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CONCENTRATION BOUNDS FOR STOCHASTIC APPROXIMATIONS

N. FRIKHA¹ AND S. MENOZZI²

Abstract. We obtain non asymptotic concentration bounds for two kinds of stochastic approximations. We first consider the deviations between the expectation of a given function of the Euler scheme of some diffusion process at a fixed deterministic time and its empirical mean obtained by the Monte-Carlo procedure. We then give some estimates concerning the deviation between the value at a given time-step of a stochastic approximation algorithm and its target. Under suitable assumptions both concentration bounds turn out to be Gaussian. The key tool consists in exploiting accurately the concentration properties of the increments of the schemes. For the first case, as opposed to the previous work [LM10], we do not have any systematic bias in our estimates. Also, no specific non-degeneracy conditions are assumed.

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1. STATEMENT OF THE PROBLEM

Let us consider a d -dimensional stochastic evolution scheme of the form

$$\xi_{n+1} = \xi_n + \gamma_{n+1}F(n, \xi_n, Y_{n+1}), \quad n \geq 0, \xi_0 = x \in \mathbf{R}^d, \quad (1.1)$$

where the $(Y_i)_{i \in \mathbf{N}^*}$ are i.i.d. \mathbf{R}^q -valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ whose law μ satisfies a Logarithmic Sobolev Inequality with constant $\alpha > 0$ (in short μ satisfies $LSI(\alpha)$), $(\gamma_n)_{n \geq 1}$ is a deterministic positive sequence of time steps and $F : \mathbf{N} \times \mathbf{R}^d \times \mathbf{R}^q \rightarrow \mathbf{R}^d$ is a measurable function satisfying some assumptions that will be specified later on.

We are interested in giving non asymptotic concentration bounds for two specific problems related to evolutions of type (1.1). Precisely, we want to control the deviations of the empirical mean associated to a function of the Euler scheme of a diffusion process at a fixed deterministic time from the real mean and to give deviation estimates between the value of a Robbins-Monro type stochastic algorithm taken at fixed time-step and its target. Concerning stochastic algorithms, our deviation results are to our best knowledge the first of this nature.

Keywords and phrases: Non asymptotic bounds, Euler scheme, Stochastic approximation algorithms, Gaussian concentration

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1.1. Euler Scheme of a Diffusion Process

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions and $(W_t)_{t \geq 0}$ be a q -dimensional $(\mathcal{F}_t)_{t \geq 0}$ Brownian motion. Let us consider a d -dimensional diffusion process $(X_t)_{t \geq 0}$ with dynamics:

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad (1.2)$$

where the coefficients b, σ are assumed to be uniformly Lipschitz continuous in space and measurable in time.

For a given Lipschitz continuous function f and a fixed deterministic time horizon T , quantities like $\mathbf{E}_x[f(X_T)]$ appear in many applications. In mathematical finance, it represents the price of a European option with maturity T when the dynamics of the underlying asset is given by (1.2). Under suitable assumptions on the function f and the coefficients b, σ , namely smoothness or non degeneracy, it can also be related to the Feynman-Kac representation of the heat equation associated to the generator of X . Two steps are needed to approximate $\mathbf{E}_x[f(X_T)]$:

- The first one consists in approximating the dynamics by a discretization scheme that can be simulated. We will consider in this work the Euler scheme with time step $\Delta = T/N$, $N \in \mathbf{N}^*$. That is the process with dynamics:

$$X_t^\Delta = x + \int_0^t b(\phi(s), X_{\phi(s)}^\Delta) ds + \int_0^t \sigma(\phi(s), X_{\phi(s)}^\Delta) dW_s, \quad (1.3)$$

where we introduced for all $s \geq 0$ the notation $\phi(s) := \sup\{t_i := i\Delta : t_i \leq s < t_{i+1}\}$ so that the dynamics of $(X_t)_{t \geq 0}$ and $(X_t^\Delta)_{t \geq 0}$ have a similar form.

- The second one consists in approximating the expectation involving the Euler scheme, $\mathbf{E}_x[f(X_T^\Delta)]$ by a Monte-Carlo estimator:

$$E_M^\Delta(x, T, f) := \frac{1}{M} \sum_{i=1}^M f((X_T^{\Delta, 0, x})^i),$$

where the $((X_T^{\Delta, 0, x})^i)_{i \in [1, M]}$ are independent copies of the Euler scheme (1.3) starting at x at time 0 and evaluated at time T .

The global error between $\mathbf{E}_x[f(X_T)]$, the quantity to estimate, and its implementable approximation $E_M^\Delta(x, T, f)$ can be decomposed as follows:

$$\begin{aligned} \mathcal{E}(\Delta, M, x, T, f) &:= (\mathbf{E}_x[f(X_T)] - \mathbf{E}_x[f(X_T^\Delta)]) + (\mathbf{E}_x[f(X_T^\Delta)] - E_M^\Delta(x, T, f)) \\ &:= \mathcal{E}_D(\Delta, x, T, f) + \mathcal{E}_S(\Delta, M, x, T, f). \end{aligned} \quad (1.4)$$

The term $\mathcal{E}_D(\Delta, x, T, f)$ corresponds to the *discretization error* and has been widely investigated in the literature since the seminal work of Talay and Tubaro [TT90]. This contribution usually yields an error of order Δ , provided the coefficients b, σ and the function f are sufficiently smooth, [TT90], or that they satisfy some non-degeneracy assumptions. For a bounded measurable f , when the diffusion coefficient is uniformly elliptic and bounded, if b, σ are also assumed to be three times continuously differentiable, the control at order Δ for $\mathcal{E}_D(\Delta, x, T, f)$ can be derived from Konakov and Mammen [KM02] (see also Bally and Talay [BT96] for an extension of this result to the hypoelliptic setting).

The term $\mathcal{E}_S(\Delta, M, x, T, f)$ corresponds to the *statistical error*. Under some usual integrability conditions, i.e. $f(X_T^\Delta) \in L^2(\mathbf{P})$, it is *asymptotically* controlled by the central limit theorem. A first non-asymptotic result is given by the Berry-Essen theorem provided $f(X_T^\Delta) \in L^3(\mathbf{P})$, but for practical purposes, the crucial quantity to control non-asymptotically is the deviation between the empirical mean $E_M^\Delta(x, T, f)$ and the real one $\mathbf{E}_x[f(X_T^\Delta)]$. Precisely, for a *fixed* M and a given threshold $r > 0$, one would like to give bounds on the quantity $\mathbf{P}[|E_M^\Delta(x, T, f) - \mathbf{E}_x[f(X_T^\Delta)]| > r]$.

In the ergodic framework and for a constant diffusion coefficient such controls have been obtained by Malrieu and Talay [MT06]. In the current context, a first attempt can be found in [LM10] where such a control is derived

up to a systematic bias independent of M which is due to the fact that we use the Aronson Gaussian bounds for the density of the Euler scheme. Also, to derive those density bounds some non-degeneracy conditions were needed.

Introduce the following assumption **(A)**:

(A.1) The coefficients $b : \mathbf{R}^+ \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ and $\sigma : \mathbf{R}^+ \times \mathbf{R}^d \rightarrow \mathbf{R}^d \otimes \mathbf{R}^q$ are bounded and Lipschitz continuous in space.

(A.2) For $q = d$, the diffusion coefficient is uniformly elliptic, i.e. there exists $\Lambda \in (0, 1]$ s.t. for all $(t, x, \xi) \in \mathbf{R}^+ \times (\mathbf{R}^d)^2$, $\langle a(t, x)\xi, \xi \rangle \geq \Lambda|\xi|^2$.

