



HAL
open science

A refinement of Günther's candle inequality

Benoit Kloeckner, Greg Kuperberg

► **To cite this version:**

Benoit Kloeckner, Greg Kuperberg. A refinement of Günther's candle inequality. 2012. hal-00688285v1

HAL Id: hal-00688285

<https://hal.science/hal-00688285v1>

Preprint submitted on 17 Apr 2012 (v1), last revised 18 Jul 2013 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A refinement of Günther's candle inequality

Benoît R. Kloeckner*

Institut Fourier, Université de Grenoble I

Greg Kuperberg†

Department of Mathematics, University of California, Davis

Dedicated to our friends Sylvain Gallot and Albert Schwarz

We introduce an upper bound on the curvature of a Riemannian manifold which is in between a sectional curvature bound and a Ricci curvature bound. We prove that this condition implies Günther's inequality on the candle function of a manifold, thus bringing the inequality closer in form to the complementary inequality due to Bishop.

1. INTRODUCTION

Two important relations between curvature and volume in differential geometry are Bishop's inequality [1, §11.10], which is an upper bound on the volume of a ball from a lower bound on Ricci curvature, and Günther's inequality [7], which is a lower bound on volume from an upper bound on *sectional* curvature. Bishop's inequality has a weaker hypothesis than Günther's inequality and can be interpreted as a stronger result. The asymmetry between these inequalities is a counter-intuitive fact of Riemannian geometry.

In this article, we will partially remedy this asymmetry. We will define another curvature statistic, the root-Ricci function, denoted $\sqrt{\text{Ric}}$, that also implies Günther's inequality¹. It is not a tensor because it involves square roots of sectional curvatures, but it shares other properties with Ricci curvature.

To motivate the $\sqrt{\text{Ric}}$ function, consider the geometry of the complex hyperbolic plane $\mathbb{C}\mathbb{H}^2$. In this 4-manifold, the volume of a ball of radius r is

$$\text{Vol}(B(r)) = \frac{\pi^2}{2} \sinh(r)^4 \sim \frac{\pi^2}{32} \exp(4r).$$

The corresponding sphere surface volume has a factor of $\sinh(2r)$ from the unique complex line containing a given geodesic γ , which has curvature -4 , and two factors of $\sinh(r)$ from the totally real planes that contain γ , which have curvature -1 . Günther's inequality and Bishop's inequality yield the estimates

$$\frac{\pi^2}{48} \exp(3\sqrt{2}r) \gtrsim \text{Vol}(B(r)) \gtrsim \frac{\pi^2}{12} \exp(3r).$$

The true volume growth of balls in $\mathbb{C}\mathbb{H}^2$ (and in some other cases, see Section 3.1) is governed by the average of the square roots of the negatives of the sectional curvatures. This is how we define the $\sqrt{\text{Ric}}$ function, for each tangent direction u at each point p in M .

More precisely, let M be a Riemannian n -manifold with sectional curvature $K \leq \rho$ for some constant $\rho \geq 0$. For any unit tangent vector $u \in UT_p M$ with $p \in M$, we define

$$\sqrt{\text{Ric}}(\rho, u) \stackrel{\text{def}}{=} \rho - \left(\frac{1}{n-1} \text{Tr}(\sqrt{\rho - R(\cdot, u, \cdot, u)}) \right)^2.$$

Here $R(u, v, w, x)$ is the Riemann curvature tensor expressed as a tetralinear form, and the square root is the positive square root of a positive semidefinite matrix or operator. The slightly complicated definition is an average of curvatures, analogous to the natural quantity $\text{Ric}/(n-1)$, but conjugated by the transformation $f(x) = \sqrt{\rho - x}$.

The best version of either Günther's or Bishop's inequality is not directly a bound on the volume of balls in M , but rather a bound on the logarithmic derivative of the *candle function* of M . Let $\gamma = \gamma_u$ be a geodesic curve in M that begins at $p = \gamma(0)$ with initial velocity $u \in UT_p M$. Then the candle function $s(\gamma, r)$ is by definition the Jacobian of the map $u \mapsto \gamma_u(r)$. In other words, it is defined by the equations

$$dq = s(\gamma_u, r) du dr \quad q = \gamma_u(r) = \exp_p(ru),$$

where dq is Riemannian measure on M , dr is Lebesgue measure on \mathbb{R} , and du is Riemannian measure on the sphere $UT_p M$. This terminology has the physical interpretation that if an observer is at the point q in M , and if a unit candle is at the point p , then $1/s(\gamma, r)$ is its apparent brightness².

