Discrete Controllability of Distributed Parameters Systems using Lattice Boltzmann Models: an application to the Shallow Water Equations

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Abstract: The Lattice Boltzmann (cellular automata) model equivalent to the 1D shallow water equations is first introduced. Then, its state-space representation with the introduction of input-output variables is presented. Finally, the discrete controllability problem is investigated. First considerations and methods suitable for large scale discrete dynamical systems Gramian computations are presented. Then controllability analysis arguments are developed based on the obtained Gramians and on the convergence of energy properties.

Keywords: Lattice Boltzmann models, controllability and observability gramians, Lyapunov equations, distributed parameters systems, Shallow water equations.

1. INTRODUCTION

Discrete models are extensively used to describe simply complex distributed parameters systems. These discrete models are often obtained from the discretization of partial differential equations. However, more recently, models have been built to represent dynamical systems directly from discrete formulations of quantitative behaviors or physical principles. Cellular automata models are examples of such an approach. Historically they have been introduced by von Neuman and Ulam Neuman (1966) as self-reproducing automata. However, more recently, they have been extensively used to model distributed parameters systems in epidemiology Ostfeld et al. (2005); Slimi et al. (2009), ecosystems and population spatial dynamics Chopard and Lagrava (2006); Hencinas et al. (2007), biomedical applications Ouared et al. (2006); Chopard et al. (2010), etc. In these application examples discrete state space are convenient. However in many distributed parameters systems based on conservation laws for macroscopic variables (in the thermodynamical meaning), continuous state spaces seem more appropriate. For this reason, Lattice Boltzmann Models (LBM) are a convenient extension of cellular automata models as they have real-valued states. LBM have been used also to represent various kinds of physical models Chopard and Droz (2005). They are now recognized as a powerful way to solve Navier-Stokes equations. In this paper, we will first introduce them in a general framework and then consider a particular application to fluid flow modeling and control. LBM have been recently developed for 1D and 2D shallow water flows Pham et al. (2010); Zhou (2004); Salmon (1999). They have proved to be very efficient alternative solutions to classical discretizations of Shallow Water equations Pham et al. (2010), especially to describe large and complex free surface water transportation systems. However, very little is known about their behavior in what concern qualitative dynamical properties. This paper is an attempt to present LBM in a state space formalism suitable for the analysis of these properties. In section 2, the LBM for shallow water equation is established. In section 3, the Gramian computation problem is developed. Finally, in section 4, the obtained results for the Gramians are analyzed

2. FROM THE LOCAL LATTICE-BOLTZMANN MODEL TO THE GLOBAL STATE SPACE REPRESENTATION

We will consider in the following the example of water flow in a regular open channel with slope I and width B, as represented in Fig.1. Under the shallow water assumptions Pham et al. (2010), the water levels and flow in such a canal reach satisfy:

$$\begin{aligned} &\partial_t h + \partial_x(hu) = 0 \\ &\partial_t(hu) + \partial_x(\frac{1}{2}gh^2 + hu^2) = F \end{aligned} \tag{1}$$

where h denotes the water depth, u the depth-averaged horizontal velocity of the flow and g the gravitational acceleration. The force term, $F = gh(I - J)$, accounts for the bed slope, I, and the bed friction, J, which is modelled with the classical Manning formula: $J = n^2u^2/\left(\frac{Bh}{B+2h}\right)^{4/3}$, with n the Manning's coefficient.

