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Simulation of BSDEs by Wiener Chaos Expansion

Philippe Briand¹ and Céline Labart²

Laboratoire de Mathématiques, CNRS UMR 5127, Université de Savoie,
Campus Scientifique, 73376 Le Bourget du Lac, France.

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Abstract

We present an algorithm to solve BSDEs based on Wiener Chaos Expansion and Picard's iterations. We get a forward scheme where the conditional expectations are easily computed thanks to chaos decomposition formulas. We use the Malliavin derivative to compute Z . Concerning the error, we derive explicit bounds with respect to the number of chaos and the discretization time step. We also present numerical experiments. We obtain very encouraging results in terms of speed and accuracy.

1 Introduction

In this paper, we are interested in the numerical approximation of solutions (Y, Z) to backward stochastic differential equations (BSDEs for short in the sequel). BSDEs have been introduced by J.-M. Bismut in [Bis73] in the linear case, whereas the nonlinear case has been considered later by É. Pardoux and S. Peng in [PP90]. A BSDE is an equation of the following form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T, \quad (1)$$

where B is a d -dimensional standard Brownian motion, the terminal condition ξ is a real-valued \mathcal{F}_T -measurable random variable where $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ stands for the augmented filtration of the Brownian motion B and the generator f is a map from $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ into \mathbb{R} . A solution to this equation is a pair of processes $\{(Y_t, Z_t)\}_{0 \leq t \leq T}$ which is required to be adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. We will assume the conditions of Pardoux and Peng to ensure existence and uniqueness of solutions.

Our main objective in this study is the numerical approximation of the solution (Y, Z) to BSDE (1) (even though there exists a large literature on this subject). The first two contributions to this topic are due to D. Chevance [Che97], who considered generators independent of Z , and V. Bally [Bal97], who used a random time mesh. J. Ma and J. Yong [MY99] proposed numerical schemes based on the link between Markovian BSDEs and semilinear partial differential equations (PDEs). Another approach, based on Donsker's theorem and close to [Che97], was proposed by F. Coquet, V. Mackevicius and J. Mémin [CMM99] in the case of a generator f independent of Z ; the general case was treated by Ph. Briand, B. Delyon and J. Mémin in [BDM01], who later extended it to a more general framework [BDM02], including the case of a "stepwise constant Brownian motion". This extension led to the formulas

$$Y_t = \mathbb{E}(Y_{t+h} | \mathcal{F}_t) + hf(t, Y_t, Z_t), \quad Z_t = h^{-1/2} \mathbb{E}(Y_{t+h} (B_{t+h} - B_t) | \mathcal{F}_t)$$

¹ philippe.briand@univ-savoie.fr.

² celine.labart@univ-savoie.fr.

known as the dynamic programming algorithm. Even though the convergence was proved in the case of path-dependent terminal condition ξ , the rate of convergence was left as an open question in [BDM02]. This problem was solved by J. Zhang [Zha04] and B. Bouchard and N. Touzi [BT04] in the case of Markovian BSDE, namely in the case of a terminal condition $\xi = g(X_T)$ where X is the solution to a stochastic differential equation. Their result was generalized by E. Gobet and C. Labart [GL07] and also by E. Gobet and A. Makhlouf [GM10].

From a numerical point of view, the main difficulty in solving BSDEs is to efficiently compute conditional expectations. Several approaches have been proposed using various tools: the Malliavin calculus [BT04], regression methods [GLW05, GLW06] and quantization technics [BP03].

Finally, let us mention that there exists some works dealing with the discretization of solutions to BSDEs in a more general framework: forward-backward SDEs [DM06] and quadratic BSDEs [Ric11].

Let us now describe briefly the main points of our approach in the case of a real-valued Brownian motion. As already used in several quoted papers, see also [BD07, GL10, BSar], our starting point is the use of Picard's iterations: $(Y^0, Z^0) = (0, 0)$ and for $q \in \mathbb{N}$,

$$Y_t^{q+1} = \xi + \int_t^T f(s, Y_s^q, Z_s^q) ds - \int_t^T Z_s^{q+1} \cdot dB_s, \quad 0 \leq t \leq T.$$

It is well-known that the sequence (Y^q, Z^q) converges exponentially fast towards the solution (Y, Z) to BSDE (1). We write this Picard scheme in a forward way

$$\begin{aligned} Y_t^{q+1} &= \mathbb{E} \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \mid \mathcal{F}_t \right) - \int_0^t f(s, Y_s^q, Z_s^q) ds, \\ Z_t^{q+1} &= D_t Y_t^{q+1} = D_t \mathbb{E} \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \mid \mathcal{F}_t \right), \end{aligned}$$

where $D_t X$ stands for the Malliavin derivative of the random variable X .

In order to compute the previous conditional expectation, we use a Wiener chaos expansion of the random variable

$$F^q = \xi + \int_0^T f(s, Y_s^q, Z_s^q) ds.$$

More precisely, we use the following orthogonal decomposition of the random variable F^q

$$F^q = \mathbb{E}[F^q] + \sum_{k \geq 1} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right),$$

where K_l denotes the Hermite polynomial of degree l , $(g_i)_{i \geq 1}$ is an orthonormal basis of $L^2(0, T)$ and, if $n = (n_i)_{i \geq 1}$ is a sequence of integers, $|n| = \sum_{i \geq 1} n_i$. $(d_k^n)_{k \geq 1, |n|=k}$ is the sequence of coefficients ensuing from the decomposition of F^q . Of course, from a practical point of view, we only keep a finite number of terms in this expansion:

- we work with a finite number of chaos, p ;
- we choose a finite number of functions g_1, \dots, g_N .

This leads to the following approximation with $n = (n_1, \dots, n_N)$

$$F^q \simeq \mathbb{E}[F^q] + \sum_{1 \leq k \leq p} \sum_{|n|=k} d_k^n \prod_{1 \leq i \leq N} K_{n_i} \left(\int_0^T g_i(s) dB_s \right).$$

One of the key point in using such a decomposition is that, for choices of simple functions g_1, \dots, g_N , there exist explicit formulas for both

$$\mathbb{E}(F^q \mid \mathcal{F}_t) \quad \text{and} \quad Z_t^{q+1} = D_t \mathbb{E}(F^q \mid \mathcal{F}_t) ; \quad (2)$$

this plays a crucial role in our algorithm. Using these formulas and starting from M trajectories of the underlying Brownian motion we are able to construct M trajectories of the solution (Y, Z) to the BSDE.

In the following, the functions g_i are chosen as step functions:

$$g_i = \mathbf{1}_{]t_{i-1}, t_i]}(t)/\sqrt{h}, \quad i = 1, \dots, N, \quad \text{where } h = \frac{T}{N}$$

and the previous formulas are really simple (see (10)-(11) and Proposition 5). Eventually, the main advantage of this method is that only one decomposition has to be computed per Picard iteration: the decomposition of F^q . Therein lies the main difference between our approach and the approach based on regression technics developed by C. Bender and R. Denk in [BD07]. In their paper, for a given Picard iteration q and for each time t_i of the mesh grid, two projections have to be computed, one for $Y_{t_i}^q$ and one for $Z_{t_i}^q$. The second difference comes from the way of computing Z^q . In our method, once the decomposition of F^q is computed, Z^q is given explicitly as the Malliavin derivative of Y^q . Let us also point out that our algorithm can handle fully path dependent terminal conditions.

The rest of the paper is organized as follows. Section 2 contains the notations and the preliminary results, Section 3 describes precisely the algorithm, Section 4 is devoted to the study of the convergence of the algorithm and finally Section 5 contains some numerical experiments. Some technical proofs are post-done to the appendix.

2 Preliminaries

2.1 Definitions and Notations

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an \mathbb{R}^d -valued Brownian motion B , we consider

- $\{(\mathcal{F}_t); t \in [0, T]\}$, the filtration generated by the Brownian motion B and augmented
- $\mathbb{E}_t(X)$ denotes $\mathbb{E}(X|\mathcal{F}_t)$ for any X in $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$.
- $L^2(\mathcal{F}_T) := L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ the space of all \mathcal{F}_T -measurable random variables (r.v. in the following) $X : \Omega \mapsto \mathbb{R}^d$ satisfying $\|X\|^2 = \mathbb{E}(|X|^2) < \infty$.
- $S_T^2(\mathbb{R}^d)$ the space of all càdlàg predictable processes $\phi : \Omega \times [0, T] \mapsto \mathbb{R}^d$ such that $\|\phi\|_{S_T^2}^2 = \mathbb{E}(\sup_{t \in [0, T]} |\phi_t|^2) < \infty$.
- $H_T^2(\mathbb{R}^d)$ the space of all predictable processes $\phi : \Omega \times [0, T] \mapsto \mathbb{R}^d$ such that $\|\phi\|_{H_T^2}^2 = \mathbb{E} \int_0^T |\phi_t|^2 dt < \infty$.
- $C_b^{k,l}$ the set of continuously differentiable functions $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$ with continuous and uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l). The function ϕ is also bounded.
- $\|\partial_{sp}^j f\|_\infty^2$ the norm of the derivatives of $f : (t, x) \in [0, T] \times \mathbb{R}^d$ w.r.t. all the space variables which sum equals j : $\|\partial_{sp}^j f\|_\infty^2 := \sum_{|k|=j} \|\partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} f\|_\infty^2$, where $|k| = k_1 + \cdots + k_d$.
- C_p^∞ the set of smooth functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ with partial derivatives of polynomial growth.
- $\|(\cdot, \cdot)\|_{L^2}^2$ the norm on the space $S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$ defined by

$$\|(Y, Z)\|_{L^2}^2 = \mathbb{E} \left(\sup_{t \in [0, T]} (|Y_t|^2) + \int_0^T |Z_t|^2 dt \right). \quad (3)$$

We also recall some useful definitions related to Malliavin calculus. We use the notations of [Nua98].

- \mathcal{S} denotes the class of random variables of the form $F = f(W(h_1), \dots, W(h_n))$, where $f \in C_p^\infty(\mathbb{R}^n, \mathbb{R})$, $(h_1, \dots, h_n) \in L^2([0, T]; \mathbb{R}^n)$ and $W(h_i) = \int_0^T h_i(t) dW_t$.
- $\|F\|_{r,2}^2$ denotes the following norm on \mathcal{S}

$$\|F\|_{r,2}^2 := \mathbb{E}|F|^2 + \sum_{q=1}^r \sum_{|\alpha|=q} \mathbb{E} \left(\int_0^T \cdots \int_0^T \left| D_{(t_1, \dots, t_q)}^\alpha F \right|^2 dt_1 \cdots dt_q \right)$$

where D^α represents the multi-index Malliavin derivative operator.

- $\mathbb{D}^{r,2}$ denotes the closure of \mathcal{S} w.r.t. $\|\cdot\|_{r,2}$ and $\mathbb{D}^{\infty,2} = \bigcap_{r=1}^\infty \mathbb{D}^{r,2}$.

2.2 Wiener Chaos Expansion

2.2.1 Notations and useful results

We refer to [Nua98] for more details on this section. Let us briefly recall the Wiener chaos expansion in the simple case of a real-valued Brownian motion. It is well known that every random variable $F \in L^2(\mathcal{F}_T)$ as an expansion of the following form:

$$\begin{aligned} F = & \mathbb{E}[F] + \int_0^T u_1(s_1) dB_{s_1} \\ & + \int_0^T \int_0^{s_2} u_2(s_2, s_1) dB_{s_1} dB_{s_2} + \dots + \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} u_n(s_n, \dots, s_1) dB_{s_1} \dots dB_{s_n} + \dots \end{aligned} \quad (4)$$

where the functions $(u_n, n \geq 1)$ are deterministic functions. There is an ambiguity for the definition of these functions u_n . We adopt in this paper the following point of view: the function u_n is defined on the simplex

$$\mathcal{S}_n(T) := \{(s_1, \dots, s_n) \in [0, T]^n : 0 < s_1 < \dots < s_n < T\}$$

We define the iterated integral for a deterministic function $f \in L^2(\mathcal{S}_n(T))$ as

$$J_n(f) := \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} f(s_n, \dots, s_1) dB_{s_1} \cdots dB_{s_n}.$$

Due to the Itô isometry, $\|J_n(f)\|^2 = \|f\|_{L^2(\mathcal{S}_n(T))}^2$ and $\mathbb{E}[J_n(f)J_m(g)] = \delta_{nm} \langle f, g \rangle_{L^2(\mathcal{S}_n(T))}$. Then, $\|F\|^2 = \sum_{n \geq 0} \|u_n\|_{L^2(\mathcal{S}_n(T))}^2$.