The candle function $s_\kappa(r)$ of a geometry of constant curvature κ is given by

$$s_\kappa(r) = \begin{cases} \left(\frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}} \right)^{n-1} & \kappa > 0 \\ r^{n-1} & \kappa = 0 \\ \left(\frac{\sinh(\sqrt{-\kappa}r)}{\sqrt{-\kappa}} \right)^{n-1} & \kappa < 0 \end{cases}.$$

Theorem 1.1. *Let M be a Riemannian n -manifold with $K \leq \rho$ for some $\rho \geq 0$, and suppose that*

$$\sqrt{\text{Ric}}(\rho, u) \leq \kappa$$

* benoit.kloeckner@ujf-grenoble.fr

† greg@math.ucdavis.edu; Supported by NSF grant CCF #1013079.

¹ We take the "ic" in the Ricci tensor Ric to mean taking a partial trace of the Riemann tensor R , but we take a square root first.

² Certain distant objects in astronomy with known luminosity are called *standard candles* and are used to estimate astronomical distances.

for some $\kappa < \rho$ and for every $u \in UTM$. Then

$$(\log s(\gamma, r))' \geq (\log s_\kappa(r))'$$

for every geodesic γ in M , when $2r\sqrt{\rho} \leq \pi$.

The prime denotes the derivative with respect to r .

When $\rho = 0$, the conclusion of Theorem 1.1 is identical to Günther's inequality for manifolds with $K \leq \kappa$, but the hypothesis is strictly weaker. When $\rho > 0$, the curvature hypothesis is weaker, but the length restriction is stronger. The usual version of the inequality holds up to a distance of $\pi/\sqrt{\kappa}$, i.e., for our distance restriction, we replace κ with ρ and divide by 2.

ACKNOWLEDGMENTS

The authors would like to thank John Hunter and Sylvain Gallot for helpful conversations.

2. RELATIONS BETWEEN CONDITIONS

2.1. Candle conditions

We first mention two interesting properties of the candle function $s(\gamma, r)$:

1. $s(\gamma, r)$ vanishes when $\gamma(0)$ and $\gamma(r)$ are conjugate points.
2. The candle function is symmetric: If $\tilde{\gamma}(t) = \gamma(r-t)$, then $s(\tilde{\gamma}, r) = s(\gamma, r)$.

The second property is not trivial to prove, but it is a folklore fact in differential geometry [17][Lem. 5] (and a standard principle in optics).

Say that a manifold M is $\text{Candle}(\kappa)$ if the inequality

$$s(\gamma, r) \geq s_\kappa(r)$$

holds for all γ, r ; or $\text{LCD}(\kappa)$, for *logarithmic candle derivative*³, if the logarithmic condition

$$(\log s(\gamma, r))' \geq (\log s_\kappa(r))'$$

holds for all γ, r ; or $\text{Ball}(\kappa)$ if the volume inequality

$$\text{Vol}(B(p, r)) \geq \text{Vol}(B_\kappa(r))$$

holds for all p and r ; here B_κ denotes a ball in the simply connected space of constant curvature κ . (If $\kappa > 0$, then the first two conditions are only meaningful up to the distance $\pi/\sqrt{\kappa}$ between conjugate points in the comparison geometry.) We also write $\text{Candle}(\kappa, \ell)$, $\text{LCD}(\kappa, \ell)$, and $\text{Ball}(\kappa, \ell)$ if the same conditions hold up to a distance of $r = \ell$.

The logarithmic derivative $(\log s(\gamma, r))'$ of the candle function has its own important geometric interpretation: it is the mean curvature of the geodesic sphere with radius r and center $p = \gamma(0)$ at the point $\gamma(r)$. So it also equals Δr , where Δ is the Laplace Beltrami operator, and r is the distance from any point to p . So if M is $\text{LCD}(\kappa)$, then we obtain the comparison $\Delta r \geq \Delta_\kappa r$, and the statement that spheres in M are more extrinsically curved than spheres in a space of constant curvature κ .

2.2. Curvature and volume comparisons

If $\kappa \leq \rho = 0$, then we can organize the comparison properties of an n -manifold M that we have mentioned as follows:

$$K \leq \kappa \implies \sqrt{\text{Ric}}(0) \leq \kappa \implies \text{LCD}(\kappa) \implies \text{Candle}(\kappa) \implies \text{Ball}(\kappa, \text{inj}(M)) \quad (1)$$

where $\text{inj}(M)$ is the injectivity radius of M . The first implication is elementary; the second one is Theorem 1.1; and the third and fourth ones are also elementary, given by integrating with respect to length r .

