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# Homologies of Algebraic Structures via Braidings and Quantum Shuffles

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## Abstract

In this paper we construct braidings characterizing different algebraic structures: a rack, an associative algebra, a Leibniz algebra and their representations. Some of these braidings seem original. This produces, via braided space (co)homology coming from quantum (co)shuffle (co)multiplication, a family of (co)chain complexes for each of these structures. One recovers Koszul, rack, bar, Hochschild and Leibniz complexes in these families. All the constructions are categorified, resulting in particular in their super- and co-versions. Loday's hyper-boundaries are efficiently treated using the "shuffle" tools. A notion of modules over braided spaces, encompassing algebra, Lie and rack modules, is introduced.

**Keywords:** algebraic structures, braidings, quantum shuffles, homology, duality, hyper-boundaries, Koszul complex, racks, (co)bar complex, Leibniz algebras, Hochschild homology, Chevalley-Eilenberg homology, (Leibniz) algebras in monoidal categories, (bi)modules over braided spaces.

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## 1 Introduction

The aim of this paper is to develop a **unifying algebraic framework for (co)homologies of algebraic structures**. The central examples considered here are those of racks (or, more generally, self-distributive systems) and of associative or Lie (more generally, Leibniz) algebras and coalgebras, as well as of their representations. These examples are quite simple but eloquent in illustrating the advantages of our general constructions. More structures (bialgebras and Hopf algebras, Hopf and Yetter-Drinfeld modules) are left for a subsequent paper. The very pedestrian approach chosen here makes our theory adapted for applications.

Let us first present the motivation behind **using braidings and quantum shuffles** (the notion was introduced by Rosso, see [25] and [26]) **for understanding (co)homologies**. Given an algebraic structure (i.e. a set  $S$  or a vector space  $V$  endowed with several operations, obeying a list of compatibility properties), there are various approaches to constructing a differential on  $T(V)$  (with  $V = \mathbb{k}S$  in the case of a set), giving (co-)homologies containing useful information about the original structure. One can think in terms of derived functors, or structure deformations and obstructions, or topology (regarding the notion of an algebra as a generalization of the algebra of functions on a space, or trying to devise a state-sum knot invariant using the fundamental quandle of a knot, cf. [3]). The formulas for the differentials obtained, when written down explicitly, are often quite simple and have the same flavor. They are signed sums of terms of the same nature, one for each component  $1, 2, \dots, n$  of  $\overline{V^{\otimes n}}$ , with the terms being local in some sense: for associative algebras, each term of the bar or Hochschild differential involves only two neighboring components, while for racks and Leibniz algebras the  $i$ -th term "sees" only the components to the left of the  $i$ -th one. Verifying that one has indeed a differential, i.e.  $d^2 = 0$ , consists of checking some local algebraic identities (which mysteriously coincide with defining properties for our algebraic structure!) coupled with a sign manipulation, no less mysterious. This procedure makes one think of quantum (co)shuffle (co)multiplication, with:

- ⊗ the same routine applied at each position (for instance, when calculating  $v_0 \Delta v_1 \otimes v_2 \otimes \dots \otimes v_n$ , with  $v_i \in V$ );
- ⊗ the sign depending on the position (if our braiding has a sign coefficient  $(-1)^{\deg v \deg w}$  for example);
- ⊗ the overall (co)associativity which is a consequence of the local Yang-Baxter identity involving three neighboring components of  $V^{\otimes n}$  only.

Braided vector spaces and their homologies allow to formalize these parallels.

The **(co)homology constructing machinery** proposed here can be roughly outlined as follows (all notions and constructions are detailed in the paper):

1. Given an algebraic structure on a vector space  $V$ , find a **braiding encoding this structure**. For racks, this turns out to be a well-known braiding coming from the topological interpretation in knot theory (cf. the foundational papers [10] and [21]). For associative and Lie algebras, the braidings presented here seem to be original.
2. Add a sign to the braiding (think of the signs  $(-1)^i$  in the formulas for differentials!).
3. This gives a (co)associative quantum (co)shuffle (co)multiplication on  $T(V)$ .
4. Find a *cut*, i.e. a "square zero" element  $v \in V$  or co-element  $\varphi \in V^*$  for this signed (co)multiplication. In concrete situations cuts will turn out to be familiar algebraic objects (units, characters, group-like elements etc.).
5. Quantum shuffle multiplication by this element on the right or on the left is thus a differential on  $T(V)$ , and similarly in the co-case.

The surprising fact is that in the associative case our braiding is **non-invertible**, which luckily never causes any problems in the constructions described. Continuing Majid's "braided" philosophy (cf. [19] and references therein), this means that, after going from permutation groups to braid groups by relaxing the idempotence condition  $\sigma_i^2 = 1$ , we move further on to the monoids of positive braids by forgetting the invertibility condition for  $\sigma_i$ 's.

The complexes easily obtained by our procedure include Koszul, rack, bar, Leibniz, Hochschild and Cartier complexes, all of these in quite a general version. For each of them one thus obtains an easy and conceptual way of proving  $d^2 = 0$ , as well as of "guessing" the right boundary map.

The appearance of **Leibniz homology** is quite remarkable here. It was defined by Loday (cf. [15],[16],[17]) as a non-commutative analogue of the Chevalley-Eilenberg homology of Lie algebras, allowing to lift the corresponding boundary map from the exterior to the tensor algebra. Our approach is an alternative way to "guess" the right lift.

Note that our construction produces two differentials – a left and a right one – for each cut. They are moreover compatible, giving a bicomplex structure. In addition, differentials for different cuts are often compatible, allowing one

to combine them. Thus we get a **family of differentials** for each algebraic structure, the familiar one being just one combination of them.

The interpretation of chain complexes for algebraic structures presented here is similar in some sense to the language of **simplicial modules**, but has the advantage of being more transparent, saying more on the nature of the  $d_i$  components and on the reasons of their compatibility. A detailed comparison is given in section 3.

Homologies of braided vector spaces are briefly mentioned in some sources, but no detailed study of appearing complexes seem to exist, perhaps because of their extreme generality. On the contrary, the step

$$\boxed{\text{”algebraic structure = braiding”}},$$

which is at the heart of this paper, seems to have never been proposed before. We allow ourselves the equality sign ”=” rather than the implication ” $\Rightarrow$ ” since in all the examples considered here the Yang-Baxter equality characterizing our braiding is equivalent to the defining properties of the original algebraic structure.

One more nice feature of our differentials is that they can be easily generalized to give **Loday’s hyper-boundaries** (cf. [15], exercise E.2.2.7), producing a convenient tool for manipulating them (see in particular theorem 2). Section 4 is dedicated to this point.

All constructions presented in the paper are raised to the **categoric level** in section 6. No special technical difficulties are encountered during such a generalization. One gets in particular categoric versions of bar and Leibniz complexes for associative or Leibniz algebras in an arbitrary (symmetric in the Leibniz case) preadditive category, in the spirit of [2] and [1]. Three applications of the categoric approach are presented: **Leibniz superalgebra** homology, chain complexes for **dual structures** (e.g. Cartier differential for coalgebras) and right-left duality for braidings and differentials. The dualities are also interpreted pictorially, with an operadic flavor. Note that the categoric translation of our constructions is often ”doubly braided”: an ”external” braiding (the one encoding an algebraic structure; it can be quite complicated) is built in a symmetric monoidal category (with a quite simple underlying ”internal” braiding – a flip or a Koszul flip in the examples). Thus one gets representations of **virtual braid groups**.

The last section presents two approaches to homologies with coefficients. The first one is quite general and goes through the notion of **modules over braided vector spaces**, generalizing simultaneously and quite surprisingly modules over associative or Leibniz algebras (cf. [15]) and racks (cf. [5]), as well as our cuts. A bicomplex is associated to a pair of such modules (a left and a right one), giving familiar homologies with coefficients in all our particular cases. Hochschild homology is obtained by introducing bimodules over braided vector spaces. The second approach is a **structure mixing technique**: chain complexes for a module  $V$  over an associative or Leibniz algebra  $A$  are built by amalgamating all the structures into one algebra structure on  $V \oplus A$ .

**Graphical calculus** (in the spirit of Baez ([2]), Majid ([20]) and other authors) is extensively used throughout this paper, giving an illustrating tool, a convenient method of presenting some proofs and also an important source of inspiration.

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**Notations and conventions.**

We work with vector spaces over a base field  $\mathbb{k}$  (most statements still remaining true over a commutative ring). All tensor products are over  $\mathbb{k}$  if nothing is explicitly mentioned. Notations

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n},$$

$$T(V)_+ := \bigoplus_{n \geq 1} V^{\otimes n}$$

are used for the (reduced) tensor algebra of  $V$ , with  $V^{\otimes 0} := \mathbb{k}$ . A simplified notation is used for its elements:

$$\bar{v} = v_1 v_2 \dots v_n := v_1 \otimes v_2 \otimes \dots \otimes v_n \in V^{\otimes n}$$

leaving the tensor product sign for

$$v_1 v_2 \dots v_n \otimes w_1 w_2 \dots w_m \in V^{\otimes n} \otimes V^{\otimes m}.$$

Sometimes the notation  $V^{\otimes n}$  will be reduced to  $V^n$ .

We will often call the vector space  $T(V)$  *the tensor space of  $V$* , emphasizing that we endow it with a multiplication different from the usual concatenation.

The dual of a vector space  $V$  is denoted by  $V^* := \text{Hom}_{\mathbb{k}}(V, \mathbb{k})$ .

The word *complex* will mean a differential (co)chain complex, and similarly for bicomplexes.

The symmetric and braid groups on  $n$  elements are denoted by  $S_n$  and  $B_n$  respectively. Inclusions  $S_n \subset S_m$  and  $B_n \subset B_m$  for  $n < m$ , implicit in the paper, are obtained by letting an  $s \in S_n$  act on the first  $n$  elements of an  $m$ -tuple, or, respectively, by adding  $m - n$  untangled strands on the right of an  $n$ -braid. We use the usual action of  $S_n$  on  $V^{\otimes n}$ :

$$\sigma(v_1 v_2 \dots v_n) := v_{\sigma^{-1}(1)} v_{\sigma^{-1}(2)} \dots v_{\sigma^{-1}(n)}.$$

## 2 Braided world: a short reminder

We recall here different facts about braided vector spaces necessary for subsequent sections. For a detailed treatment of braid groups, [12] is an excellent reference. As for quantum shuffles, we send the reader to [26] and [25].

**Definition 2.1.** A *braiding* on a vector space  $V$  is a linear map  $\sigma : V \otimes V \rightarrow V \otimes V$  satisfying the *Yang-Baxter identity*

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 : V \otimes V \otimes V \rightarrow V \otimes V \otimes V, \quad (\text{YB})$$

where  $\sigma_i$  is the braiding  $\sigma$  applied to components  $i$  and  $i + 1$  of  $V^{\otimes 3}$ .

A braiding is called *symmetric* if  $\sigma^2 = \text{Id}_{V \otimes V}$ .

A vector space endowed with a braiding is called *braided*.

Note that unlike most authors **we do not demand that the braiding be invertible**, and later on we will present some interesting highly non-invertible braidings.

A braiding on a set is defined similarly: tensor products  $\otimes$  are simply replaced by Cartesian products  $\times$ . These two settings are particular cases of a more abstract one: they both come from **braided categories**, studied in more detail in section 6.

The basic examples of braidings are the *flip*

$$\tau : v \otimes w \mapsto w \otimes v,$$

the *signed flip*

$$-\tau : v \otimes w \mapsto -w \otimes v,$$

and their generalization for graded vector spaces, the *Koszul flip*

$$\tau_{\text{Koszul}} : v \otimes w \mapsto (-1)^{\deg v \deg w} w \otimes v$$

for homogeneous  $v$  and  $w$ . This last braiding explains the **Koszul sign convention** in many settings.

Remark that in general for a braiding  $\sigma$ , its opposite  $-\sigma : v \otimes w \mapsto -\sigma(v \otimes w)$  is also a braiding.

A braiding gives an action of the braid monoid  $B_n^+$  (i.e. the monoid of positive braids, cf. [12]) on  $V^{\otimes n}$  which is best depicted in the graphical form

$$\sigma_i(\vec{v}) = \begin{array}{ccccccc} v_1 \otimes & v_2 \otimes & & v_{i-1} \otimes & \sigma(v_i \otimes v_{i+1}) & \otimes v_{i+2} & \otimes v_n \\ | & | & & | & \text{X} & | & | \\ v_1 \otimes & v_2 \otimes & \dots & v_{i-1} \otimes & v_i \otimes v_{i+1} & \otimes v_{i+2} & \dots \otimes v_n \end{array}$$

All diagrams in this paper are to be read **from bottom to top**. One could have presented the crossing as  $\diagup \diagdown$ . It is just a matter of convention, and the one used here comes from the rack theory (section 5.2).

For symmetric braidings the action above is in fact an action of the symmetric group  $S_n$ .

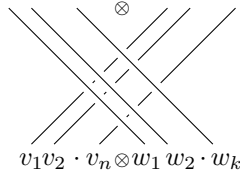
The graphical translation of the Yang-Baxter equation (YB) for braidings is the third Reidemeister move:



*Remark 2.2.* A braiding  $\sigma$  on  $V$  is naturally extended to a braiding  $\sigma$  on its tensor space  $T(V)$  by

$$\sigma(\bar{v} \otimes \bar{w}) = (\sigma_k \cdots \sigma_1) \cdots (\sigma_{n+k-2} \cdots \sigma_{n-1}) (\sigma_{n+k-1} \cdots \sigma_n) (\bar{v}\bar{w}) \in V^{\otimes k} \otimes V^{\otimes n}$$

for  $\bar{v} \in V^{\otimes n}, \bar{w} \in V^{\otimes k}$ , or graphically:



Now, the famous inclusion

$$S_n \hookrightarrow B_n$$

$$s = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_k} \longmapsto T_s := \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$$

where

- ⊗  $\tau_i \in S_n$  are transpositions of neighboring elements  $i$  and  $i+1$ , called *simple transpositions*,
- ⊗  $\sigma_i$  are the corresponding generators of  $B_n$ ,
- ⊗  $\tau_{i_1} \tau_{i_2} \cdots \tau_{i_k}$  is one of the shortest words representing  $s$ ,

is well defined and factorizes through

$$S_n \hookrightarrow B_n^+ \hookrightarrow B_n.$$

This is a set inclusion not preserving the monoid structure. A more precise result will be necessary later:

**Lemma 2.3.** We have  $T_{s_1 s_2} = T_{s_1} T_{s_2}$  if and only if for each pair of elements  $(i, j)$  reversed by  $s_2$ , their images  $(s_2(i), s_2(j))$  are not reversed by  $s_1$ .