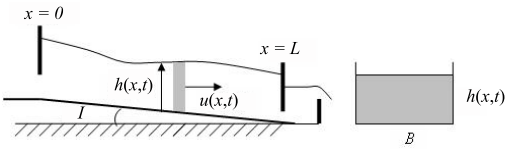

Fig. 1. Longitudinal (left) and lateral (right) views of an open rectangular channel

2.1 The Lattice Boltzmann Model

The Lattice Boltzmann (LB) method has proven to be a powerful numerical tool to simulate the fluid flows and many other physical phenomena in a fully discrete (time and space) grid Chopard and Droz (2005). The dynamics in LBM consists of an alternation of collision and streaming phases:

$$\begin{array}{ll} f_i^{out}(x,t) = f_i^{in}(x,t) + \Omega_i(f(x,t)) & \text{Collision} \\ f_i^{in}(x + v_i\delta t, t+\delta t) = f_i^{out}(x,t) & \text{Streaming} \end{array} \tag{2}$$

where f^{in} denotes the vector of all density distributions, f_i^{in}, of particles entering a site, f_i^{out} denotes those distributions which result from the collision, δx is the lattice spacing and δt is the time step. Ω_i is the collision operator, which is commonly defined by the Bhatnagar-Gross-Krook (BGK) model Bhatnagar et al. (1954): $\Omega_i = \frac{1}{\tau}(f_i^{eq} - f_i)$, where τ is a relaxation time and f_i^{eq} are the equilibrium distribution functions which depend on the physical process to be described. By combining the equations in (2), we obtain the evolution equation

$$f_i(x + v_i\delta t, t+\delta t) = f_i(x,t) + \frac{1}{\tau}(f_i^{eq} - f_i) \tag{3}$$

where f stands for f^{in} and where the spatial and temporal arguments of the term $f_i^{eq} - f_i$ is the same as the first term of the right hand side. Here we will consider a 1D lattice termed $D1Q3$ (1D spatial grid with densities or velocities in each point, see Fig. 2) with a lattice spatial spacing δx and a time step δt. This results in a lattice (or numerical) velocity $v = \frac{\delta x}{\delta t}$. Thus we assign for the three velocities in (Fig. 2) the values $v_0 = 0$, $v_1 = v$ and $v_2 = -v$. In

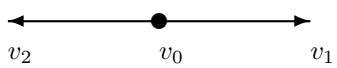

Fig. 2. Lattice D1Q3

the Shallow Water example, external forces have to be considered in the evolution equations. For a constant force, this can be done by adding the force terms F in equations (3) in the following way Chopard and Droz (2005):

$$f_i(x + v\delta t, t+\delta t) = f_i(x,t) + \frac{1}{\tau}(f_i^{eq} - f_i) + w_i\frac{\delta t}{c_s^2}v_i F \tag{4}$$

Many parameters and unknown equilibrium distributions appear in such a LBM. Usually, they all are determined by the physical assumptions required for the model. For instance, in the D1Q3 case, symmetry reasons lead to a choice of the parameters w_i and c_s^2 which must satisfy Zhou (2004)

$$\begin{array}{l} \sum_i w_i = 1, \quad \sum_i v_i w_i = 0, \quad \sum_i v_i^2 w_i = c_s^2, \\ \sum_i v_i^3 w_i = 0, \quad \sum_i v_i^4 w_i = 3c_s^4 \end{array} \tag{5}$$

This implies $w_0 = \frac{2}{3}$, $w_1 = w_2 = \frac{1}{6}$, $c_s^2 = \frac{v^2}{3}$. On another hand the equilibrium distributions may be chosen in order to give rise to the "correct" (or physical) balance equations Chopard and Droz (2005). In the 1D Shallow Water example, this lead to the following relations between microscopic (or mesoscopic) values of the equilibrium distributions and the three "macroscopic variables" which are to total mass, momentum and energy. The three balance equations read in this case Zhou (2004):

$$\begin{array}{ll} \sum_i f_i^{eq} = h, & \sum_i v_i f_i^{eq} = hu \\ \sum_i v_i^2 f_i^{eq} = & \frac{1}{2}gh^2 + hu^2 \end{array} \tag{6}$$

They lead to

$$\begin{array}{rl} f_0^{eq} = & h - \frac{1}{2v^2}gh^2 - \frac{1}{v^2}hu^2 \\ f_1^{eq} = & \frac{1}{4v^2}gh^2 + \frac{1}{2v}hu + \frac{1}{2v^2}hu^2 \\ f_2^{eq} = & \frac{1}{4v^2}gh^2 - \frac{1}{2v}hu + \frac{1}{2v^2}hu^2 \end{array} \tag{7}$$

Finally, the macroscopic properties h and u must satisfy at any time the relations

$$\begin{array}{rl} h = & \sum_i f_i = f_0 + f_1 + f_2 \\ hu = & \sum_i v_i f_i = vf_1 - vf_2 \end{array} \tag{8}$$

The variables h and u, are the variables classically used to describe Shallow Water flows.

