



# New closed-form thermoelastostatic Green function and Poisson-type integral formula for a quarter-plane

V. Seremet, Guy Bonnet

## ► To cite this version:

V. Seremet, Guy Bonnet. New closed-form thermoelastostatic Green function and Poisson-type integral formula for a quarter-plane. Mathematical and Computer Modelling, 2011, 53 (1-2), pp.347-358. 10.1016/j.mcm.2010.09.001 . hal-00688135

**HAL Id: hal-00688135**

**<https://hal.science/hal-00688135>**

Submitted on 28 Jan 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# New closed-form thermoelastostatic Green function and Poisson-type integral formula for a quarter-plane

Victor Şeremet<sup>a,\*</sup>, Guy Bonnet<sup>b</sup>

<sup>a</sup> Agrarian State University of Moldova, Mirceşti street 44, Chisinau, 2049, Republic of Moldova

<sup>b</sup> University Paris-Est, Marne la Vallée, Laboratory MSME UMR 8208 CNRS, 5 Boulevard Descartes, 77454 Marne la Vallée, cedex 2, France

## A B S T R A C T

A new Green's function and a new Poisson-type integral formula for a boundary value problem (BVP) in thermoelastostatics for a quarter-plane subject by mixed homogeneous mechanical boundary conditions are derived in this paper. The thermoelastic displacements are generated by a heat source, applied in the inner points of the quarter-plane and by temperature, prescribed on its boundary semi-straight-lines. All results, obtained in terms of elementary functions, are formulated in a special theorem. The first difficulty to obtain these results is in deriving the functions of influence of a unit concentrated force onto elastic volume dilatation  $\Theta^{(k)}$ . The second difficulty is in calculating a volume integral of the product of function  $\Theta^{(k)}$  and Green's function  $G$  in heat conduction. A closed-form solution for a particular BVP of thermoelastostatics for a quarter-plane also is included. Using the proposed approach, it is possible to extend the obtained results not only for any canonical Cartesian domain, but also for any canonical orthogonal one.

**Keywords:**

Green's functions

Thermoelastostatics

Heat conduction

Elasticity

Thermoelastic influence functions

Volume dilatation

Poisson-type integral formula

## 1. Introduction

A Green's function plays the leading role in finding solutions in integrals for boundary value problems (BVPs) in different fields of mathematical physics, including thermoelastostatics. The main objective of this paper is to prove a theorem on deriving a Green's function and a Poisson-type integral formula for a thermoelastic quarter-plane  $V \equiv (0 \leq x_1, x_2 < \infty)$ , where  $(x_1, x_2)$  are rectangular coordinates. In the case of deriving a Poisson-type integral formula, thermoelastic displacements are generated by the inner heat source  $F(x); x \equiv (x_1, x_2)$  and by temperatures  $T_1(0, y_2)$  and  $T_2(y_1, 0)$ , prescribed on the boundary semi-straight-lines  $\Gamma_{10}(y_1 = 0, 0 \leq y_2 < \infty)$  and  $\Gamma_{20}(y_2 = 0, 0 \leq y_1 < \infty)$ . The mechanical boundary conditions are homogeneous; the boundary semi-straight-line  $\Gamma_{10}$  is rigidly fixed, but the boundary semi-straight-line  $\Gamma_{20}$  is fixed without any rotation. In the case of deriving the Green's function, the heat source is described by a  $\delta$ -Dirac function. But the temperature and above-mentioned mechanical conditions on the boundary semi-straight-lines  $\Gamma_{10}$  and  $\Gamma_{20}$  are homogeneous in this case.

A stress analysis and contact problems for an elastic quarter-plane are discussed in [1]. But note, formulation of the problem, that consists in deriving the Poisson-type integral formula for a thermoelastic quarter-plane, is suggested in this paper for the first time. This formulation differs substantially from the formulation presented in [1]. This is explained by the fact, that the Poisson-type integral formula, that has to be derived, must be expressed directly in terms of a given heat source, boundary temperature and some known kernels. In such a way the use of obtained Poisson-type integral formula does not request a solution to intermediary heat conduction or to additional elasticity BVPs.

\* Corresponding author.

E-mail addresses: v.seremet@uasmd.md (V. Şeremet), guy.bonnet@univ-paris-est.fr (G. Bonnet).

To prove the above-mentioned theorem we need some basic equations of thermoelastostatics, described in Section 1.1. Also we need the general Green-type integral formula in stationary thermoelasticity and the formulae for thermoelastic Green's functions, suggested and published by the author earlier and presented in the Section 1.2.

### 1.1. Basic equations in thermoelastostatics

To date, a number of theories on thermoelasticity have been developed and described in classical scientific literature [2–5]. However, many new developments of thermoelasticity and many references are included in [6]. The best developed theory widely used in practical calculations is the theory of stationary thermal stresses, i.e., the theory of uncoupled thermoelasticity, when the temperature field does not depend on the field of elastic displacements, and when inertial terms can be ignored. According to this theory, the formulation of the BVP consists of non-homogeneous Lamé's equations

$$\mu \nabla^2 u_k(\xi) + (\lambda + \mu) \theta_{,k}(\xi) - \gamma T_{,k}(\xi) = 0, \quad \xi \equiv (\xi_1, \xi_2, \xi_3) \in V \quad (1)$$

with the respective homogeneous mechanical boundary conditions. In Eq. (1)  $\nabla^2$  is the three dimensional (3D) Laplace's differential operator;  $\theta = u_{j,j}$  is the thermoelastic volume dilatation;  $\gamma = \alpha_t(2\mu + 3\lambda)$  is the coefficient of thermoelasticity;  $\alpha_t$  is the coefficient of the linear thermal dilatation;  $\lambda, \mu$  are Lamé's constants of elasticity; the temperature field in Eq. (1) can be determined by the BVP in heat conditions that consists of Poisson's equation

$$\nabla^2 T(\xi) = -a^{-1} F(\xi) \quad (2)$$

with the respective boundary conditions for heat actions, which are: temperature  $T(y)$ , heat flux  $a(\partial T(y)/\partial n_y)$ , or for certain laws of heat exchange  $\alpha T(y) + a[\partial T(y)/\partial n_y]$  between the exterior medium and the surface of the body. In Eq. (2)  $a$  is the coefficient of temperature conductivity;  $\alpha$  is the coefficient of convective heat conductivity. To solve the BVP of stationary thermoelasticity in Eqs. (1) and (2) using traditional methods, at the first stage, we need to solve the BVP of heat conduction in Eq. (2) with the given boundary conditions. At the second stage we need to solve the BVP of thermoelasticity in Eq. (1) at the already known temperature field at some given homogeneous mechanical boundary conditions. If the displacements  $u_k(\xi)$  are known, then thermal stresses can be determined by the Duhamel–Neumann law:

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \delta_{ij}(\lambda u_{j,j} - \gamma T); \quad i, j = 1, 2, 3. \quad (3)$$

Furthermore, the two stages in obtaining the solution of the BVPs in thermoelasticity using traditional methods can be replaced by one single stage. To achieve this, the generalized Green's functions and Green-type integral formula [7–9] have to be used. The results obtained in thermoelastostatics can be also generalized with respect to dynamic uncoupled thermoelasticity [10]. Some of these formulae are presented in the next subsections of the Introduction.

### 1.2. Green-type integral formula in thermoelastostatics

On the basis of the proposed thermoelastic influence functions [7–9], the following new general Green-type integral formula to determine the fields of displacements, described by BVP in Eqs. (1)–(2), has been suggested:

$$u_k(\xi) = a^{-1} \int_V F(x) U_k(x, \xi) dV(x) - \int_{\Gamma_D} T(y) (\partial U_k(y, \xi) / \partial n_y) d\Gamma_D(y) + \int_{\Gamma_N} (\partial T(y) / \partial n_y) U_k(y, \xi) d\Gamma_N(y) + \int_{\Gamma_M} [T(y) + \alpha a^{-1} (\partial T(y) / \partial n_y)] U_k(y, \xi) d\Gamma_M(y). \quad (4)$$