The following subsets of symmetric groups will be extensively used afterwards:



**Definition 2.4.** The permutation sets

$$Sh_{p,q} := \left\{ s \in S_{p+q} \text{ s.t. } \begin{cases} s(1) < s(2) < \dots < s(p), \\ s(p+1) < s(p+2) < \dots < s(p+q) \end{cases} \right\}$$

or, more generally,

$$Sh_{p_1,p_2,\dots,p_k} := \left\{ s \in S_{p_1+p_2+\dots+p_k} \text{ s.t. } \begin{cases} s(1) < s(2) < \dots < s(p_1), \\ s(p_1+1) < \dots < s(p_1+p_2), \\ \dots, \\ s(p+1) < s(p+2) < \dots < s(p+p_k) \end{cases} \right\}$$

where  $p = p_1 + p_2 + \dots + p_{k-1}$ , are called *shuffle sets*.

The conditions from this definition mean that one permutes  $p_1 + p_2 + \dots + p_k$  elements preserving the order within  $k$  consecutive blocks of size  $p_1, p_2, \dots, p_k$ , just like when shuffling cards, which explains the name. The set  $Sh_{p_1,p_2,\dots,p_k}$  consists of  $\binom{p_1+p_2+\dots+p_k}{p_1,p_2,\dots,p_k}$  elements.

**Lemma 2.5.** Take  $p, q, r \in \mathbb{N}$  and put  $n = p + q + r$ . Viewing  $Sh_{p,q} \subseteq S_{p+q}$  and  $Sh_{q,r} \subseteq S_{q+r}$  as subsets of  $S_n$  by letting  $Sh_{p,q}$  permute the first  $p + q$  elements of an  $n$ -tuple, and, similarly, by letting  $Sh_{q,r}$  permute the last  $q + r$  elements of an  $n$ -tuple, one has the following decomposition:

$$Sh_{p,q,r} = Sh_{p+q,r} Sh_{p,q} = Sh_{p,q+r} Sh_{q,r}.$$

That is, an element of  $Sh_{p,q,r}$  can be seen, in a unique way, as an element of  $Sh_{p,q}$  followed by one from  $Sh_{p+q,r}$ , and similarly for the second decomposition.

Everything is now ready for defining quantum shuffle algebras. This structure originated in the work of Rosso ([24],[25]).

**Definition 2.6.** The *quantum shuffle multiplication* on the tensor space  $T(V)$  of a braided vector space  $(V, \sigma)$  is the  $\mathbb{k}$ -linear extension of the map

$$\begin{aligned} \Delta_\sigma &= \Delta_\sigma^{p,q} : V^{\otimes p} \otimes V^{\otimes q} \longrightarrow V^{\otimes(p+q)} \\ \bar{v} \otimes \bar{w} &\longmapsto \bar{v} \Delta_\sigma \bar{w} := \sum_{s \in Sh_{p,q}} T_s^\sigma(\bar{v}\bar{w}). \end{aligned}$$

The expression  $\bar{v}\bar{w}$  in the brackets means just the concatenation of pure tensors  $\bar{v}$  and  $\bar{w}$ . Notation  $T_s^\sigma$  stands for  $T_s \in B_n^+$  acting on  $V^{\otimes n}$  via the braiding  $\sigma$ .

Endowed with this multiplication,  $T(V)$  is called the *quantum shuffle algebra* of  $(V, \sigma)$  and denoted by  $Sh_\sigma(V)$ .

The symbol  $\Delta$  comes from a Cyrillic letter pronounced as "sh" in English.

In the case of the trivial braiding ( $\sigma = \text{flip}$ ) one speaks simply about the *shuffle algebra* of  $V$ , and a simplified notation  $\Delta$  is used.

By a *braided Hopf algebra* (in the sense of Majid, cf. Definition 2.2 in [19] for example) we will mean additional structure on a braided vector space satisfying all the axioms of a Hopf algebra except for the compatibility between the multiplication and the comultiplication, which is replaced by the braided compatibility (this last notion will be recalled in the following proposition). More generally, it is a Hopf algebra in a braided category (see section 6 for categoric notions).

The quantum shuffle multiplication can be upgraded to give an interesting example of a braided Hopf algebra structure (commutative if the initial braiding is symmetric):

**Proposition 2.7.** *Let  $(V, \sigma)$  be a braided vector space.*

1. *The multiplication  $\Delta_\sigma$  of  $Sh_\sigma(V)$  is associative.*
2. *If  $\sigma^2 = \text{Id}$ , then the multiplication  $\Delta_\sigma$  is  $\sigma$ -commutative, i.e.*

$$\Delta_\sigma(\bar{v} \otimes \bar{w}) = \Delta_\sigma(\sigma(\bar{v} \otimes \bar{w}))$$

*(with the extension of  $\sigma$  to  $T(V)$  from remark 2.2).*

3. *The element  $1 \in \mathbb{k}$  is a unit for  $Sh_\sigma(V)$ .*
4. *The deconcatenation map*

$$\begin{aligned} \Delta : v_1 v_2 \dots v_n &\longmapsto \sum_{p=0}^n v_1 v_2 \dots v_p \otimes v_{p+1} \dots v_n, \\ 1 &\longmapsto 1 \otimes 1, \end{aligned}$$

*(where an empty product means 1), and the augmentation map*

$$\begin{aligned} \varepsilon : v_1 v_2 \dots v_n &\longmapsto 0, \\ 1 &\longmapsto 1, \end{aligned}$$

*define a coalgebra structure on  $T(V)$ .*

5. *These algebra and coalgebra structures are  $\sigma$ -compatible, in the sense that*

$$\Delta \circ \Delta_\sigma = (\Delta_\sigma \otimes \Delta_\sigma) \circ \sigma_2 \circ (\Delta \otimes \Delta).$$

6. *An antipode can be given on  $Sh_\sigma(V)$  by the formula*

$$\begin{aligned} s : \bar{v} &\longmapsto (-1)^n T_{\rho_n}^\sigma(\bar{v}), \quad \bar{v} \in V^{\otimes n}, \quad \rho_n = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix} \in S_n, \\ 1 &\longmapsto 1. \end{aligned}$$

The braided vector space  $(Sh_\sigma(V), \sigma)$  is thus endowed with a braided Hopf algebra structure.

*Proof.* We will only give the proof of the most difficult statements.

1. Let  $\bar{v} \in V^p, \bar{w} \in V^q, \bar{u} \in V^r$ . We have

$$\begin{aligned} (\bar{v} \Delta_\sigma \bar{w}) \Delta_\sigma \bar{u} &= \sum_{s \in Sh_{p+q,r}, t \in Sh_{p,q}} T_s^\sigma \circ (T_t^\sigma \otimes \text{Id}_r)(\overline{vwu}) \\ &\stackrel{\text{Lemma 2.3}}{=} \sum_{\substack{s \in Sh_{p+q,r}, \\ t \in Sh_{p,q} \subseteq S_{p+q} \subseteq S_{p+q+r}}} T_{sot}^\sigma(\overline{vwu}) \\ &\stackrel{\text{Lemma 2.5}}{=} \sum_{s \in Sh_{p,q,r}} T_s^\sigma(\overline{vwu}). \end{aligned}$$

The same reasoning gives

$$\bar{v} \Delta_\sigma (\bar{w} \Delta_\sigma \bar{u}) = \sum_{s \in Sh_{p,q,r}} T_s^\sigma(\overline{vwu}),$$

so these two expressions are equal.

6. Take a  $\bar{v} \in V^{\otimes n}, n \geq 1$ . There are two types of signed summands in the expression of  $\Delta_\sigma \circ (\text{Id} \otimes s) \circ \Delta(\bar{v})$ : those where the last element in the quantum shuffle product comes from the first component of  $\Delta(\bar{v}) \in T(V) \otimes T(V)$ , and those where it comes from the second one. Each summand appears exactly once in each type, and with different signs due the sign  $(-1)^{\dots}$  in the definition of the antipode. The overall sum is therefore zero.  $\square$

The above proposition is well-known for invertible braidings ([26]); we point out that it still holds when the braiding admits no inverse.

*Remark 2.8.* If our vector space  $V$  is finite dimensional, then the graded dual of  $T(V)$  inherits a braided Hopf algebra structure from the one described above, with the usual concatenation product and the *quantum co-shuffle coproduct*:

$$\begin{aligned} \Theta_\sigma|_{V^{\otimes n}} &:= \sum_{p+q=n; p,q \geq 0} \Theta_\sigma^{p,q}, \\ \Theta_\sigma^{p,q} &:= \sum_{s \in Sh_{p,q}} T_{s^{-1}}^\sigma : W^{\otimes n} \longrightarrow W^{\otimes p} \otimes W^{\otimes q}, \quad W := V^*. \end{aligned}$$

The components  $\Theta_\sigma^{p,q}$  are often called *quantum unshuffles*. Moreover, this formula defines a coproduct on  $T(W)$  for an arbitrary (even infinite-dimensional!) vector space  $W$ , and this coproduct upgrades to a braided Hopf algebra structure "dual" to that described in the previous proposition.

The comultiplication

$$\Theta_\sigma^{red}|_{V^{\otimes n}} := \sum_{p+q=n; p,q > 0} \Theta_\sigma^{p,q}$$

is also coassociative on  $T(V)$ , as well as on  $T(V)_+$ . It is called the *reduced quantum co-shuffle coproduct*.

One should wait for section 6.3 for a detailed treatment of this duality.

### 3 (Co)homology of braided vector spaces

This section is devoted to a systematic way of constructing differentials on  $T(V)$  for a braided vector space  $(V, \sigma)$ . The compatibility of such differentials as well as their behavior under some quotients and restrictions are also studied. We create here a tool which will be applied in various familiar algebraic settings in subsequent sections and in subsequent papers.

Fix a braided vector space  $(V, \sigma)$ .

We start with distinguishing elements of  $V$  and  $V^*$  which behave with respect to the braiding  $\sigma$  as if it were just a flip  $\tau$ .

**Definition 3.1.**     $\otimes$  Two elements  $v, w \in V$  are called  $\sigma$ -compatible if

$$\sigma(v \otimes w) = w \otimes v, \quad \text{and} \quad \sigma(w \otimes v) = v \otimes w.$$

$\otimes$  A *lower cut* for a braided vector space  $(V, \sigma)$  is an element  $e \in V$  which is  $\sigma$ -compatible with itself, i.e.

$$\sigma(e \otimes e) = e \otimes e, \quad (\text{cut})$$

or, in the shuffle form,

$$e \underset{-\sigma}{\Delta} e = 0.$$

$\otimes$  Two co-elements  $f, g \in V^*$  are called  $\sigma$ -compatible if

$$(f \otimes g) \circ \sigma = g \otimes f, \quad \text{and} \quad (g \otimes f) \circ \sigma = f \otimes g.$$

$\otimes$  An *upper cut* for a braided vector space  $(V, \sigma)$  is an element  $\epsilon \in V^*$  which is  $\sigma$ -compatible with itself, i.e.

$$(\epsilon \otimes \epsilon) \circ \sigma = \epsilon \otimes \epsilon, \quad (\text{Cut})$$

or, in the shuffle form,

$$(\epsilon \otimes \epsilon)(v \underset{-\sigma}{\Delta} w) = 0 \quad \forall v, w \in V.$$

Units and counits often turn out to be cuts, which explains the notation  $e, \epsilon$ . In general, cuts turn out to be quite familiar algebraic structures. The terms come from the graphical depiction of the elements and their defining properties:



The labels  $e, \epsilon$  are often omitted when clear from the context.

Another type of compatibility with the braiding will also be used in what follows:

**Definition 3.2.** Take a braided vector space  $(V, \sigma)$ .

⊗ An element  $c \in V$  is called *left-stable* if

$$\sigma(V \otimes c) \subseteq c \otimes V, \quad (\text{lstable})$$

and *right-stable* if

$$\sigma(c \otimes V) \subseteq V \otimes c. \quad (\text{rstable})$$

⊗ Dually, a co-element  $\varphi \in V^*$  is called *left-stable* if

$$\sigma(\text{Ker } \varphi \otimes V) \subseteq V \otimes \text{Ker } \varphi, \quad (\text{Lstable})$$

and *right-stable* if

$$\sigma(V \otimes \text{Ker } \varphi) \subseteq \text{Ker } \varphi \otimes V. \quad (\text{Rstable})$$

As usual, a *stable* element means both right- and left-stable.

In the trivial case (when the braiding is merely the flip or the signed flip) each element in  $V$  or in  $V^*$  is stable.

Among left-stable elements are in particular all *strongly left  $\sigma$ -compatible* elements, i.e. such that  $\sigma \circ (\text{Id} \otimes c) = c \otimes \text{Id}$  (or, respectively,  $(\text{Id} \otimes \varphi) \circ \sigma = \varphi \otimes \text{Id}$  on  $V \otimes V$ ). In other words, these elements satisfy a half of the  $\sigma$ -compatibility condition with any other element of  $V$  (resp.  $V^*$ ). An analogue for right stability is straightforward.