Definition. Let F be a random variable in $L^2(\mathcal{F}_T)$ whose chaos expansion is given by (4). We introduce

- $P_n(F) := J_n(u_n)$ the Wiener chaos of order n of F .
- $\mathcal{C}_p(F) := \sum_{n \leq p} P_n(F)$ the chaos decomposition of F up to order p , i.e.

$$\begin{aligned} \mathcal{C}_p(F) = & \mathbb{E}[F] + \int_0^T u_1(s_1) dB_{s_1} + \int_0^T \int_0^{s_2} u_2(s_2, s_1) dB_{s_1} dB_{s_2} \\ & + \dots + \int_0^T \int_0^{s_p} \cdots \int_0^{s_2} u_p(s_p, \dots, s_1) dB_{s_1} \dots dB_{s_p}. \end{aligned} \quad (5)$$

We state two Lemmas useful for the sequel.

Lemma 1 (Nualart). $F \in \mathbb{D}^{m,2}$ if and only if $\sum_{n \geq 0} (n+m-1) \times \cdots \times n \times \mathbb{E}[|P_n(F)|^2] < \infty$. In this case, we have

$$\sum_{n \geq 0} (n+m-1) \times \cdots \times n \times \mathbb{E}[|P_n(F)|^2] \leq \|F\|_{\mathbb{D}^{m,2}}^2.$$

From Lemma 1, we deduce

Lemma 2. Let $F \in \mathbb{D}^{m,2}$. We have

$$\mathbb{E}[|F - \mathcal{C}_p(F)|^2] \leq \frac{\|F\|_{\mathbb{D}^{m,2}}^2}{(p+m) \cdots (p+1)}.$$

Proof.

$$\begin{aligned} \mathbb{E}[|F - \mathcal{C}_p(F)|^2] &= \sum_{k \geq p+1} \mathbb{E}[|P_k(F)|^2] = \sum_{k \geq p+1} (k+m-1) \cdots k \times \frac{1}{(k+m-1) \cdots k} \times \mathbb{E}[|P_k(F)|^2] \\ &\leq \frac{1}{(p+m) \cdots (p+1)} \sum_{k \geq p+1} (k+m-1) \cdots k \mathbb{E}[|P_k(F)|^2]. \end{aligned}$$

□

The following Lemma gives some useful properties of the chaos decomposition.

Lemma 3. Let F be a r.v. in $L^2(\mathcal{F}_T)$ and H be in $H_T^2(\mathbb{R})$. Then

- $\forall p \geq 1, \mathbb{E}[|\mathcal{C}_p(F)|^2] \leq \mathbb{E}[|F|^2],$
- $\mathcal{C}_p\left(\int_0^T H_s ds\right) = \int_0^T \mathcal{C}_p(H_s) ds.$
- For all $t \leq r$ $D_t \mathbb{E}_r[\mathcal{C}_p(F)] = \mathbb{E}_r[\mathcal{C}_{p-1}(D_t F)].$

2.2.2 Wiener chaos expansion and Hermite polynomials

Another approach to Wiener chaos expansion uses Hermite polynomials. This approach can be easily generalized when considering d -dimensional Brownian motions, this is then the one we consider in the following. We present it for $d = 1$. Let $\{g_i\}_{i \geq 1}$ be an orthonormal basis of $L^2(0, T)$. The Wiener chaos of order n , $\mathcal{P}_n(F)$, is the L^2 -closure of the vector field spanned by

$$\left\{ \prod_{i \geq 1} \sqrt{n_i!} K_{n_i} \left(\int_0^T g_i(s) dB_s \right) : |(n_i)_{i \geq 1}| := \sum n_i = n \right\}$$

where K_n is the Hermite polynomial of order n defined by the expansion:

$$e^{xt-t^2/2} = \sum_{n \geq 0} K_n(x) t^n.$$

with the convention $K_{-1} \equiv 0$. With this normalization, we have $K_n'(x) = K_{n-1}(x)$ for any integer n . It is well-known that $(K_n)_{n \geq 0}$ is a sequence of orthogonal polynomials in $L^2(\mathbb{R}, \mu)$, where μ denotes the reduced centered Gaussian measure. Moreover, we have

$$\int_{\mathbb{R}} K_n^2(x) \mu(dx) = \frac{1}{n!}.$$

Every square integrable random variable F , measurable with respect to \mathcal{F}_T , admits the following orthogonal decomposition

$$F = d_0 + \sum_{k \geq 1} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right), \quad (6)$$

where $n = (n_i)_{i \geq 1}$ is a sequence of positive integers and where $|n|$ stands for $\sum_{i \geq 1} n_i$. Taking into account the normalization of the Hermite polynomials we use, we get

$$d_0 = \mathbb{E}[F], \quad d_k^n = n! \mathbb{E} \left[F \times \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right) \right],$$

where $n! = \prod_{i \geq 1} n_i!$. Before describing the chaos decomposition formulas we use in the algorithm, we give a Lemma useful in the sequel.

Lemma 4. *Let $g \in L^2(0, T)$ and let $U_t = \int_0^t g^2(s) ds$. For $n \in \mathbb{N}$, let us define*

$$M_t^n = U_t^{n/2} K_n \left(B(g)_t / \sqrt{U_t} \right), \quad B(g)_t = \int_0^t g(s) dB_s.$$

Then $\{M_t^n\}_{0 \leq t \leq T}$ is a martingale and

$$dM_t^n = g(t) M_t^{n-1} dB_t.$$

2.3 Chaos decomposition formulas

These formulas are based on the decomposition (6). To get tractable formulas, we consider a finite number of chaos and a finite number of functions (g_1, \dots, g_N) . The $(g_i)_{1 \leq i \leq N}$ functions are chosen such that we can quickly compute $\mathbb{E}(F|\mathcal{F}_T)$ and $D_t \mathbb{E}(F|\mathcal{F}_T)$ (as required in (2)). We develop in this Section the case $d = 1$, we refer to Section B.2 when $d > 1$.

The first step consists in considering a finite number of chaos. In order to approximate the random variable F , we consider its projection $\mathcal{C}_p(F)$ onto the first p chaos, namely

$$\mathcal{C}_p(F) = d_0 + \sum_{1 \leq k \leq p} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right). \quad (7)$$

Of course, we still have an infinite number of terms in the previous sum and the second step consists in working with only the first N functions g_1, \dots, g_N of an orthonormal basis of $L^2(0, T)$.

Let us consider a regular mesh grid of N time steps $\mathcal{T} = \{t_i = i \frac{T}{N}, i = 0, \dots, N\}$ and the N step functions

$$g_i = \mathbf{1}_{]t_{i-1}, t_i]}(t) / \sqrt{h}, \quad i = 1, \dots, N, \quad \text{where } h := \frac{T}{N}. \quad (8)$$

We complete these N functions g_1, \dots, g_N into an orthonormal basis of $L^2(0, T)$, $(g_i)_{i \geq 1}$. For instance, one can consider the Haar basis on each interval (t_{i-1}, t_i) , $i = 1, \dots, N$. We implicitly assume that $N \geq p$. This leads to the following approximation

$$\mathcal{C}_p^N(F) = d_0 + \sum_{1 \leq k \leq p} \sum_{|n|=k} d_k^n \prod_{1 \leq i \leq N} K_{n_i} \left(\int_0^T g_i(s) dB_s \right), \quad (9)$$

where $n = (n_1, \dots, n_N)$ and $|n| = n_1 + \dots + n_N$. Due to the simplicity of the functions g_i , $i = 1, \dots, N$, we can compute explicitly

$$\int_0^T g_i(s) dB_s = G_i, \quad \text{where } G_i = \frac{B_{t_i} - B_{t_{i-1}}}{\sqrt{h}}.$$

Roughly speaking this means that P_k , the k^{th} chaos, is generated by

$$\{K_{n_1}(G_1) \dots K_{n_N}(G_N) : n_1 + \dots + n_N = k\}.$$

Thus, the approximation we will use for the random variable F is

$$\mathcal{C}_p^N(F) = d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n K_{n_1}(G_1) \dots K_{n_N}(G_N) = d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i), \quad (10)$$

where the coefficients d_0 and d_k^n are given by

$$d_0 = \mathbb{E}[F], \quad d_k^n = n! \mathbb{E}[FK_{n_1}(G_1) \dots K_{n_N}(G_N)]. \quad (11)$$

From (10), we deduce the expressions of $\mathbb{E}_t(\mathcal{C}_p^N F)$ and $D_t \mathbb{E}_t(\mathcal{C}_p^N(F))$, useful for the approximation of (Y, Z) by the chaos decomposition (see (2)).

Proposition 5. *Let F be a real random variable in $L^2(\mathcal{F}_T)$ and let r be an integer in $\{1, \dots, N\}$. For all $t_{r-1} < t \leq t_r$, we have*

$$\begin{aligned} \mathbb{E}_t(\mathcal{C}_p^N F) &= d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t-t_{r-1}}{h}\right)^{\frac{nr}{2}} K_{n_r} \left(\frac{B_t - B_{t_{r-1}}}{\sqrt{t-t_{r-1}}}\right), \\ D_t \mathbb{E}_t(\mathcal{C}_p^N(F)) &= h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n_r > 0}} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t-t_{r-1}}{h}\right)^{\frac{nr-1}{2}} K_{n_r-1} \left(\frac{B_t - B_{t_{r-1}}}{\sqrt{t-t_{r-1}}}\right), \end{aligned}$$

where, if $r \leq N$ and $n = (n_1, \dots, n_N)$, $n(r)$ stands for (n_1, \dots, n_r) .

The proof of Proposition 5 is postponed to Section B.1.

Remark 6. For $t = t_r$, Proposition 5 leads to

$$\begin{aligned} \mathbb{E}_{t_r}(\mathcal{C}_p^N F) &= d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i \leq r} K_{n_i}(G_i) \\ D_{t_r} \mathbb{E}_{t_r}(\mathcal{C}_p^N F) &= h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n_r > 0}} d_k^n \prod_{i < r} K_{n_i}(G_i) \times K_{n_r-1}(G_r). \end{aligned}$$

Let us end this subsection by some examples.

Example 7 (Case $p = 2$). From (10)-(11), we have

$$\mathcal{C}_2^N(F) = d_0 + \sum_{j=1}^N d_1^{e_j} K_1(G_j) + \sum_{j=1}^N \sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i) K_1(G_j) + \sum_{j=1}^N d_2^{2e_j} K_2(G_j),$$

where e_j denotes the unit vector whose j^{th} component is one, and $e_{ij} = e_i + e_j$. For $j = 1, \dots, N$ and $i = 1, \dots, j-1$, it holds

$$d_1^{e_j} = \mathbb{E}(FK_1(G_j)), \quad d_2^{e_{ij}} = \mathbb{E}(FK_1(G_i)K_1(G_j)), \quad d_2^{2e_j} = 2\mathbb{E}(FK_2(G_j)).$$

Remark 6 leads to

$$\begin{aligned} \mathbb{E}_{t_r}(\mathcal{C}_2^N F) &= d_0 + \sum_{j=1}^r d_1^{e_j} K_1(G_j) + \sum_{j=1}^r \sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i) K_1(G_j) + \sum_{j=1}^r d_2^{2e_j} K_2(G_j), \\ D_{t_r} \mathbb{E}_{t_r}(\mathcal{C}_2^N F) &= h^{-1/2} \left(d_1^{e_r} + d_2^{2e_r} K_1(G_r) + \sum_{i=1}^{r-1} d_2^{e_{ir}} K_1(G_i) \right). \end{aligned}$$

3 Description of the algorithm

The algorithm is based on three types of approximations : Picard's iterations, a Wiener chaos expansion up to a finite order and the truncation of a $L^2([0, T])$ basis in order to apply formulas of Proposition 5. We present the different steps of the approximation procedure in Section 3.1. The practical implementation is presented in Section 3.2.

3.1 Approximation procedure

3.1.1 Picard's iterations

The first step consists in approximating (Y, Z) — solution to (1) — by Picard's sequence $(Y^q, Z^q)_q$, built as follows : $(Y^0 = 0, Z^0 = 0)$ and for all $q \geq 1$

$$Y_t^{q+1} = \xi + \int_t^T f(s, Y_s^q, Z_s^q) ds - \int_t^T Z_s^{q+1} \cdot dB_s, \quad 0 \leq t \leq T. \quad (12)$$

From (12), under the assumptions that $\xi \in \mathbb{D}^{1,2}$ and $f \in C_b^{0,1,1}$, we express (Y^{q+1}, Z^{q+1}) as a function of the processes (Y^q, Z^q) :

$$Y_t^{q+1} = \mathbb{E}_t \left(\xi + \int_t^T f(s, Y_s^q, Z_s^q) ds \right), \quad Z_t^{q+1} = D_t Y_t^{q+1}, \quad (13)$$

which can also be written

$$Y_t^{q+1} = \mathbb{E}_t \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \right) - \int_0^t f(s, Y_s^q, Z_s^q) ds, \quad Z_t^{q+1} = D_t Y_t^{q+1}. \quad (14)$$

As recalled in the introduction, the computation of the conditional expectation is the cornerstone in the numerical resolution of BSDEs. Chaos decomposition formulas enable to circumvent this problem.