2.2 Linearized state space evolution equation

The state space in the case of a LBM is the set of values for the distribution functions vector. It depends of dimension and kind of lattice (see Chopard and Droz (2005); Zhou (2004)). In a lattice with n sites where each site has $|v|$ neighbors, the state space is $Z_{lbm} = \left((\mathbf{R}^+)^{|v|n}\right)$.

Local dynamic: The local dynamic is defined by the distribution functions related to "particles" arriving at time $t+1$ on a given site from the neighbor sites where they were at time t. In our $D1Q3$ lattice case, after normalizing the time and space steps, these dynamics read:

$$f_i(x_k, t+1) = f_i(x_{k+\bar{v}}, t) + \frac{1}{\tau}(f_i^{eq} - f_i) \tag{9}$$

with $i = 0, 1, 2$ and $\bar{v} = \frac{v_i}{v}$.

Global dynamic: The global dynamic is obtained by closing the set of local dynamics equations (9) with the closure relations (7) for the distributions equilibrium profiles. In our $D1Q3$ lattice case, these nonlinear dynamics read:

$$f_i(x_k, t+1) = A_i \begin{bmatrix} f_0(x_{k+\bar{v}}, t) \\ f_1(x_{k+\bar{v}}, t) \\ f_2(x_{k+\bar{v}}, t) \end{bmatrix} \tag{10}$$

with $i = 0, 1, 2$. Defining the state space vector as

$$x(k) = [f_0(x_1,k)\ f_1(x_1,k)\ f_2(x_1,k) \quad \cdots \quad f_0(x_n,k)\ f_1(x_n,k)\ f_2(x_n,k)]^T, \tag{11}$$

we obtain a classical evolution model of the form

$$x(k+1) = A(Fr, \Psi)x(k) \tag{12}$$

with the matrices A_i, $i = 0, 1, 2$, and A given in the appendix. It is remarkable that these matrices only depend on the flow Froude number, $Fr^2 = \frac{u^2}{gh}$, and the lattice Froude number, $\Psi^2 = \frac{v^2}{gh}$. These two dimensionless numbers characterize the kind of flow which arises locally at the considered location (see appendix A). It is important to notice that, since Fr and Ψ are functions of h and u which are in turn functions of the distributions f_i (see equations (8)), we have $A = A(x)$ and the global evolution equation (12) is nonlinear.

Linearized global dynamics: We will consider the linearization around stationary equilibrium profiles for the water levels h_e and flows $h_e u_e$ in the classical sense. Note that in general, equilibrium profiles for the water levels (h_e) or depth-averaged horizontal velocities (u_e) in the Saint-Venant equations (1) need not to be uniform stationary profiles. These equilibrium profiles correspond to distribution functions $\bar{f}_i$ such that $\sum_i \bar{f}_i = h_e$ and $\sum_i v_i \bar{f}_i = h_e u_e$. Note also that these stationary distributions are not the equilibrium distributions, f_i^{eq} in the Maxwell Boltzmann relaxation meaning which are given in equations (7). Writing $f_i = \bar{f}_i + \epsilon_i$, one obtains for the macroscopic variables:

$$\begin{aligned} h &= \sum_i f_i = h_e + \sum_i \epsilon_i \\ hu &= \sum_i v_i f_i = h_e u_e + v(\epsilon_1 - \epsilon_2) \\ u &= u_e - \frac{u_e}{h_e}\sum_i \epsilon_i + \frac{v}{h_e}(\epsilon_1 - \epsilon_2) \\ hu^2 &= h_e u_e^2 - u_e^2 \sum_i \epsilon_i + 2vu_e(\epsilon_1 - \epsilon_2) \end{aligned} \tag{13}$$