**Theorem 1.** *Let  $(V, \sigma)$  be a braided vector space.*

1. *For a lower cut  $e$ , the maps*

$$\begin{aligned} d : V^{\otimes n} &\longrightarrow V^{\otimes(n+1)} & \text{and } d_e : V^{\otimes n} &\longrightarrow V^{\otimes(n+1)} \\ \bar{v} &\longmapsto e \underset{-\sigma}{\Delta} \bar{v} & \bar{v} &\longmapsto (-1)^n \bar{v} \underset{-\sigma}{\Delta} e \end{aligned}$$

*define differentials on  $T(V)$ .*

2. *For two lower cuts  $e_1$  and  $e_2$ , one gets a differential bicomplex  $(T(V), e_1 d, d_{e_2})$ . If the cuts are moreover  $\sigma$ -compatible, then one gets a differential bicomplex  $(T(V), e_1 d, e_2 d)$ .*

3. *Similarly, for an upper cut  $\epsilon$ , the maps*

$$\begin{aligned} V^{\otimes n} &\longrightarrow V^{\otimes(n-1)} \\ \epsilon d : \bar{v} &\longmapsto (\epsilon \otimes \text{Id}_{n-1}) \underset{-\sigma}{\Theta}^{1, n-1}(\bar{v}) \\ d^\epsilon : \bar{v} &\longmapsto (-1)^{n-1} (\text{Id}_{n-1} \otimes \epsilon) \underset{-\sigma}{\Theta}^{n-1, 1}(\bar{v}) \end{aligned}$$

*define differentials on  $T(V)$ .*

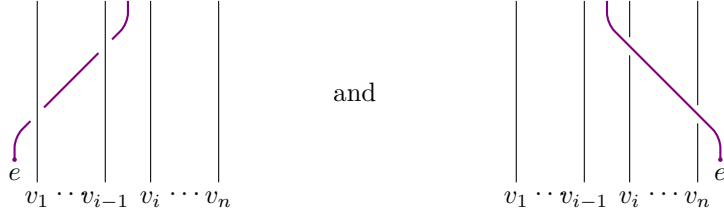
4. For two upper cuts  $\epsilon_1, \epsilon_2$ , one gets a differential bicomplex  $(T(V), \epsilon^1 d, d^{\epsilon_2})$ . If the cuts are moreover  $\sigma$ -compatible, then one gets a differential bicomplex  $(T(V), \epsilon^1 d, \epsilon^2 d)$ .
5. If an element  $c \in V$  is left-stable (or right-stable), then, for any lower cut  $e$ ,  $d$  (respectively  $d_e$ ) preserves the two-sided ideal  $I_c$  of the tensor algebra  $T(V)$  generated by  $c$  and thus descends to  $T(V)/I_c \simeq T(V/\mathbb{k}c)$ .
6. If an element  $\varphi \in V^*$  is left-stable (or right-stable), then, for any upper cut  $\epsilon$ , the subspace  $T(\text{Ker } \varphi) \subseteq T(V)$  is preserved by  $\epsilon d$  (resp.  $d^\epsilon$ ).

*Proof.* Easy verifications using the associativity of  $\Delta_{-\sigma}$ , the coassociativity of  $\Theta_{-\sigma}$  and the defining property of cuts. For example,

$$d^2(\bar{v}) = e \Delta_{-\sigma} (e \Delta_{-\sigma} \bar{v}) = (e \Delta_{-\sigma} e) \Delta_{-\sigma} \bar{v} = 0 \Delta_{-\sigma} \bar{v} = 0.$$

□

Graphically,  $d$  and  $d_e$  are presented as alternating sums of



respectively, and similarly for  $\epsilon d$  and  $d^\epsilon$ .

Observe that  $d$  and  $d_e$  increase by 1 the *degree* given by  $\deg(v_1 \dots v_n) = n$  thus defining cohomologies, while  $\epsilon d$  and  $d^\epsilon$  decrease the degree and therefore define homologies.

For readers used to reasoning in terms of **simplicial modules** (see for example [15]), the maps

$$d_i(\bar{v}) := (\epsilon \otimes \text{Id}_{n-1}) T_{s_{i,n}}^\sigma(\bar{v}),$$

where  $s_{i,n} \in S_n$  is the permutation moving the  $i$ 'th element to the leftmost position, give face maps. Our shuffle differential thus coincides with the overall canonical differential  $d := \sum_{i=1}^n (-1)^{i-1} d_i$  for the above face maps. The shuffle interpretation has two advantages over the simplicial one:

1. the positions  $i$  are controlled by the shuffle structure; it is thus sufficient to work on the local level with braidings, forgetting subscript manipulations;
2. the signs  $(-1)^i$  are hidden by letting the negative braiding  $-\sigma$  enter into play; the sign manipulations can thus be avoided as well.

Similarly, the maps

$$\delta_i(\bar{v}) := T_{s_{i+1}, n+1}^{\sigma} (e\bar{v}),$$

give face maps, forming a part of a cosimplicial structure.

An analogous treatment can be given to right differentials.

As for degeneracy maps, they seem unfortunately to have no nice shuffle interpretations.

Summarizing this section, we state that a braiding on  $V$  with a compatible lower or upper cut (i.e. a square zero element for the negative quantum shuffle product / coproduct) give a differential bicomplex structure on  $T(V)$ :

$$\boxed{\text{braiding} + \text{cut} \implies \text{differential bicomplex}}$$

## 4 Loday's hyper-boundaries

Our braided setting provides an elegant interpretation for Loday's hyper-boundaries (see [15], exercise E.2.2.7), which we redefine as generalizations of the differentials introduced above. The relations from Loday's exercise are easily proved and generalized thanks to our interpretation.

**Definition 4.1.** Let  $(V, \sigma)$  be a braided vector space with an upper cut  $\epsilon$ . The maps

$$\begin{aligned} V^{\otimes n} &\longrightarrow V^{\otimes(n-k)} \\ \epsilon, (k)d : \bar{v} &\longmapsto (\epsilon \otimes \cdots \otimes \epsilon \otimes \text{Id}_{n-k}) \Theta_{-\sigma}^{k, n-k}(\bar{v}) \\ d^{\epsilon, (k)} : \bar{v} &\longmapsto (-1)^{kn - \frac{k(k+1)}{2}} (\text{Id}_{n-k} \otimes \epsilon \otimes \cdots \otimes \epsilon) \Theta_{-\sigma}^{n-k, k}(\bar{v}) \end{aligned}$$

are called *hyper-boundaries* on  $T(V)$ .

The last sign should be understood as  $(-1)^{n-1}(-1)^{n-2} \cdots (-1)^{n-k}$ .

For  $k = 1$  one recovers the differentials  ${}^{\epsilon}d$  and  $d^{\epsilon}$ .

For a left- or right-stable  $\varphi$ , left or right hyper-boundaries can be restricted, as usual, to maps from  $(\text{Ker } \varphi)^{\otimes n}$  to  $(\text{Ker } \varphi)^{\otimes(n-k)}$ .

The next step is to understand compositions of hyper-boundaries, generalizing  $d^{(1)} \circ d^{(1)} = 0 = {}^{(1)}d \circ {}^{(1)}d$ . We start with a kind of a special case, a well-known

**Lemma 4.2.** Consider a vector space  $W$  and an element  $w \in W$ . One has

$$w^{\otimes m} \Delta_{-\tau} w^{\otimes k} = \binom{m+k}{k}_{-1} w^{\otimes(m+k)},$$

where

$$\binom{m+k}{k}_{-1} = \begin{cases} 0, & \text{if } mk \text{ is odd,} \\ \binom{\lfloor (m+k)/2 \rfloor}{\lfloor k/2 \rfloor}, & \text{otherwise,} \end{cases}$$

and the brackets  $\lfloor \cdot \rfloor$  stand for the lower integral part of a number.

*Proof.* By definition,

$$w^{\otimes m} \underset{-\tau}{\Delta} w^{\otimes k} = \sum_{s \in Sh_{m,k}} T_s^{-\tau} w^{\otimes(m+k)} = \sum_{s \in Sh_{m,k}} \text{sign}(s) w^{\otimes(m+k)},$$

where  $\text{sign}(s)$  is the sign of a permutation  $s$ . Now for each negative permutation in  $Sh_{m,k}$  we will associate a positive one in an injective way, counting the remaining positive permutations in  $Sh_{m,k}$ .

Given a negative permutation  $s \in Sh_{m,k}$ , choose, if it exists, the least  $i$  such that one of the preimages  $s^{-1}(2i-1), s^{-1}(2i)$  lies in the set  $\{1, \dots, m\}$ , while the other one lies in  $\{m+1, \dots, m+k\}$ . Such  $i$ 's will be called *split*. To such an  $s$  one associates  $\bar{s} := (s(1), \dots, s(2i-2), s(2i), s(2i-1), s(2i+1), \dots, s(m+k))$ , i.e. it is our  $s$  with the values at  $2i-1$  and  $2i$  interchanged. This constructs a bijection between negative and positive permutations for which a split  $i$  exists. It remains to count permutations without split  $i$ 's (we call such permutations *unsplit*) and to check that they are all positive.

- ⊗ If  $m+k$  is even, an unsplit permutation divides the elements  $1, \dots, m+k$  into consecutive pairs with preimages by  $s$  lying in the same set  $\{1, \dots, m\}$  or  $\{m+1, \dots, m+k\}$ . It is possible only when both  $m$  and  $k$  are even, giving  $\binom{(m+k)/2}{k/2}$  possibilities for the values of  $s^{-1}$  on  $(m+k)/2$  pairs.
- ⊗ If  $m+k$  is odd – say,  $m$  is even and  $k$  is odd – then, similarly, an unsplit permutation divides the elements  $1, \dots, m+k-1$  into consecutive pairs with preimages by  $s$  lying in the same set, and  $s^{-1}(m+k)$  lies automatically in  $\{m+1, \dots, m+k\}$ . This gives  $\binom{(m+k-1)/2}{(k-1)/2}$  possibilities.

To conclude, notice that all the unsplit permutations obtained are positive, since the sign coming from the element  $2i-1$  is "killed" by the sign coming from the element  $2i$ .  $\square$

This lemma is crucial in the calculations giving

**Theorem 2.** *Let  $(V, \sigma)$  be a braided vector space with an upper cut  $\epsilon$ . One has*

$$\begin{aligned} \epsilon^{(m)} d \circ \epsilon^{(k)} d &= \binom{m+k}{k}_{-1} \epsilon^{(m+k)} d, \\ d^{\epsilon^{(m)}} \circ d^{\epsilon^{(k)}} &= \binom{m+k}{k}_{-1} d^{\epsilon^{(m+k)}}. \end{aligned}$$

*Proof.* We will prove the first formula only. By definition,

$$\epsilon^{(m)} d \circ \epsilon^{(k)} d(\bar{v}) = (\epsilon \otimes \dots \otimes \epsilon \otimes \text{Id}_{n-k-m}) \circ (\epsilon \otimes \dots \otimes \epsilon \otimes \underset{-\sigma}{\Theta}^{m, n-k-m}) \circ \underset{-\sigma}{\Theta}^{k, n-k}(\bar{v}).$$

By the coassociativity of the co-shuffle coproduct, it equals

$$(\epsilon \otimes \dots \otimes \epsilon \otimes \text{Id}_{n-k-m}) \circ (\underset{-\sigma}{\Theta}^{k, m} \otimes \text{Id}_{n-k-m}) \circ \underset{-\sigma}{\Theta}^{m+k, n-m-k}(\bar{v}).$$

Now  $\epsilon$  is a cut, so

$$(\epsilon \otimes \epsilon) \circ \sigma = \epsilon \otimes \epsilon = (\epsilon \otimes \epsilon) \circ \tau,$$



thus

$$\epsilon, (m)d \circ \epsilon, (k)d(\bar{v}) = (\epsilon \otimes \cdots \otimes \epsilon \otimes \text{Id}_{n-k-m}) \circ (\Theta_{-\tau}^{k,m} \otimes \text{Id}_{n-k-m}) \circ \Theta_{-\sigma}^{m+k, n-m-k}(\bar{v}).$$

The dual version of the previous lemma calculates

$$(\epsilon \otimes \cdots \otimes \epsilon) \circ \Theta_{-\tau}^{k,m} = \binom{m+k}{k}_{-1} \epsilon \otimes \cdots \otimes \epsilon,$$

and one recognizes in the previous expression  $\binom{m+k}{k}_{-1} \epsilon, (m+k)d$ . □

## 5 Basic examples: familiar complexes recovered

Now we consider vector spaces  $V$  with some **algebraic structure** and we look for **braidings encoding the properties of this structure**. Invertibility conditions for such braidings will be discussed. Upper and lower cuts as well as stable elements will be determined, always up to scalar multiples. The answers to these auxiliary questions will always be familiar algebraic notions. Theorem 1 then gives numerous bicomplex structures on  $T(V)$ . We calculate explicitly some of the differentials obtained this way, recovering many familiar (co-)homologies.

### 5.1 Koszul complex

Following a nice mathematical tradition, the first example we consider is the trivial one: that of an "empty" structure. Take any vector space  $V$  and the flip  $\tau : v \otimes w \mapsto w \otimes v$  as its braiding. Each  $e \in V$  is automatically a lower cut, and each  $\epsilon \in V^*$  is an upper cut. In particular,

$$\epsilon d : v_1 \dots v_n \mapsto \sum_{i=1}^n (-1)^{i-1} \epsilon(v_i) v_1 \dots \widehat{v}_i \dots v_n$$

gives the well-known *Koszul differential*, in its simplest form.

### 5.2 Rack complex

The simplest non-trivial example of a braiding naturally coming from an algebraic structure is the following. Take a set  $S$  with a binary operation  $\triangleleft : S \times S \rightarrow S$ . Define an application

$$\begin{array}{ccc} b & a \triangleleft b & \\ & \times & \\ a & b & \end{array}$$

$$\begin{aligned} \sigma = \sigma_{\triangleleft} : S \times S &\longrightarrow S \times S \\ (a, b) &\longmapsto (b, a \triangleleft b). \end{aligned} \quad (\text{RackBraid})$$

It is very familiar to topologists, since it can be interpreted in terms of the fundamental group of the complement of a knot. See for instance the seminal paper [10], or [11] for a very readable introduction.

**Lemma 5.1.** The map  $\sigma_{\triangleleft}$  is a braiding if and only if  $\triangleleft$  is self-distributive, i.e.

$$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \quad \forall a, b, c \in S. \quad (\text{SelfDistr})$$

*Proof.* Here and in subsequent lemmas we content ourselves with graphical proofs.

Let us see what the Yang-Baxter equation (YB) means for  $\sigma_{\triangleleft}$ :

One easily recognizes (SelfDistr). □

A pair  $(S, \triangleleft)$  satisfying (SelfDistr) is called a *shelf*, or an *self-distributive system*.