3.1.2 Wiener Chaos Expansion

Computing the chaos decomposition of the r.v. $F = \xi + \int_t^T f(s, Y_s^q, Z_s^q) ds$ (appearing in (13)) in order to compute Y_t^{q+1} is not judicious. F depends on t , and then the computation of Y^{q+1} on the grid $\mathcal{T} = \{t_i = i \frac{T}{N}, i = 0, \dots, N\}$ would require $N + 1$ calls to the chaos decomposition function. To build a efficient algorithm, we need to call the chaos decomposition function as less as possible, since each call is computationally demanding and brings an approximation error due to the truncation and to the Monte-Carlo approximation (see next Sections). Then, we look for a r.v. F^q independent of t such that Y_t^{q+1} and Z_t^{q+1} can be expressed as functions of $\mathbb{E}_t(F^q)$, $D_t \mathbb{E}_t(F^q)$ and of Y^q and Z^q . Equation (14) gives a more tractable expression of Y^{q+1} . Let F^q be defined by $F^q := \xi + \int_0^T f(s, Y_s^q, Z_s^q) ds$. Then

$$Y_t^{q+1} = \mathbb{E}_t(F^q) - \int_0^t f(s, Y_s^q, Z_s^q) ds, \quad Z_t^{q+1} = D_t \mathbb{E}_t(F^q). \quad (15)$$

The second type of approximation consists in computing the chaos decomposition of F^q up to order p . Since F^q does not depend on t , the chaos decomposition function \mathcal{C}_p is called only once per Picard's iteration.

Let $(Y^{q,p}, Z^{q,p})$ denote the approximation of (Y^q, Z^q) built at step q using a chaos decomposition with order p : $(Y^{0,p}, Z^{0,p}) = (0, 0)$ and

$$Y_t^{q+1,p} = \mathbb{E}_t[\mathcal{C}_p(F^{q,p})] - \int_0^t f(s, Y_s^{q,p}, Z_s^{q,p}) ds, \quad Z_t^{q+1,p} = D_t \mathbb{E}_t[\mathcal{C}_p(F^{q,p})], \quad (16)$$

where $F^{q,p} = \xi + \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}) ds$. In the sequel, we also use the following equality

$$Z_t^{q+1,p} = \mathbb{E}_t[D_t \mathcal{C}_p(F^{q,p})]. \quad (17)$$

3.1.3 Truncation of the basis

The third type of approximation comes from the truncation of the orthonormal $L^2([0, T])$ basis used in the definition of \mathcal{C}_p (7). Instead of considering a basis of $L^2([0, T])$, we only keep the first N functions (g_1, \dots, g_N) defined by (8) to build the chaos decomposition function \mathcal{C}_p^N (9). Proposition 5 gives us explicit formulas for $\mathbb{E}_t(\mathcal{C}_p^N F)$ and $D_t \mathbb{E}_t(\mathcal{C}_p^N F)$. From (16), we build $((Y^{q,p,N}, Z^{q,p,N})_q$ in the following way : $((Y^{0,p,N}, Z^{0,p,N}) = (0, 0)$ and

$$Y_t^{q+1,p,N} = \mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N})) - \int_0^t f(s, Y_s^{q,p,N}, Z_s^{q,p,N}) ds, \quad Z_t^{q+1,p,N} = D_t(\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N}))), \quad (18)$$

where $F^{q,p,N} := \xi + \int_0^T f(s, Y_s^{q,p,N}, Z_s^{q,p,N}) ds$.

Equation (18) is tractable as soon as we know closed formulas for the coefficients d_k^n of the chaos decomposition of $\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N}))$ and $D_t(\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N})))$ (see Proposition 5). When it is not the case, we need to use a Monte-Carlo method to approximate these coefficients. The next Section is devoted to the practical implementation. In particular, we give the pseudo-code of the algorithm.

3.2 Implementation

In this Section, we first explain how to practically compute the chaos decomposition $\mathcal{C}_p^N(F)$ of a r.v. F . Then, we give the pseudo-code of the algorithm.

3.2.1 Monte-Carlo simulations of the chaos decomposition

Let F denote a r.v. of $L^2(\mathcal{F}_T)$. Practically, when we are not able to compute exactly d_0 and/or the coefficients d_k^n of the chaos decomposition (10)-(11) of F , we use Monte-Carlo simulations to approximate them. Let $(F^m)_{1 \leq m \leq M}$ be a M i.i.d. sample of F and $(G_1^m, \dots, G_N^m)_{1 \leq m \leq M}$ be a M i.i.d. sample of (G_1, \dots, G_N) . We recall d_0 and the coefficients $(d_k^n)_{1 \leq k \leq p, |n|=k}$ are given by $d_0 = \mathbb{E}[F]$ and $d_k^n = n! \mathbb{E}[F K_{n_1}(G_1) \dots K_{n_N}(G_N)]$ (see (11)). Then, they are solutions of

$$\arg \min_{\mathbf{c}=(c_0, (c_k^n)_{1 \leq k \leq p, |n|=k})} \mathbb{E}[|F - \psi(\mathbf{c}, G)|^2], \quad (19)$$

where $\psi : (\mathbf{c}, G) \mapsto c_0 + \sum_{k=1}^p \sum_{|n|=k} c_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i)$. We propose two methods to approximate $\mathbf{d} := (d_0, (d_k^n)_{1 \leq k \leq p, |n|=k})$

- the first one consists in approximating the expectations of (11) by empirical means $\widehat{\mathbf{d}}_M := (\widehat{d}_0, \widehat{d}_k^n)_{1 \leq k \leq p, |n|=k}$ where

$$\widehat{d}_0 := \frac{1}{M} \sum_{m=1}^M F^m, \quad \widehat{d}_k^n := \frac{n!}{M} \sum_{m=1}^M F^m K_{n_1}(G_1^m) \dots K_{n_N}(G_N^m),$$

- the second one is based on a sample average approximation

$$\overline{\mathbf{d}}_M := (\overline{d}_0, \overline{d}_k^n)_{1 \leq k \leq p, |n|=k} = \arg \min_{c_0, (c_k^n)_{1 \leq k \leq p, |n|=k}} \frac{1}{M} \sum_{m=1}^M |F^m - \psi(\mathbf{c}, G^m)|^2$$

Remark 8. In terms of computation time, the first method is much faster than the second one.

- The first method requires $O(M \times p)$ computations per coefficient. Since we are looking for $O(N^p)$ coefficients, its computational cost is $O(M \times p \times N^p)$.

- The second method requires $O(M \times p \times N^p)$ computations to evaluate $\frac{1}{M} \sum_{m=1}^M |F^m - \psi(c, G^m)|^2$ (in fact, it requires the same number of computations as the first method, since the function ψ contains as much as additions as coefficients, and each addition contains as much as products as the associated coefficient). We still have to compute the argmin, which computational cost depends on the method we use.

From a theoretical point of view, the second method gives better convergence results than the first one. For the first method, we only know that $\widehat{\mathbf{d}}_M$ converges to \mathbf{d} a.s.. Concerning the second method, we know that $\overline{\mathbf{d}}_M$ converges to \mathbf{d} a.s. and under regularity assumptions on ψ , the uniform strong law of large numbers gives the a.s. convergence of $\frac{1}{M} \sum_{m=1}^M |F^m - \psi(\overline{\mathbf{d}}_M, G^m)|^2$ to $\mathbb{E}[|F - \psi(\mathbf{d}, G)|^2]$.

In the following, $\mathcal{C}_p^{N,M}(F)$ denotes the approximation of the chaos decomposition of order p of F when using the first method to approximate the coefficients d_k^n :

$$\mathcal{C}_p^{N,M}(F) = \widehat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} \widehat{d}_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i).$$

$\mathbb{E}_t(\mathcal{C}_p^{N,M}(F))$ and $D_t(\mathbb{E}_t(\mathcal{C}_p^{N,M}(F)))$ denote the conditional expectations obtained in Proposition 5 when $(d_0, d_k^n)_{1 \leq k \leq p, |n|=k}$ are replaced by $(\widehat{d}_0, \widehat{d}_k^n)_{1 \leq k \leq p, |n|=k}$:

$$\begin{aligned} \mathbb{E}_t(\mathcal{C}_p^{N,M}(F)) &:= \widehat{d}_0 + \sum_{k=1}^p \sum_{|n(r)=k} \widehat{d}_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - t_{r-1}}{h} \right)^{\frac{n_r}{2}} K_{n_r} \left(\frac{B_t - B_{t_{r-1}}}{\sqrt{t - t_{r-1}}} \right), \\ D_t \mathbb{E}_t(\mathcal{C}_p^{N,M}(F)) &:= h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)=k \\ n_r > 0}} \widehat{d}_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - t_{r-1}}{h} \right)^{\frac{n_r-1}{2}} K_{n_r-1} \left(\frac{B_t - B_{t_{r-1}}}{\sqrt{t - t_{r-1}}} \right), \end{aligned}$$

Remark 9. When M samples of $\mathcal{C}_p^{N,M}(F)$ are needed, we can either use the same samples as the ones used to compute \widehat{d}_0 and \widehat{d}_k^n : $(\widehat{\mathcal{C}}_p^N(F))^m = \widehat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} \widehat{d}_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i^m)$, or use new ones. In the first case, we only require M samples of F and (G_1, \dots, G_N) . The coefficients \widehat{d}_k^n and \widehat{d}_0 are not independent of $\prod_{1 \leq i \leq N} K_{n_i}(G_i^m)$. The notation $\mathbb{E}_t(\mathcal{C}_p^{N,M}(F))$ introduced above cannot be linked to $\mathbb{E}(\mathcal{C}_p^{N,M}F|\mathcal{F}_t)$. In the second case, the coefficients \widehat{d}_k^n and \widehat{d}_0 are independent of $\prod_{1 \leq i \leq N} K_{n_i}(G_i^m)$ and we have $\mathbb{E}_t(\mathcal{C}_p^{N,M}F) = \mathbb{E}(\mathcal{C}_p^{N,M}F|\mathcal{F}_t)$. This second approach requires $2M$ samples of F and (G_1, \dots, G_N) and its variance increases with N . Practically, we use the first technique.

3.2.2 Pseudo-code of the Algorithm

In this Section, we describe in details the algorithm. We aim at computing M trajectories of an approximation of (Y, Z) on the grid $\mathcal{T} = \{t_i = i \frac{T}{N}, i = 0, \dots, N\}$. Starting from $(Y^{0,p,N}, Z^{0,p,N}) = (0, 0)$, (18) enables to get $(Y^{q,p,N}, Z^{q,p,N})$ for each Picard's iteration q on \mathcal{T} . However, if we only know the values of $(Y^{q,p,N}, Z^{q,p,N})$ on a grid and if we use a Monte Carlo procedure to compute the coefficients d_k^n , we are not able to compute $(Y^{q+1,p,N}, Z^{q+1,p,N})$ on \mathcal{T} exactly. Then, to take into account the different approximations presented before, we introduce $F^{q,p,N,M} := \xi + h \sum_{i=0}^{N-1} f(t_i, Y_{t_i}^{q,p,N,M}, Z_{t_i}^{q,p,N,M})$ and

$$\begin{aligned} Y_{t_i}^{q+1,p,N,M} &= \mathbb{E}_{t_i}(\mathcal{C}_p^{N,M}(F^{q,p,N,M})) - h \sum_{j=0}^{i-1} f(t_j, Y_{t_j}^{q,p,N,M}, Z_{t_j}^{q,p,N,M}), \\ Z_{t_i}^{q+1,p,N,M} &= D_{t_i}(\mathbb{E}_{t_i}(\mathcal{C}_p^{N,M}(F^{q,p,N,M}))). \end{aligned} \tag{20}$$

Here are the notations we use in the algorithm.