The LBM evolution equation (3) leads to the evolution equation for the variations ϵ_i

$$\epsilon_i(x_k + v_i\delta t, t + \delta t) = (1 - \frac{1}{\tau})\epsilon_i(x_k, t) + \frac{1}{\tau}(f_i^{eq} - \bar{f}_i) \tag{14}$$

Since the distribution functions f^{eq} and $\bar{f}_i$ are related, one gets the evolution equations on the linearized variables (variations ϵ_i):

$$\epsilon_i(x_k, t+1) = T_i \begin{bmatrix} \epsilon_0(x_{k+\bar{v}}, t) \\ \epsilon_1(x_{k+\bar{v}}, t) \\ \epsilon_2(x_{k+\bar{v}}, t) \end{bmatrix}, \quad i = 0, 1, 2 \tag{15}$$

Defining similarly to previously the linearized state as

$$z(k) = [\epsilon_0(x_1,k)\ \epsilon_1(x_1,k)\ \epsilon_2(x_1,k) \quad \cdots \quad \epsilon_0(x_n,k)\ \epsilon_1(x_n,k)\ \epsilon_2(x_n,k)]^T, \tag{16}$$

we obtain around the stationary profiles (h_e, u_e) a linearized autonomous state dynamical system

$$z(k+1) = Tz(k) \tag{17}$$

with the matrices T_0, T_1, T_2 and T given in the appendix.

Adding the force terms: The force term F may be linearized around (h_e, u_e) as well by neglecting the terms of order two and higher in power of ϵ_i series expansion. For a site x_k, one obtains:

$$\epsilon_i(x_k, t+1) = (T_i + \bar{v}N) \begin{bmatrix} \epsilon_0(x_{k+\bar{v}}, t) \\ \epsilon_1(x_{k+\bar{v}}, t) \\ \epsilon_2(x_{k+\bar{v}}, t) \end{bmatrix} + \bar{v}gh_e(I - J_e)$$

where $J_e = J(h_e, u_e)$, and N is given in the appendix. Then the linearized LBM with force terms may be written:

$$z(k+1) = T_f z(k) \tag{18}$$

where the state $z(k)$ is defined as before (see (16)) and with T_f given in appendix.

2.3 Inputs and outputs

From a control point of view, once a global state space model has been obtained from the LBM, inputs and outputs variables need still to be defined. In this paper we will focus on the input variables denoted hereafter $\mathbf{u}$ (to make the difference with the macroscopic velocity u clear). We wish to define these input variables for general (N-dimensional) lattices, for various kind of controls (such as, for instance, distributed or boundary controls) and in such a way that the forced solution exists for any admissible controls. For this, we consider:

- a lattice $\mathcal{T}$, which covers the whole spatial domain of interest;
- a region $\mathcal{T}^x \subset \mathcal{T}$ where the lattice sites can be excited;
- a state space Z_{lbm} containing $x(k)$ or $z(k)$ here above
- a control space $\mathcal{U} = l_2(\mathcal{T}^x, \mathbf{R})$ with

  $$l_2(\mathcal{T}^x, \mathbf{R}) = \left\{ \mathbf{u} : \mathcal{T}^x \to \mathbf{R} / \sum_{x_j \in \mathcal{T}^x} \mathbf{u}(x_j) * \mathbf{u}(x_j) < \infty \right\}$$