The "if and only if" formulation of the lemma shows that the braiding  $\sigma_{\triangleleft}$  encodes the defining property of a shelf, just as we wanted.

Fix a shelf  $(S, \triangleleft)$  until the end of this section.

The braiding  $\sigma_{\triangleleft}$  is invertible if and only if the application  $a \mapsto a \triangleleft b$  is a bijection on  $S$  for every  $b \in S$ , that is if there exists an application  $\tilde{\triangleleft} : S \times S \rightarrow S$  such that

$$(a \triangleleft b) \tilde{\triangleleft} b = (a \tilde{\triangleleft} b) \triangleleft b = a \quad \forall a, b \in S. \quad (\text{Rack})$$

A triple  $(S, \triangleleft, \tilde{\triangleleft})$  satisfying (SelfDistr) and (Rack) is called a *rack*.

Now *linearize* a shelf  $(S, \triangleleft)$ : put  $V = \mathbb{k}S$  and extend the braiding  $\sigma_{\triangleleft}$  to  $V$  linearly. Lower cuts  $e = \sum_{i \in I} \alpha_i a_i \in V$ , where  $\{a_i\}_{i \in I}$  is a finite collection of pairwise distinct elements of  $S$ , and  $\alpha_i \in \mathbb{k}^*$ , are then characterized by  $e \triangleleft a_i = e \quad \forall i \in I$ . All  $a \in S$  are cuts if and only if  $S$  is a *quandle*, i.e. all its elements are idempotents. Upper cuts  $\epsilon \in V^*$  are characterized by  $\epsilon(a \triangleleft b) = \epsilon(a), \forall a, b \in S$  such that  $\epsilon(b) \neq 0$ . In particular,

$$\epsilon_0 : a \mapsto 1, \quad \forall a \in S$$

is an upper cut.

Every element of  $S$  is left-stable. For an arbitrary element  $v = \sum_{i \in I} \alpha_i a_i \in V$  as above, the left stability condition is  $a \triangleleft a_i = a \triangleleft a_j \quad \forall i, j \in I, a \in S$ . The right stability condition for  $v \in V$  is

$$v \triangleleft b = v, \quad \forall b \in S.$$

Similarly, for a  $\varphi \in V^*$  the left stability condition is

$$(\text{Ker } \varphi) \triangleleft b \subseteq \text{Ker } \varphi, \quad \forall b \in S.$$

The right stability condition for a co-element has no simple formulations. We just note here that the "Dirac elements"

$$\varphi_a(b) = \begin{cases} 1, & \text{if } b = a, \\ 0, & \text{for other } b \in S \end{cases}$$

are right-stable.

Let us conclude by calculating the most interesting differentials:

**Proposition 5.2.** *Take a shelf  $(A, \triangleleft)$ . The braiding  $\sigma_{\triangleleft}$  defined by (RackBraid) and the upper cut  $\epsilon_0 : a \mapsto 1, \forall a \in S$  define, via theorem 1, a chain bicomplex structure on  $T(\mathbb{k}A)$  by*

$$\begin{aligned} \epsilon^0 d(a_1 \dots a_n) &= \sum_{i=1}^n (-1)^{i-1} (a_1 \triangleleft a_i) \dots (a_{i-1} \triangleleft a_i) a_{i+1} \dots a_n, \\ d^{\epsilon_0}(a_1 \dots a_n) &= \sum_{i=1}^n (-1)^{i-1} a_1 \dots \widehat{a_i} \dots a_n. \end{aligned}$$

The differential  $\epsilon^0 d - d^{\epsilon_0}$  on  $T(\mathbb{Q}S)$  gives the *rack homology*. It was first defined by Fenn, Rourke and Sanderson in [8], and its cycles can be used to produce knot invariants.

### 5.3 Bar complex

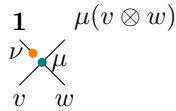
Take a vector space  $V$  with a bilinear operation  $\mu : V \otimes V \rightarrow V$  and a distinguished element  $\mathbf{1} \in V$ , sometimes regarded as a linear map

$$\nu : \mathbb{k} \rightarrow V, \quad \nu(\alpha) := \alpha \mathbf{1}.$$

Morally one should think about modeling **unital associative algebras**. We will construct quite an exotic braiding on  $V$ , encoding the associativity of  $\mu$ .

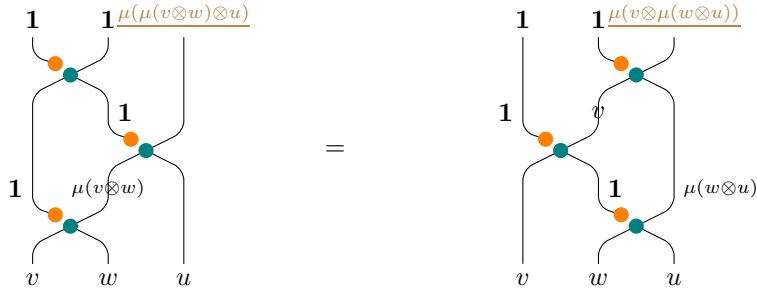
Consider the application

$$\begin{aligned} \sigma = \sigma_{\mu} : V \otimes V &\longrightarrow V \otimes V \\ v \otimes w &\longmapsto \mathbf{1} \otimes \mu(v \otimes w). \end{aligned} \quad (\text{AssBraid})$$



**Lemma 5.3.** Suppose that  $\mathbf{1}$  is a right unit for  $\mu$ , i.e.  $\mu(v \otimes \mathbf{1}) = v \quad \forall v \in V$ . Then the map  $\sigma_\mu$  is a braiding if and only if  $\mu$  is associative on  $V$ .

*Proof.* Graphically, the equation (YB) means

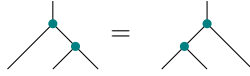


This is equivalent to the associativity condition

$$\mu(\mu(v \otimes w) \otimes u) = \mu(v \otimes \mu(w \otimes u)) \quad \forall v, w, u \in V. \quad (\text{Ass})$$

□

The associativity condition is graphically depicted as follows:



We thus get, like in the case of shelves, a braiding subtly encoding the algebraic structure "associative algebra".

Fix an associative algebra  $(V, \mu)$  with a right unit  $\mathbf{1}$  until the end of this section.

The braiding  $\sigma_\mu$  is **highly non-invertible**. More precisely,  $\sigma_\mu^2 = \sigma_\mu$  if  $\mathbf{1}$  is moreover a left unit.

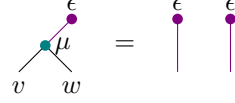
The right unit  $\mathbf{1}$  satisfies

$$\sigma_\mu(v \otimes \mathbf{1}) = \mathbf{1} \otimes v \quad \forall v \in V$$

and is therefore strongly left  $\sigma_\mu$ -compatible; it is the only left-stable element. For a right-stable  $v$ , there exists a  $l_v \in V^*$  such that  $\mu(v \otimes w) = l_v(w)v \quad \forall w \in V$ ; such a  $v$  is called an *integral* in  $V$ .

Left stability for  $\varphi \in V^*$  is equivalent to  $\text{Ker } \varphi$  being stable by the right multiplication by  $V$ , which is automatic when for example  $\varphi$  respects the multiplication  $\mu$ . Right stability reads  $\mathbf{1} \in \text{Ker } \varphi$ , i.e.  $\varphi(\mathbf{1}) = 0$ . If  $\mathbf{1}$  is moreover a left unit, then  $\mu(\mathbf{1} \otimes V) = V$ , so a stable upper cut would have the whole  $V$  in its kernel, being thus trivial.

Working, as usual, up to scalar multiples, the only lower cut in  $V$  is  $\mathbf{1}$ , while upper cuts turn out to be *characters*, i.e. maps  $\epsilon \in V^*$  respecting the multiplication:



$$\epsilon(\mu(v \otimes w)) = \epsilon(v)\epsilon(w) \quad \forall v, w \in V. \quad (\text{Char})$$

Note that  $\epsilon(\mathbf{1}) = 1$  for a non-zero character  $\epsilon$ .

Cohomologies obtained this way are not interesting, while homologies are:

**Proposition 5.4.** *A character  $\epsilon$  on a unital associative algebra  $(V, \mu, \mathbf{1})$  defines a chain bicomplex structure on  $T(V)$  by*

$${}^{\epsilon}d(v_1 \dots v_n) = \epsilon(v_1)v_2 \dots v_n + \epsilon(\mathbf{1}) \sum_{i=1}^{n-1} (-1)^i v_1 \dots v_{i-1} \mu(v_i \otimes v_{i+1}) v_{i+2} \dots v_n,$$

$$d^{\epsilon}(v_1 \dots v_n) = (-1)^{n-1} \epsilon(v_n) v_1 \dots v_{n-1} + \sum_{i=0}^{n-2} (-1)^i \epsilon(v_{i+1}) \dots \epsilon(v_n) v_1 \dots v_i \mathbf{1} \dots \mathbf{1}.$$

The left differential  ${}^{\epsilon}d$  restricts moreover to  $T(\text{Ker } \epsilon)$  and, in general, to any  $T(\text{Ker } \varphi)$  for a character  $\varphi$ .

In the **non-unital case**, i.e. when  $V$  is endowed with a bilinear operation  $\mu$  only, one enriches  $V$  with a formal unit:  $\tilde{V} := V \oplus \mathbb{k}\mathbf{1}$ , extending  $\mu$  by

$$\mu(\mathbf{1} \otimes v) = \mu(v \otimes \mathbf{1}) = v \quad \forall v \in \tilde{V}.$$

Due to the equivalence of the associativity of  $\mu$  on  $V$  and on  $\tilde{V}$ , lemma 5.3 affirms that  $\sigma_{\mu}$  is a braiding on  $\tilde{V}$  if and only if  $\mu$  is associative on  $V$ . Taking the character  $\epsilon_0(V) \equiv 0, \epsilon_0(\mathbf{1}) = 1$  on  $\tilde{V}$  and restricting  ${}^{\epsilon_0}d$  to  $T(\text{Ker } \epsilon_0) \simeq T(V)$ , one recovers the well-known *bar (or standard) differential*:

$${}^{\epsilon_0}d(v_1 \dots v_n) = \sum_{i=1}^{n-1} (-1)^i v_1 \dots v_{i-1} \mu(v_i \otimes v_{i+1}) v_{i+2} \dots v_n.$$

Another interesting differential is obtained by feeding two different characters in our machinery.

**Proposition 5.5.** *Two characters  $\epsilon_1, \epsilon_2$  on an associative algebra  $(V, \mu)$ , extended to  $\tilde{V} = V \oplus \mathbb{k}\mathbf{1}$  by  $\epsilon_1(\mathbf{1}) = 1 = \epsilon_2(\mathbf{1})$ , define a chain complex structure on  $T(\tilde{V})$  by*

$$\begin{aligned} ({}^{\epsilon_1}d - d^{\epsilon_2})(v_1 \dots v_n) = & \epsilon_1(v_1)v_2 \dots v_n \\ & + \sum_{i=1}^{n-1} (-1)^i v_1 \dots v_{i-1} \mu(v_i \otimes v_{i+1}) v_{i+2} \dots v_n \\ & + (-1)^n \epsilon_2(v_n) v_1 \dots v_{n-1} \\ & - \sum_{i=0}^{n-2} (-1)^i \epsilon_2(v_{i+1}) \dots \epsilon_2(v_n) v_1 \dots v_i \mathbf{1} \dots \mathbf{1}. \end{aligned}$$

The same formula, without the last sum, induces a differential on  $T(V)$ .

This differential will be denoted by  $\epsilon_1 d^{\epsilon_2}$ .

*Proof.* The only statement which does not follow directly from our general theory is the one about the induced differential on  $T(V)$ .

First restrict  $\epsilon_1 d^{\epsilon_2}$  to  $T(V) \otimes T(\mathbb{k}\mathbf{1}) \subset T(\tilde{V})$ . Let us then check the legitimacy of quotienting out  $T(V) \otimes T(\mathbb{k}\mathbf{1})_+$ . It is clear that  $\epsilon_1 d^{\epsilon_2}(v_1 \dots v_i \mathbf{1} \dots \mathbf{1}) \in T(V) \otimes T(\mathbb{k}\mathbf{1})_+$  if there are at least two copies of  $\mathbf{1}$ ; for elements of the form  $v_1 \dots v_i \mathbf{1}$ , one notices that

$$v_1 \dots v_{i-1} \mu(v_i \otimes \mathbf{1}) = v_1 \dots v_{i-1} v_i = \epsilon_2(\mathbf{1}) v_1 \dots v_{i-1} v_i;$$

thus  $T(V) \otimes T(\mathbb{k}\mathbf{1})_+$  is indeed a subcomplex. The differential obtained on  $T(V) \simeq (T(V) \otimes T(\mathbb{k}\mathbf{1})) / (T(V) \otimes T(\mathbb{k}\mathbf{1})_+)$  is  $\epsilon_1 d^{\epsilon_2}$  without the terms containing the  $\mathbf{1}$ 's.  $\square$

This trick of adding formal elements will often be handy in what follows.

## 5.4 Leibniz complex

Leibniz algebras are "non-commutative" versions of Lie algebras. They were discovered by A.Bloh in 1965, but it was Loday who woke the general interest in this structure in 1989 by, firstly, lifting the classical Chevalley-Eilenberg boundary map from the exterior to the tensor algebra, which yields a new interesting chain complex, and, secondly, by observing that the antisymmetry condition could be omitted (cf. [15],[16],[17],[6]). Here we recover Loday's complex guided by our "braided" considerations. Our interpretation explains the somewhat mysterious element ordering and signs in the formula given by Loday.

Like in the previous section, let  $V$  be a vector space with a bilinear operation, denoted by  $[\cdot, \cdot] : V \otimes V \longrightarrow V$  this time, and a distinguished element  $\mathbf{1} \in V$ . Morally, think about modeling **Lie algebras**. The braiding we construct in this setting is also quite exotic. It is inspired by the braiding for associative algebras.