We show in [LM10] that there exist $c := c(\mathbf{A}) \in (0, 1]$, $C := C(\mathbf{A}, T) \geq 1$ s.t. denoting by $p^\Delta(T, x, \cdot)$ the density of the Euler scheme:

$$\frac{C^{-1}}{T^{d/2}} \exp\left(-c^{-1} \frac{|x-y|^2}{T}\right) \leq p^\Delta(T, x, y) \leq \frac{C}{T^{d/2}} \exp\left(-c \frac{|x-y|^2}{T}\right). \quad (1.5)$$

The main problem is that we have to handle a constant of the form C^M with $C \geq 1$ when tensorizing the density estimates in order to control the difference $E_M^\Delta(x, T, f) - \mathbf{E}_x[f(X_T^\Delta)]$. This is what yields to an estimate of the form $\mathbf{P}[E_M^\Delta(x, T, f) - \mathbf{E}_x[f(X_T^\Delta)] > r + \delta] \leq \exp(-\tilde{c}Mr^2/T)$, $\tilde{c} := \tilde{c}(\mathbf{A})$, with a systematic bias $\delta > 0$.

In this work, under the sole assumption **(A.1)**, we actually manage to remove the previous bias, see Theorem 2.1. The key tool consists in exploiting recursively that the increments of the Euler scheme (1.3) are Gaussian and that as a process the Euler scheme enjoys a semi-Markov property, or equivalently that taken at the discretization times it defines a Markov chain. We also need to control the Lipschitz constants in space of the function $v^\Delta(t_i, x) := \mathbf{E}[f(X_T^\Delta) | X_{t_i}^\Delta = x]$, $(i, x) \in \llbracket 0, N \rrbracket \times \mathbf{R}^d$. This is done using standard stability arguments and yields an additional exponential dependence in T for the concentration bound, i.e. with the above notation $\tilde{c} := \tilde{c}(T, |\sigma|_\infty, [f]_1, [b]_1, [\sigma]_1)$ where \tilde{c} depends exponentially on T and for a Lipschitz continuous function φ , $[\varphi]_1$ stands for its Lipschitz modulus of continuity.

1.2. Robbins-Monro Stochastic Approximation Algorithm

Besides our considerations for the Euler scheme, we derive non asymptotic bounds for stochastic approximation algorithms, namely recursive schemes defined by

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1}), \quad n \geq 0, \quad \theta_0 \in \mathbf{R}^d, \quad (1.6)$$

where $H : \mathbf{R}^d \times \mathbf{R}^q \rightarrow \mathbf{R}^d$ is a Borel function, $(Y_n)_{n \geq 1}$ is an i.i.d. \mathbf{R}^q -valued sequence of \mathcal{F}_n -adapted random variables with law μ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ and $(\gamma_n)_{n \geq 1}$ is a sequence of non-negative deterministic steps satisfying the usual assumption

$$\sum_{n \geq 1} \gamma_n = +\infty, \quad \text{and} \quad \sum_{n \geq 1} \gamma_n^2 < +\infty. \quad (1.7)$$

In the sequel, we suppose that the mean function $h(\cdot) = \mathbb{E}[H(\cdot, Y)]$ is continuous and the sequence defined by (1.6) *a.s.* converges towards its unique target θ^* , that is $\{h = 0\} = \{\theta^*\}$. We refer to [Duf96], [KY03] for some general convergence results for (1.6) under the existence of a so-called *Lyapunov function*; see also [LP12] for a convergence theorem under the existence of a *pathwise Lyapunov function*.

In the following, we assume the following conditions on the function H and the step sequence $(\gamma_n)_{n \geq 1}$ in (1.6) are in force:

(HL) The map $(\theta, y) \in \mathbf{R}^d \times \mathbf{R}^q \mapsto H(\theta, y)$ is uniformly Lipschitz continuous.

(HUA) The map $h : \theta \in \mathbf{R}^d \mapsto \mathbf{E}[H(\theta, Y)]$ is continuously differentiable in θ and there exists $\bar{\alpha} > 0$, $\underline{\alpha} > 0$ s.t. $\forall \theta \in \mathbf{R}^d, \forall \xi \in \mathbf{R}^d, \underline{\alpha}|\xi|^2 \leq \langle Dh(\theta)\xi, \xi \rangle \leq \bar{\alpha}|\xi|^2$ (*Uniform Attractivity*).

Note that under **(HUA)** and the *linear growth assumption*

$$\forall \theta \in \mathbf{R}^d, \quad \mathbf{E} \left[|H(\theta, Y)|^2 \right] \leq C(1 + |\theta - \theta^*|^2),$$

(which is satisfied if **(HL)** holds and $Y \in L^2(\mathbb{P})$) the function $L : \theta \mapsto \frac{1}{2} |\theta - \theta^*|^2$ is a Lyapunov function for the recursive procedure defined by (1.6) so that one easily deduces that $\theta_n \rightarrow \theta^*$, *a.s.* as $n \rightarrow +\infty$.

2. MAIN RESULTS

2.1. Deviations on the Euler Scheme

Theorem 2.1 (Concentration Bounds for the Euler scheme). *Let f be a uniformly Lipschitz continuous function and the coefficients b, σ of the diffusion (1.2) be also uniformly Lipschitz continuous in space uniformly in time. Assume as well that σ is bounded. Denote by X_T^Δ the value at time T of the Euler scheme (1.3) associated to the diffusion (1.2). For all $M \in \mathbf{N}^*$ and all $r \geq 0$, one has*

$$\mathbf{P}_x \left[\left| \frac{1}{M} \sum_{i=1}^M f((X_T^\Delta)^i) - \mathbf{E}_x[f(X_T^\Delta)] \right| \geq r \right] \leq 2 \exp\left(-\frac{r^2 M}{T \Psi(T, f, b, \sigma, q)}\right),$$

$$\Psi(T, f, b, \sigma, q) := 8[f]_1^2 |\sigma|_\infty^2 \exp(2([b]_1 + c[\sigma]_1(1 \vee c[\sigma]_1))T),$$

where q is the dimension of the underlying Brownian motion in (1.2) and (1.3).

Note that in the above theorem, we do not need any non-degeneracy condition on the diffusion coefficient. To handle the previous quantity we are going to rewrite $D := f(X_T^\Delta) - \mathbf{E}_x[f(X_T^\Delta)] := \sum_{i=1}^N \mathbf{E}[f(X_T^\Delta)|\mathcal{F}_{t_i}] - \mathbf{E}[f(X_T^\Delta)|\mathcal{F}_{t_{i-1}}]$. The main idea then consists in controlling the local increments that can be interpreted as a difference between a function of the current Brownian increment and its mean. That is $D := \sum_{i=1}^N f_i^\Delta(X_{t_{i-1}}^\Delta, W_{t_i} - W_{t_{i-1}}) - \mathbf{E}[f_i^\Delta(X_{t_{i-1}}^\Delta, W_{t_i} - W_{t_{i-1}})|\mathcal{F}_{t_{i-1}}]$, with $f_i^\Delta(x, w) := \mathbf{E}[f(X_T^\Delta)|X_{t_i}^\Delta = x + b(t_{i-1}, x)\Delta + \sigma(t_{i-1}, x)w]$, for all $(i, x, w) \in \llbracket 1, N \rrbracket \times \mathbf{R}^d \times \mathbf{R}^q$. If at some point along the time-discretization the process has a degenerate diffusion term, we can see that the term $f_i^\Delta(X_{t_{i-1}}^\Delta, W_{t_i} - W_{t_{i-1}}) - \mathbf{E}[f_i^\Delta(X_{t_{i-1}}^\Delta, W_{t_i} - W_{t_{i-1}})|\mathcal{F}_{t_{i-1}}]$ will not give any additional contribution in the global deviation.