  The space $l_2(\mathcal{T}^x, \mathbf{R})$ may be, in the finite dimensional case (finite number of excitable sites in the lattice), identified with the space $\mathbf{R}^p$ where $p := card(\mathcal{T}^x)$. The l_2 convergence is then trivial. We will consider this case in the following.
- an input operator $G : \mathcal{U} \to Z_{lbm}$, $\mathbf{u} \to G\mathbf{u}$. Usually, in LB models, local input maps g_{x_j} may be defined describing how the states corresponding to the excited site $x_j \in \mathcal{T}^x$ are locally controlled. The global input operator may then be defined as

  $$G\mathbf{u} = \begin{cases} g_{x_j}(\mathbf{u}(x_j)) \text{ if } x_j \in \mathcal{T}^x \\ 0 \text{ if not} \end{cases} \tag{19}$$

In the general case, these definitions result in a LBM controlled system

$$x_{k+1} = \mathcal{F}_{lbm}(x_k, \mathbf{u}_k),\ x_0 \in Z_{lbm} \tag{20}$$

In the linear case it reduces to a classical state space system

$$z_{k+1} = F_{lbm} z_k + G_{lbm}\mathbf{u}_k,\ z_0 \in Z_{lbm}^r \tag{21}$$

Z_{lbm}^r is a reduced state space where controlled states have been removed. F_{lbm} is the matrix obtained when the rows corresponding to excited states (considered as inputs) have been removed from the dynamic matrix

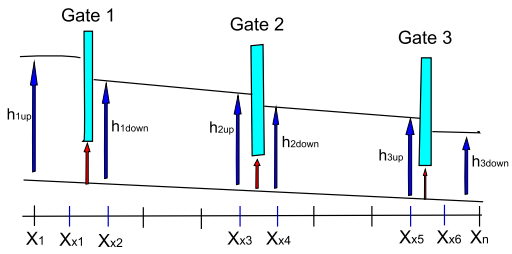

Fig. 3. Interconnected scheme

T_f: $F_{lbm} = M^T T_f M$ where M is the projection matrix corresponding to the cancellation of the states used in the control. G_{lbm} is the control matrix defined as: $G_{lbm} = M^T T_f M_c$, where M_c is the projection matrix selecting those columns corresponding to the controlled states. Naturally, G_{lbm} may be viewed as a control operator and F_{lbm} as the infinitesimal generator of a semi-group $\{F_{lbm}^k\}_{k\geq 0}$, Gibson (1972); Packel (1972). In the example of the canal described by the (Fig. 3), we assume n sites and 3 gates. We consider the whole canal as the domain $\mathcal{T}$. The two sites around each gate as those belonging to $\mathcal{T}^x$. When we introduce a gate in the canal, we can impose some distribution functions to the lattice, see Pham et al. (2010). We can for instance define a control such that:

- if $x_j \in \mathcal{T}^x$ is before the gate, $\mathbf{u}(x_j) = \varepsilon_2(x_j, k)$;
- if $x_j \in \mathcal{T}^x$ is after the gate, $\mathbf{u}(x_j) = \varepsilon_1(x_j, k)$.

And this, for each gate. If the sites before the gates are, for example, x_2, x_8 and x_{14}, we shall have as control:

$$\mathbf{u} = [\varepsilon_2(x_2)\ \varepsilon_1(x_3)\ \varepsilon_2(x_8)\ \varepsilon_1(x_9)\ \varepsilon_2(x_{14})\ \varepsilon_1(x_{15})]^T \quad (22)$$

Matrices F_{lbm} and G_{lbm} for this example are easily derived by noticing that the rows to be withdrawn from the dynamic matrix T_f for form the matrix F_{lbm} are : 6, 8, 24, 26, 42 and 44, and the matrix M is obtained by withdrawing the columns corresponding in the identity matrix.

3. CONTROLLABILITY FOR LARGE SCALE LB CONTROL SYSTEMS

3.1 Analytical or direct methods

To determine the controllability of a system, either the controllability matrix or the controllability Gramian may be used. For the LB control systems (21), the controllability matrix is given by

$$\mathcal{C} = [G_{lbm} \quad F_{lbm}G_{lbm} \quad \dots \quad F_{lbm}^{m-1}G_{lbm}] \quad (23)$$

where m is the dimension of F_{lbm}. The LB control systems is then controllable if this controllability matrix is full row rank. Controllability Gramian defined as $\mathcal{C}\mathcal{C}^T$ may be computed as

$$W_c = \sum_{\tau=0}^{m-1} F_{lbm}^{\tau} G_{lbm} G'_{lbm} (F'_{lbm})^{\tau} \quad (24)$$

Equivalently, the LB control systems will be controllable if W_c is full rank.