Consider the application

$$\begin{aligned} \sigma = \sigma_{[\cdot, \cdot]} : V \otimes V &\longrightarrow V \otimes V \\ v \otimes w &\longmapsto w \otimes v + \mathbf{1} \otimes [v, w]. \end{aligned} \quad (\text{LeiBraid})$$

**Lemma 5.6.** Suppose that  $\mathbf{1}$  is central in  $V$ , i.e.

$$[\mathbf{1}, v] = [v, \mathbf{1}] = 0 \quad \forall v \in V. \quad (\text{LieUnit})$$

Then the map  $\sigma_{[\cdot, \cdot]}$  is a braiding if and only if

$$[v, [w, u]] = [[v, w], u] - [[v, u], w] \quad \forall v, w, u \in V. \quad (\text{Lei})$$

*Proof.* We omit the details of the calculations here; they are quite easy with our graphical calculus. One gets:

$$\begin{aligned} \sigma_1 \sigma_2 \sigma_1(v \otimes w \otimes u) &= u \otimes w \otimes v + \mathbf{1} \otimes w \otimes [v, u] + \mathbf{1} \otimes [w, u] \otimes v + \\ &\quad + u \otimes \mathbf{1} \otimes [v, w] + \mathbf{1} \otimes \mathbf{1} \otimes \underline{[[v, w], u]}, \end{aligned}$$

$$\begin{aligned} \sigma_2 \sigma_1 \sigma_2(v \otimes w \otimes u) &= u \otimes w \otimes v + \mathbf{1} \otimes w \otimes [v, u] + \mathbf{1} \otimes [w, u] \otimes v + \\ &\quad + u \otimes \mathbf{1} \otimes [v, w] + \mathbf{1} \otimes \mathbf{1} \otimes \underline{([v, [w, u]] + [[v, u], w])}. \end{aligned}$$

So (YB) for  $\sigma_{[\cdot]}$  is equivalent to (Lei) for  $[\cdot]$ .  $\square$

The condition (Lei) is graphically depicted as follows:

**Definition 5.7.** A pair  $(V, [\cdot])$  satisfying (Lei) is called a *Leibniz algebra*. A central element  $\mathbf{1} \in V$  (i.e. satisfying (LieUnit)) will often be called a *Lie unit*.

Remark that one gets the notion of a *Lie algebra* when adding the antisymmetry condition.

Lemma 5.6 means that once again we get a braiding encoding an algebraic structure – it is the Leibniz algebra structure this time.

Fix a Leibniz algebra  $(V, [\cdot])$  with a Lie unit  $\mathbf{1}$  until the end of this section.

The braiding  $\sigma_{[\cdot]}$  is invertible, the inverse given by

$$\sigma_{[\cdot]}^{-1} : v \otimes w \mapsto w \otimes v - [w, v] \otimes \mathbf{1}.$$

*Remark 5.8.* The invertibility of  $\sigma_{[\cdot]}$  means that this braiding allows to construct braid invariants out of any unital Leibniz algebra. It would be interesting to explore the nature of these invariants.

One has

$$\sigma_{[\cdot]}(\mathbf{1} \otimes v) = v \otimes \mathbf{1}, \quad \sigma_{[\cdot]}(v \otimes \mathbf{1}) = \mathbf{1} \otimes v \quad \forall v \in V,$$

hence  $\mathbf{1}$  is strongly left and right  $\sigma$ -compatible and thus stable. In general, stable elements with respect to our braiding are central in some sense: the left stability condition for  $c$  reads  $[V, c] = 0$ , coinciding with that for strong left  $\sigma_{[\cdot]}$ -compatibility, and the right-stability condition is  $[c, V] \subseteq \text{kc}$ . Left stability for  $\varphi \in V^*$  is equivalent to  $\text{Ker } \varphi$  being stable by the right “bracket-multiplication” by  $V$ , which is automatic when for example  $\varphi$  respects the bracket  $[\cdot]$ . Right stability means either  $\varphi(\mathbf{1}) = 0$  or  $[V, \text{Ker } \varphi] = 0$ .

An  $e \in V$  is a lower cut if and only if  $[e, e] = 0$ . Upper cuts  $\epsilon \in V^*$  are characterized by:

$$\otimes \text{ either } \epsilon(\mathbf{1}) = 0,$$

⊗ or  $\epsilon$  is a *Lie character*, i.e. respects the bracket:

$$\epsilon([v, w]) = 0 \quad \forall v, w \in V. \quad (\text{LieChar})$$

The differentials obtained are the most interesting in the last case:

**Proposition 5.9.** *A Lie character  $\epsilon$  on a Leibniz algebra  $(V, [,])$  with a Lie unit  $\mathbf{1}$  defines a chain complex structure on  $T(V)$  by*

$$\begin{aligned} \epsilon d(v_1 \dots v_n) &= \epsilon(\mathbf{1}) \sum_{1 \leq i < j \leq n} (-1)^{j-1} v_1 \dots v_{i-1} [v_i, v_j] v_{i+1} \dots \widehat{v}_j \dots v_n + \\ &+ \sum_{1 \leq j \leq n} (-1)^{j-1} \epsilon(v_j) v_1 \dots \widehat{v}_j \dots v_n. \end{aligned}$$

*This differential restricts to  $T(\text{Ker } \epsilon)$  and, in general, to any  $T(\text{Ker } \varphi)$  for a Lie character  $\varphi$ .*

In the **non-unital case**, i.e. when  $V$  is endowed with a bilinear operation only, one adds a formal Lie unit,  $\widetilde{V} := V \oplus \mathbb{k}\mathbf{1}$ , extending  $[,]$  by

$$[\mathbf{1}, v] = [v, \mathbf{1}] = 0 \quad \forall v \in \widetilde{V}.$$

Due to the equivalence of the Leibniz condition (**Lei**) for  $[,]$  on  $V$  and on  $\widetilde{V}$ , lemma 5.6 affirms that  $\sigma_{[,]}$  is a braiding on  $\widetilde{V}$  if and only if  $[,]$  is Leibniz on  $V$ . Taking the character  $\epsilon_0(V) \equiv 0, \epsilon_0(\mathbf{1}) = 1$  on  $\widetilde{V}$  and restricting  $\epsilon_0 d$  to  $T(\text{Ker } \epsilon_0) \simeq T(V)$ , one recovers the *Leibniz complex*:

$$\epsilon_0 d(v_1 \dots v_n) = \sum_{1 \leq i < j \leq n} (-1)^{j-1} v_1 \dots v_{i-1} [v_i, v_j] v_{i+1} \dots \widehat{v}_j \dots v_n.$$

Taking a Lie algebra and descending to the exterior algebra  $\Lambda(V)$ , one recognizes the *Chevalley-Eilenberg complex* for Lie algebras.

Let us summarize the last two subsections before proceeding to their categorical and then dual versions:

**Theorem 3.** *1. A unital associative (or Leibniz) algebra  $V$  can be endowed with a braiding  $\sigma_\mu$  (resp.  $\sigma_{[,]}$ ) defined by the formula (**AssBraid**) (resp. (**LeiBraid**)).*

*2. Any (Lie) character  $\varphi$  is a left-stable cut for this braiding.*

Together with theorem 1, this gives

**Corollary 5.10.** *Any (Lie) character on a unital associative (or Leibniz) algebra  $V$  produces a degree  $-1$  differential on  $T(V)$  and on its subspace  $T(\text{Ker } \varphi)$ .*



## 6 An upper world: categories

In this section we show how to raise the most important constructions from previous sections to the categoric level, illustrating the advantages of this approach with three examples:

1. Leibniz superalgebra homology;
2. homologies of dual structures: (Leibniz) coalgebras etc.;
3. right-left duality.

Only basic tools of category theory will be used here; Mac Lane's and Turaev's famous books [18] and [27] are excellent references for general and, respectively, braided aspects of category theory. We also recommend Westrich's preprint [29] where most of the categoric notions used here are nicely presented and illustrated.

### 6.1 Categorifying everything

We start with recalling some classical definitions from category theory.

**Definition 6.1.**  $\otimes$  A strict *monoidal* (or *tensor*) *category* is a category  $\mathcal{C}$  endowed with

- a bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  satisfying the associativity condition;
- an object  $\mathbf{I}$  which is a left and right identity for  $\otimes$ .

$\otimes$  A strict monoidal category  $\mathcal{C}$  is called *braided* if it is endowed with a *braiding* (or a *commutativity constraint*), i.e. a natural family of isomorphisms,

$$c = \{c_{V,W} : V \otimes W \simeq W \otimes V\}, \quad \forall V, W \in Ob(\mathcal{C}),$$

satisfying

$$\begin{aligned} c_{V,W \otimes U} &= (\text{Id}_W \otimes c_{V,U}) \circ (c_{V,W} \otimes \text{Id}_U), \\ c_{V \otimes W, U} &= (c_{V,U} \otimes \text{Id}_W) \circ (\text{Id}_V \otimes c_{W,U}) \end{aligned}$$

for any triple of objects  $V, W, U$ . "Natural" means here

$$c_{V',W'} \circ (f \otimes g) = (g \otimes f) \circ c_{V,W}$$

for all  $V, W, V', W' \in Ob(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(V, V')$ ,  $g \in \text{Hom}_{\mathcal{C}}(W, W')$ .

$\otimes$  A braided category  $\mathcal{C}$  is called *symmetric* if its braiding is symmetric:

$$c_{V,W} \circ c_{W,V} = \text{Id}_{W \otimes V}, \quad \forall V, W \in Ob(\mathcal{C}).$$

We will omit the part "monoidal" of the usual terms "braided monoidal" and "symmetric monoidal" in what follows.

- ⊗ A category  $\mathcal{C}$  is called *preadditive* if all its morphism sets  $\text{Hom}_{\mathcal{C}}(V, W)$  are abelian groups, the composition of morphisms being  $\mathbb{Z}$ -bilinear. For a preadditive and monoidal category to be called *preadditive monoidal*, its tensor product should be bilinear on morphisms. The same condition is imposed on braided (and in particular symmetric) preadditive categories.
- ⊗ A *unital associative algebra* (abbreviated as UAA) in a strict monoidal category  $\mathcal{C}$  is an object  $V$  together with morphisms  $\mu : V \otimes V \rightarrow V$  and  $\nu : \mathbf{I} \rightarrow V$ , satisfying associativity condition

$$\mu \circ (\text{Id}_V \otimes \mu) = \mu \circ (\mu \otimes \text{Id}_V)$$

and unit condition.

- ⊗ A *unital Leibniz algebra* (abbreviated as ULA) in a symmetric preadditive category  $\mathcal{C}$  is an object  $V$  together with morphisms  $[\cdot, \cdot] : V \otimes V \rightarrow V$  and  $\nu : \mathbf{I} \rightarrow V$ , satisfying generalized Leibniz condition

$$[\cdot, \cdot] \circ (\text{Id}_V \otimes [\cdot, \cdot]) = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_V) - [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_V) \circ (\text{Id}_V \otimes c_{V, V}) : V^{\otimes 3} \rightarrow V$$

and Lie unit condition.

See for instance [2] and [19] for the definition of algebras in a monoidal category, and [9] for a survey on Lie algebras in a symmetric preadditive category.

We work only with **strict** monoidal categories here for the sake of simplicity; according to a theorem of Mac Lane ([18]), each monoidal category is monoidally equivalent to a strict one. This justifies in particular notations  $V \otimes W \otimes U$  and  $V^{\otimes n}$ . The word "strict" will be omitted but always implied in what follows.

Note that to define a unital Leibniz algebra, one needs more structure on the underlying category than for associative algebras.

Here are some basic examples sufficient for this paper:

1. The category of sets **Set** is monoidal, with the Cartesian product  $\times$  as its tensor product, and a one-element set  $\mathbf{I}$  as its identity object. An identification of  $(A \times B) \times C$  with  $A \times (B \times C)$  and of  $\mathbf{I} \times A$  with  $A \times \mathbf{I}$  and with  $A$  for any sets  $A, B, C$ , which will be implicitly done in what follows, gives a strict monoidal category. This category is symmetric, with the braiding given by the usual flip isomorphism.
2. The category of  $\mathbb{k}$ -vector spaces **Vect** $_{\mathbb{k}}$  is symmetric preadditive, with the usual tensor product, the one-dimensional space  $\mathbb{k}$  as its identity object and the flip or the signed flip as its braiding. It is denoted by **Vect** $_{\mathbb{k}}^-$  if the signed flip is chosen. Identifications similar to those for sets are implicit here to assure the strictness. The linearization map  $\text{Lin} : A \mapsto \mathbb{k}A$  gives a functor of symmetric monoidal categories  $\text{Lin} : \mathbf{Set} \rightarrow \mathbf{Vect}_{\mathbb{k}}$ .
3. The category of graded  $\mathbb{k}$ -vector spaces **VectGrad** $_{\mathbb{k}}$  is symmetric preadditive, with the usual graded tensor product, the one-dimensional zero-graded space  $\mathbb{k}$  as its identity object and the Koszul flip as its braiding. Necessary identifications are effectuated to assure the strictness.

4. Any braided category  $\mathcal{C}$  has two interesting subcategories:

- (a) The subcategory  $\mathbf{Alg}(\mathcal{C})$  of UAAs and unital algebra morphisms (i.e. morphisms respecting  $\mu$  and  $\nu$ ) in  $\mathcal{C}$  is a monoidal subcategory. In particular, it is stable by tensor products since

$$\mu_{V \otimes W} := (\mu_V \otimes \mu_W) \circ (\mathrm{Id}_V \otimes c_{W,V} \otimes \mathrm{Id}_W)$$

gives a multiplication on  $V \otimes W$  for UAAs  $V$  and  $W$ , and it includes  $\mathbf{I}$  with identities as algebra structures, which will be the default UAA structure on  $\mathbf{I}$  in what follows.  $\mathbf{Alg}(\mathcal{C})$  is moreover a braided subcategory if  $\mathcal{C}$  is symmetric, since the symmetry of the braiding guarantees its compatibility with algebra structures.

For a preadditive  $\mathcal{C}$ ,  $\mathbf{Alg}(\mathcal{C})$  is not a preadditive subcategory in general, since  $f + g$  is not necessarily an algebra morphism even if  $f$  and  $g$  are.