If $\kappa \leq \rho > 0$, then

$$K \leq \kappa \implies \sqrt{\text{Ric}}(\rho) \leq \kappa \implies \text{LCD}(\kappa, \frac{\pi}{2\sqrt{\rho}}) \implies \text{Candle}(\kappa, \frac{\pi}{2\sqrt{\rho}}) \implies \text{Ball}(\kappa, \ell),$$

where

$$\ell = \min(\text{inj}(M), \frac{\pi}{2\sqrt{\rho}}).$$

Finally, for all $\ell > 0$,

$$\text{Candle}(\kappa, \ell) \implies \text{Ric} \leq (n-1)\kappa g,$$

where g is the metric on M , because

$$s(\gamma, r) = r^{n-1} - \text{Ric}(\gamma'(0))r^n + O(r^{n+1}). \quad (2)$$

In particular, in two dimensions, all of the implications in (1) are equivalences.

2.3. Curvature bounds

If $\rho' > \rho$, then $\sqrt{\text{Ric}}(\rho) \geq \sqrt{\text{Ric}}(\rho')$. In addition, the root-Ricci function converges to Ricci curvature for large ρ :

$$\lim_{\rho \rightarrow +\infty} \sqrt{\text{Ric}}(\rho, u) = \frac{1}{n-1} \text{Ric}(u, u) \quad \forall u \in UTM.$$

The corresponding limit $\rho \rightarrow \infty$ in Theorem 1.1 has the interpretation that the upper bound looks more and more like a bound based on Ricci curvature at short distances. This is an optimal limit in the sense that Ricci curvature is the first non-trivial derivative of $s(\gamma, r)$ at $r = 0$ by (2). On the other

³ And not to be confused with liquid crystal displays.

hand, without the length restriction, the limit $\rho \rightarrow \infty$ is impossible. That limit would be exactly Bishop's inequality with Ricci curvature, but such an inequality is not generally true.

Finally we can deduce a root-Ricci upper bound from a combination of sectional curvature and Ricci bounds. The concavity of the square root function implies that given the value of $\text{Ric}(u, u)$, the largest possible value of $\sqrt{\text{Ric}(\rho, u)}$ is achieved when $R(\cdot, u, \cdot, u)$ has one small eigenvalue and all other eigenvalues equal. For all $\kappa \leq \alpha \leq \rho$, we then get a number $\beta = \beta(\kappa, \alpha, \rho)$, decreasing in α , such that

$$K \leq \alpha \text{ and } \text{Ric} \leq \beta g \implies \sqrt{\text{Ric}(\rho)} \leq \kappa. \quad (3)$$

An explicit computation yields the optimal value

$$\beta = \rho + (n-2)\alpha - ((n-1)\sqrt{\rho - \kappa} - (n-2)\sqrt{\rho - \alpha})^2.$$

In particular,

$$\begin{aligned} \beta(\kappa, \kappa, \rho) &= (n-1)^2\kappa - n(n-1)\rho \\ \beta(\kappa, \kappa, \rho) &= (n-1)\kappa. \end{aligned}$$

In order to deduce $\sqrt{\text{Ric}(\rho)} \leq \kappa$ from classical curvature upper bounds, we can therefore ask for the strong condition $K \leq \kappa$ (which implies $\text{Ric} \leq (n-1)\kappa g$), or ask for the weaker $K \leq \rho$ together with $\text{Ric} \leq \beta(\kappa, \rho, \rho)g$, or choose from a continuum of combined bounds on K and Ric . Moreover, the above calculation holds pointwise, so that in (3), α can be a function on UTM instead of a constant.

3. APPLICATIONS

Most of the established applications of Günther's inequality are also applications of Theorem 1.1. The subtlety is that different applications use different criteria in the chain of implications (1). We give some examples. In general, let \tilde{M} denote the universal cover of M .

3.1. Exponential growth of balls

The first application of our result is to estimate the rate of growth of balls, as already given by (1). This is related to the volume entropy of a closed Riemannian manifold M , which is by definition

$$h_{\text{vol}}(M) \stackrel{\text{def}}{=} \lim_{r \rightarrow +\infty} \frac{\log \text{Vol} B_{\tilde{M}}(p, r)}{r}.$$