Due to the large scale of the considered LB control system, it is very difficult to obtain any analytical result for the structural behavior of the controllability matrix. In fact we do not even succeed in finding some qualitative structural information about this matrix. Similarly, we neither obtain analytical expressions for the controllability Gramians. Therefore we focused on the numerical solution for controllability Gramians which are (semi)positive solutions of Lyapunov equations:

$$\begin{cases} F_{lbm} W_c F'_{lbm} + G_{lbm} G'_{lbm} = W_c \\ W_c \geq 0 \end{cases} \quad (25)$$

Hence various LMI/LME tools may be tried to compute the controllability Gramians. We tried the *Yalmip* toolbox for *Matlab* (see Lofberg (2004)). However this tool is not suited for large scale (sparse) dynamical systems where it provides meaningless values for the controllability Gramians (complex or negative eigenvalues).

The *Matlab*'s toolbox *Lyapack* may be used to compute the Lyapunov equation solution for larger scale and sparse problems Penzl (2000). Originally it is designed for continuous problems. However the Cayley's transform (see Haynes (1991)) may be used to convert the discrete Lyapunov equation (25) into a continuous Lyapunov equation of the form $AW_c + W_c A^T = D$ with

$$\begin{cases} A = (F_{lbm} - I)(F_{lbm} + I)^{-1} \\ D = -0.5 * (I - F_{lbm}) G_{lbm} G_{lbm}^T (I - F_{lbm}^T) \end{cases} \quad (26)$$

However, using this approach (with $n = 200$ sites) gives large maximal eigenvalues for the residuals which are respectively 1.33 and $1.59e + 02$ in discrete Lyapunov equation (25) and in the continuous Lyapunov equation obtained from the Cayley transform (26). Therefore, we finally adopted a fixed point iterative method to solve our Gramian computation problem.

3.2 An iterative method for solving Stein's equation

Stein's equations are equations of the general form

$$AXA' - X + C = 0 \quad (27)$$

Lyapunov's equation is thus a particular case. Since they are fixed point problems, iterative schemes such as Smith's iterations (see Benner et al. (1999))

$$\begin{aligned} X_0 &= C \\ X_{k+1} &= C + AX_kA' \text{ for } k = 0, 1, 2, ... \end{aligned} \quad (28)$$

may be used to solve numerically the problem. One of the advantages of such an iterative method is that it is perfectly suited to sparse matrix problems. This reduces drastically the complexity for large scale sparse problems like the Gramian computation of LM control systems. Moreover, in our case, the error estimation in Frobenius norm $\|.\|_F$

$$\rho_k = \|W_{c,k} - W_{c,k-1}\|_F \quad (29)$$

rapidly converges to $1.0e - 17$ (the machine accuracy).

From Benner et al. (1999), if $\sigma(A) < 1$, there exist real constants $0 < L$ and $0 < r < 1$ such that

$$\|X - X_k\|_2 \leq L\|C\|_2 (1 - r)^{-1} r^{2^k} \quad (30)$$

This shows that the method converges for all equations with a Schur stable coefficient matrix A. And with this, we

tried to see how vary the controllability, or how grow the energy to give to the system for make it achieves a wanted state. The results are presented in the next section.