- (b) If  $\mathcal{C}$  is moreover symmetric preadditive, one also has the subcategory  $\mathbf{Lei}(\mathcal{C})$  of ULAs and unital Leibniz algebra morphisms (i.e. morphisms respecting  $[\cdot, \cdot]$  and  $\nu$ ) in  $\mathcal{C}$ . It is neither preadditive not even monoidal in general. It includes  $\mathbf{I}$  with a zero bracket and  $\nu = \mathrm{Id}_{\mathbf{I}}$ , which will be the default ULA structure on  $\mathbf{I}$  in what follows.

In particular,  $\mathbf{Alg}(\mathbf{Vect}_{\mathbb{k}})$  and  $\mathbf{Lei}(\mathbf{Vect}_{\mathbb{k}})$  are the familiar categories of  $\mathbb{k}$ -linear unital associative and Leibniz algebras respectively.

Now we introduce several new categoric notions, necessary for categorifying our constructions.

**Definition 6.2.**  $\otimes$  An object  $V$  in a monoidal category  $\mathcal{C}$  is called *weakly braided* if it is endowed with a *weak braiding*, i.e. a morphism

$$\sigma = \sigma_V : V \otimes V \rightarrow V \otimes V,$$

satisfying a categorified version of (YB):

$$(\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V).$$

- $\otimes$  In a monoidal category, a morphism  $\epsilon : V \rightarrow \mathbf{I}$  is called an *upper cut* for a weakly braided object  $V$  if

$$\epsilon \otimes \epsilon = (\epsilon \otimes \epsilon) \circ \sigma_V : V \otimes V \rightarrow \mathbf{I} \otimes \mathbf{I} = \mathbf{I}.$$

- $\otimes$  A *degree  $-1$  tensor differential* for an object  $V$  of a preadditive monoidal category  $\mathcal{C}$  is a family of morphisms  $\{d_n : V^{\otimes n} \rightarrow V^{\otimes(n-1)}\}_{n>0}$ , where  $V^{\otimes 0} := \mathbf{I}$ , satisfying  $d_{n-1} \circ d_n = 0 \forall n > 1$ .

- $\otimes$  A *character* for an object  $V$  of  $\mathbf{Alg}(\mathcal{C})$  is an algebra morphism  $\epsilon \in \mathrm{Hom}_{\mathbf{Alg}(\mathcal{C})}(V, \mathbf{I})$ . A *Lie character* for an object  $V$  of  $\mathbf{Lei}(\mathcal{C})$  is a Leibniz algebra morphism  $\epsilon \in \mathrm{Hom}_{\mathbf{Lei}(\mathcal{C})}(V, \mathbf{I})$ , i.e. it satisfies  $\epsilon \circ [\cdot, \cdot] = 0$  and  $\epsilon \circ \nu = \mathrm{Id}_{\mathbf{I}}$ .

Every object in a braided category  $\mathcal{C}$  is weakly braided with  $\sigma_V = c_{V,V}$ , since the Yang-Baxter equality is automatic in  $\mathcal{C}$ . The idea of working with "local" braidings on  $V$  instead of demanding the whole category  $\mathcal{C}$  to be "globally" braided is similar to what is done in [9], where self-invertible YB operators are considered in order to define YB-Lie algebras in an additive monoidal  $\mathcal{C}$ . Note however that, contrary to their operator, our weak braiding is not necessarily invertible.

Observe that any monoidal (and braided and/or preadditive when necessary) functor preserves all the structures from the previous two definitions.

We are now sufficiently armed to attack a categorification of theorem 1.

Any weakly braided object  $(V, \sigma)$  in a monoidal category comes with an evident action of the monoid  $B_n^+$  on  $V^{\otimes n}$  for each  $n \geq 1$ . If the category is moreover preadditive, one can mimic the construction of quantum (co)shuffle (co)product to get morphisms  $\Delta_{\sigma}^{p,q} : V^{\otimes n} = V^{\otimes p} \otimes V^{\otimes q} \rightarrow V^{\otimes n}$  and  $\Theta_{\sigma}^{p,q} : V^{\otimes n} \rightarrow V^{\otimes p} \otimes V^{\otimes q} = V^{\otimes n}$ . Here  $n = p + q$ . Still in the preadditive context,  $-\sigma$  is well defined and gives a new weak braiding for  $V$ . Theorem 1 (with its proof!) is then generalized as follows:

**Theorem 1<sup>cat</sup>.** *Let  $(\mathcal{C}, \otimes, \mathbf{I})$  be a preadditive monoidal category. For any weakly braided object  $(V, \sigma)$  with an upper cut  $\epsilon$ , the family of morphisms*

$$({}^{\epsilon}d)_n := (\epsilon \otimes \text{Id}_{n-1}) \circ \Theta_{-\sigma}^{1, n-1}$$

*defines a degree  $-1$  tensor differential for  $V$ .*

One can also categorify the notions of bicomplexes,  $\sigma$ -compatible cuts, stable cuts, and tensor differentials for  $V$  which can be "restricted" to tensor differentials for "Ker( $\epsilon$ )" (quotation marks are used to emphasize that the corresponding notions are to be interpreted (in a non-trivial way!) in the categoric language), getting a categorification of the remaining points of theorem 1. This is quite technical but presents no conceptual difficulties.

The versions of the theorem for lower cuts and for "right" differentials  $d^{\epsilon}$  will be obtained later via different types of categoric dualities.

Start with the example of shelves. Lemma 5.1 tells that every shelf  $A \in \text{Ob}(\mathbf{Set})$  is endowed with a weak braiding  $(a, b) \mapsto (b, a \triangleleft b)$ . Since the one-element set  $\mathbf{I}$  is a final object in  $\mathbf{Set}$ , the unique morphism  $A \rightarrow \mathbf{I}$  is necessarily an upper cut. The monoidal functor  $\text{Lin}$  provides then the linearization  $\mathbb{k}A$  of our shelf with a weak braiding and an upper cut, and thus, according to theorem 1<sup>cat</sup>, with a degree  $-1$  tensor differential.

Next we take more complicated "algebraic" braidings from theorem 3.

**Theorem 3<sup>cat</sup>.**

1. *Take a unital associative algebra  $(V, \mu, \nu)$  in a monoidal category  $(\mathcal{C}, \otimes, \mathbf{I})$ .*

(a)  *$V$  can be endowed with a weak braiding*

$$\sigma_V := \nu \otimes \mu : V \otimes V = \mathbf{I} \otimes V \otimes V \rightarrow V \otimes V. \quad (\sigma_{UAA})$$

- (b) Any character  $\epsilon \in \text{Hom}_{\mathbf{Alg}(\mathcal{C})}(V, \mathbf{I})$  is an upper cut for  $(V, \sigma_V)$ .
2. Take a unital Leibniz algebra  $(V, [, ], \nu)$  in a symmetric preadditive category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ .

(a)  $V$  can be endowed with a weak braiding

$$\sigma_V := c_{V,V} + \nu \otimes [, ]. \quad (\sigma_{ULA})$$

(b) Any Lie character  $\epsilon \in \text{Hom}_{\mathbf{Lei}(\mathcal{C})}(V, \mathbf{I})$  is an upper cut for  $(V, \sigma_V)$ .

Observe that in the Leibniz algebra setting, the naturality (with respect to morphisms  $\nu$  and  $[,]$  in particular) and the symmetry of the braiding  $c$  are essential in proving that  $\sigma_V$  is indeed a weak braiding, while the naturality of  $c$  with respect to  $\epsilon$  shows that  $\epsilon$  is an upper cut for  $(V, c_{V,V})$  (which implies that it is an upper cut for  $(V, \sigma_V)$  if it preserves the Leibniz structure).

*Remark 6.3.* According to the theorem, a ULA  $V$  provides an example of a "doubly braided" object (cf. a remark in the introduction):  $\sigma_V$  and  $c_{V,V}$  are indeed two distinct weak braidings for  $V$ . One can say more: the two braidings endow tensor powers of  $V$  with an action of the **virtual braid group** (cf. the foundational paper of the virtual knot theory [13], and [28], where the virtual braid group was introduced and studied). The close connections between weak braidings and virtual braid groups will be studied in detail in a separate paper.

Working in  $\mathbf{Vect}_{\mathbb{k}}$  in section 5, we have noticed that the braidings obtained for (Leibniz) algebras **encode** underlying algebraic structures (cf. lemmas 5.3 and 5.6). It is still true in the categoric setting:

- Lemma 6.4.** 1. Take an object  $V$  in a monoidal category  $(\mathcal{C}, \otimes, \mathbf{I})$  endowed with two morphisms  $\mu : V \otimes V \rightarrow V$  and  $\nu : \mathbf{I} \rightarrow V$ , with  $\nu$  being a two-sided unit for  $\mu$ . The morphism  $\sigma_V$  defined by  $(\sigma_{UAA})$  is a weak braiding **if and only if**  $\mu$  is associative.
2. Take an object  $V$  in a symmetric preadditive category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$  endowed with two morphisms  $[,] : V \otimes V \rightarrow V$  and  $\nu : \mathbf{I} \rightarrow V$ , with  $\nu$  being a Lie unit for  $[,]$ . Additionally suppose the existence of a morphism  $\gamma : V \rightarrow \mathbf{I}$  such that  $\gamma \circ \nu = \text{Id}_{\mathbf{I}}$ . The morphism  $\sigma_V$  defined by  $(\sigma_{ULA})$  is a weak braiding **if and only if**  $[,]$  satisfies the Leibniz condition.

*Proof.* One follows the proofs of lemmas 5.3 and 5.6. The only non-trivial step is to show that  $f = g : V^{\otimes 3} \rightarrow V$  is equivalent to  $\nu \otimes \nu \otimes f = \nu \otimes \nu \otimes g : V^{\otimes 3} \rightarrow V^{\otimes 3}$ . When  $\nu$  is a unit for  $\mu$ , this is done by applying  $\mu \circ (\text{Id}_V \otimes \mu)$  to both sides of the second identity. In the Leibniz case one applies  $\gamma \otimes \gamma \otimes \text{Id}_V$ .  $\square$

The weak braidings constructed in the previous theorem enjoy a *naturality* property, providing moreover a characterization of (Leibniz) algebra morphisms:

**Proposition 6.5.** 1. In the settings of theorem  $\mathcal{G}^{\text{cat}}$ , one has

$$(f \otimes f) \circ \sigma_V = \sigma_W \circ (f \otimes f) : V \otimes V \rightarrow W \otimes W \quad (\text{Nat})$$

for any morphism  $f : V \rightarrow W$  in  $\mathbf{Alg}(\mathcal{C})$  (resp.  $\mathbf{Lei}(\mathcal{C})$ ).

2. Suppose additionally, for the Leibniz case, the existence of a morphism  $\gamma : V \rightarrow \mathbf{I}$  in  $\mathcal{C}$  such that  $\gamma \circ \nu = \text{Id}_{\mathbf{I}}$ .

Then any normalized morphism  $f : V \rightarrow W$  (i.e. respecting the units:  $f \circ \nu_V = \nu_W$ ) in  $\mathcal{C}$ , compatible with  $\sigma$ 's in the sense of **(Nat)**, necessarily respects the multiplications. In other words, such an  $f$  is a morphism in  $\mathbf{Alg}(\mathcal{C})$  (resp.  $\mathbf{Lei}(\mathcal{C})$ ).

*Proof.* The first point is easy. For the second one, since  $f$  is normalized, **(Nat)** means

$$\nu_W \otimes (f \circ \mu_V) = \nu_W \otimes (\mu_W \circ (f \otimes f)),$$

and similarly - with  $\mu$  replaced by  $[\cdot, \cdot]$  - in the Leibniz case, as the braiding  $c$  is natural. Now, as in the proof of lemma 6.4, apply  $\mu_W$  (resp.  $\gamma \otimes \text{Id}_W$ ) to both sides of this relation.  $\square$

Note that, contrary to the naturality of the braiding in a braided category, one cannot take two distinct morphisms  $f, g : V \rightarrow W$  here.

Theorems 1<sup>cat</sup> and 3<sup>cat</sup> put together give

**Corollary 6.6.** *For any  $\alpha : V \otimes V \rightarrow V$ , put*

$$\alpha_i := \text{Id}_V^{\otimes(i-1)} \otimes \alpha \otimes \text{Id}_V^{\otimes(n-i-1)} : V^{\otimes n} \rightarrow V^{\otimes(n-1)}.$$

1. Any character  $\epsilon : V \rightarrow \mathbf{I}$  for a UAA  $(V, \mu, \nu)$  in a preadditive monoidal category  $\mathcal{C}$  produces a degree  $-1$  tensor differential for  $V$ , given by

$$(\epsilon d)_n := \epsilon \otimes \text{Id}_{n-1} + \sum_{i=1}^{n-1} (-1)^i \mu_i.$$

2. Any Lie character  $\epsilon : V \rightarrow \mathbf{I}$  for a ULA  $(V, [\cdot, \cdot], \nu)$  in a symmetric preadditive category  $\mathcal{C}$  produces a degree  $-1$  tensor differential for  $V$ , given by

$$\begin{aligned} (\epsilon d)_n &:= (\epsilon \otimes \text{Id}_{n-1}) \circ \left( \sum_{1 \leq j \leq n} (-1)^{j-1} c_{V^{\otimes(j-1)}, V} \right) \\ &+ \sum_{1 \leq i < j \leq n} (-1)^{j-1} [\cdot, \cdot]_i \circ (\text{Id}_V^{\otimes(i)} \otimes c_{V^{\otimes(j-i-1)}, V} \otimes \text{Id}_V^{\otimes(n-j)}). \end{aligned}$$

## 6.2 The super trick

The first bonus one generally obtains when passing to abstract symmetric categories is the possibility to derive graded and super versions of algebraic results for free, thanks to the Koszul flip  $\tau_{\text{Koszul}}$ . One clearly sees where to put signs, which is otherwise quite difficult to guess. Here is a nice example.