By abuse of notation, we will use this same volume entropy expression when $M = \tilde{M}$ is simply connected rather than closed. Since a hyperbolic space of curvature $\kappa < 0$ and dimension n has volume entropy $\sqrt{-\kappa}(n-1)$, Theorem 1.1 implies that when $K \leq 0$,

$$h_{\text{vol}}(M) \geq (n-1)\sqrt{-\kappa^+} \quad (4)$$

where

$$\kappa^+ = \sup_u \sqrt{\text{Ric}(0, u)}.$$

The estimate (4) is sharp for every rank one symmetric space. (Recall that the rank one symmetric spaces are the generalized hyperbolic spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$, and $\mathbb{O}H^2$.) The reason is that the operator $R(\cdot, \gamma', \cdot, \gamma')$ is constant along any geodesic γ . So by the Jacobi field equation (Section 4), the volume of $B(p, r)$ has factors of $\sinh \sqrt{\lambda_k} r$ for each eigenvalue λ_k of $R(\cdot, \gamma', \cdot, \gamma')$. So we obtain the estimate

$$\text{Vol} B(p, r) \propto \prod_k (\sinh \sqrt{\lambda_k} r) \sim \exp(\kappa^+).$$

Theorem 1.1 as stated does not yield an optimal estimate of the volume entropy of higher rank, nonpositively curved symmetric spaces. In this case $R(\cdot, \gamma', \cdot, \gamma')$ is still constant, but the growth of balls is governed by the directions in which the root-Ricci curvature is the most negative, and $\sqrt{\text{Ric}(0)}$ is not constant. For example, if $M = \mathbb{R}H^2 \times \mathbb{R}H^2$ and u makes an angle of $0 \leq \theta < \pi/2$ with either factor of $\mathbb{R}H^2$, then

$$\sqrt{\text{Ric}(0, u)} = -\frac{1}{9}(\sin \theta + \cos \theta)^2.$$

In this case, $h_{\text{vol}}(M)$ is determined by the angle $\theta = \pi/4$.

However, we can apply Theorem 1.1 to a cone of geodesic rays γ instead of to an entire ball. Then a bound on $\sqrt{\text{Ric}(0, \gamma'(t))}$ yields the LCD property for $s(\gamma, \cdot)$. We get that if M is a non-positively curved locally symmetric space, then

$$h_{\text{vol}}(M) = (n-1)\sqrt{-\kappa^-}, \quad (5)$$

where

$$\kappa^- = \inf_u \sqrt{\text{Ric}(0, u)}.$$

This estimate can be computed directly, and shows that Theorem 1.1 is sharp in this modified sense.

Manning [10] showed that if M is a closed Riemannian manifold, then the topology entropy $h_{\text{top}}(M)$ of its geodesic flow is at least $h_{\text{vol}}(M)$. He also showed that if M is non-positively curved, then the two entropies are equal. These are theorems about the condition $\text{Ball}(\kappa)$ as applied to \tilde{M} . So, Theorem 1.1 says that when $K \leq 0$ (which is necessary to define $\kappa^+ = \sup \sqrt{\text{Ric}(0)}$),

$$h_{\text{top}}(M) \geq (n-1)\sqrt{-\kappa^+}.$$

Another consequence of the exponential growth of balls in \tilde{M} is that if M is closed, then $\pi_1(M)$ has exponential growth. In particular, if $K \leq \kappa < 0$, then $\pi_1(M)$ has exponential growth. This result was found independently by Milnor [12] and earlier by Schwarz [13]. In his paper, Milnor conjectured that the $K \leq \kappa$ condition can be weakened. In particular, he asked whether it is enough for $\text{Ric} \leq (n-1)\kappa g$. It is not clear whether he meant this hypothesis alone or in conjunction with $K \leq 0$. In the former case, Lohkamp [9] showed that every manifold of dimension $n \geq 3$ has a Ricci-negative metric. In the latter case, Eschenburg and O'Sullivan [5] may have been the first to establish that \tilde{M} has exponential volume growth; see also [2, 14, 16]. Theorem 1.1 is another proof of this fact by (3).

3.2. Isoperimetric inequalities

Yau [3, 17] established that if M is complete, simply connected, and has $K \leq \kappa < 0$, and $D \subseteq M$ is a domain, then D satisfies a linear isoperimetric inequality:

$$\text{Vol}(\partial D) \geq (n-1)\sqrt{-\kappa}\text{Vol}(D).$$

His proof only uses a weakening of condition $\text{LCD}(\kappa)$, namely that

$$(\log s(\gamma, r))' \geq (n-1)\sqrt{-\kappa}.$$

So Theorem 1.1 yields Yau's inequality when $\sqrt{\text{Ric}}(0) \leq \kappa$.