4. PRESENTATION OF RESULTS

4.1 Simulation setup

We consider the canal from figure 3 discretized with a n sites $D1Q3$ lattice. Numerical values for the canal parameters are those from an experimental plant in the LCIS laboratory in Valence, France. We used the gravitational acceleration $g = 9.81ms^{-2}$; the width $B = 0.1m$, the length $L = 8m$, the Manning's coefficient $n_m = 0.01s/m^{1/3}$ and the relaxation time $\tau = 1.2s$. The parameters of the model are defined such that $\delta x = L/(n-1)$ and $\delta t = \delta x/2$. This corresponds to a numerical lattice velocity of 2 which make the method suitable for the subcritical Shallow Water Flows.

4.2 Threshold of controllability

To determine controllability, it seems natural to check if the controllability matrix is *full row rank*. Under the considered configuration, it appears that the system is controllable (under Kalman's criterion) until the number of sites reaches $n = 30$. The lost of controllability corresponds to lower eigenvalues of the controllability matrix under the floating point accuracy. It is thus possible in this example to define a numerical threshold of controllability which will correspond to a maximum number of sites. In the next subsections, the effect on the controllability of growing of the number of sites beyond this maximum will be investigated.

4.3 Energy convergence

We consider the given system in a zero initial state and will compute the minimum input signal energy needed to reach another state, given by $x_r = 0.033 * [1 \quad 1 \quad \cdots \quad 1]^T$ with $dim(x_r) = m$ and $m = 3 * n - 6$, the dimension of the state space Z^r_{lbm} . We also consider for simplicity periodic boundary conditions (distributions in the first and last sites are made equals, see Pham et al. (2010)). These boundary conditions allow to compensate the dissipated energy in the canal, injecting it at the upstream end.

We expect that for a controllable LB control system, the state x_r may be reached with a control signal with finite energy, whatever the number of sites is. Indeed the model is constructed on invariant macroscopic properties (total mass, momentum, energy) which are indeed invariant quantities in the numerical simulations. However, we observe numerically (see Fig. 4) that the energy grows quickly with the number of sites. The same remark applies for the Gramian trace represented in (Fig. 5). The two previous figures are obtained with Gramians computed with the Smith's iteration algorithm with residual error in the discrete Lyapunov equation (25) near the machine's precision .

5. CONCLUDING REMARKS

In this paper we considered LB models for distributed parameters systems. It has been shown how the local evolution equation and equilibrium closure equations in LBM may be used to obtain a global state space model for the microscopic distributions.

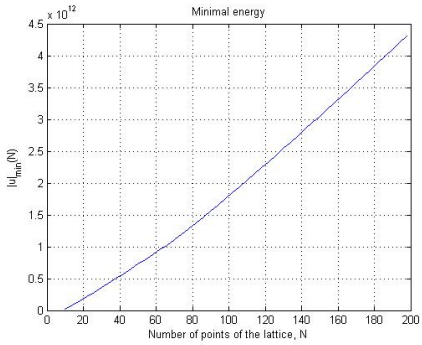

Fig. 4. Energy of system

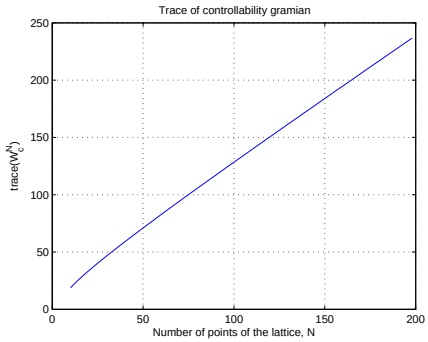

Fig. 5. Trace of Gramian

We shown how a linearization around a macroscopic stationary profile may be derived on the shallow water equations example to derive a linearized state space model for the distributions small variations.

We also shown on this shallow water example how input variables may be introduced to derive a LBM control system similar to classical finite dimensional approximations of boundary control systems.

We also shown that the derived LB control system numerically looses its controllability property when the number of sites increases beyond a limit. Neither the minimum energy to reach a given state, nor the trace of the controllability Gramian seem to be bounded when this number of sites increases.