- d : dimension of the Brownian motion
- q : index of Picard's iteration
- K_{it} : number of Picard's iterations
- M : number of Monte–Carlo samples
- N : number of time steps used for the discretization of Y and Z
- p : order of the chaos decomposition
- $\mathbf{Y}^q \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $Y^{q,p,N,M}$ computed on the grid \mathcal{T} .
- For all $l \in \{1, \dots, d\}$, $(\mathbf{Z}^q)_l \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $(Z^{q,p,N,M})_l$ computed on the grid \mathcal{T} .

Since $\xi \in L^2(\mathcal{F}_T)$, ξ can be written as a measurable function of the Brownian path. Then, one gets one sample of ξ from one sample of (G_1, \dots, G_N) (where G_i represents $\frac{B_{t_i} - B_{t_{i-1}}}{\sqrt{h}}$).

For the sake of clearness, we detail the algorithm for $d = 1$.

Algorithm 1 Iterative algorithm

- 1: Pick at random $N \times M$ values of standard Gaussian r.v. stored in \mathbf{G} .
 - 2: Using \mathbf{G} , compute $(\xi^m)_{1 \leq m \leq M}$.
 - 3: $\mathbf{Y}^0 \equiv 0$, $\mathbf{Z}^0 \equiv 0$.
 - 4: **for** $q = 0 : K_{it} - 1$ **do**
 - 5: **for** $m = 1 : M$ **do**
 - 6: Compute $(F^q)^m = \xi^m + h \sum_{i=1}^N f(t_i, (\mathbf{Y}^q)_{i,m}, (\mathbf{Z}^q)_{i,m})$
 - 7: **end for**
 - 8: Compute the vector $\mathbf{d} = (\widehat{d}_0, (\widehat{d}_k^n)_{1 \leq k \leq p, |n|=k})$ of the chaos decomposition of F^q
 - 9: $\widehat{d}_0 := \frac{1}{M} \sum_{m=1}^M (F^q)^m$, $\widehat{d}_k^n = \frac{n!}{M} \sum_{m=1}^M (F^q)^m K_{n_1}(G_1^m) \dots K_{n_N}(G_N^m)$
 - 10: **for** $j = 0 : N - 1$ **do**
 - 11: **for** $m = 1 : M$ **do**
 - 12: Compute $(\mathbb{E}_{t_j}(\mathcal{C}_p^{N,M} F^q))^m$, $(D_{t_j}(\mathbb{E}_{t_j}(\mathcal{C}_p^{N,M} F^q)))^m$
 - 13: $(\mathbf{Y}^{q+1})_{j,m} = (\mathbb{E}_{t_j}(\mathcal{C}_p^{N,M} F^q))^m + \sum_{i=1}^j f(t_i, (\mathbf{Y}^q)_{i,m}, (\mathbf{Z}^q)_{i,m})$
 - 14: $(\mathbf{Z}^{q+1})_{j,m} = (D_{t_j}(\mathbb{E}_{t_j}(\mathcal{C}_p^{N,M} F^q)))^m$
 - 15: **end for**
 - 16: **end for**
 - 17: **end for**
-

Let us now deal with the complexity of the algorithm :

For each q :

- the computation of the vector F^q (loop line 5) requires $O(M \times N)$ computations,
- the computation of the vector \mathbf{d} (line 8) requires $O(M \times p \times (N \times d)^p)$ computations, (in dimension d we have $O((N \times d)^p)$ coefficients, and the computation of each coefficient requires $O(M \times p)$ computations (see Remark 8)),
- for each N and M (lines 10-11)
 - the computation of $(\mathbb{E}_{t_j}(\mathcal{C}_p^{N,M} F^q))^m$ and of $(D_{t_j}^l(\mathbb{E}_{t_j}(\mathcal{C}_p^{N,M} F^q)))_{1 \leq l \leq d}^m$ (line 12) requires $O(d \times p \times (N \times d)^p)$ computations
 - the computation of $(\mathbf{Y}^{q+1})_{j,m}$ (loop line 13) requires $O(N)$ computations and the computation of $((\mathbf{Z}^{q+1})_{j,m}^l)_{1 \leq l \leq d}$ requires $O(d)$ computations.

The complexity of the algorithm is then $O(K_{it} \times M \times p \times (N \times d)^{p+1})$.

4 Convergence results

We aim at bounding the error between (Y, Z) — the solution of (1) — and $(Y^{q,p,N}, Z^{q,p,N})$ defined by (20). Before stating the main result of the paper, we introduce some hypotheses.

Hypothesis 10. Let F denote a r.v. in $\mathbb{D}^{m,2}$ such that for all integer $r \leq m$ the function

$$(s_r, \dots, s_1) \mapsto \mathbb{E}(D_{s_1, \dots, s_r}^{(r)} F)$$

is Hölder of order α_F , i.e. $\exists k_r^F$ s.t. $|\mathbb{E}(D_{s_1, \dots, s_r}^{(r)} F) - \mathbb{E}(D_{t_1, \dots, t_r}^{(r)} F)| \leq k_r^F (|s_1 - t_1|^{\alpha_F} + \dots + |s_r - t_r|^{\alpha_F})$. In the following, K_m^F denotes $\sup_{r \leq m} k_r^F$.

The following Hypothesis is the generalization of Hypothesis 10 to the case $m = \infty$.

Hypothesis 11. Let F denote a r.v. in $\mathbb{D}^{\infty,2}$ such that for all integer r the function

$$(s_r, \dots, s_1) \mapsto \mathbb{E}(D_{s_1, \dots, s_r}^{(r)} F)$$

is Hölder of order α_F , i.e. $\exists k_r^F$ s.t. $|\mathbb{E}(D_{s_1, \dots, s_r}^{(r)} F) - \mathbb{E}(D_{t_1, \dots, t_r}^{(r)} F)| \leq k_r^F (|s_1 - t_1|^{\alpha_F} + \dots + |s_r - t_r|^{\alpha_F})$.

Theorem 12. Let k be an integer s.t. $k \leq p$. Assume that $f \in C_b^{0,p,p}$ and ξ satisfies Hypothesis 10 when $m = p$. We have

$$\|(Y - Y^{q,p,N}, Z - Z^{q,p,N})\|_{L^2}^2 \leq \frac{A_0}{2^q} + \frac{A_1(q, k)}{(p+1)^k} + A_2(q, p) \left(\frac{T}{N}\right)^{2\alpha_\xi \wedge 1},$$

where A_0 is given in Section 4.1, A_1 is given in Proposition 16, and A_2 is given in Proposition 20.

If $f \in C_b^{0,\infty,\infty}$ and ξ satisfies Hypothesis 11, we get

$$\lim_{q \rightarrow \infty} \lim_{p \rightarrow \infty} \lim_{N \rightarrow \infty} \|(Y - Y^{q,p,N}, Z - Z^{q,p,N})\|_{L^2}^2 = 0.$$

Proof. We split the error in 3 terms :

1. Picard's iterations : $\mathcal{E}^q = \|(Y - Y^q, Z - Z^q)\|_{L^2}^2$, where (Y^q, Z^q) is defined by (12),
2. the truncation of the chaos decomposition : $\mathcal{E}^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|_{L^2}^2$, where $(Y^{q,p}, Z^{q,p})$ is defined by (16),
3. the truncation of the $L^2([0, T])$ basis : $\mathcal{E}^{q,p,N} = \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|_{L^2}^2$, where $(Y^{q,p,N}, Z^{q,p,N})$ is defined by (18).

We have

$$\|(Y - Y^{q,p,N}, Z - Z^{q,p,N})\|_{L^2}^2 \leq 3(\mathcal{E}^q + \mathcal{E}^{q,p} + \mathcal{E}^{q,p,N}).$$

It remains to combine (21), Proposition 16 and Proposition 20 to get the first result. \square

4.1 Picard's iterations

The first type of error has already been studied in [PP92] and [EPQ97], we only recall the main result.

Hypothesis 13. We assume

- the generator $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is Lipschitz continuous: there exists a constant L_f such that for all $t \in \mathbb{R}^+$, $y, z \in \mathbb{R}$ and $p, q \in \mathbb{R}^d$

$$|f(t, y, z) - f(t, p, q)| \leq L_f (|y - p| + |z - q|),$$

- $\mathbb{E}[\|\xi\|^2 + \int_0^T |f(s, 0, 0)|^2 ds] < \infty$.

From [EPQ97, Corollary 2.1], we know that under Hypothesis 13, the sequence $(Y^q, Z^q)_q$ defined by (12) converges to (Y, Z) $d\mathbb{P} \times dt$ a.s. and in $S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$. Moreover, we have

$$\mathcal{E}^q := \|(Y - Y^q, Z - Z^q)\|_{L^2}^2 \leq \frac{A_0}{2^q}, \quad (21)$$

where A_0 depends on T , $\|\xi\|^2$ and on $\|f(\cdot, 0, 0)\|_{L^2([0, T])}^2$.

For the following, we also need

Lemma 14. ([EPQ97, Proof of Proposition 5.3]) *Let $m \in \mathbb{N}^*$. Assume that $f \in C_b^{0, m, m}$ and $\xi \in \mathbb{D}^{m, 2}$. For all $q \in \mathbb{N}$, (Y^q, Z^q) belongs to $L^2([0, T], \mathbb{D}^{m, 2} \times (\mathbb{D}^{m, 2})^d)$.*

In [EPQ97], the proof of Lemma 14 is done for $m = 1$, but it can be easily generalized for any integer $m \geq 1$.

4.2 Error due to the truncation of the chaos decomposition

We assume that the integrals are computed exactly, as well as expectations. The error is only due to the truncation of the chaos decomposition \mathcal{C}_p introduced in (5).

Lemma 15. *Assume that $f \in C_b^{0, m, m}$ and $\xi \in \mathbb{D}^{m, 2}$. For all $q \in \mathbb{N}$, $(Y^{q, p}, Z^{q, p})$ belongs to $L^2([0, T], \mathbb{D}^{m, 2} \times (\mathbb{D}^{m, 2})^d)$.*

Proof of Lemma 15. Assume that $(Y^{q, p}, Z^{q, p})$ belongs to $L^2([0, T]; \mathbb{D}^{m, 2} \times (\mathbb{D}^{m, 2})^d)$ and let us show that $(Y^{q+1, p}, Z^{q+1, p})$ belongs to $L^2([0, T]; \mathbb{D}^{m, 2} \times (\mathbb{D}^{m, 2})^d)$. Since $F^{q, p} \in L^2(\mathcal{F}_T)$, $\mathcal{C}_p(F^{q, p}) \in \mathbb{D}^{m, 2}$ (in fact, we have $\mathcal{C}_p(F^{q, p}) \in \mathbb{D}^{\infty, 2}$). Then, $\mathbb{E}_t[\mathcal{C}_p(F^{q, p})] \in \mathbb{D}^{m, 2}$. We deduce from (16) that $Y^{q+1, p} \in L^2([0, T]; \mathbb{D}^{m, 2})$. It remains to prove that $Z^{q+1, p} \in L^2([0, T]; (\mathbb{D}^{m, 2})^d)$. From (17) and the Clark-Ocone formula, we get $\int_0^t Z_s^{q+1, p} dB_s = \mathbb{E}_t[\mathcal{C}_p(F^{q, p})] - \mathbb{E}[\mathcal{C}_p(F^{q, p})]$. The r.h.s. belongs to $\mathbb{D}^{m, 2}$. We get the result by using [PP92, Lemma 2.3], which proves that if an Itô integral is differentiable in the Malliavin sense, its integrand is so. \square

Proposition 16. *Let $m \in \mathbb{N}^*$. Assume that $f \in C_b^{0, m, m}$ and $\xi \in \mathbb{D}^{m, 2}$. We recall $\mathcal{E}^{q, p} = \|(Y^q - Y^{q, p}, Z^q - Z^{q, p})\|_{L^2}^2$. We get*

$$\mathcal{E}^{q+1, p} \leq C_1 T(T+1) L_f^2 \mathcal{E}^{q, p} + \frac{K_1(m)}{(p+1) \cdots (p+m)} \quad (22)$$

where $K_1(m)$ depends on $\sup_{k \leq m} (\|\partial_{sp}^k f\|_\infty)$, $\|\xi\|_{\mathbb{D}^{m, 2}}$, T and on $\sup_{q \in \mathbb{N}} \|(Y^q, Z^q)\|_{L^2([0, T]; \mathbb{D}^{m, 2} \times (\mathbb{D}^{m, 2})^d)}^2$ and C_1 is a scalar.

Since $\mathcal{E}^{0, p} = 0$, we deduce from (22) that $\mathcal{E}^{q, p} \leq \frac{A_1(q, m)}{(p+1)^m}$ where $A_1(q, m) := K_1(m) \frac{(C_1 T(T+1) L_f^2)^q - 1}{C_1 T(T+1) L_f^2 - 1}$. Then, $(Y^{p, q}, Z^{p, q})$ converges to (Y^q, Z^q) when p tends to ∞ in $\|(\cdot, \cdot)\|_{L^2}$ (see (3) for the Definition of the norm).