McKean [11] showed that the same weak $\text{LCD}(\kappa)$ condition also implies a spectral gap

$$\lambda_0(\tilde{M}) \geq \frac{-\kappa n^2}{4}$$

for the first eigenvalue of the positive Laplace-Beltrami operator acting on $L^2(M)$. This spectral gap is deduced from a Poincaré inequality that is of interest in and of itself:

$$\int_M f^2 \leq \frac{4}{-\kappa n^2} \int_M |\nabla f|^2$$

for all smooth, compactly supported functions f . McKean stated his result under the hypothesis $K \leq \kappa$; it has been generalized by Setti [14] and Borbély [2] to mixed sectional and Ricci bounds; Theorem 1.1 provides a further generalization. Note in particular that Borbély's result is optimal for complex hyperbolic spaces (and we get the same bound in this case), but we get better bounds for quaternionic and octonionic hyperbolic spaces.

Croke [4] establishes the isoperimetric inequality for a compact non-positively curved 4-manifold M with unique geodesics. In other words, if B is a Euclidean 4-ball with

$$\text{Vol}(M) = \text{Vol}(B),$$

then

$$\text{Vol}(\partial M) \geq \text{Vol}(\partial B).$$

His proof only uses the condition $\text{Candle}(0)$, in fact only for maximal geodesics between boundary points⁴. So, Croke's theorem also holds if

$$\sqrt{\text{Ric}}\left(\left(\frac{\pi}{2L}\right)^2\right) \leq 0,$$

where L is the maximal length of a geodesic; for any given L , this curvature bound is weaker than $K \leq 0$. It is a well-known conjecture that if M is n -dimensional and non-positively curved, then the isoperimetric inequality holds. The conjecture can be attributed to Weil [15], because his proof in

dimension $n = 2$ initiated the subject. More recently, Kleiner [8] established the case $n = 3$. We are led to ask whether Weil's isoperimetric conjecture still holds for $\text{Candle}(0)$ or $\text{LCD}(0)$ manifolds.

In a forthcoming paper, we will partially generalize Croke's result to signed curvature bounds. In this generalization, $\text{Candle}(\kappa)$ and $\text{LCD}(\kappa)$ play an essential role and Theorem 1.1 yields a weakening of the needed curvature assumptions.

4. THE PROOF

In this section, we will prove Theorem 1.1. The basic idea is to analyze the energy functional that arises in a standard proof of Günther's inequality, with the aid of the change of variables $R = A^2 - \rho I$.

Using the Jacobi field model, Theorem 1.1 is really a result about linear ordinary differential equations. The normal bundle to the geodesic $\gamma(t)$ can be identified with \mathbb{R}^{n-1} using parallel transport. Then an orthogonal vector field $y(t)$ along γ is a Jacobi field if it satisfies the differential equation

$$y'' = -R(t)y, \quad (6)$$

where

$$R(t) = R(\cdot, u(t), \cdot, u(t))$$

is the sectional curvature matrix and $u(t) = \gamma'(t)$ is the unit tangent to γ at time t . By the first Bianchi identity, $R(t)$ is a symmetric matrix. The candle function $s(r) = s(\gamma, r)$ is determined by a matrix solution

$$Y'' = -R(t)Y \quad Y(0) = 0 \quad (7)$$

by the formula

$$s(r) = \frac{\det Y(r)}{\det Y'(0)}.$$

Its logarithmic derivative is given by

$$(\log s(r))' = \frac{s'(r)}{s(r)} = \frac{(\det Y)'(r)}{\det Y(r)}.$$

All invertible solutions $Y(r)$ to (7) are equivalent by right multiplication by a constant matrix, and yield the same value for $s(r)$ and its derivative. In particular, if we let $Y(r) = I$, then the logarithmic derivative simplifies to

$$(\log s(r))' = \text{Tr}(Y'(r)).$$

Following a standard proof of Günther's inequality [6][Thm. 3.101], we define an energy functional whose minimum, remarkably, both enforces (7) and minimizes the objective $(\log s(r))'$. Namely, we assume Dirichlet boundary conditions

$$y(0) = 0 \quad y(r) = v,$$

⁴ We credit [4] as our original motivation for this article.

and we let

$$E(R, y) = \int_0^r (\langle y', y' \rangle - \langle y, Ry \rangle) dt.$$

By a standard argument from calculus of variations, the critical points of $E(R, y)$ are exactly the solutions to (6) with the given boundary conditions.