These numerical results clearly invite us to a deeper analysis and understanding of what numerical convergence of the LB models (which has been proven to be quadratic in space and time for the state, Zhou (2004)) means in terms of controllability or observability properties. More precisely, it appears that the meaning of controllability itself has to be restricted in an appropriate way if we expect convergence results. From one side, one has to consider energy supplied through the inputs for clearly identified scenarii and from the other side it seems that local (regional) controllability property has to be considered with respect to some controllability numerical threshold.

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Appendix A. MATRICES

In the sequel we will use the notations $Fr^2 = \frac{u^2}{gh}$ and $\Psi^2 = \frac{v^2}{gh}$ to denote respectively the real Froude number and a kind of lattice Froude number. The real Froude number is an a-dimensional parameter characterizing locally the kind of flow arising at the considered location. It expresses the quotient between the kinetic and potential energies. $Fr < 1$ corresponds two a "fluvial" flow, $Fr > 1$ represents a "torrential" flow and $Fr = 1$ corresponds to a critical flow.

$$A_0 = \frac{1}{\tau}\left[\tau - \frac{1}{2\Psi^2} \quad 1 - \frac{1}{2\Psi^2} - \frac{F_r}{\Psi} \quad 1 - \frac{1}{2\Psi^2} + \frac{F_r}{\Psi}\right]$$

$$A_1 = \frac{1}{\tau}\left[\frac{1}{4\Psi^2} \quad \frac{1}{4\Psi^2} + \frac{F_r}{2\Psi} + \tau - \frac{1}{2} \quad \frac{1}{4\Psi^2} - \frac{F_r}{2\Psi} - \frac{1}{2}\right]$$

$$A_2 = \frac{1}{\tau}\left[\frac{1}{4\Psi^2} \quad \frac{1}{4\Psi^2} + \frac{F_r}{2\Psi} - \frac{1}{2} \quad \frac{1}{4\Psi^2} - \frac{F_r}{2\Psi} + \tau - \frac{1}{2}\right]$$

and by letting $a = \frac{1}{\Psi_e^2}(1 - Fr_e^2)$ and $b = I + J_e + \frac{4J_eR_e}{3h_e}$, we have

$$T_0 = \frac{1}{\tau}[\tau - a \quad 1 - a - 2\frac{Fr_e}{\Psi_e} \quad 1 - a + 2\frac{Fr_e}{\Psi_e}]$$

$$T_1 = \frac{1}{\tau}[\frac{a}{2} \quad \tau - \frac{1}{2} + \frac{a}{2} + \frac{Fr_e}{\Psi_e} \quad \frac{a}{2} - \frac{1}{2} - \frac{Fr_e}{\Psi_e}]$$

$$T_2 = \frac{1}{\tau}[\frac{a}{2} \quad \frac{a}{2} - \frac{1}{2} + \frac{Fr_e}{\Psi_e} \quad \tau - \frac{1}{2} + \frac{a}{2} - \frac{Fr_e}{\Psi_e}]$$

$$N = \frac{\delta t}{2v} g[b \quad (b - \frac{2J_e v}{u_e}) \quad (b + \frac{2J_e v}{u_e})]$$

$$A = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_1 \\ 0 & A_2 & 0 & \cdots & 0 & 0 \\ 0 & A_0 & 0 & \cdots & 0 & 0 \\ A_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_2 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & A_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_2 \\ 0 & 0 & 0 & \cdots & 0 & A_0 \\ 0 & 0 & 0 & \cdots & A_1 & 0 \\ A_2 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The matrix T is obtained by replacing in the matrix A each matrix A_i by a matrix T_i with the same direction i.

$$T_f = \begin{bmatrix} T_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & T_1 + N \\ 0 & T_2 - N & 0 & \cdots & 0 & 0 \\ 0 & T_0 & 0 & \cdots & 0 & 0 \\ T_1 + N & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & T_2 - N & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & T_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & T_2 - N \\ 0 & 0 & 0 & \cdots & 0 & T_0 \\ 0 & 0 & 0 & \cdots & T_1 + N & 0 \\ T_2 - N & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$