In Eq. (4)  $\Gamma_D$ ,  $\Gamma_N$  and  $\Gamma_M$  are parts of the entire surface  $\Gamma \equiv \Gamma_D \cup \Gamma_N \cup \Gamma_M$  on which the Dirichlet's, the Neumann's and the mixed boundary conditions are prescribed, respectively. The functions  $U_k(x, \xi)$ ,  $\partial U_k(y, \xi) / \partial n_y$ ;  $y \in \Gamma_D$ ,  $U_k(y, \xi)$ ;  $y \in \Gamma_N$  and  $U_k(y, \xi)$ ;  $y \in \Gamma_M$  represent influence of an inner unit point heat source  $Q = \delta(x - \xi) \in V$ , unit point temperature  $T = \delta(y - \xi) \in \Gamma_D$ , unit heat flux  $\partial T(y) / \partial n_y = a^{-1} \delta(y - \xi) \in \Gamma_N$  and unit point law  $T(y) + \alpha a^{-1} (\partial T(y) / \partial n_y) = a^{-1} \delta(y - \xi) \in \Gamma_M$  of heat exchange between the exterior medium and surface of the body, on the thermoelastic displacements, respectively. They are determined by the following general integral formulas [7–9]:

$$\begin{aligned} U_k(x, \xi) &= \gamma \int_V G(x, z) \Theta^{(q)}(z, \xi) dV(z); \quad z \equiv (z_1, z_2, z_3); \\ \frac{\partial U_k(y, \xi)}{\partial n_y} &= \lim_{x \rightarrow y} \frac{\partial U_k(x, \xi)}{\partial n_x} = \gamma \int_V \frac{\partial G(y, z)}{\partial n_y} \Theta^{(k)}(z, \xi) dV(z); \quad x, \xi, z \in V, y \in \Gamma_D; \\ U_k(y, \xi) &= \lim_{x \rightarrow y} U_k(x, \xi) = \lim_{x \rightarrow y} \gamma \int_V G(y, z) \Theta^{(k)}(z, \xi) dV(z); \quad x, \xi, z \in V, y \in \Gamma_N; \\ U_k(y, \xi) &= \lim_{x \rightarrow y} U_k(x, \xi) = \lim_{x \rightarrow y} \gamma \int_V G(y, z) \Theta^{(k)}(z, \xi) dV(z); \quad x, \xi, z \in V, y \in \Gamma_M. \end{aligned} \quad (5)$$

The functions  $\Theta^{(k)}(z, \xi)$  and  $G(x, z)$  in Eq. (5) are the functions of influence of a unit point concentrated body force onto volume dilatation of BVP in elasticity and the Green's function of BVP in heat conduction, respectively. The influence functions  $U_k(x, \xi)$  have physical meanings as the displacements in an inner point of observation  $x \equiv (x_1, x_2, x_3)$ , generated by a unit heat source, described by Dirac's  $\delta$  function, applied in an inner point,  $\xi \equiv (\xi_1, \xi_2, \xi_3)$ . The influence functions  $U_k(x, \xi)$  satisfy two types of homogeneous boundary conditions: for elasticity BVP with respect to point  $\xi \equiv (\xi_1, \xi_2, \xi_3)$  and for heat conduction problems with respect to point  $x \equiv (x_1, x_2, x_3)$  [7–10].

## 2. Deriving the Green's functions and Poisson-type integral formula for a thermoelastic quarter-plane

**Theorem.** Let the field of displacements  $u_k(\xi)$  at inner points  $\xi \equiv (\xi_1, \xi_2)$  of the elastic quarter-plane  $V(0 \leq x_1, x_2 < \infty)$  be determined by non-homogeneous Lamé's equations

$$\mu \nabla^2 u_k(\xi) + (\lambda + \mu) \theta_{,k}(\xi) - \gamma T_{,k}(\xi) = 0; \quad k = 1, 2 \quad (6)$$

and in the points  $y \equiv (0, y_2)$  and  $y \equiv (y_1, 0)$  of its boundary semi-straight-lines  $\Gamma_{10}(y_1 = 0, 0 \leq y_2 < \infty)$  and  $\Gamma_{20}(y_2 = 0, 0 \leq y_1 < \infty)$  the following homogeneous mechanical conditions are given:

(a) on the absolutely rigidly fixed boundary semi-straight-line  $\Gamma_{10}(y_1 = 0, 0 \leq y_2 < \infty)$

$$u_1(\xi_1 = 0, \xi_2 = y_2) = u_2(\xi_1 = 0, \xi_2 = y_2) = 0 \quad (7)$$

and

(b) locally mixed boundary conditions on the boundary semi-straight-line  $\Gamma_{20}(y_2 = 0, 0 \leq y_1 < \infty)$

$$u_1(\xi_1 = y_1, \xi_2 = 0) = \sigma_{22}(\xi_1 = y_1, \xi_2 = 0) = 0, \quad (8)$$

where  $\sigma_{22}$  are the normal stresses which are determined by the Duhamel–Neumann law (3). Let the temperature field  $T(\xi)$  in Eq. (6), generated by the inner heat source  $F(\xi)$  and boundary temperature (Dirichlet's boundary condition), also satisfy the following BVP in heat conduction:

$$\nabla^2 T(\xi) = -a^{-1} F(\xi), \quad \xi \in V; \quad T = T_1(y); \quad y \equiv (0, y_2) \in \Gamma_{10}; \quad T = T_2(y); \quad y \equiv (y_1, 0) \in \Gamma_{20}. \quad (9)$$

If the inner heat source  $F$  and boundary temperature  $T$  satisfy the conditions:

$$\int_0^\infty \int_0^\infty |F(x)| dx_1 dx_2 < \infty; \quad \int_0^\infty |T_1(0, y_2)| dy_2 < \infty; \quad \int_0^\infty |T_2(y_1, 0)| dy_1 < \infty, \quad (10)$$

then the solution of this BVP in Eqs. (6)–(9) of thermoelasticity for searched displacements  $u_k(\xi)$  can be presented by the following Poisson-type integral formula, written in the matrix form:

$$\mathbf{u}(\xi) = \frac{1}{a} \int_0^\infty \int_0^\infty F(x) \mathbf{U}(x, \xi) dx_1 dx_2 - \int_0^\infty T_1(0, y_2) \mathbf{Q}_1(0, y_2; \xi) dy_2 - \int_0^\infty T_2(y_1, 0) \mathbf{Q}_2(y_1, 0; \xi) dy_2. \quad (11)$$

The matrices  $\mathbf{U}(x, \xi)$  and  $\mathbf{Q}_1(0, y_2; \xi) = \partial \mathbf{U}(0, y_2; \xi) / \partial n_{y_1}$ ,  $\mathbf{Q}_2(y_1, 0; \xi) = \partial \mathbf{U}(y_1, 0; \xi) / \partial n_{y_2}$  of influence of an inner unit point heat source and unit point temperature given on the boundary semi-straight-lines onto thermoelastic displacements, also the matrix  $\mathbf{u}(\xi)$  of searched displacements in Eq. (11), are determined as follows:

(a) for the matrix  $\mathbf{U}(x, \xi)$

$$\mathbf{U}(x, \xi) = \begin{pmatrix} U_1(x, \xi) \\ U_2(x, \xi) \end{pmatrix} = \frac{\gamma}{4\pi(\lambda + 2\mu)} \times \begin{pmatrix} -(x_1 - \xi_1) \ln \frac{r_1 r_2}{r r_{12}} + 2B^{-1} x_1 \xi_1 (x_1 + \xi_1) \left( \frac{1}{r_1^2} - \frac{1}{r_{12}^2} \right) \\ (x_2 - \xi_2) \left( \ln \frac{r}{r_1} - 2B^{-1} \frac{x_1 \xi_1}{r_1^2} \right) + (x_2 + \xi_2) \left( \ln \frac{r_2}{r_{12}} - 2B^{-1} \frac{x_1 \xi_1}{r_{12}^2} \right) \end{pmatrix}, \quad (12)$$

where  $r = r(x, \xi) = |x - \xi|$ ;  $r_1(x, \xi) = |x - \xi_1^*|$ ;  $r_2 = r_2(x, \xi) = |x - \xi_2^*|$ ;  $x \equiv (x_1, x_2)$ ;  $\xi \equiv (\xi_1, \xi_2)$ ;  $\xi_1^* \equiv (-\xi_1, \xi_2)$ ;  $\xi_2^* \equiv (\xi_1, -\xi_2)$ ;  $r_{12}(x, \xi) = |x - \xi_{12}^*|$ ;  $\xi_{12}^* \equiv (-\xi_1, -\xi_2)$ ;  $B = [(\lambda + 3\mu)/(\lambda + \mu)]$ ;

(b) for the matrix  $\mathbf{Q}_1(0, y_2; \xi)$

$$\mathbf{Q}_1(0, y_2; \xi) = \begin{pmatrix} Q_{11}(0, y_2; \xi) \\ Q_{12}(0, y_2; \xi) \end{pmatrix} = \frac{\gamma}{\pi(\lambda + 3\mu)} \begin{pmatrix} -\xi_1^2 \left( \frac{1}{r^2} - \frac{1}{r_2^2} \right) \\ \xi_1 \left( \frac{y_2 + \xi_2}{r_2^2} + \frac{y_2 - \xi_2}{r^2} \right) \end{pmatrix}, \quad (13)$$

where  $r = r(y, \xi) = |y - \xi|$ ;  $r_2 = r_2(y, \xi) = |y - \xi_2^*|$ ;  $y \equiv (0, y_2)$ ;



and  
(c) for the matrix  $\mathbf{Q}_2(y_1, 0; \xi)$

$$\mathbf{Q}_2(y_1, 0; \xi) = \begin{pmatrix} Q_{21}(y_1, 0; \xi) \\ Q_{22}(y_1, 0; \xi) \end{pmatrix} = \frac{\gamma}{2\pi(\lambda + 2\mu)} \begin{pmatrix} \xi_2(y_1 - \xi_1) \left( \frac{1}{r^2} - \frac{1}{r_1^2} \right) - \frac{4B^{-1}y_1\xi_1\xi_2(y_1 + \xi_1)}{r_1^4} \\ \ln \frac{r_1}{r} + \xi_2^2 \left( \frac{1}{r^2} - \frac{1}{r^2} \right) + \frac{2B^{-1}y_1\xi_1}{r_1^2} \left( 1 - \frac{2\xi_2^2}{r_1^2} \right) \end{pmatrix}; \quad (14)$$

$$\mathbf{u}(\xi) = \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix},$$

where  $r = r(y, \xi) = |y - \xi|$ ;  $r_1(y, \xi) = |y - \xi_1^*|$ ;  $y \equiv (y_1, 0)$ .