Take a *graded Leibniz algebra*  $(V, [\cdot, \cdot], \nu)$ , i.e. an object of  $\mathbf{Lei}(\mathbf{VectGrad}_{\mathbb{k}})$ . Recall that the category  $\mathbf{VectGrad}_{\mathbb{k}}$  comes with the symmetric braiding  $\tau_{\text{Koszul}}$ . Leibniz condition in this setting is

$$[v, [w, u]] = [[v, w], u] - (-1)^{\deg v \deg w} [[v, u], w]$$

for any homogeneous elements  $v, w, u \in V$ . Note that on the picture illustrating (Lei), the crossing on the right corresponds to the "internal" braiding  $c_{V,V} = \tau_{Koszul}$  on  $V$ .

Theorem 1<sup>cat</sup> gives a weak braiding for  $V$ :

$$\sigma_V : v \otimes w \longmapsto (-1)^{\deg v \deg w} w \otimes v + \mathbf{1} \otimes [v, w],$$

which, together with a Lie character  $\epsilon : V_0 \rightarrow \mathbb{k}$  (respecting degrees, it has to be zero on other components of  $V$ ), can be fed into the machinery from theorem 3<sup>cat</sup> to give

**Proposition 6.7.** *A  $\mathbb{k}$ -linear graded Leibniz algebra  $(V, [, ], \nu)$  with a Lie character  $\epsilon$  can be endowed with a  $-1$  tensor differential by the formula*

$$\begin{aligned} \epsilon d(v_1 \dots v_n) &= \epsilon(\mathbf{1}) \sum_{1 \leq i < j \leq n} (-1)^{j-1+\alpha_{i,j}} v_1 \dots v_{i-1} [v_i, v_j] v_{i+1} \dots \widehat{v}_j \dots v_n + \\ &+ \sum_{1 \leq j \leq n} (-1)^{j-1+\alpha_{0,j}} \epsilon(v_j) v_1 \dots \widehat{v}_j \dots v_n, \end{aligned}$$

where  $\alpha_{i,j} := \deg(v_j) \sum_{i < k < j} \deg(v_k)$ . All elements  $v, w, v_i$  are taken homogeneous here.

Observe that the  $(-1)^{\alpha_{i,j}}$  part of the sign comes from the Koszul braiding, while  $(-1)^{j-1}$  appears because we take the opposite braiding when defining  $(\epsilon d)_n := (\epsilon \otimes \text{Id}_{n-1}) \circ \Theta_{-\sigma}^{1,n-1}$  in theorem 1<sup>cat</sup>.

*Leibniz superalgebras* are treated similarly: one has just to work in the category of super vector spaces over  $\mathbb{k}$ . The reader is sent to [14] and other papers on the subject for details. One thus recovers the *Leibniz superalgebra homology*, which is a lift of the Lie superalgebra homology.

Note that one also gets for free the *color Leibniz algebra homology* (cf. [7], or [23] for a Lie version), since color Leibniz algebras are particular cases of Leibniz algebras in a symmetric preadditive category. See also [29] for an excellent survey of different types of braided Lie algebras.

### 6.3 Co-world, or the world upside down

One more nice feature of the categoric approach is an automatic treatment of **dualities**. The most common notion of duality, the "upside-down" one, is described here, with the cobar complex for coalgebras (first defined by Cartier in [4]) providing a nice example. In the monoidal context one has two more dualities, the "right-left" and the combined ones. They will be treated in the next section.

**Definition 6.8.** Given a category  $\mathcal{C}$ , its *dual (or opposite) category*  $\mathcal{C}^{\text{op}}$  is constructed by keeping the objects of  $\mathcal{C}$  and reversing all the arrows. In other words, the domain and codomain of each morphism change places. One writes  $f^{\text{op}} \in \text{Hom}_{\mathcal{C}^{\text{op}}}(W, V)$  for the morphism in  $\mathcal{C}^{\text{op}}$  corresponding to  $f \in \text{Hom}_{\mathcal{C}}(V, W)$ .

We will sometimes call  $\mathcal{C}^{\text{op}}$  a *co-category*, to avoid confusion with other notions of duality. Observe that this construction is involutive:  $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$ .

A typical example comes from the full subcategory  $\mathbf{vect}_{\mathbb{k}}$  of  $\mathbf{Vect}_{\mathbb{k}}$  consisting of finite dimensional vector spaces. The usual duality functor sending  $V$  to  $V^* := \text{Hom}_{\mathbb{k}}(V, \mathbb{k})$  and  $f$  to  $f^*$  gives an equivalence of symmetric preadditive categories  $\mathbf{vect}_{\mathbb{k}}$  and  $(\mathbf{vect}_{\mathbb{k}})^{\text{op}}$ .

The *duality principle* (cf. [18], section II.2) tells that a "categoric" theorem for  $\mathcal{C}$  implies a dual theorem for  $\mathcal{C}^{\text{op}}$  by reversing all arrows and the order of arrows in each composition.

Categoric structures also admit dual ones in the same sense. For example,

**Definition 6.9.** A *counital coassociative coalgebra* (abbreviated as co-UAA) in a strict monoidal category  $\mathcal{C}$  is an object  $V$  together with morphisms  $\Delta : V \rightarrow V \otimes V$  and  $\varepsilon : V \rightarrow \mathbf{I}$ , such that  $(V, \Delta^{\text{op}}, \varepsilon^{\text{op}})$  is a UAA in  $\mathcal{C}^{\text{op}}$ .

The associativity condition is then "reversed" to the *coassociativity* condition

$$(\text{Id}_V \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}_V) \circ \Delta,$$

and the unit condition to the counit condition.

*Counital co-Leibniz coalgebras* (abbreviated as co-ULA) are defined similarly; cf. [22] where Lie coalgebras are introduced. The subcategory of co-UAAs and co-ULAs in  $\mathcal{C}$  are denoted by  $\mathbf{coAlg}(\mathcal{C})$  and  $\mathbf{coLei}(\mathcal{C})$  respectively. (*Lie*) *co-characters*, *lower cuts* and *degree 1 tensor differentials*  $d^n$  are also defined via dualities. A lower cut  $e : \mathbf{I} \rightarrow V$  is described for example by the condition

$$e \otimes e = \sigma_V \circ (e \otimes e) : \mathbf{I} = \mathbf{I} \otimes \mathbf{I} \rightarrow V \otimes V.$$

A convenient way to handle the "upside-down" duality is the graphical one: changing from  $\mathcal{C}$  to  $\mathcal{C}^{\text{op}}$  consists simply in **turning all diagrams upside down**, or, in other words, taking a horizontal mirror image. Here by "diagrams" we mean those scattered throughout the paper. For instance, the *co-Leibniz* condition

$$(\text{Id}_V \otimes \partial) \circ \partial = (\partial \otimes \text{Id}_V) \circ \partial - (\text{Id} \otimes c_{V,V}) \circ (\partial \otimes \text{Id}) \circ \partial$$

is graphically depicted as



Now let us make a list of dualities for categoric structures. Some dualities for theorems will follow.



a monoidal structure	a monoidal structure
a braiding	a braiding
a symmetric braiding	a symmetric braiding
a preadditive structure	a preadditive structure
a unital associative algebra $(V, \mu, \nu)$	a co-UAA $(V, \mu^{\text{op}}, \nu^{\text{op}})$
a unital Leibniz algebra $(V, [, ], \nu)$	a co-ULA $(V, [, ]^{\text{op}}, \nu^{\text{op}})$
a character $\varphi$ for a UAA $(V, \mu, \nu)$	a co-character $\varphi^{\text{op}}$ for $(V, \mu^{\text{op}}, \nu^{\text{op}})$
a Lie character $\varphi$ for a ULA $(V, [, ], \nu)$	a Lie co-character $\varphi^{\text{op}}$ for $(V, [, ]^{\text{op}}, \nu^{\text{op}})$
a weak braiding $\sigma$ for $V$	a weak braiding $\sigma^{\text{op}}$ for $V$
an upper cut $\epsilon$ for $(V, \sigma)$	a lower cut $\epsilon^{\text{op}}$ for $(V, \sigma^{\text{op}})$
a degree $-1$ tensor differential for $V$	a degree $1$ tensor differential for $V$

Note also that for a weakly braided  $(V, \sigma)$  and the action of  $B_n^+$  on  $V^{\otimes n}$  coming from  $\sigma$ , one has

$$(T_s^\sigma)^{\text{op}} = T_{s^{-1}}^{(\sigma^{\text{op}})} \in \text{End}_{\mathcal{C}^{\text{op}}}(V^{\otimes n}) \quad \forall s \in S_n,$$

since a decomposition of  $s^{-1}$  into simple transpositions can be obtained from one for  $s$  by simply reversing the order in the decomposition. Therefore, assuming the category preadditive, the definition of the quantum co-shuffle coproduct (remark 2.8) is translated as

$$(\Delta_\sigma^{p,q})^{\text{op}} = \Theta_{\sigma^{\text{op}}}^{p,q}.$$

In particular, the results of remark 2.8 follow from this duality.

Everything is now ready for dualizing theorems  $1^{\text{cat}}$  and  $3^{\text{cat}}$ .

**Theorem 1<sup>co</sup>.** *Let  $(\mathcal{C}, \otimes, \mathbf{I})$  be a preadditive monoidal category. For any weakly braided object  $(V, \sigma)$  with a lower cut  $e$ , the family of morphisms*

$$(\mathcal{d})^n := \Delta_{-\sigma}^{1,n} \circ (e \otimes \text{Id}_n)$$

*defines a degree 1 tensor differential for  $V$ .*

**Theorem 3<sup>co</sup>.**

1. Take a counital coassociative coalgebra  $(V, \Delta, \varepsilon)$  in a monoidal category  $(\mathcal{C}, \otimes, \mathbf{I})$ .

(a)  $V$  can be endowed with a weak braiding

$$\sigma_V := \varepsilon \otimes \Delta : V \otimes V \rightarrow \mathbf{I} \otimes V \otimes V = V \otimes V.$$

(b) Any co-character  $e \in \text{Hom}_{\mathbf{coAlg}(\mathcal{C})}(\mathbf{I}, V)$  is a lower cut for  $(V, \sigma_V)$ .

2. Take a counital co-Leibniz coalgebra  $(V, \partial, \varepsilon)$  in a symmetric preadditive category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ .

(a)  $V$  can be endowed with a weak braiding

$$\sigma_V := c_{V,V} + \varepsilon \otimes \partial.$$

(b) Any Lie co-character  $e \in \text{Hom}_{\mathbf{coLie}(C)}(\mathbf{1}, V)$  is a lower cut for  $(V, \sigma_V)$ .

A graphical depiction of  $\sigma_V$  for, for instance, a co-UAA  $V$  is by construction the horizontal mirror image of the diagram one had for UAAs:

$$\begin{array}{c} \Delta(w) \\ \diagup \quad \diagdown \\ \varepsilon \quad \Delta \\ \diagdown \quad \diagup \\ v \quad w \end{array}$$

A co-version of corollary 6.6 is then formulated in the evident way, with dual explicit formulas. Lemma 6.4 and proposition 6.5 are also dualized directly. In particular, the braidings from the previous theorem encode the co-associativity (resp. co-Leibniz) condition.

Let us finish this section with some remarks proper to our favorite category  $\mathbf{Vect}_{\mathbb{k}}$ .

**Lemma 6.10.** In  $\mathbf{Vect}_{\mathbb{k}}$ , a map  $e : \mathbb{k} \rightarrow V, \alpha \mapsto \alpha \mathbf{e}$  for a co-UAA  $(V, \Delta, \epsilon)$  is a co-character if and only if  $\mathbf{e} \in V$  is *group-like*, i.e.  $\Delta(\mathbf{e}) = \mathbf{e} \otimes \mathbf{e}$ , while a Lie co-character for a co-ULA  $(V, \partial, \epsilon)$  corresponds to an  $\mathbf{e} \in \text{Ker}(\partial)$ .

Further, "non-unital parts" of sections 5.3 and 5.4 admit co-versions. To create a counit for a coassociative or co-Leibniz coalgebra  $(V, \delta)$  (resp.  $(V, \partial)$ ), one extends it by adding a formal element:  $\tilde{V} := V \oplus \mathbb{k}\mathbf{1}$ , modifying the coproduct:

$$\Delta(v) = \delta(v) + \mathbf{1} \otimes v + v \otimes \mathbf{1} \quad \forall v \in V,$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$$

in the coassociative coalgebra case, and

$$\partial(\mathbf{1}) = 0,$$

keeping the original  $\partial$  on  $V$ , in the co-Leibniz case. Thus the application  $\varepsilon_0 \in \tilde{V}^*$  given by  $\varepsilon_0(V) \equiv 0, \varepsilon_0(\mathbf{1}) = 1$  is a (Lie) counit for  $\Delta$  (resp.  $\partial$ ), and  $\mathbf{1}$  is a group-like element (resp.  $\mathbf{1} \in \text{Ker}(\partial)$ ). One easily checks the following

**Lemma 6.11.** The new coproduct  $\Delta$  (resp.  $\partial$ ) is coassociative (resp. co-Leibniz) if and only if the original  $\delta$  (resp.  $\partial$ ) is.

To conclude, we write down the differentials obtained in this particular setting:

**Proposition 6.12.** *Given a  $\mathbb{k}$ -linear coalgebra  $(V, \delta)$ , extend it to a counital one  $(\tilde{V}, \Delta, \varepsilon_0)$  as described above. Then the group-like  $\mathbf{1}$  gives, via theorem 1<sup>co</sup>, the following differential on  $T(\tilde{V})$ :*

$$\mathbf{1}d(v_1 \dots v_n) = \mathbf{1}v_1 \dots v_n + \sum_{i=1}^n (-1)^i v_1 \dots v_{i-1} \Delta(v_i) v_{i+1} \dots v_n$$

The differential induced on  $T(V) \simeq T(\tilde{V}/\mathbb{k}\mathbf{1})$  is

$$\tilde{\mathbf{1}}d(v_1 \dots v_n) = \sum_{i=1}^n (-1)^i v_1 \dots v_{i-1} \Delta(v_i) v_{i+1} \dots v_n.$$

One eagerly recognizes the *cobar differential* for coalgebras.