Remark 17. We deduce from Proposition 16 that for all T and L_f , we have $\lim_{q \rightarrow \infty} \lim_{p \rightarrow \infty} \mathcal{E}^{q, p} = 0$. When $C_1 T(T+1) L_f^2 < 1$, i.e. for T small enough, we also get $\lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} \mathcal{E}^{q, p} = 0$.

Proof of Proposition 16. For the sake of clearness, we assume $d = 1$ and $m = 1$. In the following, one notes $\Delta Y_t^{q, p} := Y_t^{q, p} - Y_t^q$, $\Delta Z_t^{q, p} := Z_t^{q, p} - Z_t^q$ and $\Delta f_t^{q, p} := f(t, Y_t^{q, p}, Z_t^{q, p}) - f(t, Y_t^q, Z_t^q)$. Firstly, we deal with $\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1, p}|^2]$. From (15) and (16) we get

$$\begin{aligned} \Delta Y_t^{q+1, p} &= \mathbb{E}_t[\mathcal{C}_p(F^{q, p}) - F^q] - \int_0^t \Delta f_s^{q, p} ds, \\ &= \mathbb{E}_t[\mathcal{C}_p(\xi) - \xi] + \mathbb{E}_t \left[\mathcal{C}_p \left(\int_0^T f(s, Y_s^{q, p}, Z_s^{q, p}) ds \right) - \int_0^T f(s, Y_s^q, Z_s^q) ds \right] - \int_0^t \Delta f_s^{q, p} ds. \end{aligned}$$

We introduce $\pm \mathcal{C}_p \left(\int_0^T f(s, Y_s^q, Z_s^q) ds \right)$ in the second conditional expectation. This leads to

$$\begin{aligned} \Delta Y_t^{q+1,p} = & \mathbb{E}_t[\mathcal{C}_p(\xi) - \xi] + \mathbb{E}_t \left[\mathcal{C}_p \left(\int_0^T \Delta f_s^{q,p} ds \right) \right] + \mathbb{E}_t \left[\int_0^T \mathcal{C}_p(f(s, Y_s^q, Z_s^q)) - f(s, Y_s^q, Z_s^q) ds \right] \\ & - \int_0^t \Delta f_s^{q,p} ds, \end{aligned}$$

where we have used the second property of Lemma 3 to rewrite the third term.

From the previous equation, we bound $\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2]$ by using Doob's inequality and the Lipschitz property of f

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2 \right] & \leq 16 \mathbb{E}[|\mathcal{C}_p(\xi) - \xi|^2] + 16 \mathbb{E} \left[\left| \mathcal{C}_p \left(\int_0^T \Delta f_s^{q,p} ds \right) \right|^2 \right] \\ & + 16T \int_0^T \mathbb{E} \left[|\mathcal{C}_p(f(s, Y_s^q, Z_s^q)) - f(s, Y_s^q, Z_s^q)|^2 \right] ds + 8TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2] ds. \end{aligned}$$

To bound the second expectation of the previous inequality, we use the first property of Lemma 3 and the Lipschitz property of f . Then, we bring together this term with the last one to get

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2 \right] & \leq 16 \mathbb{E}[|\mathcal{C}_p(\xi) - \xi|^2] + 16T \int_0^T \mathbb{E} \left[|\mathcal{C}_p(f(s, Y_s^q, Z_s^q)) - f(s, Y_s^q, Z_s^q)|^2 \right] ds \\ & + 24TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2] ds. \end{aligned} \quad (23)$$

Let us now upper bound $\mathbb{E}[\int_0^T |\Delta Z_s^{q+1,p}|^2 ds]$. To do so, we use the Itô isometry $\mathbb{E}[\int_0^T |\Delta Z_s^{q+1,p}|^2 ds] = \mathbb{E}[(\int_0^T \Delta Z_s^{q+1,p} dB_s)^2]$. Using the Definitions (15)-(17) of Z_t^{q+1} and $Z_t^{q+1,p}$ and the Clark-Ocone Theorem leads to

$$\begin{aligned} \int_0^T \Delta Z_s^{q+1,p} dB_s & = F^q - \mathbb{E}(F^q) - (\mathcal{C}_p(F^{q,p}) - \mathbb{E}(\mathcal{C}_p(F^{q,p}))), \\ & = Y_T^{q+1} + \int_0^T f(s, Y_s^q, Z_s^q) ds - Y_0^{q+1} - \left(Y_T^{q+1,p} + \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}) ds - Y_0^{q+1,p} \right) \end{aligned}$$

Rearranging this summation makes appear $\Delta Y_T^{q+1,p}$ and $(\Delta Y_0^{q+1,p})$. Young's inequality gives

$$\mathbb{E} \left[\int_0^T |\Delta Z_s^{q+1,p}|^2 ds \right] \leq \frac{1}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2 \right] + 32TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2] ds. \quad (24)$$

Since $\int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2] ds \leq (T+1)\mathcal{E}^{q,p}$, by combining (23) and (24) we obtain

$$\frac{1}{2} \mathcal{E}^{q+1,p} \leq 16 \mathbb{E}[|\mathcal{C}_p(\xi) - \xi|^2] + 16T \int_0^T \mathbb{E} \left[|\mathcal{C}_p(f(s, Y_s^q, Z_s^q)) - f(s, Y_s^q, Z_s^q)|^2 \right] ds + 56T(T+1)L_f^2 \mathcal{E}^{q,p}.$$

Since $\xi \in \mathbb{D}^{1,2}$, $f \in C_b^{0,1,1}$ and $(Y^q, Z^q) \in L^2([0, T], \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ (see Lemma 14), Lemma 2 gives

$$\begin{aligned} \mathcal{E}^{q+1,p} & \leq \frac{32}{p+1} \|D\xi\|_{L^2(\Omega \times [0, T])}^2 + \frac{32T \|\partial_{sp} f\|_{\infty}^2}{p+1} \left(\int_0^T \|D_t Y^q\|_{L^2(\Omega \times [0, T])}^2 + \|D_t Z^q\|_{L^2(\Omega \times [0, T])}^2 dt \right) \\ & + 112T(T+1)L_f^2 \mathcal{E}^{q,p}. \end{aligned}$$

Since $(D_t Y^q, D_t Z^q)$ converges to $(D_t Y, D_t Z)$ in $L^2([0, T]; H_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}))$ (see [EPQ97, Proof of Proposition 5.3]), (22) follows. \square

4.3 Error due to the truncation of the basis

We are now interested in bounding the error between $(Y^{q,p}, Z^{q,p})$ defined by (16) and $(Y^{q,p,N}, Z^{q,p,N})$ defined by (18).

Before giving an upper bound for the error, we measure the error between \mathcal{C}_p and \mathcal{C}_p^N for a r.v. in $\mathbb{D}^{m,2}$, when $m \geq p$.

Remark 18. Let $m \in \mathbb{N}^*$, ξ satisfies Hypothesis 10 and $f \in C_b^{0,m,m}$. Then, for all integers p and q , $I_{q,p} := \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}) ds$ satisfies Hypothesis 10 where $\alpha_{I_{q,p}} = \frac{1}{2} \wedge \alpha_\xi$ and $K_m^{I_{q,p}}$ depends on K_m^ξ , $\sup_{k \leq m} (\|\partial_{sp}^k f\|_\infty)$, T and on $\sup_{q' \leq q} \|(Y^{q',p}, Z^{q',p})\|_{L^2([0,T]; \mathbb{D}^{m,2} \times (\mathbb{D}^{m,2})^d)}$.

We refer to Section A.1 for the proof of Remark 18.

Lemma 19. *Let F denote a r.v. in $L^2(\mathcal{F}_T)$ satisfying Hypothesis 10 for any integer $m \geq p$. We have*

$$\mathbb{E}(|(\mathcal{C}_p^N - \mathcal{C}_p)(F)|^2) \leq (K_p^F)^2 \left(\frac{T}{N}\right)^{2\alpha_F} \sum_{i=1}^p i^2 \frac{T^i}{i!} \leq (K_p^F)^2 \left(\frac{T}{N}\right)^{2\alpha_F} T(1+T)e^T,$$

where K_p^F is defined in Hypothesis 10.

We refer to Section A.2 for the proof of the Lemma.

Proposition 20. *Let m be an integer s.t. $m \geq p$. Assume that $f \in C_b^{0,m,m}$ and ξ satisfies Hypothesis 10. We recall $\mathcal{E}^{q,p,N} := \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|_{L^2}^2$. We get*

$$\mathcal{E}^{q+1,p,N} \leq C_2 T(T+1) L_f^2 \mathcal{E}^{q,p,N} + K_2(q,p) \left(\frac{T}{N}\right)^{2\alpha_\xi \wedge 1} \quad (25)$$

where C_2 is a scalar and $K_2(q,p)$ depends on K_p^ξ , $\sup_{k \leq p} (\|\partial_{sp}^k f\|_\infty^2)$, T and on $\sup_{q' \leq q} \|(Y^{q',p}, Z^{q',p})\|_{L^2([0,T], \mathbb{D}^{p,2} \times (\mathbb{D}^{p,2})^d)}$.

Since $\mathcal{E}^{0,p,N} = 0$, we deduce from (25) that $\mathcal{E}^{q,p,N} \leq A_2(q,p) \left(\frac{T}{N}\right)^{2\alpha_\xi \wedge 1}$, where $A_2(q,p) := K_2(q,p) T(T+1) e^T \frac{(C_2 T(T+1) L_f^2)^{q-1}}{C_2 T(T+1) L_f^2 - 1}$. Then, $(Y^{p,q,N}, Z^{p,q,N})$ converges to $(Y^{q,p}, Z^{q,p})$ when N tends to ∞ in $\|(\cdot, \cdot)\|_{L^2}$ (see (3) for the Definition of the norm).

Proof of Proposition 20. For the sake of clearness, we assume $d = 1$. In the following, one notes $\Delta Y_t^{q,p,N} := Y_t^{q,p,N} - Y_t^{q,p}$, $\Delta Z_t^{q,p,N} := Z_t^{q,p,N} - Z_t^{q,p}$ and $\Delta f_t^{q,p,N} := f(t, Y_t^{q,p,N}, Z_t^{q,p,N}) - f(t, Y_t^{q,p}, Z_t^{q,p})$. Firstly, we deal with $\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N}|^2]$. From (16) and (18) we get

$$\Delta Y_t^{q+1,p,N} = \mathbb{E}_t[\mathcal{C}_p^N(F^{q,p,N}) - \mathcal{C}_p(F^{q,p})] + \int_0^t \Delta f_s^{q,p,N} ds.$$

Following the same steps as in the proof of Proposition 16, one gets

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N}|^2\right] &\leq 16\mathbb{E}[|\mathcal{C}_p^N(\xi) - \mathcal{C}_p(\xi)|^2] + 16\mathbb{E}\left[\left|(\mathcal{C}_p^N - \mathcal{C}_p)\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}) ds\right)\right|^2\right] \\ &\quad + 24TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p,N}|^2 + |\Delta Z_s^{q,p,N}|^2] ds. \end{aligned} \quad (26)$$

Let us now upper bound $\mathbb{E}[\int_0^T |\Delta Z_s^{q+1,p,N}|^2 ds]$. Following the same steps as in the proof of Proposition 16, one gets

$$\mathbb{E}\left[\int_0^T |\Delta Z_s^{q+1,p,N}|^2 ds\right] \leq \frac{1}{2}\mathbb{E}\left[\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N}|^2\right] + 32TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p,N}|^2 + |\Delta Z_s^{q,p,N}|^2] ds. \quad (27)$$

Combining (26) and (27) we obtain

$$\begin{aligned} \frac{1}{2}\mathcal{E}^{q+1,p,N} \leq & 16\mathbb{E}[|(\mathcal{C}_p^N - \mathcal{C}_p)(\xi)|^2] + 16\mathbb{E}\left[\left|(\mathcal{C}_p^N - \mathcal{C}_p)\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p})ds\right)\right|^2\right] \\ & + 56T(T+1)L_f^2\mathcal{E}^{q,p,N}. \end{aligned}$$

Since ξ and $I_{q,p} := \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p})ds$ satisfy Hypothesis 10 (see Remark 18), Lemma 19 gives

$$\mathcal{E}^{q+1,p,N} \leq 16\left(\frac{T}{N}\right)^{2\alpha_\xi \wedge 1} T(T+1)e^T ((K_p^\xi)^2 + (K_p^{I_{q,p}})^2) + 112T(T+1)L_f^2\mathcal{E}^{q,p,N},$$

and (25) follows. \square

5 Numerical Examples

The computations have been done on a PC INTEL Core 2 Duo P9600 2.53 GHz with 4Gb of RAM.