**Proof.** First, the Green's function  $G$  for Poisson's equation for a quarter-plane with Dirichlet's boundary conditions is rewritten, and then, in the Section 2.1, we derive the volume dilatation  $\Theta^{(k)}(x, \xi)$ . In Section 2.2 it is shown how to derive thermoelastic influence functions  $U_k(x, \xi)$ . Finally, in Section 2.3, on the base of the functions  $U_k(x, \xi)$ , the Poisson-type integral formula for stated BVP of thermoelasticity is derived.  $\square$

To obtain the matrix  $\mathbf{U}(x, \xi)$  in Eq. (12) for the BVP in Eqs. (6)–(9) we use the integral formula in Eq. (5). The functions  $G(x, \xi)$  and  $\Theta^{(k)}(x, \xi)$  in this equation must be the Green's function of the Dirichlet problem in heat conduction and, respectively, the influence functions of a unit concentrated body force  $\delta_{ik}\delta(x - \xi)$  onto volume dilatation in theory of elasticity for the quarter-plane  $V$ ;  $\delta_{ik}$  is Kronecker's symbol and  $\delta(x - \xi)$  is Dirac's function.

So, to get the Green's function  $G(x, \xi)$  we have to solve the BVP, which consists of the equation in the heat conduction with the homogeneous boundary conditions similar to those in Eq. (9):

$$\begin{aligned} \nabla^2 G(x, \xi) &= -\delta(x - \xi); \quad x, \xi \in V; \\ G &= 0; \quad x_1 = 0, \quad 0 \leq x_2 < \infty; \quad G = 0; \quad x_2 = 0, \quad 0 \leq x_1 < \infty. \end{aligned} \quad (15)$$

Here we recall this Green's function from the handbook [8] (see problem and answer 10.P.1):

$$\begin{aligned} G(x, \xi) &= \frac{1}{2\pi} \ln \frac{r_1 r_2}{r r_{12}}; \quad r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}; \quad r_1 = \sqrt{(x_1 + \xi_1)^2 + (x_2 - \xi_2)^2}; \\ r_2 &= \sqrt{(x_1 - \xi_1)^2 + (x_2 + \xi_2)^2}; \quad r_{12} = \sqrt{(x_1 + \xi_1)^2 + (x_2 + \xi_2)^2}. \end{aligned} \quad (16)$$

This Green's function is constructed using the image method in [11]. An interesting alternative method to derive Poisson integral formula using the special developed image concept and its applications are presented in [12,13].

## 2.1. Deriving the volume dilatation $\Theta^{(k)}(x, \xi)$ inside the quarter-plane $V$

This section consists of three subsections. In Section 2.1.1 we formulate the BVP for volume dilatation deriving. Deriving the volume dilatation  $\Theta^{(k)}$  on the boundaries straight lines  $\Gamma_{10}$  and  $\Gamma_{20}$  is presented in the Section 2.1.2. Finally, in the Section 2.1.3 the volume dilatation  $\Theta^{(k)}(x, \xi)$  inside the quarter-plane  $V$  is derived.

### 2.1.1. The formulation of BVP for volume dilatation deriving. Integral representation for $\Theta^{(k)}(x, \xi)$

To get the influence functions  $\Theta^{(k)}(x, \xi)$ , usually, we have to solve the following BVP, which consists of Lamé's equations in the theory of elasticity and homogeneous boundary conditions as in Eqs. (7) and (8):

$$\begin{aligned} \mu \nabla^2 U_i^{(k)}(x, \xi) + (\lambda + \mu) \Theta_i^{(k)}(x, \xi) &= -\delta_{ik} \delta(x - \xi); \quad x, \xi \in V; \quad i, k = 1, 2; \\ U_1^{(k)} &= U_2^{(k)} = 0; \quad x_1 = 0, \quad 0 \leq x_2 < \infty; \quad \sigma_{22}^{(k)} = U_1^{(k)} = 0, \quad 0 \leq x_1 < \infty, \quad x_2 = 0 \end{aligned} \quad (17)$$

and then, on the base of displacements  $U_i^{(k)}(x, \xi)$ , to compute the volume dilatation

$$\Theta^{(k)}(x, \xi) = U_{j,j}^{(k)}(x, \xi), \quad j = 1, 2. \quad (18)$$

But as it will be shown below, in the case of the boundary condition in Eq. (17), we can derive the volume dilatation  $\Theta^{(k)}(x, \xi)$  using the equation

$$\nabla^2 \Theta^{(k)}(x, \xi) = -(\lambda + 2\mu)^{-1} (\partial \delta(x - \xi) / \partial x_k) \quad (19)$$

and its integral representation via respective Green's function  $G_\Theta(x, \xi)$ :

$$\Theta^{(k)}(x, \xi) = -\frac{1}{\lambda + 2\mu} \frac{\partial}{\partial \xi_k} G_\Theta(x, \xi) + \int_\Gamma \left[ \frac{\partial \Theta^{(k)}(y, \xi)}{\partial n_\Gamma} - \Theta^{(k)}(y, \xi) \frac{\partial}{\partial n_\Gamma} \right] G_\Theta(y, x) d\Gamma(y), \quad (20)$$

where  $\Gamma$  is the surface of the quarter-plane that consists of two boundary semi-straight-lines:  $\Gamma_{10}(y_1 = 0, 0 \leq y_2 < \infty)$  and  $\Gamma_{20}(0 \leq y_1 < \infty, y_2 = 0)$ ;  $G_\Theta(x, \xi)$  is the Green's function for Poisson's equation that follows homogeneous boundary conditions similar to volume dilatation  $\Theta^{(k)}(x, \xi)$ . More details about the BVP for  $G_\Theta(x, \xi)$  can be seen in Eq. (44).

### 2.1.2. Deriving the volume dilatation $\Theta^{(k)}$ on the boundary semi-straight-lines $\Gamma_{10}$ and $\Gamma_{20}$

To show this, first prove that the boundary conditions in Eq. (17) lead to the following values of the volume dilatation:

(a) On the boundary semi-straight-line  $\Gamma_{10}(y_1 = 0, 0 \leq y_2 < \infty)$ :

$$\Theta^{(k)}(y, \xi) = -\frac{1}{\lambda + 2\mu} L_{\Theta}^{(k)} \frac{1}{\pi} \frac{\partial}{\partial \xi_1} \ln \frac{r(y, \xi)}{r_2(y, \xi)}; \quad y \equiv (0, y_2); \quad L_{\Theta}^{(k)} = B^{-1} \xi_1 \frac{\partial}{\partial \xi_k} - \delta_{1k} \quad (21)$$

and

(b) On the boundary semi-straight-line  $\Gamma_{20}(0 \leq y_1 < \infty, y_2 = 0)$ :

$$\Theta^{(k)}(y, \xi) = 0; \quad y \equiv (y_1, 0). \quad (22)$$

Let us prove that the second boundary conditions in Eq. (17) lead to zero volume dilatation on the boundary semi-straight-line  $\Gamma_{20}(0 \leq y_1 < \infty, y_2 = 0)$ , it means

$$\sigma_{22}^{(k)} = U_1^{(k)} = 0; \quad 0 \leq x_1 < \infty, \quad x_2 = 0 \rightarrow \Theta^{(k)}(y, \xi) = 0. \quad (23)$$

To do this, first of all, from formula for normal stresses  $\sigma_{22}^{(k)}$  determined by Hooke's law:

$$\sigma_{ij}^{(k)} = \mu(U_{i,j}^{(k)} + U_{j,i}^{(k)}) + \lambda \delta_{ij} \Theta^{(k)}; \quad i, j, k = 1, 2 \quad (24)$$

and formula (18) for  $\Theta^{(k)}$  we obtain:

$$\sigma_{22}^{(k)} = 2\mu U_{2,2}^{(k)} + \lambda \Theta^{(k)} = (\lambda + 2\mu) \Theta^{(k)} - 2\mu U_{1,1}^{(k)}. \quad (25)$$

Next, from boundary conditions

$$\sigma_{22}^{(k)} = U_1^{(k)} = 0; \quad 0 \leq x_1 < \infty, \quad x_2 = 0 \quad (26)$$

in Eq. (23) it follows

$$\sigma_{22}^{(k)} = U_{1,1}^{(k)} = 0; \quad 0 \leq x_1 < \infty, \quad x_2 = 0. \quad (27)$$

Finally from Eqs. (25) and (27) it follows Eq. (22). So we proved that on the boundary  $\Gamma_{20}$  the volume dilatation is equal to zero.