We can repeat the same calculation with the matrix solution

$$Y(0) = 0 \quad Y(r) = I,$$

with the analogous energy

$$E(R, Y) = \int_0^r (\langle Y', Y' \rangle - \langle Y, RY \rangle) dt.$$

Here the inner product of two matrices is the Hilbert-Schmidt inner product

$$\langle A, B \rangle = \text{Tr}(A^T B).$$

Moreover, if Y is a solution to (7), then $E(R, Y)$ simplifies to $(\log s(r))'$ by integration by parts:

$$\begin{aligned} E(R, Y) &= \int_0^r (\langle Y', Y' \rangle - \langle Y, RY \rangle) dt \\ &= \langle Y(r), Y'(r) \rangle - \langle Y(0), Y'(0) \rangle - \int_0^r \langle Y, Y'' + RY \rangle dt \\ &= \langle I, Y'(r) \rangle - 0 - 0 = \text{Tr}(Y'(r)) = (\log s(r))'. \end{aligned}$$

Thus, our goal is to minimize $E(R, Y)$ with respect to both Y and R . We want to minimize with respect to Y in order to solve (7). Then for that Y , we want to minimize with respect to R to prove Theorem 1.1.

The following proposition tells us that (6) or (7) has a unique solution with Dirichlet boundary conditions, and that it is an energy minimum. Here and below, recall the matrix notation $A \leq B$ (which was already used for Ricci curvature in the introduction) to express the statement that $B - A$ is positive semidefinite.

Proposition 4.1. *If $R \leq \rho I$, and if y is continuous with an L^2 derivative, then $E(R, y)$ is a positive definite quadratic function of y when $\sqrt{\rho}r < \pi$, with the Dirichlet boundary conditions $y(0) = y(r) = 0$.*

Proof. Let

$$E(\rho, y) = E(\rho I, y) = \int_0^r (\langle y', y' \rangle - \rho \langle y, y \rangle) dt$$

be the corresponding energy of the comparison case with constant curvature ρ . (Recall that the ultimate comparison is with constant curvature κ , but to get started we use ρ instead.) Then

$$E(\rho, y) \leq E(R, y),$$

so it suffices to show that $E(\rho, y)$ is positive definite. When $\rho = 0$, $E(\rho, y)$ is manifestly positive definite. Otherwise $E(\rho, y)$ is diagonalized in the basis of functions

$$y_k(t) = \sin\left(\frac{\pi kt}{r}\right)$$

with $k \geq 1$. A direct calculation yields

$$E(\rho, y_k) = \frac{\pi^2 k^2 - r^2 \rho}{r} > 0,$$

as desired. \square

Remark. There is also a geometric reason that the comparison case $E(\rho, y)$ is positive definite: When $\rho = 0$, a straight line segment in Euclidean space is a minimizing geodesic; when $\rho > 0$, the same is true of a geodesic arc of length $r < \pi/\sqrt{\rho}$ on a sphere with curvature $\sqrt{\rho}$. We give a direct calculation to stay in the spirit of ODEs.

Proposition 4.2. *Let ρ and $r < \pi/\sqrt{\rho}$ be fixed and suppose that $R \leq \rho I$. Then $s(r)$ and $(\log s(r))'$ are both bounded below.*

Proof. We will simply prove the usual Günther inequality for As in the proof of Proposition 4.1,

$$E(R, Y) \geq E(\rho, Y)$$

for all R and Y with $Y(0) = 0$ and $Y(r) = I$. For each fixed R , the minimum of the left side is $(\log s(r))'$. The minimum of the right side (which may occur for a different Y , but no matter) is $(\log s_\rho(r))'$, which is a positive number. We obtain the same conclusion for $s(r)$ by integration. \square

Proposition 4.3. *Assume the hypotheses of Proposition 4.2. If R is L^∞ , then the solution Y to (7) is bounded uniformly, i.e., with a bound that depends only on $\|R\|$ (and r and ρ). Also Y' is uniformly bounded and Lipschitz, and Y'' is uniformly bounded and L^∞ .*

Proof. In this proposition and nowhere else, it is more convenient to assume the initial conditions

$$\hat{Y}(0) = 0 \quad \hat{Y}'(0) = I$$

rather than Dirichlet boundary conditions. The fact that \hat{Y} and its derivatives are uniformly bounded, with these initial conditions, is exactly Grönwall's inequality. To convert back to Dirichlet boundary conditions, we want to instead bound

$$Y(t) = \hat{Y}(t)\hat{Y}(r)^{-1}.$$

This follows from Proposition 4.2 by the formula

$$\hat{Y}(r)^{-1} = \text{adj}(\hat{Y}(r)) \det(\hat{Y}(r))^{-1},$$

where adj denotes the adjugate of a matrix.

