2006 IEEE International Conference on, 0-0 2006, pp. 868 –873.
16. A. Zomorodian and G. Carlsson, *Computing persistent homology*, Discrete & Computational Geometry **33** (2005), 249–274, 10.1007/s00454-004-1146-y.

E-mail address: laurent.decreasefond@telecom-paristech.fr

E-mail address: philippe.martins@telecom-paristech.fr

E-mail address: anais.vergne@telecom-paristech.fr

INSTITUT TELECOM, TELECOM PARISTECH, CNRS LTCI, PARIS, FRANCE