5.1 Non linear driver and path-dependent terminal condition

We consider the case $d = 1$, $f(t, y, z) = \cos(y)$ and $\xi = \sup_{0 \leq t \leq T} B_t$.

- **Convergence in p .** Table 1 represents the evolution of $Y_0^{q,p,N,M}$ and $Z_0^{q,p,N,M}$ w.r.t q (Picard's iteration index), when $p = 2$ and $p = 3$. We fix $M = 10^5$ and $N = 20$. The seed of the generator is also fixed.

iterations	1	2	3	4	5	6	CPU time
$p = 2$	1.656357	1.017117	1.237135	1.186691	1.195462	1.194256	14.06
$p = 3$	1.656357	1.012091	1.234398	1.183544	1.192367	1.191173	174.09

Table 1: Evolution of $Y_0^{q,p,N,M}$ w.r.t. Picard's iterations, $M = 10^5$, $N = 20$ and CPU time

iterations	1	2	3	4	5	6	CPU time
$p = 2$	0.969128	0.249148	0.525273	0.459326	0.470069	0.469117	14.06
$p = 3$	0.969128	0.242977	0.523846	0.455827	0.466903	0.465939	174.09

Table 2: Evolution of $Z_0^{q,p,N,M}$ w.r.t. Picard's iterations, $M = 10^5$, $N = 20$ and CPU time

One notes that the difference between the values of $Y_0^{q,2,N,M}$ and $Y_0^{q,3,N,M}$ (resp. $Z_0^{q,2,N,M}$ and $Z_0^{q,3,N,M}$) doesn't exceed 0.2% (resp. 0.6%). This is due to the fast convergence of the algorithm in p . The CPU time is 12 times higher when $p = 3$ than when $p = 2$. Then, the use of order 3 in the chaos decomposition is not necessary. In the following, we take $p = 2$.

- **Convergence in M .** Figure 1 illustrates the evolution of $Y_0^{q,p,N,M}$ and $Z_0^{q,p,N,M}$ w.r.t. q when $p = 2$ and $N = 20$ for different values of M . The seed of the generator is random. When M equals 10^4 and 10^5 the algorithm stabilizes after very few iterations. When $M = 10^3$, there is no convergence.
- **Convergence in N .** Figure 2 illustrates the evolution of $Y_0^{q,p,N,M}$ and $Z_0^{q,p,N,M}$ w.r.t. q when $p = 2$ and $M = 10^5$ for different values of N . The seed of the generator is random. The algorithm converges even when $N = 5$, but $Y_0^{6,p,5,M}$ is quite below $Y_0^{6,p,40,M}$.

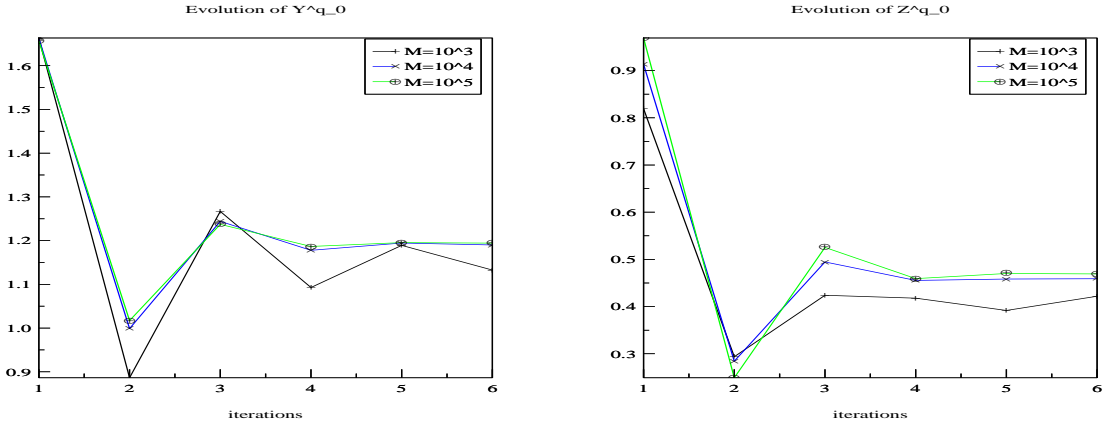


Figure 1: Evolution of $Y_0^{q,p,N,M}$ and $Z_0^{q,p,N,M}$ w.r.t. q and M when $N = 20$, $p = 2 - \xi = \sup_{0 \leq t \leq T} B_t$, $f(t, y, z) = \cos(y)$.

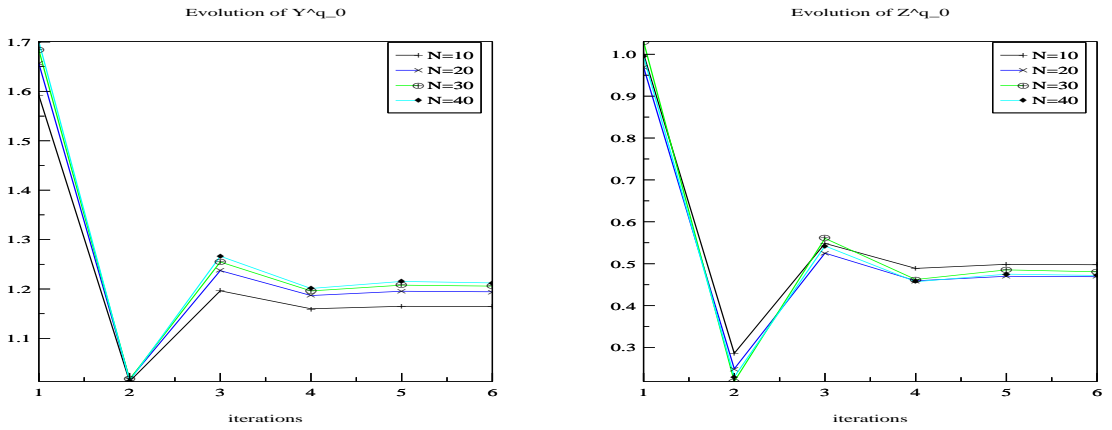


Figure 2: Evolution of $Y_0^{q,p,N,M}$ and $Z_0^{q,p,N,M}$ w.r.t. N when $M = 10^5$, $p = 2 - \xi = \sup_{0 \leq t \leq T} B_t$, $f(t, y, z) = \cos(y)$.

5.2 Linear Driver - Financial Benchmark

We consider the case of pricing and hedging a Discrete Down and Out Barrier Call option, i.e. $f(t, y, z) = -ry$ and $\xi := (S_T - K)^+ \mathbf{1}_{\forall n \in [0, N] S_{t_n} \geq L}$, where S represents the Black-Scholes diffusion

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}, \quad \forall t \in [0, T].$$

The option parameters are $r = 0.01$, $\sigma = 0.2$, $T = 1$, $K = 0.9$, $L = 0.85$, $S_0 = 1$ and $N = 20$ (N is also the number of time discretizations of the chaos decomposition).

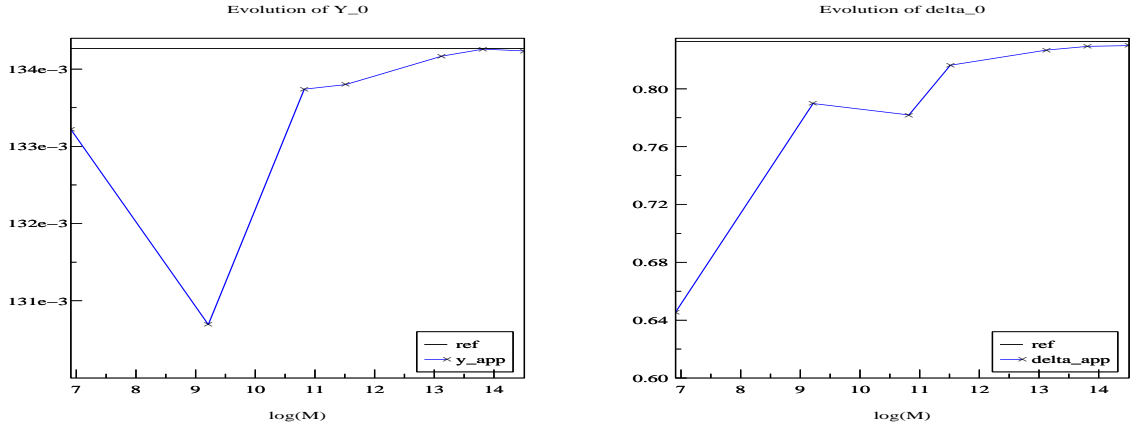


Figure 3: Evolution of $Y_0^{q,p,N,M}$ and $\delta_0 := \frac{Z_0^{q,p,N,M}}{\sigma S_0}$ w.r.t. $\log(M)$ when $N = 20$, $p = 2$, $q = 5$ -Discrete Down and Out Barrier Call option

We can get a benchmark for Y_0 and Z_0 by using a variance reduction Monte Carlo method. For this set of parameters, the reference values are $Y_0 = 0.134267$ with a confidence interval $7.9468e - 05$ and $\delta_0 = \frac{Z_0}{\sigma S_0} = 0.8327$. We compare them with $Y_0^{q,p,N,M}$ and $\frac{Z_0^{q,p,N,M}}{\sigma S_0}$ when $N = 20$, $p = 2$, $q = 5$ (we choose the first value of q from which the algorithm has converged) for different values of M . Figure 3 represents the evolution of $Y_0^{5,p,N,M}$ and $\delta_0^{5,p,N,M}$ w.r.t. $\log(M)$. One notices that for $M = 10^6$ the computed values are very close to the reference ones.

5.3 Non linear driver in dimension 5 - Financial Benchmark

We consider the pricing and hedging of a Put Basket option in dimension 5, i.e. $\xi = (K - \frac{1}{5} \sum_{i=1}^5 S_T^i)_+$, where

$$\forall i = 1, \dots, 5 \quad S_t^i = S_0^i e^{(\mu^i - \frac{\sigma^i}{2})t + \sigma B_t^i}.$$

μ^i (resp. σ^i) represents the trend (resp. the volatility) of the i^{th} asset. $B = (B^1, \dots, B^5)$ is a 5-dimensional Brownian motion such that $\langle B^i, B^j \rangle_t = \rho t \mathbf{1}_{i \neq j} + t \mathbf{1}_{i=j}$. We suppose that $\rho \in (-\frac{1}{4}, 1)$, which ensures that the matrix $C = (\rho \mathbf{1}_{i \neq j} + \mathbf{1}_{i=j})_{1 \leq i, j \leq 5}$ is positive definite. We also assume that the borrowing rate R is higher than the bond one r . In such a case, pricing and hedging the Put Basket option is equivalent to solving a BSDE with terminal condition ξ and with driver f defined by $f(t, y, z) = -ry - \theta \cdot z + (R - r)(y - \sum_{i=1}^5 (\Sigma^{-1} z)_i)^-$, where $\theta := \Sigma^{-1}(\mu - r \mathbf{1})$ ($\mathbf{1}$ is the vector whose every component is one) and Σ is the matrix defined by $\Sigma_{ij} = \sigma^i L_{ij}$ (L denote the lower triangular matrix involved in the Cholesky decomposition $C = LL^*$). We refer to [EPQ97][Example 1.1] for more details. Figure 4 represents the evolution of $Y_0^{5,p,N,M}$, the approximated price at time 0, and the relative error on $\delta_0^1 := \frac{(\Sigma^{-1} Z_0^{5,p,N,M})^1}{S_0^1}$ — the quantity of asset 1 to possess at time 0 — w.r.t. $\log(M)$. We compare our results with the ones obtained

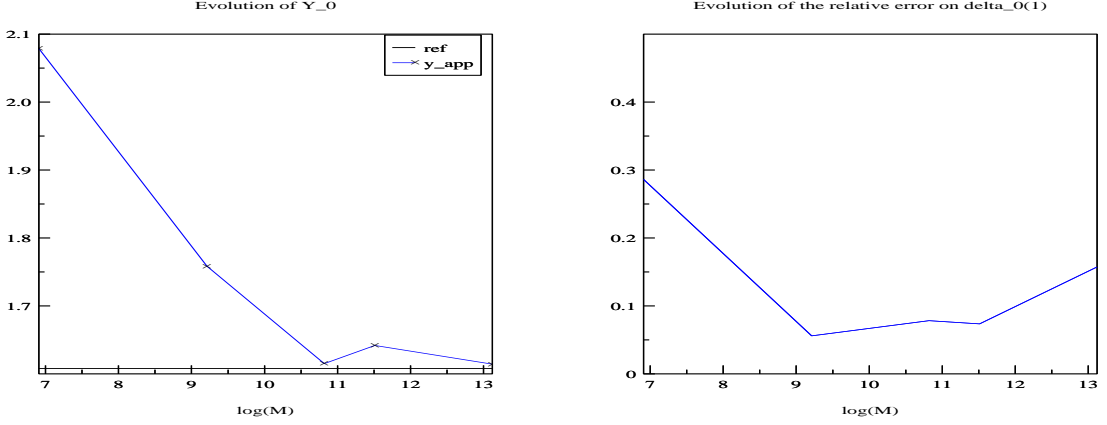


Figure 4: Evolution of $Y_0^{q,p,N,M}$ and $\delta_0(1)$ w.r.t. $\log(M)$ when $N = 20$, $p = 2$, $q = 5$, $d = 5$ - Basket Put option with different interest and borrowing rates

using the Algorithm proposed in [GL10] (cited here as reference values). The CPU time needed to compute price and delta when $M = 50000$ and $N = 20$ is 161s. One notices that the convergence is very fast and quite accurate for $M = 50000$.