Finally, $Y''(t)$ is L^∞ and uniformly bounded because $Y(t)$ satisfies (7). Also $Y'(0) = \hat{Y}'(0)\hat{Y}(r)^{-1}$ is uniformly bounded, so we can integrate to conclude that $Y'(t)$ is uniformly bounded and Lipschitz. \square

To prove Theorem 1.1, we want to minimize $(\log s(r))'$ or $E(R, Y)$ over all R such that

$$R \leq \rho I \quad \text{Tr}(\sqrt{\rho I - R}) \geq \alpha \stackrel{\text{def}}{=} \sqrt{\rho - \kappa}. \quad (8)$$

To better understand this minimization problem, we make a change of variables. Let $A(t)$ be a symmetric matrix such that

$$R(t) = \rho I - A(t)^2 \quad \text{Tr}(A(t)) \geq \alpha.$$

In order to know that every $R(t)$ is realized, we can let

$$A = \sqrt{\rho I - R}$$

be the positive square root of $\rho I - R$. Even if A is not positive semidefinite, $R(t)$ still satisfies (8). This simplifies the optimization problem: in the new variable A , the semidefinite hypothesis can be waived.

Now the energy function becomes:

$$\begin{aligned} E(A, Y) &= \int_0^r (\langle Y', Y' \rangle - \langle Y, (\rho - A^2)Y \rangle) dt \\ &= \int_0^r (\text{Tr}((Y')^T Y') + \text{Tr}(Y^T A^2 Y) - \rho \text{Tr}(Y^T Y)) dt. \end{aligned}$$

For the moment, fix Y and let $Z = YY^T$. Then as a function of A ,

$$E(A) = \int_0^r \text{Tr}(A^2 YY^T) dt + \text{constant}.$$

Since YY^T is symmetric and strictly positive definite, E is a positive-definite quadratic function of A , and we can directly solve for the minimum as

$$A = \frac{\alpha(YY^T)^{-1}}{\text{Tr}((YY^T)^{-1})}. \quad (9)$$

Even though we waived the assumption that A is positive semidefinite, minimization restores it as a conclusion. Moreover,

$$\text{Tr}(A) = \text{Tr}(\sqrt{\rho I - R}) = \alpha. \quad (10)$$

Proposition 4.4. *With the hypotheses (8), and if $r < \pi/\sqrt{\rho}$, a minimum of $(\log s(r))'$ exists. Equivalently, a joint minimum of $E(A, Y)$ or $E(R, Y)$ exists.*

Proof. The above calculation lets us assume (10), which means that R is uniformly bounded. By Proposition 4.3, so is Y'' . We can restrict to a set of pairs (R, Y'') of class L^∞ , which is compact in the weak-* topology by the Banach-Alaoglu theorem. Equivalently, we can restrict to a uniformly bounded, uniformly Lipschitz set of pairs $(\int R, Y')$, which is compact in the uniform topology by the Arzela-Ascoli theorem. By integration by parts, we can write

$$\begin{aligned} E(R, Y) &= \int_0^r (\langle Y', Y' \rangle - \langle Y, RY \rangle) dt \\ &= [\langle Y, (\int R)Y \rangle]_0^r + \int_0^r (\langle Y', Y' \rangle dt + 2\langle Y', (\int R)Y \rangle) dt. \end{aligned}$$

Thus the energy is continuous as a function of $\int R$ and Y' and has a minimum on a compact family. \square

Proposition 4.4 reduces Theorem 1.1 to the equations (7) and (9) combine to make a non-linear matrix ODE:

$$\begin{aligned} Y'' &= (A^2 - \rho)Y & A &= \frac{\alpha(YY^T)^{-1}}{\text{Tr}((YY^T)^{-1})} \\ Y(0) &= 0 & Y(r) &= I. \end{aligned}$$

Proposition 4.4 tells us that this ODE has at least one solution; we will proceed by finding all solutions with the given boundary conditions. First, if we suppress the boundary condition $Y(r) = I$, the solutions $Y(t)$ are invariant under both left and right multiplication by $O(n-1)$. So we can write

$$Y(t) = U\hat{Y}(t)V,$$

where $\hat{Y}'(0)$ is diagonal with positive entries. In this case $\hat{A}(0)$ is also diagonal, and we obtain that $\hat{Y}(t)$ is diagonal for all t , and with positive entries because the entries cannot cross 0. Therefore $UV = I$, because the identity is the only diagonal orthogonal matrix with positive entries.