With respect to the previous work [LM10], we got rid off the systematic bias. Anyhow, the concentration constants now depend on the Lipschitz constant of the function $v^\Delta(0, x) := \mathbf{E}[f(X_T^\Delta)|X_0^\Delta = x]$ which has order $\Psi(T, f, b, \sigma, q)^{1/2}$. This magnitude corresponds to the product of the Lipschitz constant of the final function f and the mean of the Lipschitz constant for the flow of the Euler scheme, which gives the exponential dependence in time, see Proposition 4.1 and its proof for details.

2.2. Deviations for Robbins-Monro algorithms

Theorem 2.2 (Concentration Bounds for Robbins-Monro algorithms). *Assume that the function H of the recursive procedure $(\theta_n)_{n \geq 0}$ (with starting point $\theta_0 \in \mathbf{R}^d$) defined by (1.6) satisfies **(HL)** and **(HUA)**, and that the step sequence $(\gamma_n)_{n \geq 1}$ satisfies (1.7). Suppose that the law μ of the innovation satisfies $LSI(\alpha)$, $\alpha > 0$. Then, for all $N \in \mathbf{N}^*$ and all $r \geq 0$,*

$$\mathbf{P}(|\theta_N - \theta^*| \geq r + \delta_N) \leq \exp\left(-\frac{r^2}{\alpha[H]_1^2 \Pi_N \sum_{k=1}^N \gamma_k^2 / \Pi_k}\right)$$

where $\Pi_N := \prod_{k=0}^{N-1} (1 - 2\underline{\alpha}\gamma_{k+1} + [H]_1^2 \gamma_{k+1}^2)$ and $\delta_N := \mathbf{E}[|\theta_N - \theta^*|]$. Moreover, the bias δ_N at step N satisfies

$$\delta_N \leq \exp(-\underline{\alpha}\Gamma_N) |\theta_0 - \theta^*| + [H]_1 \sigma_Y \left(\gamma_N N^{1/2} + \sum_{k=1}^{N-1} e^{-\alpha(\Gamma_N - \Gamma_{k+1})} (\gamma_k - \gamma_{k+1}) + \sum_{k=1}^{N-1} e^{-\alpha(\Gamma_N - \Gamma_{k+1})} \gamma_k \gamma_{k+1} \bar{\alpha} \right),$$

where $\Gamma_N := \sum_{k=1}^N \gamma_k$, $\sigma_Y := \mathbf{E} [F^2(Y)]^{1/2} < +\infty$, with $F : y \mapsto \mathbf{E} [|y - Y|]$.

Concerning the choice of the step sequence $(\gamma_n)_{n \geq 1}$ and its impact on the concentration rate and bias, we obtain the following results:

- If we choose $\gamma_n = \frac{c}{n}$, with $c > 0$. Then $\delta_N \rightarrow 0$, $N \rightarrow +\infty$, $\Gamma_N = c \log(N) + c'_1 + r_N$, $c'_1 > 0$ and $r_N \rightarrow 0$, so that $\Pi_N = \mathcal{O}(N^{-2c\alpha})$.
 - If $c < \frac{1}{2\alpha}$, the series $\sum_{k=1}^N \gamma_k^2 / \Pi_k$ converges so that we obtain $\Pi_N \sum_{k=1}^N \gamma_k^2 / \Pi_k = \mathcal{O}(N^{-2c\alpha})$.
 - If $c > \frac{1}{2\alpha}$, a comparison between the series and the integral yields $\Pi_N \sum_{k=1}^N \gamma_k^2 / \Pi_k = \mathcal{O}(N^{-1})$.

Let us notice that we find the same critical level for the constant c as in the Central Limit Theorem for stochastic algorithms. Indeed, if $c > \frac{1}{2\mathcal{R}e(\lambda_{min})}$ where λ_{min} denotes the eigenvalue of $Dh(\theta^*)$ with the smallest real part then we know that a Central Limit Theorem holds for $(\theta_n)_{n \geq 1}$ (see e.g. [Duf96]). However, this local condition on the Jacobian matrix of h at the equilibrium is replaced by a uniform assumption in our framework. This is quite natural since we want to derive non-asymptotic bounds for the stochastic approximation (1.6).

Concerning the bias we have the following bound:

$$\delta_N \leq K \left(\frac{|\theta_0 - \theta^*|}{N^{\alpha c}} + \frac{[H]_1 \sigma_Y (1 + \bar{\alpha})}{N} \right) + [H]_1 \sigma_Y \frac{c}{N^{1/2}}, \quad K := K(c).$$

- If we choose $\gamma_n = \frac{c}{n^\rho}$, $c > 0$, $\frac{1}{2} < \rho < 1$, then $\delta_N \rightarrow 0$, $\Gamma_N \sim \frac{c}{1-\rho} N^{1-\rho}$ as $N \rightarrow +\infty$ and elementary computations show that there exists $C > 0$ s.t. for all $N \geq 1$, $\Pi_N \leq C \exp(-2\underline{\alpha} \frac{c}{1-\rho} N^{1-\rho})$. Hence, for all $\epsilon \in (0, 1 - \rho)$ we have:

$$\begin{aligned} \Pi_N \sum_{k=1}^N \gamma_k^2 \Pi_k^{-1} &\leq c^2 \left\{ \Pi_N \Pi_{N-N^{\rho+\epsilon}}^{-1} \sum_{k=1}^{N-N^{\rho+\epsilon}} \frac{1}{k^{2\rho}} + \sum_{k=N-N^{\rho+\epsilon}+1}^N \frac{1}{k^{2\rho}} \right\} \\ &\leq c^2 \left\{ C \exp(-2\underline{\alpha} \frac{c}{1-\rho} (N^{1-\rho} - (N - N^{\rho+\epsilon})^{1-\rho})) + \frac{N^{\rho+\epsilon}}{(N - N^{\rho+\epsilon} + 1)^{2\rho}} \right\} \\ &\leq c^2 \left\{ C \exp(-2\underline{\alpha} c N^\epsilon) + \frac{1}{N^{\rho-\epsilon}} \right\}. \end{aligned}$$

Up to a modification of ϵ , this yields $\Pi_N \sum_{k=1}^N \gamma_k^2 \Pi_k^{-1} = o(N^{-\rho+\epsilon})$, $\epsilon \in (0, 1 - \rho)$.

Concerning the bias, from Kronecker's Lemma, we have the following bound:

$$\delta_N \leq K \exp\left(-\frac{\underline{\alpha} c}{1-\rho} N^{1-\rho}\right) |\theta_0 - \theta^*| + [H]_1 \sigma_Y o(1) \left(\frac{1}{N^{\rho-\epsilon}} + \bar{\alpha} \frac{1}{N^{2\rho-\epsilon-1}} \right) + [H]_1 \sigma_Y \frac{c}{N^{\rho-1/2}}, \quad K := K(c), \quad \forall \epsilon > 0.$$

Since each step is bigger compared to the case $\gamma_n = \frac{c}{n}$, the impact of the initial difference $|\theta_0 - \theta^*|$ is exponentially small.

3. ABSTRACT CONCENTRATION PROPERTIES FOR A GENERAL EVOLUTION SCHEME

In this section we assume that $(Y_i)_{i \in \mathbf{N}^*}$ is a sequence of i.i.d. \mathbf{R}^q -valued random variables whose law μ satisfies $LSI(\alpha)$. We recall that μ satisfies $LSI(\alpha)$, $\alpha > 0$ if for all $f \in H^1(d\mu) := \{g \in L^2(d\mu) : \int |\nabla g|^2 d\mu < +\infty\}$, we have

$$\text{Ent}_\mu(f^2) \leq \alpha \int |\nabla f|^2 d\mu, \quad (3.1)$$

where for all $\varphi \geq 0$, $\text{Ent}_\mu(\varphi) := \int \varphi \log(\varphi) d\mu - \int \varphi d\mu \log(\int \varphi d\mu)$. Also, for all $i \in \mathbf{N}^*$, we denote by $\mathcal{F}_i := \sigma(Y_j, j \leq i)$ the natural σ -field associated to the $(Y_j)_{j \in \llbracket 1, i \rrbracket}$.