To prove Eq. (21) we write Lamé's Eq. (17) (at  $i = 1$ ) in the following form:

$$\nabla_x^2 \left[ U_1^{(k)}(x, \xi) + \frac{\lambda + \mu}{2\mu} x_1 \Theta^{(k)}(x, \xi) \right] = -\frac{1}{\mu} \left[ \delta_{1k} + \frac{\lambda + \mu}{2(\lambda + 2\mu)} \frac{\partial}{\partial x_k} \right] \delta(x - \xi); \quad i, k = 1, 2, \quad (28)$$

where Eq. (19) and the Euler's relationship

$$\nabla^2(f \cdot \varphi) = f \nabla^2 \varphi + 2f_{,i} \varphi_{,i} + \varphi \nabla^2 f \quad (29)$$

were used.

Eq. (28) can be regarded as an equation of Poisson-type and so its solutions can be found in terms of Green's functions of the respective BVP. Hence, having calculated the multiple integrals taken over the quarter plane  $V$  by parts, and the integral

$$\int_V f(z) \delta(z - x) dV(z) = f(x) \quad (30)$$

we represent the solutions of Eq. (28) as follows

$$\begin{aligned} U_1^{(k)}(x, \xi) + \frac{\lambda + \mu}{2\mu} x_1 \Theta^{(k)}(x, \xi) = & A \left( B \delta_{1k} - \xi_1 \frac{\partial}{\partial \xi_k} \right) G_1(x, \xi) - \int_{\Gamma_{10}} \left[ U_1^{(k)}(y, \xi) + \frac{\lambda + \mu}{2\mu} y_1 \Theta^{(k)}(y, \xi) \right] \\ & \times \frac{\partial G_1(y, x)}{\partial n_{y1}} d\Gamma_{10}(y) - \int_{\Gamma_{20}} \left[ U_1^{(k)}(y, \xi) + \frac{\lambda + \mu}{2\mu} y_1 \Theta^{(k)}(y, \xi) \right] \frac{\partial G_1(y, x)}{\partial n_{y2}} d\Gamma_{20}(y), \end{aligned} \quad (31)$$

where  $G_1(x, \xi)$  is the Green's function, having boundary conditions analogical for displacements  $U_1^{(k)}(x, \xi)$  and

$$A = [(\lambda + \mu)/2\mu(\lambda + 2\mu)]; \quad B = [(\lambda + 3\mu)/(\lambda + \mu)]. \quad (32)$$

According to Eq. (17) we have the following boundary conditions for  $U_1^{(k)}$ :

$$U_1^{(k)} = 0; \quad x_1 = 0, 0 \leq x_2 < \infty; \quad U_1^{(k)} = 0; \quad 0 \leq x_1 < \infty, x_2 = 0. \quad (33)$$

So, for deriving the function  $G_1(x, \xi)$  we have the BVP (15) in the quarter-plane  $V(0 \leq x_1, x_2 < \infty)$ .

However, the Green's functions for Poisson's equation for the quarter-plane under the above Dirichlet's boundary conditions are given in the Answer to Problem 10.P.1 [10]. So the Green's function  $G_1(x, \xi) = G(x, \xi)$  is determined by the expression in Eq. (16).

Taking into consideration the first boundary condition in Eq. (33) for displacements  $U_1^{(k)}$ , we then obtain:

$$U_1^{(k)} = 0 \Rightarrow U_1^{(k)} + [(\lambda + \mu)/2\mu]y_1\Theta^{(k)} = 0, \quad (34)$$

because on the boundary semi-straight-line  $\Gamma_{10}$  we have  $\Theta^{(k)} \neq 0$ , but  $y_1 = 0$ . Analogically, on the boundary semi-straight-line  $\Gamma_{20}$  we have the equalities

$$U_1^{(k)} = 0; \quad \Theta^{(k)}(y, \xi) = 0 \Rightarrow U_1^{(k)} + [(\lambda + \mu)/2\mu]y_1\Theta^{(k)} = 0 \quad (35)$$

that follows from Eqs. (22) and (33). So, due Eqs. (34) and (35) we find that the integrals in Eq. (31) are zero. So, the representation in Eq. (31) takes the simple form:

$$U_1^{(k)}(x, \xi) + \frac{\lambda + \mu}{2\mu}x_1\Theta^{(k)}(x, \xi) = A \left( B\delta_{1k} - \xi_1 \frac{\partial}{\partial \xi_k} \right) G_1(x, \xi). \quad (36)$$

Let us calculate the derivative with respect to the variable  $x_1$  in Eq. (36) and then proceed to the limit  $x \rightarrow y$ ;  $x \in V$ ,  $y \in \Gamma_{10}$ ;  $x \equiv (x_1, x_2)$ ;  $y \equiv (0, y_2)$ :

$$\lim_{x \rightarrow y \equiv (0, y_2)} \left\{ U_{1,1}^{(k)}(x, \xi) + \frac{\lambda + \mu}{2\mu}[\Theta^{(k)}(x, \xi) + x_1\Theta_{,1}^{(k)}(x, \xi)] \right\} = \lim_{x \rightarrow y \equiv (0, y_2)} \left[ A \left( B\delta_{1k} - \xi_1 \frac{\partial}{\partial \xi_k} \right) \frac{\partial}{\partial x_1} G(x, \xi) \right]. \quad (37)$$

Then, taking into consideration the fact, that due to the boundary conditions in Eq. (17), the tangential derivatives  $U_{2,2}^{(k)}$  on the boundary semi-straight-line  $\Gamma_{10} \equiv (y_1 = 0, 0 \leq y_2 < \infty)$  are equal to zero, it means

$$U_{2,2}^{(k)}(y, \xi) = 0, \quad (38)$$

and from Eq. (18) we come to the conclusion that on  $\Gamma_{10}$

$$U_{1,1}^{(k)}(y, \xi) = \Theta^{(k)}(y, \xi). \quad (39)$$

Let us calculate the limit from the left part in Eq. (37):

$$\begin{aligned} \lim_{x \rightarrow y \equiv (0, y_2)} \left\{ U_{1,1}^{(k)}(x, \xi) + \frac{\lambda + \mu}{2\mu}[\Theta^{(k)}(x, \xi) + x_1\Theta_{,1}^{(k)}(x, \xi)] \right\} &= U_{1,1}^{(k)}(y, \xi) + \frac{\lambda + \mu}{2\mu}[\Theta^{(k)}(y, \xi) + y_1\Theta_{,1}^{(k)}(x, \xi)] \\ &= U_{1,1}^{(k)}(y, \xi) + \frac{\lambda + \mu}{2\mu}\Theta^{(k)}(y, \xi) \\ &= \frac{\lambda + 3\mu}{2\mu}\Theta^{(k)}(y, \xi), \end{aligned} \quad (40)$$

where Eq. (39) was used.

Analogically, calculating the limit of the right part in Eq. (37), we obtain the following formula:

$$\begin{aligned} \lim_{x \rightarrow y \equiv (0, y_2)} \left[ A \left( B\delta_{1k} - \xi_1 \frac{\partial}{\partial \xi_k} \right) \frac{\partial}{\partial x_1} G_1(x, \xi) \right] &= \lim_{x \rightarrow y \equiv (0, y_2)} \left\{ A \left( B\delta_{1k} - \xi_1 \frac{\partial}{\partial \xi_k} \right) \frac{1}{2\pi} \left[ \frac{\partial}{\partial \xi_1} \ln \frac{r(x, \xi)r_1(x, \xi)}{r_2(x, \xi)r_{12}(x, \xi)} \right] \right\} \\ &= A \left( B\delta_{1k} - \xi_1 \frac{\partial}{\partial \xi_k} \right) \frac{1}{\pi} \frac{\partial}{\partial \xi_1} \ln \frac{r(y_2, \xi)}{r_2(y_2, \xi)} \\ &= -ABL_{\Theta}^{(k)} \frac{1}{\pi} \frac{\partial}{\partial \xi_1} \ln \frac{r(x, \xi)}{r_2(x, \xi)}; \quad L_{\Theta}^{(k)} = B^{-1}\xi_1 \frac{\partial}{\partial \xi_k} - \delta_{1k}, \end{aligned} \quad (41)$$

where the expressions in Eq. (16) for Green's function  $G_1(x, \xi) = G(x, \xi)$  and the relation

$$\frac{\partial}{\partial x_1} G_1(x, \xi) = (2\pi)^{-1} \frac{\partial}{\partial \xi_1} \ln \frac{r(x, \xi)r_1(x, \xi)}{r_2(x, \xi)r_{12}(x, \xi)} \quad (42)$$

were used.