A Technical results of Section 4.3

A.1 Proof of Remark 18

We prove the result for $m = 1$, i.e. we show that if ξ satisfies Kypothesis 10 for $m = 1$ and $f \in C_b^{0,1,1}$, then $\Phi : t \mapsto \mathbb{E}[D_t \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}) ds]$ is Hölder of order $\alpha := \frac{1}{2} \wedge \alpha_\xi$ with a constant K depending on K_1^ξ , $\|\partial_{sp} f\|_\infty$, T and on $\sup_{q' \leq q} \|(Y^{q',p}, Z^{q',p})\|_{L^2([0,T]; \mathbb{D}^{1,2} \times (\mathbb{D}^{1,2})^d)}$. Let us first prove the following Lemma.

Lemma 21. *Assume that ξ satisfies Hypothesis 10 for $m = 1$ and $f \in C_b^{0,1,1}$. For all s, t such that $0 \leq s \leq t$ and $|t - s| \leq 1$, we have*

$$\Delta_{t,s}^{q,p} := \mathbb{E} \left[\sup_{t \leq r \leq T} (D_t Y_r^{q,p} - D_s Y_r^{q,p})^2 \right] + \int_t^T \mathbb{E} [(D_t Z_r^{q,p} - D_s Z_r^{q,p})^2] dr \leq C_1(q,p)(t-s)^{1 \wedge 2\alpha_\xi}.$$

where $C_1(q,p)$ depends on K_1^ξ , T , $\|\partial_{sp} f\|_\infty$ and on $\sup_{q' \leq q} \|(Y^{q',p}, Z^{q',p})\|_{L^2([0,T], \mathbb{D}^{1,2} \times (\mathbb{D}^{1,2})^d)}$.

Proof of Lemma 21. Let s, t, r be such that $0 \leq s \leq t \leq r$ and $|t - s| \leq 1$. Let us introduce some notations: $\Delta_{ts} Y_r^{q,p} := D_t Y_r^{q,p} - D_s Y_r^{q,p}$, $\Delta_{ts} Z_r^{q,p} := D_t Z_r^{q,p} - D_s Z_r^{q,p}$ and $f(\theta_r^q) := f(r, Y_r^{q,p}, Z_r^{q,p})$. From (16) and Lemma 3, we get $D_t Y_r^{q,p} = \mathbb{E}_r[\mathcal{C}_{p-1}(D_t F^{q,p})] - \int_t^r D_t f(\theta_u^{q-1}) du$. Then,

$$\Delta_{ts} Y_r^{q,p} = \mathbb{E}_r[\mathcal{C}_{p-1}(D_t F^{q,p} - D_s F^{q,p})] - \int_t^r D_t f(\theta_u^{q-1}) - D_s f(\theta_u^{q-1}) du + \int_s^t D_s f(\theta_u^{q-1}) du.$$

By using Doob's inequality, Lemma 3 and the previous equation, we bound $\mathbb{E}[\sup_{t \leq r \leq T} (\Delta_{ts} Y_r^{q,p})^2]$:

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq r \leq T} (\Delta_{ts} Y_r^{q,p})^2 \right] &\leq 12 \mathbb{E} [|D_t F^{q,p} - D_s F^{q,p}|^2] \\ &\quad + 3T \int_t^T \mathbb{E} [|D_t f(\theta_u^{q-1}) - D_s f(\theta_u^{q-1})|^2] du + 3(t-s) \int_s^t \mathbb{E} [|D_s f(\theta_u^{q-1})|^2] du. \end{aligned}$$

Using the definition of $F^{q,p}$ (see (16)), we get $D_t F^{q,p} - D_s F^{q,p} = D_t \xi - D_s \xi + \int_t^T D_t f(\theta_u^{q-1}) - D_s f(\theta_u^{q-1}) du - \int_s^t D_s f(\theta_u^{q-1}) du$. Plugging this result in the previous inequality yields

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq r \leq T} (\Delta_{ts} Y_r^{q,p})^2 \right] &\leq 36 \mathbb{E} [|D_t \xi - D_s \xi|^2] \\ &\quad + 39T \int_t^T \mathbb{E} [|D_t f(\theta_u^{q-1}) - D_s f(\theta_u^{q-1})|^2] du + 39(t-s) \int_s^t \mathbb{E} [|D_s f(\theta_u^{q-1})|^2] du. \end{aligned}$$

Since $D_t f(\theta_u^{q-1}) = \partial_y f(\theta_u^{q-1}) D_t Y_u^{q-1,p} + \partial_z f(\theta_u^{q-1}) D_t Z_u^{q-1,p}$ and $\partial_y f$ and $\partial_z f$ are bounded, we obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq r \leq T} (\Delta_{ts} Y_r^{q,p})^2 \right] &\leq 36 \mathbb{E} [|D_t \xi - D_s \xi|^2] + 78T \|\partial_y f\|_\infty^2 \int_t^T \mathbb{E} [|\Delta_{ts} Y_u^{q-1,p}|^2] du \\ &\quad + 78T \|\partial_z f\|_\infty^2 \int_t^T \mathbb{E} [|\Delta_{ts} Z_u^{q-1,p}|^2] du + 78C_2(q,p)(t-s), \end{aligned} \quad (28)$$

where $C_2(q,p) := \|\partial_y f\|_\infty^2 \sup_{0 \leq s \leq T} \|D_s Y^{q,p}\|_{\mathbb{H}_T^2}^2 + \|\partial_z f\|_\infty^2 \sup_{0 \leq s \leq T} \|D_s Z^{q,p}\|_{\mathbb{H}_T^2}^2$. Let us now upper bound $\int_t^T \mathbb{E} [|\Delta_{ts} Z_r^{q,p}|^2] dr$. Using (17) and the Clark-Ocone formula gives $\int_0^T Z_r^{q,p} dB_r = C_p(F^{q-1,p}) - \mathbb{E}(C_p(F^{q-1,p}))$. Hence, we have $\int_t^T Z_r^{q,p} dB_r = C_p(F^{q-1,p}) - \mathbb{E}_t(C_p(F^{q-1,p})) = Y_T^{q,p} + \int_t^T f(\theta_u^{q-1}) du - Y_t^{q,p}$. Then, since $s \leq t \leq r$, we get

$$\int_t^T \Delta_{ts} Z_r^{q,p} dB_r = \Delta_{ts} Y_T^{q,p} - \Delta_{ts} Y_t^{q,p} + \int_t^T (D_t f(\theta_u^{q-1}) - D_s f(\theta_u^{q-1})) du.$$

Young's inequality gives

$$\int_t^T \mathbb{E} [|\Delta_{ts} Z_r^{q,p}|^2] dr \leq \frac{1}{2} \mathbb{E} \left[\sup_{t \leq r \leq T} |\Delta_{ts} Y_r^{q,p}|^2 \right] + 32T \int_t^T \mathbb{E} [|D_t f(\theta_u^{q-1}) - D_s f(\theta_u^{q-1})|^2] du.$$

As above, we develop the Malliavin derivatives of $f(\theta_u^{q-1})$ and we use that $\partial_y f$ and $\partial_z f$ are bounded. We obtain

$$\begin{aligned} \int_t^T \mathbb{E} [|\Delta_{ts} Z_r^{q,p}|^2] dr &\leq \frac{1}{2} \mathbb{E} \left[\sup_{t \leq r \leq T} |\Delta_{ts} Y_r^{q,p}|^2 \right] + 64T \|\partial_y f\|_\infty^2 \int_t^T \mathbb{E} [|\Delta_{ts} Y_u^{q-1,p}|^2] du \\ &\quad + 64T \|\partial_z f\|_\infty^2 \int_t^T \mathbb{E} [|\Delta_{ts} Z_u^{q-1,p}|^2] du \end{aligned} \quad (29)$$

Combining (28) and (29) and using the Hypothesis 10 satisfied by ξ yields

$$\frac{1}{2} \Delta_{t,s}^{q,p} \leq (78C_2(q,p) + 36(K_1^\xi)^2)(t-s)^{1 \wedge 2\alpha_\xi} + 284T \|\partial_{sp} f\|_\infty^2 \Delta_{t,s}^{q-1,p}.$$

Since $\Delta_{t,s}^{0,p} = 0$, we get Lemma 21 by induction. \square

We are now able to prove that Φ is Hölder. Let s, t be such that $0 \leq s \leq t$ and $|t-s| \leq 1$, we have

$$\Phi(t) - \Phi(s) = \int_t^T \mathbb{E} [D_t f(\theta_r^q) - D_s f(\theta_r^q)] dr - \int_s^t \mathbb{E} [D_s f(\theta_r^q)] dr.$$

As above, we develop the Malliavin derivatives of $f(\theta_r^q)$

$$\begin{aligned} \Phi(t) - \Phi(s) &= \int_t^T \mathbb{E} [\partial_y f(\theta_r^q) \Delta_{ts} Y_r^{q,p}] dr + \int_t^T \mathbb{E} [\partial_z f(\theta_r^q) \Delta_{ts} Z_r^{q,p}] dr \\ &\quad - \int_s^t \mathbb{E} [\partial_y f(\theta_r^q) D_s Y_r^{q,p} + \partial_z f(\theta_r^q) D_s Z_r^{q,p}] dr. \end{aligned}$$

Cauchy-Schwarz inequality gives $\mathbb{E}[\partial_y f(\theta_r^q) \Delta_{ts} Y_r^{q,p}] \leq \|\partial_y f\|_\infty \sqrt{\mathbb{E}[(\Delta_{ts} Y_r^{q,p})^2]}$. Using the same argument to bound $\mathbb{E}[\partial_z f(\theta_r^q) \Delta_{ts} Z_r^{q,p}]$ leads to

$$|\Phi(t) - \Phi(s)|^2 \leq 6T \|\partial_{sp} f\|_\infty^2 \Delta_{t,s}^{q,p} + 6C_2(q,p)(t-s), \quad (30)$$

where $\Delta_{t,s}^{q,p}$ (resp. $C_2(q,p)$) has been introduced in Lemma 21 (resp. in the proof of Lemma 21). Combining (30) and Lemma 21 ends the proof.