We first recall a basic property of measures satisfying a logarithmic Sobolev inequality, that is they enjoy Gaussian concentration.

Proposition 3.1 (Gaussian concentration for a measure satisfying $LSI(\alpha)$). *Let μ be a measure on \mathbf{R}^q satisfying $LSI(\alpha)$, $\alpha > 0$, and let f be a real valued 1-Lipschitz function on \mathbf{R}^q then,*

$$\forall \lambda \geq 0, \mathbf{E}_\mu[\exp(\lambda f)] \leq \exp(\lambda \mathbf{E}_\mu(f) + \frac{\alpha \lambda^2}{4}), \forall r \geq 0, \mu(f \geq r + \mathbf{E}_\mu(f)) \leq \exp(-r^2/\alpha). \quad (3.2)$$

The above Proposition is standard. It is usually proved using the so-called *Herbst argument*. For the sake of completeness we provide a proof in Appendix A.1. We refer to Ledoux [Led99] for additional related properties.

Proposition 3.2 (Gaussian concentration for a stochastic evolution scheme). *Fix $N \in \mathbf{N}^*$. Let $a \in \mathbf{R}$ be a given scalar. Define $\mathcal{D}_{i+1} := f_i(X_i, Y_{i+1}) - \mathbf{E}[f_i(X_i, Y_{i+1}) | \mathcal{F}_i]$ for some \mathcal{F}_i -measurable random variables X_i where the real-valued functions $(f_i)_{i \in \llbracket 1, N \rrbracket}$ are Lipschitz continuous in the y variable with constants $([f_i]_1)_{i \in \llbracket 1, N \rrbracket} > 0$ uniformly in x . Let $(\gamma_i)_{i \in \llbracket 1, N \rrbracket}$ be a given sequence of time steps. For all $r \geq a$, we have:*

$$\mathbf{P}[a + \sum_{i=1}^N \gamma_i \mathcal{D}_i \geq r] \leq \exp\left(-\frac{(r-a)^2}{\alpha \sum_{i=1}^N ([f_i]_1 \gamma_i)^2}\right).$$

Proof. Set $\mathcal{P}(r) := \mathbf{P}[a + \sum_{i=1}^N \gamma_i \mathcal{D}_i \geq r]$. We first do an exponential Tchebychev inequality. For $\lambda \geq 0$ to be specified later on, we get:

$$\begin{aligned} \mathcal{P}(r) &\leq \exp(-\lambda(r-a)) \mathbf{E}\left[\exp\left(\lambda \left[\sum_{i=1}^N \gamma_i \mathcal{D}_i\right]\right)\right] \\ &\leq \exp(-\lambda(r-a)) \mathbf{E}\left[\exp\left(\lambda \sum_{i=1}^{N-1} \gamma_i \mathcal{D}_i\right) \mathbf{E}[\exp(\lambda \gamma_N \mathcal{D}_N) | \mathcal{F}_{N-1}]\right]. \end{aligned} \quad (3.3)$$

Observe now that working with regular conditional expectations, we have

$$\begin{aligned} &\mathbf{E}[\exp(\lambda \gamma_N \mathcal{D}_N) | \mathcal{F}_{N-1}] \\ &= \mathbf{E}_\mu[\exp(\lambda \gamma_N (f_{N-1}(x, Y) - \mathbf{E}_\mu[f_{N-1}(x, Y)])) |_{x=X_{N-1}}, \end{aligned}$$

where Y is a random variable with law μ . From Proposition 3.1, we derive

$$\mathbf{E}[\exp(\lambda \gamma_N \mathcal{D}_N) | \mathcal{F}_{N-1}] \leq \exp(\alpha ([f_N]_1 \gamma_N \lambda)^2 / 4).$$

Plugging this estimate in (3.3) and iterating the procedure we derive

$$\mathcal{P}(r) \leq \exp(-\lambda(r-a)) \exp\left(\frac{\alpha \lambda^2}{4} \sum_{i=1}^N ([f_i]_1 \gamma_i)^2\right),$$

and optimizing w.r.t λ , we obtain: $\mathcal{P}(r) \leq \exp\left(-\frac{(r-a)^2}{\alpha \sum_{i=1}^N ([f_i]_1 \gamma_i)^2}\right)$.

4. EULER SCHEME: PROOF OF THE MAIN RESULTS

We consider in this section the case of the Euler scheme of a non-degenerate diffusion process introduced in (1.3). Observe that the Euler scheme (1.3) enters the framework of equation (1.1) setting for all $i \in \mathbf{N}^*$, $\gamma_i = \Delta$

and for all $(i, x, y) \in \llbracket 1, N \rrbracket \times \mathbf{R}^d \times \mathbf{R}^q$, $F(i, x, y) = b(t_i, x) + \gamma_{i+1}^{-1/2} \sigma(t_i, x)y$. We suppose Assumption **(A.1)** of Section 1.1 is in force.

In the following we denote by $X_T^{\Delta, t_i, x}$ the value at time T of the Euler scheme starting from $x \in \mathbf{R}^d$ at time $t_i := i\Delta$, $i \in \llbracket 1, N \rrbracket$. Thus $\mathbf{P}_x[f(X_T^\Delta) - \mathbf{E}_x[f(X_T^\Delta)] \geq r] = \mathbf{P}[f(X_T^{\Delta, 0, x}) - \mathbf{E}[f(X_T^{\Delta, 0, x})] \geq r]$. Let us now rewrite the deviation $f(X_T^{\Delta, 0, x}) - \mathbf{E}[f(X_T^{\Delta, 0, x})]$ in order to apply Proposition 3.2. Recalling that for $t \in [0, T]$, $\mathcal{F}_t := \sigma(W_s, s \leq t)$ stands for the filtration of the Brownian motion, we get:

$$\begin{aligned} f(X_T^{\Delta, 0, x}) - \mathbf{E}[f(X_T^{\Delta, 0, x})] &:= \sum_{i=1}^N \mathbf{E}[f(X_T^{\Delta, 0, x}) | \mathcal{F}_{t_i}] - \mathbf{E}[f(X_T^{\Delta, 0, x}) | \mathcal{F}_{t_{i-1}}] \\ &:= \sum_{i=1}^N \mathbf{E}[f(X_T^{\Delta, 0, x}) | X_{t_i}^{\Delta, 0, x}] - \mathbf{E}[\mathbf{E}[f(X_T^{\Delta, 0, x}) | X_{t_i}^{\Delta, 0, x}] | \mathcal{F}_{t_{i-1}}], \end{aligned} \quad (4.1)$$

using the Markov property for the last equality. Set now for all $(i, x) \in \llbracket 0, N-1 \rrbracket \times \mathbf{R}^d$, $v^\Delta(t_i, x) = \mathbf{E}[f(X_T^{\Delta, t_i, x})]$ and $v^\Delta(T, x) = f(x)$. Equation (4.1) rewrites:

$$\begin{aligned} f(X_T^{\Delta, 0, x}) - \mathbf{E}[f(X_T^{\Delta, 0, x})] &= \sum_{i=1}^N v^\Delta(t_i, X_{t_i}^{\Delta, 0, x}) - \mathbf{E}[v^\Delta(t_i, X_{t_i}^{\Delta, 0, x}) | \mathcal{F}_{t_{i-1}}] \\ &= \sum_{i=1}^N \left\{ v^\Delta(t_i, X_{t_i}^{\Delta, 0, x} + b(t_{i-1}, X_{t_{i-1}}^{\Delta, 0, x})\Delta + \sigma(t_{i-1}, X_{t_{i-1}}^{\Delta, 0, x})(W_{t_i} - W_{t_{i-1}})) \right. \\ &\quad \left. - \mathbf{E}[v^\Delta(t_i, X_{t_i}^{\Delta, 0, x} + b(t_{i-1}, X_{t_{i-1}}^{\Delta, 0, x})\Delta + \sigma(t_{i-1}, X_{t_{i-1}}^{\Delta, 0, x})(W_{t_i} - W_{t_{i-1}})) | \mathcal{F}_{t_{i-1}}] \right\} \\ &:= \sum_{i=1}^N f_i^\Delta(X_{t_{i-1}}^{\Delta, 0, x}, W_{t_i} - W_{t_{i-1}}) - \mathbf{E}[f_i^\Delta(X_{t_{i-1}}^{\Delta, 0, x}, W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_{t_{i-1}}] \end{aligned}$$

where

$$\forall (i, x, y) \in \llbracket 1, N \rrbracket \times \mathbf{R}^d \times \mathbf{R}^q, f_i^\Delta(x, y) := v^\Delta(t_i, x + b(t_{i-1}, x)\Delta + \sigma(t_{i-1}, x)y). \quad (4.2)$$

Remark 4.1. *The above procedure is in some sense similar to what is usually done to analyze the weak discretization error, see [TT90], [BT96]. The main difference is that we do not introduce a telescopic sum involving the underlying PDE but the conditional expectations with functions of the Euler scheme (whereas for the discretization error one considers conditional expectations involving functions of the diffusion that starts from the current value of the Euler scheme). In order to proceed, all we need is to have a control on the Lipschitz modulus in the variable y of the functions $f_i^\Delta(x, y)$ defined in (4.2), uniformly in x .*

Under the current assumptions we have the following Proposition which is proved in Section A.2.

Proposition 4.1 (Control of the Lipschitz constants). *Assume the coefficients b, σ in (1.3) are uniformly Lipschitz continuous in space (uniformly in time) and denote their respective Lipschitz constants by $[b]_1, [\sigma]_1$. We also suppose that σ is bounded and denote its supremum by $|\sigma|_\infty$. Then for all $i \in \llbracket 1, N \rrbracket$, the functions f_i^Δ introduced in (4.2) are uniformly Lipschitz continuous in the space variable y uniformly in x and we have that there exists $c := c(q)$ (dimension of the underlying Brownian motion) s.t:*

$$[f_i^\Delta]_1 := \sup_{x \in \mathbf{R}^d, y \neq y'} \frac{|f_i^\Delta(x, y) - f_i^\Delta(x, y')|}{|y - y'|} \leq 2[f]_1 |\sigma|_\infty \exp(\{[b]_1 + c[\sigma]_1(1 \vee c[\sigma]_1)\}(T - t_i)).$$

where $[f]_1$ stands for the Lipschitz constants of the function f .

Hence, we are now in position to apply Proposition 3.2. Taking $a = 0$, $\gamma_i = \Delta$, $\mathcal{D}_i = f_i^\Delta(X_{t_{i-1}}^\Delta, W_{t_i} - W_{t_{i-1}}) - \mathbf{E}[f_i^\Delta(X_{t_{i-1}}^\Delta, W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_{t_{i-1}}]$, $\forall i \in \llbracket 1, N \rrbracket$, recalling that the random variable $W_{t_i} - W_{t_{i-1}}$ satisfies a $LSI(2/\Delta)$, and using Proposition 4.1 we obtain:

$$\begin{aligned} \mathbf{P}_x[f(X_T^\Delta) - \mathbf{E}_x[f(X_T^\Delta)] \geq r] &\leq \exp\left(-\frac{r^2}{\frac{2}{\Delta} \sum_{i=1}^N [f_i^\Delta]_1^2 \Delta^2}\right) \leq \exp\left(-\frac{r^2}{8T[f]_1^2 |\sigma|_\infty^2 \exp(2([b]_1 + c[\sigma]_1)(1 \vee c[\sigma]_1))T)}\right) \\ &:= \exp\left(-\frac{r^2}{T\Psi(T, f, b, \sigma, q)}\right). \end{aligned} \quad (4.3)$$

To derive the deviation bound between the empirical mean and the real one, the key idea is now to apply for fixed M (total number of Monte-Carlo sample paths) the previous arguments to the real valued function $F_M : \mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_M) \in \mathbf{R}^{Md} \mapsto F_M(\mathbf{x}) = \frac{1}{\sqrt{M}} \sum_{i=1}^M f(\mathbf{x}_i)$. Denoting by $|\cdot|$ the Euclidean norm of \mathbf{R}^{Md} , the Cauchy-Schwarz inequality directly gives that $[F_M]_1 := \sup_{\mathbf{x} \neq \mathbf{x}'}, \frac{\frac{1}{\sqrt{M}} \sum_{i=1}^M |f(\mathbf{x}_i) - f(\mathbf{x}'_i)|}{|\mathbf{x} - \mathbf{x}'|} \leq [f]_1$. Setting for all $(i, \mathbf{x}, \mathbf{y}) \in \llbracket 1, N \rrbracket \times (\mathbf{R}^{Md})^2$, $F_{M,i}^\Delta(\mathbf{x}, \mathbf{y}) := \frac{1}{\sqrt{M}} \sum_{j=1}^M v^\Delta(t_i, \mathbf{x}_j + b(t_{i-1}, \mathbf{x}_j)\Delta + \sigma(t_{i-1}, \mathbf{x}_j)\mathbf{y}_j)$, one derives from Proposition 4.1 that $\forall i \in \llbracket 1, N \rrbracket$, $[F_{M,i}^\Delta]_1 \leq [f_i^\Delta]_1$. Observe now that for all $r \geq 0$,

$$\begin{aligned} \mathcal{P}_M(r, x, T, f) &:= \mathbf{P}\left[\frac{1}{M} \sum_{i=1}^M f((X_T^{\Delta,0,x})^i) - \mathbf{E}_x[f(X_T^{\Delta,0,x})] \geq r\right] \\ &= \mathbf{P}\left[\frac{1}{\sqrt{M}} \sum_{i=1}^M (f((X_T^{\Delta,0,x})^i) - \mathbf{E}[f((X_T^{\Delta,0,x})^i)]) \geq r\sqrt{M}\right] \\ &= \mathbf{P}[F_M(\mathbf{X}_T^\Delta) - \mathbf{E}[F_M(\mathbf{X}_T^{\Delta,0,x})] \geq r\sqrt{M}], \end{aligned}$$

where $\mathbf{X}_T^{\Delta,0,x} := ((X_T^{\Delta,0,x})^1, \dots, (X_T^{\Delta,0,x})^M)$. Thus,

$$\begin{aligned} \mathcal{P}_M(r, x, T, f) &= \mathbf{P}\left[\sum_{i=1}^N F_{M,i}^\Delta(\mathbf{X}_{t_{i-1}}^\Delta, \mathbf{W}_{t_i} - \mathbf{W}_{t_{i-1}}) - \mathbf{E}[F_{M,i}^\Delta(\mathbf{X}_{t_{i-1}}^\Delta, \mathbf{W}_{t_i} - \mathbf{W}_{t_{i-1}}) | \mathcal{F}_{t_{i-1}}] \geq r\sqrt{M}\right], \\ \mathbf{W}_{t_i} - \mathbf{W}_{t_{i-1}} &:= (W_{t_i}^1 - W_{t_{i-1}}^1, \dots, W_{t_i}^M - W_{t_{i-1}}^M), \end{aligned}$$

where the $((W_t^i)_{t \geq 0})_{i \in \llbracket 1, M \rrbracket}$ are independent d -dimensional Brownian motions. Recalling that the random variable $\mathbf{W}_{t_i} - \mathbf{W}_{t_{i-1}}$ satisfies a $LSI(2/\Delta)$ (the logarithmic Sobolev inequality tensorizes) and that for all $i \in \llbracket 1, N \rrbracket$, $[F_{M,i}^\Delta]_1 \leq [f_i^\Delta]_1$, we derive from Proposition 3.2 (see also (4.3)) that

$$\mathcal{P}_M(r, x, T, f) := \mathbf{P}\left[\frac{1}{M} \sum_{i=1}^M f((X_T^{\Delta,0,x})^i) - \mathbf{E}_x[f(X_T^{\Delta,0,x})] \geq r\right] \leq \exp\left(-\frac{Mr^2}{T\Psi(T, f, b, \sigma, q)}\right),$$

which proves the Theorem.