Finally, substituting expressions in Eqs. (40) and (41) into the formula in Eq. (37), we come to the following formula for the volume dilatation on the boundary semi-straight-line  $\Gamma_{10}$  of the quarter-plane  $V$ :

$$\Theta^{(k)}(y, \xi) = -\frac{1}{\lambda + 2\mu}L_{\Theta}^{(k)} \frac{1}{\pi} \frac{\partial}{\partial \xi_1} \ln \frac{r(y, \xi)}{r_2(y, \xi)}; \quad L_{\Theta}^{(k)} = B^{-1}\xi_1 \frac{\partial}{\partial \xi_k} - \delta_{1k} \quad (43)$$

that coincides with the formula in Eq. (21).

### 2.1.3. Deriving the volume dilatation $\Theta^{(k)}(x, \xi)$ inside the quarter-plane $V$

So, as for the BVP for  $\Theta^{(k)}$ , we have Eqs. (19), (21) and (22), then the BVP for respective Green's function  $G_\Theta$  for the considered quarter-plane can be written in the form:

$$\begin{aligned} \nabla_x^2 G_\Theta(x, \xi) &= -\delta(x - \xi); \quad x, \xi \in V; \\ G_\Theta(y, \xi) &= 0; \quad y_1 = 0, \quad 0 \leq y_2 < \infty; \quad G_\Theta(y, \xi) = 0; \quad 0 \leq y_1 < \infty, \quad y_2 = 0. \end{aligned} \quad (44)$$

But for Green's function  $G_\Theta$ , described by (44), we have the expression that coincides with BVP, described by Eq. (15), this is why the function  $G_\Theta = G$  is determined by the expression in Eq. (16).

Finally, if we substitute the expressions in Eqs. (21) and (22) in the representation (20), rewritten for a quarter-plane in the form:

$$\begin{aligned} \Theta^{(k)}(x, \xi) &= -\frac{1}{\lambda + 2\mu} \frac{\partial}{\partial \xi_k} G_\Theta(x, \xi) - \int_0^\infty \left[ \frac{\partial \Theta^{(k)}(0, y_2; \xi)}{\partial y_1} - \Theta^{(k)}(0, y_2; \xi) \frac{\partial}{\partial y_1} \right] G_\Theta(0, y_2; x) dy_2 \\ &\quad - \int_0^\infty \left[ \frac{\partial \Theta^{(k)}(y_1, 0; \xi)}{\partial y_2} - \Theta^{(k)}(y_1, 0; \xi) \frac{\partial}{\partial y_2} \right] G_\Theta(y_1, 0; x) dy_1 \\ &= -\frac{1}{\lambda + 2\mu} \frac{\partial}{\partial \xi_k} G_\Theta(x, \xi) + \int_0^\infty \Theta^{(k)}(y_1, 0; \xi) \frac{\partial}{\partial y_2} G_\Theta(y_1, 0; x) dy_1, \end{aligned} \quad (45)$$

then, calculating the integral:

$$\begin{aligned} \int_0^\infty \Theta^{(k)}(0, y_2; \xi) \frac{\partial}{\partial y_2} G_\Theta(0, y_2; x) dy_2 &= -\frac{1}{(\lambda + 2\mu)\pi} L_\Theta^{(k)} \int_0^\infty \frac{\partial}{\partial \xi_1} \ln \frac{r(0, y_2; \xi)}{r_2(0, y_2; \xi)} \frac{\partial G(0, y_2; x)}{\partial y_1} dy_2 \\ &= \frac{1}{(\lambda + 2\mu)\pi} L_\Theta^{(k)} \frac{\partial}{\partial \xi_1} \int_0^\infty \ln \frac{r(0, y_2; \xi)}{r_2(0, y_2; \xi)} \frac{1}{\pi} \left[ \frac{x_1}{r(0, y_2; x)} - \frac{x_1}{r_2(0, y_2; x)} \right] dy_2 \\ &= \frac{1}{(\lambda + 2\mu)\pi} L_\Theta^{(k)} \frac{\partial}{\partial \xi_1} \ln \frac{r_1(x, \xi)}{r_{12}(x, \xi)}; \\ L_\Theta^{(k)} &= B^{-1} \xi_1 \frac{\partial}{\partial \xi_k} - \delta_{1k}; \quad r(0, y_2; x) = \sqrt{x_1^2 + (y_2 - x_2)^2}; \quad r_2(0, y_2; x) = \sqrt{x_1^2 + (y_2 + x_2)^2} \end{aligned} \quad (46)$$

we obtain the searched volume dilatation of the elastic BVP in Eq. (17) for quarter-plane, written in the form:

$$\Theta^{(k)}(x, \xi) = -\frac{1}{2\pi(\lambda + 2\mu)} \left[ \frac{\partial}{\partial \xi_k} \ln \frac{r_1 r_2}{r r_{12}} + 2L_\Theta^{(k)} \frac{\partial}{\partial \xi_1} \ln \frac{r_1}{r_{12}} \right]; \quad L_\Theta^{(k)} = B^{-1} \xi_1 \frac{\partial}{\partial \xi_k} - \delta_{1k}. \quad (47)$$

### 2.2. Deriving the thermoelastic influence functions $U_k(x, \xi)$

Now we have both functions  $G(x, \xi)$  and  $\Theta^{(k)}(x, \xi)$ , needed for deriving the thermoelastic influence functions  $U_k(x, \xi)$ , using the Eq. (12). So, substituting functions  $G(x, \xi)$  and  $\Theta^{(k)}(x, \xi)$  from Eqs. (16) and (47) in the Eq. (5) rewritten for quarter-plane, then, calculating the volume integral

$$\begin{aligned} U_k(x, \xi) &= \gamma \int_0^\infty \int_0^\infty G(x, z) \Theta^{(k)}(z, \xi) dz_1 dz_2 = -\frac{\gamma}{2\pi(\lambda + 2\mu)} \int_{-\infty}^\infty dz_3 \int_0^\infty \int_0^\infty \frac{1}{2\pi} \ln \frac{r_1(x, z) r_2(x, z)}{r(x, z) r_{12}(x, z)} \\ &\quad \times \left[ \frac{\partial}{\partial \xi_k} \ln \frac{r_1(z, \xi) r_2(z, \xi)}{r(z, \xi) r_{12}(z, \xi)} + 2L_\Theta^{(k)} \frac{\partial}{\partial \xi_1} \ln \frac{r_1(z, \xi)}{r_{12}(z, \xi)} \right] dz_1 dz_2 \end{aligned} \quad (48)$$

in the special way (see Appendix), we obtain the following expression for functions  $U_k(x, \xi)$ :

$$U_k(x, \xi) = \frac{\gamma}{8\pi(\lambda + 2\mu)} \left\{ \frac{\partial}{\partial \xi_k} [r_1^2 (\ln r_1 - 1) + r_2^2 (\ln r_2 - 1) - r^2 (\ln r - 1) - r_{12}^2 (\ln r_{12} - 1)] + 4x_1 L_\Theta^{(k)} \ln \frac{r_1}{r_{12}} \right\} \quad (49)$$

that, being presented in the matrix form, coincide with the matrix  $\mathbf{U}(x, \xi)$  in Eq. (12).

### 2.3. Deriving the Poisson-type integral formula

The next step is to calculate the other influence functions in Eq. (4), rewritten for the quarter-plane in the form:

$$u_k(\xi) = a^{-1} \int_0^\infty \int_0^\infty F(x) U_k(x, \xi) dx_1 dx_2 - \int_0^\infty T_1(0, y_2) \frac{\partial U_k(0, y_2; \xi)}{\partial n_{y_1}} dy_2 - \int_0^\infty T_2(y_1, 0) \frac{\partial U_k(y_1, 0; \xi)}{\partial n_{y_2}} dy_1. \quad (50)$$

These functions are:  $\partial U_k(0, y_2; \xi)/\partial n_{y_1}$  and  $\partial U_k(y_1, 0; \xi)/\partial n_{y_2}$  on the boundary semi-straight-lines  $\Gamma_{10}(y_1 = 0, 0 \leq y_2 \leq \infty)$  and  $\Gamma_{20}(0 \leq y_1 \leq \infty, y_2 = 0)$ . So, to get the Poisson-type integral formula we have to use the formula (50) for the quarter-plane. The influence functions  $U_k(x, \xi)$  in this formula are determined by Eq. (49). The functions  $\partial U_k(0, y_2; \xi)/\partial n_{y_1}$  and  $\partial U_k(y_1, 0; \xi)/\partial n_{y_2}$ , as functions of influence, given on the boundary semi-straight-lines, have to be determined using Eq. (5) and the functions  $U_k(x, \xi)$  in Eq. (49) as follows:

(a) Function of influence of a unit point temperature on semi-straight-line  $\Gamma_{10}$  onto thermoelastic displacements:

$$\frac{\partial U_k(y, \xi)}{\partial n_{y_1}} = \lim_{x \rightarrow y=(0, y_2)} \frac{\partial U_k(x, \xi)}{\partial n_{x_1}} = \frac{\gamma}{4\pi(\lambda + 2\mu)} \left[ \frac{\partial^2}{\partial \xi_k \partial \xi_1} (r_2^2(\ln r_2 - 1) - r^2(\ln r - 1)) - 2L_\Theta^{(k)} \ln \frac{r}{r_2} \right];$$

$$r = \sqrt{\xi_1^2 + (y_2 - \xi_2)^2}; \quad r_2 = \sqrt{\xi_1^2 + (y_2 + \xi_2)^2} \quad (51)$$

and

(b) Function of influence of a unit point temperature on semi-straight-line  $\Gamma_{20}$  onto thermoelastic displacements:

$$\frac{\partial U_k(y, \xi)}{\partial n_{y_2}} = \lim_{x \rightarrow y=(y_1, 0)} \frac{\partial U_k(x, \xi)}{\partial n_{x_2}} = \frac{\gamma}{4\pi(\lambda + 2\mu)} \frac{\partial}{\partial \xi_2} \left[ \frac{\partial}{\partial \xi_k} (r_1^2(\ln r_1 - 1) - r^2(\ln r - 1)) + 4y_1 L_\Theta^{(k)} \ln r_1 \right];$$

$$r = \sqrt{(y_1 - \xi_1)^2 + \xi_2^2}; \quad r_1 = \sqrt{(y_1 + \xi_1)^2 + \xi_2^2}. \quad (52)$$

Substituting the influence functions in Eqs. (51) and (52) into the formula (50) we obtain the following Poisson-type integral formula:

$$U_k(\xi) = \frac{\gamma}{8\pi(\lambda + 2\mu)} \left\{ a^{-1} \int_0^\infty \int_0^\infty F(x) \left[ \frac{\partial}{\partial \xi_k} [r_1^2(x, \xi)(\ln r_1(x, \xi) - 1) + r_2^2(x, \xi)(\ln r_2(x, \xi) - 1) - r^2(x, \xi)(\ln r(x, \xi) - 1) - r_{12}^2(x, \xi)(\ln r_{12}(x, \xi) - 1)] + 4x_1 L_\Theta^{(k)} \ln \frac{r_1(x, \xi)}{r_{12}(x, \xi)} \right] dx_1 dx_2 \right.$$

$$- 2 \int_0^\infty T_1(0, y_2) \left[ \frac{\partial^2}{\partial \xi_k \partial \xi_1} (r_2^2(0, y_2; \xi)(\ln r_2(0, y_2; \xi) - 1) - r^2(0, y_2; \xi)(\ln r(0, y_2; \xi) - 1)) - 2L_\Theta^{(k)} \ln \frac{r(0, y_2; \xi)}{r_2(0, y_2; \xi)} \right] dy_2 - 2 \int_{-\infty}^\infty T_2(y_1, 0) \frac{\partial}{\partial \xi_2} \left[ \frac{\partial}{\partial \xi_k} (r_1^2(y_1, 0; \xi)(\ln r_1(y_1, 0; \xi) - 1) - r^2(y_1, 0; \xi)(\ln r(y_1, 0; \xi) - 1)) + 4y_1 L_\Theta^{(k)} \ln r_1(y_1, 0; \xi) \right] dy_1 \Big\}. \quad (53)$$

Finally, it should be noted that integrals with semi-bounded intervals in Eq. (53) exist, which means that the displacements  $|u_k(\xi)| < \infty$ , when the following conditions are satisfied:

$$\int_0^\infty \int_0^\infty |F(x)| dx_1 dx_2 < \infty; \quad \int_0^\infty |T_1(0, y_2)| dy_2 < \infty; \quad \int_0^\infty |T_2(y_1, 0)| dy_1 < \infty. \quad (54)$$

The conditions in Eq. (54) will be satisfied in the case when the functions  $F(x)$  and  $T(y)$  are given on the bounded domains or bounded intervals. If we calculate the derivatives in Eqs. (49), (51) and (52) and present the obtained results in matrix form, then we can be sure that the influence matrices  $\mathbf{U}(x, \xi)$ ;  $\partial \mathbf{U}(0, y_2; \xi)/\partial n_{y_1} = \mathbf{Q}_1(0, y_2; \xi)$ ;  $\partial \mathbf{U}(y_1, 0; \xi)/\partial n_{y_2} = \mathbf{Q}_2(y_1, 0; \xi)$  and Poisson-type integral formula in Eq. (53) coincide with the results in Eqs. (11)–(14). The investigations show that the obtained displacements, described by Poisson-type integral formula (53), satisfy the stated in the theorem BVP of thermoelasticity, described by Eqs. (6)–(10).

### 3. An example of application of the obtained Poisson-type integral formula

Let us solve the Eqs. (6)–(9) and determine the thermoelastic displacements  $u_k(\xi)$  in quarter-plane if on its boundary semi-straight-lines  $\Gamma_{10}(y_1 = 0; 0 \leq y_2 < \infty)$  and  $\Gamma_{20}(0 \leq y_1 < \infty, y_2 = 0)$  the following temperature is given:

$$T(y) = \begin{cases} T_1(y) = T_0 = \text{const}, & y \in [y_1 = 0, a \leq y_2 \leq b]; y \in \Gamma_{10}; \\ T_1(y) = 0, & y \in (y_1 = 0, 0 < y_2 < a) \cup (y_1 = 0, b < y_2 < \infty); y \in \Gamma_{10}; \\ T_2(y) = 0, & y \in [0 \leq y_1 < \infty, y_2 = 0]; y \in \Gamma_{20}. \end{cases} \quad (55)$$

To solve this BVP we use the formula in Eqs. (11)–(14) for  $F(x) = 0$  and for temperature given in Eq. (55):

$$u_k(\xi) = -T_0 \int_a^b Q_{1k}(0, y_2; \xi) dy_2; \quad k = 1, 2, \quad (56)$$



where  $Q_{1k}(0, y_2; \xi)$  is determined by the matrix (13). So, taking the respective integrals over the boundary segment  $[y_1 = 0, a \leq y_2 \leq b]$ , we obtain the following thermoelastic displacements at an arbitrary inner point  $\xi \equiv (\xi_1, \xi_2)$ :

$$\begin{aligned} u_1(\xi) &= \frac{\gamma T_0 \xi_1^2}{\pi(\lambda + 3\mu)} \int_a^b \left( \frac{1}{r^2} - \frac{1}{r_2^2} \right) dy_2 = \frac{\gamma T_0 \xi_1}{\pi(\lambda + 3\mu)} \left[ \operatorname{arctg} \frac{y_2 - \xi_2}{\xi_1} - \operatorname{arctg} \frac{y_2 + \xi_2}{\xi_1} \right]_{y_2=a}^{y_2=b}; \\ u_2(\xi) &= -\frac{\gamma T_0 \xi_1}{\pi(\lambda + 3\mu)} \int_a^b \left( \frac{y_2 + \xi_2}{r_2^2} + \frac{y_2 - \xi_2}{r^2} \right) dy_2 \\ &= -\frac{\gamma T_0 \xi_1}{\pi(\lambda + 3\mu)} \int_a^b \frac{\partial}{\partial y_2} \ln r r_2 dy_2 = -\frac{\gamma T_0 \xi_1}{\pi(\lambda + 3\mu)} \ln r r_2 \Big|_{y_2=a}^{y_2=b}; \\ r &= r(0, y_2; \xi) = \sqrt{\xi_1^2 + (y_2 - \xi_2)^2}; \quad r_2 = r_2(0, y_2; \xi) = \sqrt{\xi_1^2 + (y_2 + \xi_2)^2}. \end{aligned} \quad (57)$$

The final expressions for thermoelastic displacements are written in the form:

$$\begin{aligned} u_1(\xi) &= \frac{\gamma T_0 \xi_1}{\pi(\lambda + 3\mu)} \left( \operatorname{arctg} \frac{b - \xi_2}{\xi_1} - \operatorname{arctg} \frac{b + \xi_2}{\xi_1} - \operatorname{arctg} \frac{a - \xi_2}{\xi_1} + \operatorname{arctg} \frac{a + \xi_2}{\xi_1} \right); \\ u_2(\xi) &= -\frac{\gamma T_0 \xi_1}{\pi(\lambda + 3\mu)} \ln \frac{r(b; \xi) r_2(b; \xi)}{r(a; \xi) r_2(a; \xi)}; \\ r(a; \xi) &= \sqrt{\xi_1^2 + (a - \xi_2)^2}; \quad r(b; \xi) = \sqrt{\xi_1^2 + (b - \xi_2)^2}; \\ r_2(a; \xi) &= \sqrt{\xi_1^2 + (a + \xi_2)^2}; \quad r_2(b; \xi) = \sqrt{\xi_1^2 + (b + \xi_2)^2}. \end{aligned} \quad (58)$$