A.2 Proof of Lemma 19

We prove the result by induction. Lemma 19 is true for $p = 0$, since $\mathcal{C}_0^N(F) = \mathcal{C}_0(F)$. Assume that $\mathbb{E}(|(\mathcal{C}_{p-1}^N - \mathcal{C}_{p-1})(F)|^2) \leq (K_{p-1}^F)^2 \left(\frac{T}{N}\right)^{2\alpha_F} \sum_{i=1}^{p-1} i^2 \frac{T^i}{i!}$. Since we have

$$(\mathcal{C}_p^N - \mathcal{C}_p)(F) = (\mathcal{C}_{p-1}^N - \mathcal{C}_{p-1})(F) + (P_p^N - P_p)(F),$$

it remains to show that $\mathbb{E}(|(P_p^N - P_p)(F)|^2) \leq (k_p^F)^2 \left(\frac{T}{N}\right)^{2\alpha_F} p^2 \frac{T^p}{p!}$. We recall

$$P_p(F) = \int_0^T \int_0^{s_p} \cdots \int_0^{s_2} u_p(s_p, \dots, s_1) dB_{s_1} \cdots dB_{s_p}, \quad \text{where } u_p : s_p, \dots, s_1 \mapsto \mathbb{E}(D_{s_1 \dots D_{s_p}}^{(p)} F), \quad (31)$$

$$P_p^N(F) = \sum_{|n|=p} d_p^n \prod_{1 \leq i \leq N} K_{n_i}(G_i), \quad \text{where } d_p^n = n! \mathbb{E} \left(F \prod_{1 \leq i \leq N} K_{n_i}(G_i) \right). \quad (32)$$

Let us rewrite $P_p^N(F)$ as a sum of stochastic integrals. Let $r \in \mathbb{N}$. Applying Lemma 4 to $g : t \mapsto \mathbf{1}_{]t_{i-1}, t_i]}(t)$ yields $M_i^r := h^{r/2} K_r \left(\frac{B_t - B_{t_{i-1}}}{\sqrt{h}} \right)$ is a martingale and $M_i^r = \int_{t_{i-1}}^t M_s^{r-1} dB_s$. Then, $M_i^r = \int_{t_{i-1}}^t \int_{t_{i-1}}^{s_r} \cdots \int_{t_{i-1}}^{s_2} M_{s_1}^0 dB_{s_1} \cdots dB_{s_r}$. For $r = n_i$ and $t = t_i$, we get

$$K_{n_i}(G_i) = \frac{1}{h^{\frac{n_i}{2}}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s_{n_i}} \cdots \int_{t_{i-1}}^{s_2} dB_{s_1} \cdots dB_{s_{n_i}}.$$

For $|n| := n_1 + \cdots + n_N = p$, we obtain

$$\prod_{1 \leq i \leq N} K_{n_i}(G_i) = \frac{1}{h^{\frac{p}{2}}} \underbrace{\int_{t_{N-1}}^T \cdots \int_{t_{N-1}}^{s_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{t_1}^{t_2} \cdots \int_{t_1}^{s_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{t_1} \cdots \int_0^{s_2}}_{n_1 \text{ integrals}} dB_{s_1} \cdots dB_{s_p}, \quad (33)$$

$$d_p^n = n! \frac{1}{h^{\frac{p}{2}}} \underbrace{\int_{t_{N-1}}^T \cdots \int_{t_{N-1}}^{l_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{t_1}^{t_2} \cdots \int_{t_1}^{l_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{t_1} \cdots \int_0^{l_2}}_{n_1 \text{ integrals}} u_p(l_p, \dots, l_1) dl_1 \cdots dl_p. \quad (34)$$

To compare $P_p(F)$ and $P_p^N(F)$, we split the integrals in (31)

$$P_p(F) = \sum_{|n|=p} \underbrace{\int_{t_{N-1}}^T \cdots \int_{t_{N-1}}^{s_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{t_1}^{t_2} \cdots \int_{t_1}^{s_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{t_1} \cdots \int_0^{s_2}}_{n_1 \text{ integrals}} u_p(s_p, \dots, s_1) dB_{s_1} \cdots dB_{s_p}. \quad (35)$$

Combining (32)-(33)-(34) and (35) yields $\mathbb{E}(|(P_p^N - P_p)(F)|^2) =$

$$\sum_{|n|=p} \underbrace{\int_{t_{N-1}}^T \cdots \int_{t_{N-1}}^{s_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{t_1}^{t_2} \cdots \int_{t_1}^{s_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{t_1} \cdots \int_0^{s_2}}_{n_1 \text{ integrals}} \left| \frac{d_p^n}{h^{\frac{p}{2}}} - u_p(s_p, \dots, s_1) \right|^2 ds_1 \cdots ds_p, \quad (36)$$

Moreover, $\frac{d^n}{h^{\frac{n}{2}}} - u_p(s_p, \dots, s_1) =$

$$\frac{n!}{h^p} \underbrace{\int_{t_{N-1}}^T \dots \int_{t_{N-1}}^{l_{N-1}+1}}_{n_N \text{ integrals}} \dots \underbrace{\int_{t_1}^{t_2} \dots \int_{t_1}^{l_{n_1}+1}}_{n_2 \text{ integrals}} \underbrace{\int_0^{t_1} \dots \int_0^{l_2}}_{n_1 \text{ integrals}} (u_p(l_p, \dots, l_1) - u_p(s_p, \dots, s_1)) dl_1 \dots dl_p.$$

Since u_p satisfies Hypothesis 10, we get $|u_p(l_p, \dots, l_1) - u_p(s_p, \dots, s_1)| \leq k_p^F (|l_p - s_p|^{\alpha_F} + \dots + |l_1 - s_1|^{\alpha_F}) \leq p k_p^F h^{\alpha_F}$. Then $\left| \frac{d^n}{h^{\frac{n}{2}}} - u_p(s_p, \dots, s_1) \right| \leq p k_p^F h^{\alpha_F}$. Plugging this result in (36) ends the proof.

B Wiener chaos expansion formulas

B.1 Proof of Proposition 5

Firstly, we compute $\mathbb{E}_t(\mathcal{C}_p^N(F))$ for $t \in]t_{r-1}, t_r]$. From (10), we get

$$\mathbb{E}_t(\mathcal{C}_p^N(F)) = d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \mathbb{E}_t \left(\prod_{i \geq r} K_{n_i}(G_i) \right).$$

Since Brownian increments are independent, we get $\mathbb{E}_{t_r}(\prod_{i \geq r} K_{n_i}(G_i)) = K_{n_r}(G_r) \prod_{i > r} \mathbb{E}[K_{n_i}(G_i)]$, which is null as soon as $n_{r+1} + \dots + n_N > 0$. Then, nested conditional expectations give

$$\mathbb{E}_t(\mathcal{C}_p^N(F)) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \mathbb{E}_t(K_{n_r}(G_r)).$$

By applying Lemma 4 when $g : t \mapsto \mathbf{1}_{]t_{r-1}, t_r]}(t)$, we get $\mathbb{E}_t(K_{n_r}(G_r)) = \left(\frac{t - t_{r-1}}{h} \right)^{n_r/2} K_{n_r} \left(\frac{B_t - B_{t_{r-1}}}{\sqrt{t - t_{r-1}}} \right)$, which yields the first result. Since $K'_n(x) = K_{n-1}(x)$, the second result follows.

B.2 Wiener chaos expansion formulas in \mathbb{R}^d

We want to approximate $F \in L^2(\mathcal{F}_T)$ using its chaos decomposition up to order p . We assume $N \geq dp$. We consider the following truncated basis of $L^2([0, T]; \mathbb{R}^d)$

$$\frac{\mathbf{1}_{[t_{i-1}, t_i]}(t)}{\sqrt{h}} e_j, \quad i = 1, \dots, N, \quad j = 1, \dots, d, \quad \text{where } h = \frac{T}{N}$$

where $\{t_i := ih, i = 0, \dots, N\}$ is a regular mesh grid and $(e_j)_{1 \leq j \leq d}$ represents the canonical basis of \mathbb{R}^d . P_k , the k^{th} chaos, is generated by

$$\left\{ \prod_{j=1}^d \prod_{i=1}^N K_{n_i^j}(G_i^j) : \sum_{j=1}^d \sum_{i=1}^N n_i^j = k \right\}, \quad G_i^j = \frac{\Delta_i^j}{\sqrt{h}}, \quad \Delta_i^j = B_{t_i}^j - B_{t_{i-1}}^j.$$

For $j = 1, \dots, d$, $n^j = (n_1^j, \dots, n_N^j)$, one notes $|n^j| = n_1^j + \dots + n_N^j$, $n^j! = n_1^j! \dots n_N^j!$ and for $r \leq N$, $n^j(r) = (n_1^j, \dots, n_r^j)$. $n = (n^1, \dots, n^d)^*$, $|n| = |n^1| + \dots + |n^d|$, $n! = n^1! \dots n^d!$ and $n(r) = (n^1(r), \dots, n^d(r))^*$. Since the r.v. $\left(\prod_{1 \leq j \leq d} \prod_{1 \leq i \leq N} K_{n_i^j}(G_i^j) \right)_n$ are orthogonal ones, the projection of F is given by

$$\mathcal{C}_p^N(F) = d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{1 \leq j \leq d} \prod_{1 \leq i \leq N} K_{n_i^j}(G_i^j),$$

where the coefficients d_k^n are given by

$$d_k^n = n! \mathbb{E} \left[F \prod_{1 \leq j \leq d} \prod_{1 \leq i \leq N} K_{n_i^j} (G_i^j) \right].$$

Proposition 22. For $t_{r-1} < t \leq t_r$, we have

$$\mathbb{E}_t(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i < r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \times \prod_{1 \leq j \leq d} \left(\frac{t - t_{r-1}}{h} \right)^{\frac{n_r^j}{2}} K_{n_r^j} \left(\frac{B_t^j - B_{t_{r-1}}^j}{\sqrt{t - t_{r-1}}} \right).$$

and for $l = 1, \dots, d$,

$$D_t^l(\mathbb{E}_t(\mathcal{C}_p^N F)) = \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n_r^l > 0}} d_k^n h^{-1/2} \prod_{i < r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \times \left(\frac{t - t_{r-1}}{h} \right)^{\frac{n_r^l - 1}{2}} K_{n_r^l - 1} \left(\frac{B_t^l - B_{t_{r-1}}^l}{\sqrt{t - t_{r-1}}} \right) \prod_{j \neq l} \left(\frac{t - t_{r-1}}{h} \right)^{\frac{n_r^j}{2}} K_{n_r^j} \left(\frac{B_t^j - B_{t_{r-1}}^j}{\sqrt{t - t_{r-1}}} \right).$$

Remark 23. In particular, for $t = t_r$ and $l = 1, \dots, d$,

$$\begin{aligned} \mathbb{E}_{t_r}(\mathcal{C}_p^N F) &= d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i \leq r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \\ D_{t_r}^l(\mathbb{E}_{t_r}(\mathcal{C}_p^N F)) &= \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n_r^l > 0}} d_k^n h^{-1/2} \prod_{i < r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \times K_{n_r^l - 1} (G_r^l) \prod_{j \neq l} K_{n_r^j} (G_r^j). \end{aligned}$$

Proof of Proposition 22. We first compute $\mathbb{E}_t(\mathcal{C}_p^N F)$ for $t \in]t_{r-1}, t_r]$. We have

$$\mathbb{E}_t(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{i < r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \times \mathbb{E}_t \left(\prod_{i \geq r} \prod_{1 \leq j \leq d} K_{n_i^j} (W_i^j) \right)$$

Since Brownian motions and their increments are independents, we get

$$\mathbb{E}_{t_r} \left(\prod_{i \geq r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \right) = \prod_{1 \leq j \leq d} K_{n_r^j} (G_r^j) \prod_{i > r} \prod_{1 \leq j \leq d} \mathbb{E} \left[K_{n_i^j} (G_i^j) \right];$$

which is null as soon as $n_{r+1}^1 + \dots + n_N^1 + \dots + n_{r+1}^d + \dots + n_N^d > 0$. Then, nested conditional expectations give

$$\mathbb{E}_t(F) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i < r} \prod_{1 \leq j \leq d} K_{n_i^j} (G_i^j) \times \mathbb{E}_t \left(\prod_{1 \leq j \leq d} K_{n_r^j} (G_r^j) \right).$$

From Lemma 4, for $j = 1, \dots, d$ $M_t^{n_r^j} := (t - t_{r-1})^{n_r^j/2} K_{n_r^j} \left(\frac{B_t^j - B_{t_{r-1}}^j}{\sqrt{t - t_{r-1}}} \right)$ is a martingale and $dM_t^{n_r^j} = M_t^{n_r^j - 1} \mathbf{1}_{]t_{r-1}, t_r]}(t) dB_t^j$. Then, $\prod_{1 \leq j \leq d} (t - t_{r-1})^{n_r^j/2} K_{n_r^j} \left(\frac{B_t^j - B_{t_{r-1}}^j}{\sqrt{t - t_{r-1}}} \right)$ is also a martingale and the first result follows. Since $K_{n_r^l}'(x) = K_{n_r^l - 1}(x)$, we get the second result. \square

Conclusion. In this paper, we use Wiener chaos expansions together with the Picard procedure to compute the solution to (1). Once computed the chaos decomposition of F^q , we get explicit formulas for both conditional expectations and the Malliavin derivative of conditional expectations. This enable to easily compute (Y^q, Z^q) . Numerically, we obtain fast and accurate results, which encourage us to extend these results to other type of BSDEs, like 2-BSDEs. It is also possible to couple these Wiener chaos expansions together with the dynamic programming approach. This will be the subject of a forthcoming publication.

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