5. ROBBINS-MORO ALGORITHM: PROOF OF THE MAIN RESULTS

Set first for all $n \in \mathbf{N}$, $z_n := \theta_n - \theta^*$. As in the previous section, we want to control the deviations of $|z_n|$. We use the same strategy introducing a telescopic sum of conditional expectations. For all $n \in \mathbf{N}^*$, write

$$\begin{aligned} |z_n| - \mathbf{E}[|z_n|] &= \sum_{i=1}^n \mathbf{E}[|z_n| | \mathcal{F}_i] - \mathbf{E}[|z_n| | \mathcal{F}_{i-1}] = \sum_{i=1}^n v_i(\theta_i) - \mathbf{E}[v_i(\theta_i) | \mathcal{F}_{i-1}], \\ &= \sum_{i=1}^n f_i^\gamma(\theta_{i-1}, Y_i) - \mathbf{E}[f_i^\gamma(\theta_{i-1}, Y_i) | \mathcal{F}_{i-1}], \end{aligned}$$

where we used the Markov property for the second equality and we introduced the notations $v_i(\theta) := \mathbf{E}[|\theta_n - \theta^*| | \theta_i = \theta]$, $\forall (i, \theta) \in \llbracket 1, n \rrbracket \times \mathbf{R}^d$, $f_i^\gamma(\theta, y) = v_i(\theta - \gamma_i H(\theta, y))$.

Now we have the following control concerning the Lipschitz constant of f_i^γ .

Proposition 5.1 (Controls of the Lipschitz constants). *For all $i \in \llbracket 1, n \rrbracket$, the function f_i^γ satisfies:*

$$[f_i^\gamma]_1 := \sup_{\theta \in \mathbf{R}^d, y \neq y'} \frac{|f_i^\gamma(\theta, y) - f_i^\gamma(\theta, y')|}{|y - y'|} \leq (\Pi_n \Pi_i^{-1})^{1/2} \gamma_i [H]_1.$$

where $\Pi_n := \prod_{k=0}^{n-1} (1 - 2\underline{\alpha}\gamma_{k+1} + [H]_1^2 \gamma_{k+1}^2)$, $n \geq 1$.

The proof is postponed to Section A.3.

Hence, we are now in position to apply Proposition 3.2 to derive the non-asymptotic bound for stochastic approximation algorithms. Taking $a = \delta_N$, $\mathcal{D}_i = f_i^\gamma(\theta_{i-1}, Y_i) - \mathbf{E}[f_i^\gamma(\theta_{i-1}, Y_i) | \mathcal{F}_{i-1}]$, recalling that the random variable Y satisfies a $LSI(\alpha)$. We obtain for all $r \geq 0$:

$$\mathbf{P}(|\theta_N - \theta^*| \geq r + \delta_N) \leq \exp\left(-\frac{r^2}{\alpha [H]_1^2 \Pi_N \sum_{k=1}^N \gamma_k^2 / \Pi_k}\right).$$

Contrary to the result concerning the Euler scheme, a bias appears in the non-asymptotic bound for the stochastic approximation algorithm. Consequently, it is crucial to have a control on it. At step n of the algorithm, it is equal to $\delta_n := \mathbf{E}[|\theta_n - \theta^*|]$. Under the current assumptions **(HL)** of *Lipschitz continuity* of H and **(HUA)** of *uniform attractivity*, we have the following proposition.

Proposition 5.2 (Control of the bias). *For all $n \in \mathbf{N}$, we have*

$$\delta_n \leq \exp(-\underline{\alpha}\Gamma_n) |\theta_0 - \theta^*| + [H]_1 \sigma_Y \left(\gamma_n n^{1/2} + \sum_{k=1}^{n-1} e^{-\underline{\alpha}(\Gamma_n - \Gamma_{k+1})} (\gamma_k - \gamma_{k+1}) + \sum_{k=1}^{n-1} e^{-\underline{\alpha}(\Gamma_n - \Gamma_{k+1})} \gamma_k \gamma_{k+1} \bar{\alpha} \right),$$

where $\Gamma_n := \sum_{k=1}^n \gamma_k$, $\sigma_Y := \mathbf{E}[F^2(Y)]^{1/2} < +\infty$, with $F : y \mapsto \mathbf{E}[|y - Y|]$.

Proof. With the notations of Section 1.2, we define for all $n \geq 1$, $\Delta M_n := h(\theta_{n-1}) - H(\theta_{n-1}, Y_n) = \mathbf{E}[H(\theta_{n-1}, Y_n) | \mathcal{F}_{n-1}] - H(\theta_{n-1}, Y_n)$. Recalling that $(Y_i)_{i \in \mathbf{N}^*}$ is a sequence of i.i.d. random variables we have that $S_n := \sum_{i=1}^n \Delta M_i$ is a martingale w.r.t. the natural filtration $\mathcal{F}_n := \sigma(Y_i, i \leq n)$.

From the dynamics (1.6), write now for all $n \in \mathbf{N}$,

$$\begin{aligned} z_{n+1} &:= \theta_{n+1} - \theta^* = \theta_n - \theta^* - \gamma_{n+1} \{h(\theta_n) - \Delta M_{n+1}\} \\ &= \theta_n - \theta^* - \gamma_{n+1} \int_0^1 d\lambda D h(\theta^* + \lambda(\theta_n - \theta^*)) (\theta_n - \theta^*) + \gamma_{n+1} \Delta M_{n+1}, \end{aligned}$$

where we used that $h(\theta^*) = 0$ for the last equality. Setting $J_n := \int_0^1 d\lambda Dh(\theta^* + \lambda(\theta_n - \theta^*))$, we derive inductively that

$$z_{n+1} = (I - \gamma_{n+1}J_n)z_n + \gamma_{n+1}\Delta M_{n+1} = \prod_{k=1}^{n+1} (I - \gamma_k J_{k-1})z_0 + \sum_{k=1}^{n+1} \gamma_k \left\{ \prod_{p=k+1}^{n+1} (I - \gamma_p J_{p-1}) \right\} \Delta M_k.$$

Hence, setting for all $n \in \mathbf{N}^*$, $\tilde{\Pi}_n \tilde{\Pi}_k^{-1} := \prod_{p=k+1}^n (I - \gamma_p J_{p-1})$, with the classic convention $\prod_{\emptyset} = 1$ and using an Abel transform, we have:

$$z_n = \tilde{\Pi}_n z_0 + \sum_{k=1}^n \gamma_k \tilde{\Pi}_n \tilde{\Pi}_k^{-1} \Delta M_k = \tilde{\Pi}_n z_0 + \gamma_n S_n - \sum_{k=1}^{n-1} \Delta \tilde{\gamma}_{k+1} S_k, \quad (5.1)$$

where $\tilde{\gamma}_k := \gamma_k \Pi_n \Pi_k^{-1}$ and $\Delta \tilde{\gamma}_{k+1} := \tilde{\gamma}_{k+1} - \tilde{\gamma}_k = (\gamma_{k+1} - \gamma_k) \Pi_n \Pi_{k+1}^{-1} + \gamma_k (\Pi_n \Pi_{k+1}^{-1} - \Pi_n \Pi_k^{-1})$.