So we can assume that $Y = \hat{Y}$, with diagonal entries

$$\lambda_1(t), \lambda_2(t), \dots, \lambda_{n-1}(t) > 0.$$

Each of these entries satisfies the same scalar ODE,

$$w'' = \beta(t)w^{-1} - \rho w \quad w(0) = 0 \quad w(r) = 1, \quad (11)$$

where

$$\beta(t) = \frac{\alpha}{\text{Tr}((Y(t)Y(t)^T)^{-1})^2}.$$

We claim that if $w > 0$, then $w' > 0$ as well. If $\rho = 0$, then this is immediate. Otherwise, a positive solution $w(t)$ satisfies

$$w(t) > \frac{\sin(\sqrt{\rho}t)}{\sin(\sqrt{\rho}r)} \quad w'(t) > \frac{\sqrt{\rho} \cos(\sqrt{\rho}t)}{\sin(\sqrt{\rho}r)},$$

because the right side is the solution to $w'' = -\rho w$ with the same boundary conditions. So we obtain that $w' > 0$ provided that

$$r < \frac{\pi}{2\sqrt{\rho}}.$$

(This is where we need half of the distance allowed in the usual form of Günther's inequality.)

To complete the proof, consider the phase diagram in the strip $[0, 1] \times (0, \infty)$ of the positive solutions $(w(t), w'(t))$ to (11). If we let $x = w(t)$, then the total elapsed time to reach $x = 1$ is

$$r = \int_0^1 \frac{dt}{dx} dx = \int_0^1 \frac{dx}{w'(w^{-1}(x))},$$

which is a positive integral. On the other hand, if w_1 and w_2 are two distinct solutions with

$$w_1(0) = w_2(0) = 0 \quad w_1'(0) > w_2'(0),$$

then the solutions cannot intersect in the phase diagram; we must have

$$w'_1(w_1^{-1}(x)) > w'_2(w_2^{-1}(x)) > 0.$$

So two distinct, positive solutions to (11) cannot reach $w(t) = 1$ at the same time, which means with given the boundary conditions that there is only one solution. Thus, the diagonal entries $\lambda_k(t)$ of $Y(t)$ are all equal. In conclusion, Y , A , and R all are isotropic at the minimum of the logarithmic candle derivative $(\log s(r))'$.

-
- [1] Richard L. Bishop and Richard J. Crittenden, *Geometry of manifolds*, Pure and Applied Mathematics, vol. XV, Academic Press, 1964.
- [2] Albert Borbély, *On the spectrum of the Laplacian in negatively curved manifolds*, *Studia Sci. Math. Hungar.* **30** (1995), no. 3-4, 375–378.
- [3] Dmitri Burago and Victor A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 285, Springer-Verlag, 1988, Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics.
- [4] Christopher B. Croke, *A sharp four-dimensional isoperimetric inequality*, *Comment. Math. Helv.* **59** (1984), no. 2, 187–192.
- [5] Jost-Hinrich Eschenburg and John J. O'Sullivan, *Jacobi tensors and Ricci curvature*, *Math. Ann.* **252** (1980), no. 1, 1–26.
- [6] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, second ed., Universitext, Springer-Verlag, 1990.
- [7] Paul Günther, *Einige Sätze über das Volumenelement eines Riemannschen Raumes*, *Publ. Math. Debrecen* **7** (1960), 78–93.
- [8] Bruce Kleiner, *An isoperimetric comparison theorem*, *Invent. Math.* **108** (1992), no. 1, 37–47.
- [9] Joachim Lohkamp, *Metrics of negative Ricci curvature*, *Ann. of Math. (2)* **140** (1994), no. 3, 655–683.
- [10] Anthony Manning, *Topological entropy for geodesic flows*, *Ann. of Math. (2)* **110** (1979), no. 3, 567–573.
- [11] Henry P. McKean, *An upper bound to the spectrum of Δ on a manifold of negative curvature*, *J. Differential Geometry* **4** (1970), 359–366.
- [12] John Milnor, *A note on curvature and fundamental group*, *J. Differential Geometry* **2** (1968), 1–7.
- [13] Albert Schwarz, *A volume invariant of coverings*, *Dokl. Akad. Nauk SSSR (N.S.)* **105** (1955), 32–34.
- [14] Alberto G. Setti, *A lower bound for the spectrum of the Laplacian in terms of sectional and Ricci curvature*, *Proc. Amer. Math. Soc.* **112** (1991), no. 1, 277–282.
- [15] André Weil, *Sur les surfaces a courbure negative*, *C. R. Acad. Sci. Paris* **182** (1926), 1069–1071.
- [16] Yi-Hu Yang, *On the growth of fundamental groups of nonpositive curvature manifolds*, *Bull. Austral. Math. Soc.* **54** (1996), no. 3, 483–487.
- [17] Shing-Tung Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, *Ann. Sci. École Norm. Sup. (4)* **8** (1975), no. 4, 487–507.