To determinate the expressions for thermal stresses we need to determine preliminarily the temperature field in the quarter-plane, created by temperature in Eq. (55), using the Poisson's integral formula for heat conduction problems:

$$\begin{aligned} T(\xi) &= -T_0 \int_a^b \frac{\partial}{\partial n_{y_1}} G(0, y_2; \xi) dy_2 = \frac{T_0}{\pi} \int_a^b \xi_1 \left( \frac{1}{r^2} - \frac{1}{r_2^2} \right) dy_2 = \frac{T_0}{\pi} \left( \operatorname{arctg} \frac{y_2 - \xi_2}{\xi_1} - \operatorname{arctg} \frac{y_2 + \xi_2}{\xi_1} \right)_{y_2=a}^{y_2=b}; \\ T(\xi) &= \frac{T_0}{\pi} \left( \operatorname{arctg} \frac{b - \xi_2}{\xi_1} - \operatorname{arctg} \frac{b + \xi_2}{\xi_1} - \operatorname{arctg} \frac{a - \xi_2}{\xi_1} + \operatorname{arctg} \frac{a + \xi_2}{\xi_1} \right), \end{aligned} \quad (59)$$

where the expression in Eq. (16) for the Green function was used.

It is easy to see that temperature in Eq. (59) satisfies the BVP in heat conduction, described by Poisson Eq. (9) [for  $F(\xi) = 0$ ] and boundary condition in Eq. (55).

Now, to determinate the expressions for thermal stresses we use the Duhamel–Neumann law in Eq. (3) at  $i, j = 1, 2$ , rewritten in the form:

$$\begin{aligned} \sigma_{11} &= 2\mu u_{1,1} + \lambda u_{j,j} - \gamma T; \quad i, j = 1, 2; \rightarrow \sigma_{11} = (\lambda + 2\mu) u_{1,1} + \lambda u_{2,2} - \gamma T; \\ \sigma_{22} &= 2\mu u_{2,2} + \lambda u_{j,j} - \gamma T; \rightarrow \sigma_{22} = (\lambda + 2\mu) u_{2,2} + \lambda u_{1,1} - \gamma T; \quad \sigma_{12} = \mu(u_{1,2} + u_{2,1}). \end{aligned} \quad (60)$$

Finally, calculating the derivatives with respect to displacements (58) and substituting them and temperature (59) in Eq. (60), we obtain the following thermal stresses of the stated in this section BVP in the explicit form:

a. for normal thermal stresses

$$\begin{aligned} \sigma_{11}(\xi) &= \frac{\gamma \mu T_0}{\pi(\lambda + 3\mu)} \left[ 2\xi_1 \left( \frac{b + \xi_2}{r_2^2(b; \xi)} - \frac{b - \xi_2}{r^2(b; \xi)} - \frac{a + \xi_2}{r_2^2(a; \xi)} + \frac{a - \xi_2}{r^2(a; \xi)} \right) \right. \\ &\quad \left. + \left( \operatorname{arctg} \frac{b + \xi_2}{\xi_1} - \operatorname{arctg} \frac{a + \xi_2}{\xi_1} - \operatorname{arctg} \frac{b - \xi_2}{\xi_1} + \operatorname{arctg} \frac{a - \xi_2}{\xi_1} \right) \right]; \\ \sigma_{22}(\xi) &= -\frac{\gamma \mu T_0}{\pi(\lambda + 3\mu)} \left[ 2\xi_1 \left( \frac{b + \xi_2}{r_2^2(b; \xi)} - \frac{b - \xi_2}{r^2(b; \xi)} - \frac{a + \xi_2}{r_2^2(a; \xi)} + \frac{a - \xi_2}{r^2(a; \xi)} \right) \right. \\ &\quad \left. + 3 \left( \operatorname{arctg} \frac{b + \xi_2}{\xi_1} - \operatorname{arctg} \frac{a + \xi_2}{\xi_1} - \operatorname{arctg} \frac{b - \xi_2}{\xi_1} + \operatorname{arctg} \frac{a - \xi_2}{\xi_1} \right) \right] \end{aligned} \quad (61)$$

and

b. for tangential thermal stresses

$$\sigma_{12} = -\mu \frac{\gamma T_0}{\pi(\lambda + 3\mu)} \left[ 2\xi_1^2 \left( \frac{1}{r^2(b; \xi)} + \frac{1}{r_2^2(b; \xi)} - \frac{1}{r^2(a; \xi)} - \frac{1}{r_2^2(a; \xi)} \right) + \ln \frac{r(b; \xi) r_2(b; \xi)}{r(a; \xi) r_2(a; \xi)} \right]. \quad (62)$$

The investigations have shown that the obtained solutions in elementary functions in Eq. (58) — for thermal displacements and in Eqs. (61) and (62) — for thermal stresses, satisfy the BVP in Eqs. (6)–(9) at the boundary conditions (55). All solutions vanish at infinity.

#### 4. Conclusions

1. The Poisson-type integral formula, obtained in this paper, is new, useful and completely ready to be efficiently applied for computing of the thermoelastic displacements  $u_k(\xi)$  and thermal stresses  $\sigma_{ij}(\xi)$  in quarter-planes (see example of its application given in Section 3). The main advantage of the obtained integral formula is that the searches in quarter-plane thermoelastic displacements and thermal stresses are expressed directly via given inside heat source, boundary temperature and known kernels.
2. The expressions for thermoelastic influence functions  $U_k(x, \xi)$ , given in Eqs. (19) and (49), the functions  $\partial U_k(y, \xi)/\partial n_{y1}$  and  $\partial U_k(y, \xi)/\partial n_{y2}$ , given in Eqs. (13) and (14) or (51) and (52), as well as the Poisson-type integral formula, presented in Eqs. (11)–(14) or (53), play a very important role in the development and examination of accuracy of the numerical methods employed to obtain solution to the respective BVP for thermoelastic quarter-planes. Furthermore, on the basis of the above-mentioned results for quarter-planes, a special boundary element for obtaining a solution to the BVP of thermoelasticity using BEM [14] can be elaborated;
3. The most difficult problems in the proposed method here are the problems of deriving the Green's functions  $G(x, \xi)$  in heat conduction and the functions of influence for volume dilatation  $\Theta^{(k)}(x, \xi)$  in the elasticity theory. Note, that for canonical Cartesian domains these problems were solved successfully in the handbook [8]. The computing of the integral over volume in Eq. (5) of the product of functions  $G(x, \xi)$  and  $\Theta^{(k)}(x, \xi)$  is also solved successfully [see the recommendations of the Appendix]. So, for canonical Cartesian domains, the proposed method works successfully. This means that the presented paper has opened great possibilities for researchers to derive many new Poisson-type integral formulae, not only for quarter-planes, but also for many other canonical Cartesian domains.
4. The approach presented in this paper in thermoelasticity for Cartesian canonical domains can be extended to domains of many orthogonal systems of coordinates [15–18]. This extension will be done when the lists of the respective functions  $G(x, \xi)$  and  $\Theta^{(k)}(x, \xi)$  are completed. Also, this approach is valid for other physical phenomena such as electroelasticity, magnetoelasticity and poroelasticity, described by the same BVP as thermoelasticity.

#### Acknowledgements

Dr Şeremet is grateful to the University Paris-Est, Marne-la-Vallée, Laboratory MSME UMR 8208 CNRS, France and to the Weierstrass Institute for Applied Analysis and Stochastic, Berlin, Germany for their support of research visits. Also he expresses many thanks to the reviewers of this paper, whose comments have contributed substantially to its improvement.