Now, exploiting assumption **(HUA)** (uniform attractivity of the Jacobian matrix of h), and taking the sequence of steps s.t. $\underline{\alpha} \sup_{n \geq 1} \gamma_n < 1$, we obtain:

$$\|\Pi_n \Pi_k^{-1}\| \leq \prod_{p=k+1}^n (1 - \gamma_p \underline{\alpha}) \leq \exp(-\underline{\alpha}(\Gamma_n - \Gamma_k)), \quad \Gamma_n := \sum_{i=1}^n \gamma_i,$$

where $\|\cdot\|$ stands for the matrix norm $\mathbf{R}^d \otimes \mathbf{R}^d$. Finally, we obtain from (5.1),

$$\mathbf{E}[|z_n|] \leq e^{-\underline{\alpha}\Gamma_n} |z_0| + \gamma_n \mathbf{E}[|S_n|] + \sum_{k=1}^{n-1} e^{-\underline{\alpha}(\Gamma_n - \Gamma_{k+1})} |\Delta \tilde{\gamma}_{k+1}| \mathbf{E}[|S_k|] + \sum_{k=1}^{n-1} e^{-\underline{\alpha}(\Gamma_n - \Gamma_{k+1})} \gamma_k \gamma_{k+1} \mathbf{E}[|S_k|] \bar{\alpha}.$$

Now we inspect the behavior of $\mathbf{E}[|S_n|]$ itself. Using the Cauchy-Schwarz inequality yields

$$\begin{aligned} \mathbf{E}[|S_n|] &\leq \mathbf{E}[|S_n|^2]^{1/2} = \mathbf{E}\left[\left|\sum_{i=1}^n \Delta M_i\right|^2\right]^{1/2} = \mathbf{E}\left[\sum_{i=1}^n \left\{ |\Delta M_i|^2 + 2 \sum_{j=i+1}^n \langle \Delta M_i, \Delta M_j \rangle \right\}\right]^{1/2} \\ &= \left(\sum_{i=1}^n \mathbf{E}[|\Delta M_i|^2]\right)^{1/2} = \left(\sum_{k=1}^n \mathbf{E}[|H(\theta_{k-1}, Y_k) - h(\theta_{k-1})|^2]\right)^{1/2}. \end{aligned}$$

Assumption **(HL)** implies $\forall \theta \in \mathbf{R}^d, \forall y \in \mathbf{R}^q, |H(\theta, y) - h(\theta)| = |\mathbf{E}[H(\theta, y) - H(\theta, Y)]| \leq [H]_1 F(y)$. We thus obtain

$$\mathbf{E}[|S_n|] \leq n^{1/2} [H]_1 \mathbf{E}[F^2(Y)]^{1/2} = [H]_1 n^{1/2} \sigma_Y,$$

which completes the proof.

APPENDIX A. TECHNICAL RESULTS

A.1. Herbst argument

In this Section, we prove Proposition 3.1. Recall we have assumed that μ satisfies $LSI(\alpha)$, $\alpha > 0$. Let f be a smooth and bounded function such that $|\nabla f| \leq 1$. Define the function $g^2 := \exp(\lambda f)$, $\lambda \in \mathbf{R}$. First, it is clear that

$$\mathbf{E}_\mu[|\nabla g|^2] = \frac{\lambda^2}{4} \mathbf{E}_\mu[|\nabla f|^2 e^{\lambda f}] \leq \frac{\lambda^2}{4} \mathbf{E}_\mu[\exp(\lambda f)].$$

Define $H(\lambda) := \mathbf{E}_\mu[\exp(\lambda f)]$, $\lambda \in \mathbf{R}$, applying (3.1) to g , we obtain

$$\lambda H'(\lambda) - H(\lambda) \log H(\lambda) \leq \alpha \frac{\lambda^2}{4} H(\lambda),$$

so that the function $K(\lambda) := \frac{1}{\lambda} \log H(\lambda)$ s.t. $K(0) = \frac{H'(0)}{H(0)} = \mathbf{E}_\mu[f]$ satisfies for every $\lambda \in \mathbf{R}$,

$$K'(\lambda) \leq \frac{\alpha}{4}$$

and $K(\lambda) \leq K(0) + \frac{\alpha}{4} \lambda = \mathbf{E}_\mu[f] + \frac{\alpha}{4} \lambda$ which yields

$$H(\lambda) = \mathbf{E}_\mu[\exp(\lambda f)] \leq \exp(\lambda \mathbf{E}_\mu[f] + \frac{\alpha}{4} \lambda^2).$$

Now, one may replace f by a smooth convolution in order to extend the previous inequality to all Lipschitz functions with Lipschitz constant less than 1. Using the Tchebychev exponential inequality, one has for every $\lambda \geq 0$ and $r \geq 0$

$$\mu(f \geq \mathbf{E}_\mu[f] + r) \leq \exp(-\lambda r + \alpha \lambda^2 / 4)$$

and optimizing in $\lambda \geq 0$ (r being non-negative), we obtain

$$\mu(f \geq \mathbf{E}_\mu[f] + r) \leq \exp(-r^2 / \alpha).$$

A.2. Proof of Proposition 4.1

The proof follows from usual stochastic analysis arguments that we now recall for the sake of completeness. For $i = N$ we directly get from the definition of $f_N^\Delta(x, y)$ that $[f_N^\Delta]_1 \leq [f]_1 |\sigma|_\infty$.

For $i \in \llbracket 1, N-1 \rrbracket$, let us first observe that for $y \neq y'$ the quantity

$$D_{t_i}^\Delta(T, x, y, y') := \sup_{s \in [t_i, T]} \frac{|X_s^{\Delta, t_i, G_{i-1}^\Delta(x, y)} - X_s^{\Delta, t_i, G_{i-1}^\Delta(x, y')}|}{|y - y'|}, \quad \forall z \in \mathbf{R}^d, \quad G_{i-1}^\Delta(x, z) := b(t_{i-1}, x)\Delta + \sigma(t_{i-1}, x)z$$

belongs to $L^1(\mathbf{P})$ (because both quantities $\mathbf{E}[|X_T^{\Delta, t_i, G_{i-1}^\Delta(x, y)}|]$, $\mathbf{E}[|X_T^{\Delta, t_i, G_{i-1}^\Delta(x, y')}|]$ are finite, see e.g. [BT96]). Write now:

$$\begin{aligned} D_{t_i}^\Delta(T, x, y, y') &\leq \left\{ \frac{|G_{i-1}^\Delta(x, y) - G_{i-1}^\Delta(x, y')|}{|y - y'|} + [b]_1 \left(\int_{t_i}^T du \frac{|X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y)} - X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y')}|}{|y - y'|} \right) \right. \\ &\quad \left. + \sup_{s \in [t_i, T]} \left| \int_{t_i}^s \left(\frac{\sigma(\phi(u), X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y)}) - \sigma(\phi(u), X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y')})}{|y - y'|} \right) dW_u \right| \right\}. \end{aligned}$$

Taking the expectation we derive:

$$\begin{aligned}
\mathbf{E}[D_{t_i}^\Delta(T, x, y, y')] &\leq \left\{ |\sigma|_\infty + [b]_1 \int_{t_i}^T du \mathbf{E} \left[\frac{|X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y)} - X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y')}|}{|y - y'|} \right] \right. \\
&\quad \left. + \mathbf{E} \left[\sup_{s \in [t_i, T]} \left| \int_{t_i}^s \frac{\sigma(\phi(u), X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y)}) - \sigma(\phi(u), X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y')})}{|y - y'|} dW_u \right| \right] \right\} \\
&\leq |\sigma|_\infty + [b]_1 \int_{t_i}^T du \mathbf{E}[(D_{t_i}^\Delta(u, x, y, y'))^2] \\
&\quad + c[\sigma]_1 \mathbf{E} \left[\left(\int_{t_i}^T du \frac{|X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y)} - X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y')}|^2}{|y - y'|^2} \right)^{1/2} \right], \quad c := c(q), \quad (\text{A.2})
\end{aligned}$$

using the the Burkholder-Davis-Gundy inequality for the last inequality. Observe now that

$$\begin{aligned}
\mathbf{E} \left[\left(\int_{t_i}^T du \frac{|X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y)} - X_{\phi(u)}^{\Delta, t_i, G_{i-1}^\Delta(x, y')}|^2}{|y - y'|^2} \right)^{1/2} \right] &\leq \mathbf{E} \left[D_{t_i}^\Delta(T, x, y, y')^{1/2} \left(\int_{t_i}^T du D_{t_i}^\Delta(u, x, y, y') \right)^{1/2} \right] \\
&\leq \eta \mathbf{E}[D_{t_i}^\Delta(T, x, y, y')] + \eta^{-1} \int_{t_i}^T du \mathbf{E}[D_{t_i}^\Delta(u, x, y, y')], \quad \forall \eta \in (0, 1),
\end{aligned}$$

which plugged into (A.2) yields thanks to the Gronwall Lemma

$$(1 - c[\sigma]_1 \eta) \mathbf{E}[D_{t_i}^\Delta(T, x, y, y')] \leq |\sigma|_\infty \exp \left(\{ [b]_1 + c[\sigma]_1 \eta^{-1} \} (T - t_i) \right), \quad \eta \in \left(0, \frac{1}{c[\sigma]_1} \wedge 1 \right).$$

Taking $\eta := \frac{(c[\sigma]_1)^{-1} \wedge 1}{2}$ we obtain

$$\mathbf{E}[D_{t_i}^\Delta(T, x, y, y')] \leq 2|\sigma|_\infty \exp \left(\{ [b]_1 + 2c[\sigma]_1 (1 \vee c[\sigma]_1) \} (T - t_i) \right),$$

which recalling $[f_i^\Delta]_1 \leq [f]_1 \sup_{y \neq y'} \mathbf{E}[|D_{t_i}^\Delta(T, x, y, y')|]$ completes the proof.

A.3. Proof of Proposition 5.1

From the definitions in Section 5, it suffices to control the difference $\mathbf{E}[|\theta_n^{\theta, i} - \theta_n^{\theta', i}|]$, that is the sensitivity of the algorithm w.r.t. the starting point at time $i \in \llbracket 1, n \rrbracket$. Write for all $j \in \llbracket i, n-1 \rrbracket$:

$$\begin{aligned}
|\theta_{j+1}^{\theta, i} - \theta_{j+1}^{\theta', i}|^2 &= |\theta_j^{\theta, i} - \theta_j^{\theta', i} - \gamma_{j+1} \{ H(\theta_j^{\theta, i}, Y_{j+1}) - H(\theta_j^{\theta', i}, Y_{j+1}) \}|^2 \\
&= |\theta_j^{\theta, i} - \theta_j^{\theta', i}|^2 - 2\gamma_{j+1} \langle \theta_j^{\theta, i} - \theta_j^{\theta', i}, H(\theta_j^{\theta, i}, Y_{j+1}) - H(\theta_j^{\theta', i}, Y_{j+1}) \rangle \\
&\quad + \gamma_{j+1}^2 |H(\theta_j^{\theta, i}, Y_{j+1}) - H(\theta_j^{\theta', i}, Y_{j+1})|^2 \\
&= |\theta_j^{\theta, i} - \theta_j^{\theta', i}|^2 - 2\gamma_{j+1} \langle \theta_j^{\theta, i} - \theta_j^{\theta', i}, h(\theta_j^{\theta, i}) - h(\theta_j^{\theta', i}) \rangle \\
&\quad - 2\gamma_{j+1} \langle \theta_j^{\theta, i} - \theta_j^{\theta', i}, \Delta M_{j+1}^{\theta, i} - \Delta M_{j+1}^{\theta', i} \rangle + \gamma_{j+1}^2 |H(\theta_j^{\theta, i}, Y_{j+1}) - H(\theta_j^{\theta', i}, Y_{j+1})|^2
\end{aligned}$$

where we introduced the martingale increments $\Delta M_{j+1}^{\theta,i} = H(\theta_j^{\theta,i}, Y_{j+1}) - h(\theta_j^{\theta,i})$ and $\Delta M_{j+1}^{\theta',i} = H(\theta_j^{\theta',i}, Y_{j+1}) - h(\theta_j^{\theta',i})$, $j \geq 0$ in the last equality. Now, using **(HL)** and **(HUA)** yields:

$$|\theta_{j+1}^{\theta,i} - \theta_{j+1}^{\theta',i}|^2 \leq |\theta_j^{\theta,i} - \theta_j^{\theta',i}|^2 (1 - 2\underline{\alpha}\gamma_{j+1} + [H]_1^2 \gamma_{j+1}^2) - 2\gamma_{j+1} \langle \theta_j^{\theta,i} - \theta_j^{\theta',i}, \Delta M_{j+1}^{\theta,i} - \Delta M_{j+1}^{\theta',i} \rangle,$$

and, by induction on j , we easily obtain:

$$\begin{aligned} |\theta_n^{\theta,i} - \theta_n^{\theta',i}|^2 &\leq |\theta - \theta'|^2 \prod_{j=i}^{n-1} (1 - 2\underline{\alpha}\gamma_{j+1} + [H]_1^2 \gamma_{j+1}^2) \\ &\quad - 2 \left(\prod_{j=i}^{n-1} (1 - 2\underline{\alpha}\gamma_{j+1} + [H]_1^2 \gamma_{j+1}^2) \right) \sum_{j=i}^{n-1} \tilde{\gamma}_{j+1} \langle \theta_j^{\theta,i} - \theta_j^{\theta',i}, \Delta M_{j+1}^{\theta,i} - \Delta M_{j+1}^{\theta',i} \rangle \quad (\text{A.3}) \end{aligned}$$

where $\tilde{\gamma}_{j+1} := \gamma_{j+1} / \prod_{k=i}^j (1 - 2\underline{\alpha}\gamma_{k+1} + [H]_1^2 \gamma_{k+1}^2)$. Taking the expectation in (A.3), we derive:

$$\frac{\mathbf{E}[|\theta_n^{i,\theta} - \theta_n^{i,\theta'}|^2]}{|\theta - \theta'|^2} \leq \prod_{j=i}^{n-1} (1 - 2\underline{\alpha}\gamma_{j+1} + [H]_1^2 \gamma_{j+1}^2).$$

Now:

$$\begin{aligned} |f_i^\gamma(\theta, y) - f_i^\gamma(\theta, y')| &= |\mathbf{E}[|\theta_n^{i,\theta} - \theta_n^{i,\theta'}|^2] - \mathbf{E}[|\theta_n^{i,\theta} - \theta_n^{i,\theta'}|^2]] \leq \mathbf{E}[|\theta_n^{i,\theta} - \theta_n^{i,\theta} - \theta_n^{i,\theta} + \theta_n^{i,\theta'}|] \\ &\leq \left(\prod_{j=i}^{n-1} (1 - 2\underline{\alpha}\gamma_{j+1} + [H]_1^2 \gamma_{j+1}^2) \right)^{1/2} \gamma_i [H]_1 |y - y'| \\ &= (\Pi_n \Pi_i^{-1})^{1/2} \gamma_i [H]_1 |y - y'|, \end{aligned}$$

which completes the proof of the Proposition.

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