#### Appendix

The improper integral in Eq. (34) was taken using the relations

$$\begin{aligned} \nabla_z^2 \frac{1}{4} [r^2(x, z)(\ln r(x, z) - 1)] &= \ln r(x, z); & \nabla_z^2 \frac{1}{4} [r_1^2(x, z)(\ln r_1(x, z) - 1)] &= \ln r_1(x, z); \\ \nabla_z^2 \frac{1}{4} [r_2^2(x, z)(\ln r_2(x, z) - 1)] &= \ln r_2(x, z); & \nabla_z^2 \frac{1}{4} [r_{12}^2(x, z)(\ln r_{12}(x, z) - 1)] &= \ln r_{12}(x, z), \end{aligned} \quad (\text{A.1})$$

the property of Dirac's function

$$\int_V f(x) \delta(x - \xi) dV(x) = f(\xi) \quad (\text{A.2})$$

and the Green's formula inside the quarter-plane

$$\int_V (\varphi \nabla^2 f - f \nabla^2 \varphi) dV = \int_\Gamma [\varphi (\partial f / \partial n) - f (\partial \varphi / \partial n)] d\Gamma, \quad (\text{A.3})$$

where the functions  $f$  and  $\varphi$  are determined as follows:

$$\begin{aligned} f = \frac{1}{4} \frac{1}{2\pi} \left\{ \frac{\partial}{\partial \xi_k} [r_1^2(x, z)(\ln r_1(x, z) - 1) + r_2^2(x, z)(\ln r_2(x, z) - 1) - r^2(x, z)(\ln r(x, z) - 1) \right. \\ \left. - r_{12}^2(x, z)(\ln r_{12}(x, z) - 1)] + 2L_{\Theta}^{(k)} \frac{\partial}{\partial \xi_1} [r_1^2(x, z)(\ln r_1(x, z) - 1) - r_{12}^2(x, z)(\ln r_{12}(x, z) - 1)] \right\} \end{aligned} \quad (\text{A.4})$$

and

$$\varphi = \frac{1}{2\pi} \ln \frac{r_1(x, z)r_2(x, z)}{r(x, z)r_{12}(x, z)}. \quad (\text{A.5})$$

As on the boundary semi-straight-lines  $\Gamma_{10}$ ,  $\Gamma_{20}$  for the functions  $\varphi$  and  $f$  we have the values:

$$\begin{aligned} \varphi(x; z_1 = 0, z_2) = \varphi(x; z_1; z_2 = 0) = 0; \quad f(z_1 = 0, z_2; \xi) = -\frac{1}{2\pi} 2L_{\Theta}^{(k)} \frac{\partial}{\partial \xi_1} \ln \frac{r(z_1 = 0, z_2; \xi)}{r_2(z_1 = 0, z_2; \xi)}; \\ f(z_1; z_2 = 0; \xi) = 0 \end{aligned} \quad (\text{A.6})$$

the formula (A.3) can be rewritten as follows:

$$\int_V (\varphi \nabla^2 f - f \nabla^2 \varphi) dV = - \int_{\Gamma_{10}} f (\partial \varphi / \partial n) d\Gamma_{10}. \quad (\text{A.7})$$

So, using in full the recommended Eqs. (A.1)–(A.7), we obtain:

$$\begin{aligned} U_k(x, \xi) &= \gamma \int_0^\infty \int_0^\infty G(x, z) \Theta^{(k)}(z, \xi) dz_1 dz_2 = -(\lambda + 2\mu)^{-1} \gamma \int_0^\infty \int_0^\infty \frac{1}{2\pi} \ln \frac{r_1(x, z) r_2(x, z)}{r(x, z) r_{12}(x, z)} \\ &\quad \times \frac{1}{2\pi} \left[ \frac{\partial}{\partial \xi_k} \ln \frac{r_1(z, \xi) r_2(z, \xi)}{r(z, \xi) r_{12}(z, \xi)} + 2L_{\Theta}^{(k)} \frac{\partial}{\partial \xi_1} \ln \frac{r_1(z, \xi)}{r_{12}(z, \xi)} \right] dz_1 dz_2 \\ &= -(\lambda + 2\mu)^{-1} \gamma \int_0^\infty \int_0^\infty \varphi(x, z) \nabla_z^2 f(z, \xi) dz_1 dz_2 \\ &= -(\lambda + 2\mu)^{-1} \gamma \int_0^\infty \int_0^\infty f(z, \xi) \nabla_z^2 \varphi(x, z) dz_1 dz_2 + I(x, \xi) \\ &= -(\lambda + 2\mu)^{-1} \gamma \int_0^\infty \int_0^\infty -\delta(x - z) f(z, \xi) dz_1 dz_2 + I(x, \xi). \end{aligned} \quad (\text{A.8})$$

Calculating the integral  $I$  in Eq. (A.8)

$$\begin{aligned} I(x, \xi) &= -(\lambda + 2\mu)^{-1} \gamma \int_{\Gamma_{10}} f (\partial \varphi / \partial n) d\Gamma_{10} = -(\lambda + 2\mu)^{-1} \frac{1}{8\pi} \gamma \int_0^\infty 2L_{\Theta}^{(k)} \frac{\partial}{\partial \xi_1} [r^2(y, \xi) (\ln r(y, \xi) - 1) \\ &\quad - r_2^2(y, \xi) (\ln r_2(y, \xi) - 1)] \frac{\partial}{\partial n_{y_1}} \frac{1}{2\pi} \ln \frac{r_1(x, y) r_2(x, y)}{r(x, y) r_{12}(x, y)} dy_2 \\ &= \frac{(\lambda + 2\mu)^{-1} \gamma}{8\pi} \left\{ 2L_{\Theta}^{(k)} \frac{\partial}{\partial \xi_1} [r_1^2 (\ln r_1 - 1) - r_{12}^2 (\ln r_{12} - 1)] + 4x_1 L_{\Theta}^{(k)} \ln \frac{r_1}{r_{12}} \right\} \end{aligned} \quad (\text{A.9})$$

and taking into account the property (A.2) of Dirac's function, we obtain the following final expression for searched thermoelastic influence functions

$$\begin{aligned} U_k(x, \xi) &= \frac{\gamma}{8\pi(\lambda + 2\mu)} \left\{ \frac{\partial}{\partial \xi_k} [r_1^2 (\ln r_1 - 1) + r_2^2 (\ln r_2 - 1) - r^2 (\ln r - 1) \right. \\ &\quad \left. - r_{12}^2 (\ln r_{12} - 1)] + 4x_1 L_{\Theta}^{(k)} \ln \frac{r_1}{r_{12}} \right\}, \end{aligned} \quad (\text{A.10})$$

which coincide with the final expressions in Eq. (49).

## References

- [1] M.T. Hanson, L.M. Keer, Stress analysis and contact problems for an elastic quarter-plane, *Quart. J. Mech. Appl. Math.* 42 (3) (1989) 364–383.
- [2] B.A. Boley, I.H. Weiner, *Theory of Thermal Stresses*, Wiley, New York, 1960.
- [3] E. Melan, H. Parkus, *Thermoelastic Stresses Caused by the Stationary Heat Fields*, Fizmatgiz, Moscow, 1958 (in Russian).
- [4] W. Nowacki, *Thermoelasticity*, Pergamon Press and Polish Sc. Publishers, Oxford, Warszawa, 1962.
- [5] W. Nowacki, *The Theory of Elasticity*, Mir, Moscow, 1975 (in Russian).
- [6] B. Hetnarski, M. Eslami, *Thermal Stresses — Advanced Theory and Applications*, Springer, 2009.
- [7] V.D. Seremet, New results in 3-D thermoelasticity, in: *Proc. 14th U.S. National Congress of Theoretical and Applied Mechanics*, Virginia Tech, 2002.
- [8] V.D. Seremet, *Handbook on Green's Functions and Matrices*, WIT Press, Southampton, 2003.
- [9] V. Sheremet, Generalization of Green's Formulae in Thermoelasticity. Collection: *Multiscale Green's Functions for Nanostructures*, National Science Digital Library of USA, NSF, 2003, 4p.
- [10] V. Sheremet, New formulae for dynamical thermal stresses, *J. Thermal Stresses* 25 (2) (2002) 123–153.
- [11] V.M. Budac, A.A. Samarskii, A.N. Tikhonov, *Set of Problems on Mathematical Physics*, Gostehizdat, Moscow, 1980 (in Russian).
- [12] J.T. Chen, C.S. Wu, Alternative derivations for the Poisson integral formula, *Int. J. Math. Edu. Sci. Technol.* 37 (2) (2006) 165–185.
- [13] J.T. Chen, Y.T. Lee, S.R. Yu, S.C. Shieh, Equivalence between Trefftz method and method of fundamental solution for the annular Green's function using the addition theorem and image concept, *Eng. Anal. Bound. Elem.* 33 (2009) 678–688.
- [14] C.A. Brebbia, *The Boundary Element Method for Engineers*, Wiley, New York, 1978.
- [15] V. Seremet, G. Bonnet, in: Chisinau (Ed.), *Encyclopedia of Domain Green's Functions (Thermo-Magneto-Electrostatics of Solids in Rectangular and Polar Coordinates)*, Center, Agrar. Univ. of Moldova, 2008.
- [16] V. Sheremet, D. Precupan, I. Vlad, A. Sheremet, The constructing of Green's matrices in cylindrical coordinates, in: *Proc. 17th Eng. Mech. Conference of the ASCE*, University of Delaware, USA, 2004, 9 p.
- [17] V. Seremet, G. Bonnet, T. Speianu, New results in construction of the Green's matrices in spherical coordinates, in: *Proc. of The Inaug. Internat. Conf. of the Engineering Mechanics Institute-EM08*, ASCE, University of Minnesota, USA, 2008 7 p.
- [18] V. Seremet, G. Bonnet, T. Speianu, Influence functions and integral formulae for spherical thermoelastic bodies, in: *Proc. 22nd International Congress of Theoretical and Applied Mechanics, ICTAM2008*, Adelaide University, Australia, 2008, 2 p.