## 6.4 Right-left duality

One more notion of duality is available for a monoidal category  $(\mathcal{C}, \otimes, \mathbf{I})$ . One can simply change its tensor product to the opposite one:

$$V \otimes^{\text{op}} W := W \otimes V$$

for objects, and similarly for morphisms. We call this new monoidal category *monoidally dual* to  $\mathcal{C}$ , denoting it by  $\mathcal{C}^{\otimes \text{op}}$  (there seem to be no universally accepted notation, some authors even using  $\mathcal{C}^{\text{op}}$  here and another notation for co-categories). Graphically, the categories  $\mathcal{C}$  and  $\mathcal{C}^{\otimes \text{op}}$  differ by a vertical mirror symmetry for all diagrams.

Applying monoidal duality to a co-category  $\mathcal{C}^{\text{op}}$ , one gets  $\mathcal{C}^{\text{op}, \otimes \text{op}} := (\mathcal{C}^{\text{op}})^{\otimes \text{op}}$ . Graphically it corresponds to the central symmetry.

Similarly to what we have seen for  $\mathcal{C}^{\text{op}}$ , all "categoric" notions and theorems have monoidally dual versions in  $\mathcal{C}^{\otimes \text{op}}$ . This gives in particular "right differentials",

$$(d^\epsilon)_n := (\text{Id}_{n-1} \otimes \epsilon) \circ \Theta_{-\sigma}^{n-1, 1},$$

$$(d_e)^n := \Delta_{-\sigma}^{n, 1} \circ (\text{Id}_n \otimes e),$$

monoidally dual to the "left" ones from theorems 1<sup>cat</sup> and 1<sup>co</sup>. Note that these differentials should be endowed with a sign (cf. theorem 1) if one wants a bidifferential structure.

One also has "right braidings", monoidally dual to those from theorems 3<sup>cat</sup> and 3<sup>co</sup>. In particular, a new braiding emerges for UAAs:

$$\sigma_V := \mu \otimes \nu : V \otimes V = V \otimes V \otimes \mathbf{I} \rightarrow V \otimes V.$$

Its diagram is a vertical mirror symmetry of what one had in the "left" case:



Remark that the Leibniz algebra structure is not right-left symmetric: a Leibniz algebra in  $\mathcal{C}^{\otimes \text{op}}$  is in fact a *left Leibniz algebra* in  $\mathcal{C}$  (cf. [17]).

## 7 (Co)homologies with coefficients

### 7.1 Modules over braided objects

We introduce here the notions of modules and bimodules over a weakly braided object  $V$  in a monoidal category (in particular over a braided vector space), generalizing, in quite an unexpected manner, modules over associative and Leibniz algebras (cf. [15]), as well as rack modules (cf. [5], where rack modules, called shadows by the authors, are studied and applied to the construction of knot invariants). Since at the same time a module over a weakly braided object generalizes an upper cut, one naturally arrives to homologies of weakly braided objects with coefficients. As particular cases we point out Hochschild and Chevalley-Eilenberg complexes. We also endow each tensor power  $V^{\otimes n}$  with a  $V$ -module structure, recovering in particular tensor powers of the adjoint representation of Leibniz algebras. All these facts suggest that our notion of modules is the "correct" one.

Fix a monoidal category  $(\mathcal{C}, \otimes, \mathbf{I})$ .

**Definition 7.1.** A *right module* over a weakly braided object  $(V, \sigma)$  is an object  $M \in \text{Ob}(\mathcal{C})$  equipped with a morphism  $\rho : M \otimes V \rightarrow M$  satisfying

$$\rho \circ (\rho \otimes \text{Id}_V) = \rho \circ (\rho \otimes \text{Id}_V) \circ (\text{Id}_M \otimes \sigma) : M \otimes V \otimes V \rightarrow M. \quad (\text{BrMod})$$

A *left module* is a right one in  $\mathcal{C}^{\text{op}}$ . A right (or left) *comodule* is a right (resp. left) module in  $\mathcal{C}^{\text{op}}$ .

Graphically condition (BrMod) is depicted as follows:

Start as usual with a trivial example: in a preadditive category, any object  $M$  equipped with the zero map  $M \otimes V \rightarrow M$  is a module over any weakly braided object  $(V, \sigma)$ .

Let us now interpret our new notions in more complicated settings from section 5.

1. When the braiding is simply a (signed) flip, one recovers the notion of (anti)commuting operators.
2. Take  $\mathcal{C} = \mathbf{Set}$ , and as a weak braiding on a set  $A$  take the  $\sigma_{\triangleleft}$  from (RackBraid), coming from a self-distributive operation  $\triangleleft$ . Condition (BrMod) becomes

$$(m \triangleleft a) \triangleleft b = (m \triangleleft b) \triangleleft (a \triangleleft b) \quad \forall m \in M, a, b \in A,$$

which defines precisely a *rack module* (cf. [5]).

3. Any UAA  $(V, \mu, \nu)$  in  $\mathcal{C}$  comes with the weak braiding  $\sigma_\mu$  from **(AssBraid)**. Take a right module  $(M, \rho)$  which we suppose *normalized* here, i.e.

$$\rho \circ (\text{Id}_M \otimes \nu) = \text{Id}_M$$

(morally, "the unit acts by identity"). Condition **(BrMod)** becomes

$$\rho \circ (\rho \otimes \text{Id}_V) = \rho \circ (\text{Id}_M \otimes \mu).$$

One recognizes the familiar notion of right modules over associative algebras.

4. Take a ULA  $(V, [, ], \nu)$  in a symmetric preadditive category  $\mathcal{C}$ . Endow  $V$  with the weak braiding  $\sigma_{[, ]}$  from **(LeiBraid)**. Take a normalized right module  $(M, \rho)$ . Condition **(BrMod)** becomes

$$\rho \circ (\rho \otimes \text{Id}_V) = \rho \circ (\rho \otimes \text{Id}_V) \circ (\text{Id}_M \otimes c_{V,V}) + \rho \circ (\text{Id}_M \otimes [, ]).$$

One recognizes the familiar notion of *right modules over Leibniz algebras* (cf. [15]), raised to the categoric level.

Note that dually left modules over associative or left Leibniz algebras are particular cases of left modules over weakly braided objects.

Summarizing,

**Proposition 7.2.** *In the particular cases of shelves, UAAs and ULAs, the notion of modules over corresponding weakly braided objects coincides with the usual notions of modules.*

Now, returning to the general categoric setting, we try a special choice of  $M$ , putting  $M = \mathbf{I}$ .

**Lemma 7.3.** For a morphism  $\epsilon : V = \mathbf{I} \otimes V = V \otimes \mathbf{I} \rightarrow \mathbf{I}$ , the following conditions are equivalent:

1.  $\epsilon$  defines a right module;
2.  $\epsilon$  defines a left module;
3.  $\epsilon$  is an upper cut.

Thus an upper cut for  $V$  defines a right and left  $V$ -module structure on  $\mathbf{I}$ . This observation can be generalized to endow each tensor power of  $V$  with a  $V$ -module structure:

**Proposition 7.4.** *Given a weakly braided object  $(V, \sigma)$  with an upper cut  $\epsilon$ , the map*

$${}^\epsilon \rho := (\epsilon \otimes \text{Id}_V^{\otimes n}) \circ \sigma_{V^{\otimes n}, V} : V^{\otimes n} \otimes V \rightarrow V^{\otimes n}$$

*defines a right module structure on  $V^{\otimes n}$ . The braiding  $\sigma$  is extended here to arbitrary powers of  $V$  as in remark 2.2.*

*Proof.* The definition of  $\epsilon\rho$  and repeated applications of equation (YB) give

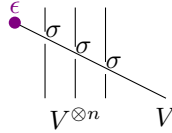
$$\begin{aligned} & \epsilon\rho \circ (\epsilon\rho \otimes \text{Id}_V) \circ (\text{Id } V^{\otimes n} \otimes \sigma) = \\ & (\epsilon \otimes \epsilon \otimes \text{Id}_V^{\otimes n}) \circ \sigma_{V^{\otimes n}, V^{\otimes 2}} \circ (\text{Id } V^{\otimes n} \otimes \sigma) = \\ & ((\epsilon \otimes \epsilon) \circ \sigma) \otimes \text{Id}_V^{\otimes n} \circ \sigma_{V^{\otimes n}, V^{\otimes 2}} \end{aligned}$$

which is, by the definition of an upper cut, the same as

$$(\epsilon \otimes \epsilon \otimes \text{Id}_V^{\otimes n}) \circ \sigma_{V^{\otimes n}, V^{\otimes 2}} = \epsilon\rho \circ (\epsilon\rho \otimes \text{Id}_V).$$

□

Here is a pictorial presentation of this action:



We will call these modules *adjoint* – the motivation will be clear from examples, where one recognizes familiar actions on  $T(V)$ :

1. For a shelf  $A$  in  $\mathcal{C} = \mathbf{Set}$  and, as an upper cut, the only map  $\epsilon_0$  from  $A$  to the one-element set  $\mathbf{I}$ ,  $A^{\times n}$  becomes an  $A$ -module via

$$(a_1 \dots a_n) \triangleleft b = (a_1 \triangleleft b) \dots (a_n \triangleleft b).$$

2. For a UAA  $V$  in  $\mathcal{C}$  with a normalized character  $\varphi$  ("normalized" in the sense of modules over UAAs, that is  $\varphi \circ \nu = \text{Id}_{\mathbf{I}}$ ), only the rightmost component of  $V^{\otimes n}$  is affected by the adjoint action if  $n > 0$ :

$$\varphi\rho = \text{Id}^{\otimes(n-1)} \otimes \mu.$$

In  $\mathbf{Vect}_{\mathbf{k}}$  it means

$$(v_1 \dots v_n) \cdot w = v_1 \dots v_{n-1} \mu(v_n \otimes w).$$

3. For a ULA  $V$  in  $\mathcal{C}$  with a normalized Lie character  $\varphi$ , one gets

$$\varphi\rho = \sum_{i=1}^n [ , ]_i \circ c_{V^{\otimes(n-i)}, V} + (\varphi \otimes \text{Id}^{\otimes n}) \circ c_{V^{\otimes n}, V}$$

(cf. the notations of corollary 6.6). Working in  $\mathbf{Vect}_{\mathbf{k}}$ , it gives

$$[v_1 \dots v_n, w] = \sum_{i=1}^n v_1 \dots v_{i-1} [v_i, w] v_{i+1} \dots v_n + \varphi(w) v_1 \dots v_n.$$

Starting with a non necessarily unital Leibniz algebra, adding a formal unit, taking the character  $\epsilon_0$  and then passing to its kernel (cf. section 5.4), one gets rid of the last term and arrives to the usual adjoint action of a Lie algebra  $V$  on  $T(V)$ .

We have seen that a module over a weakly braided object is a generalization of an upper cut. Observe that moreover it picks the right property for a generalized version of theorem 1<sup>cat</sup> (where we replace the  $V$ -module  $\mathbf{I}$  by arbitrary modules) to hold:

**Theorem 1<sup>coeffs</sup>.** *Let  $(\mathcal{C}, \otimes, \mathbf{I})$  be a preadditive monoidal category,  $(V, \sigma)$  a weakly braided object in  $\mathcal{C}$ , and  $(M, \rho)$  and  $(N, \lambda)$  a right and a left modules over  $V$  respectively. Then two families of morphisms*

$$({}^\rho d)_n := (\rho \otimes \text{Id}_V^{n-1} \otimes \text{Id}_N) \circ (\text{Id}_M \otimes \underline{\Theta}_{-\sigma}^{1, n-1} \otimes \text{Id}_N),$$

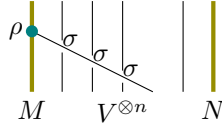
$$(d^\lambda)_n := (-1)^{n-1} (\text{Id}_M \otimes \text{Id}_V^{n-1} \otimes \lambda) \circ (\text{Id}_M \otimes \underline{\Theta}_{-\sigma}^{n-1, 1} \otimes \text{Id}_N),$$

define a bidegree  $-1$  tensor bidifferential for  $V$  with coefficient in  $M$  and  $N$ .

The complicated expression a bidegree  $-1$  tensor bidifferential for  $V$  with coefficient in  $M$  and  $N$  hides what one naturally expects: it means two families of morphisms  $d_n, d'_n : M \otimes V^n \otimes N \rightarrow M \otimes V^{n-1} \otimes N$ , satisfying

$$d_{n-1} \circ d_n = d'_{n-1} \circ d'_n = d'_{n-1} \circ d_n + d_{n-1} \circ d'_n = 0 \quad \forall n > 1.$$

Pictorially,  $({}^\rho d)_n$  for example is a signed sum of terms of the form



The proof of this result is a direct generalization of that of theorem 1.

*Remark 7.5.* Taking as  $M$  or  $N$  the unit object  $\mathbf{I}$  with a zero module structure, one obtains a degree  $-1$  tensor differential for  $V$  with coefficient in the left module  $N$  (resp. right module  $M$ ) only.

As usual, everything described here can be dualized, in any of the three senses described in sections 6.3 and 6.4.

Having the Hochschild homology in mind, one should also categorify the notion of bimodules.

**Definition 7.6.** A *bimodule* over a weakly braided object  $(V, \sigma)$  is an object  $M \in \text{Ob}(\mathcal{C})$  equipped with two morphisms  $\rho : M \otimes V \rightarrow M$  and  $\lambda : V \otimes M \rightarrow M$ , turning  $M$  into a left and right module and satisfying the following compatibility condition:

$$\rho \circ (\lambda \otimes \text{Id}_V) = \lambda \circ (\text{Id}_V \otimes \rho) : V \otimes M \otimes V \rightarrow M.$$

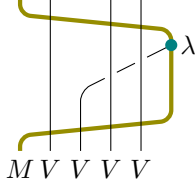
The bidifferential structure from theorem 1<sup>coeffs</sup> can be nicely adapted to bimodules:

**Proposition 7.7.** *Let  $(\mathcal{C}, \otimes, \mathbf{I}, c)$  be a symmetric preadditive category,  $(V, \sigma)$  a weakly braided object in  $\mathcal{C}$ , and  $(M, \rho, \lambda)$  a bimodule over  $V$ . Then the family of morphisms*

$$\begin{aligned} (\rho d)_n &:= (\rho \otimes \text{Id}_V^{n-1}) \circ (\text{Id}_M \otimes \Theta_{-\sigma}^{1, n-1}), \\ (d^\lambda)_n &:= (-1)^{n-1} c_{M, V^{n-1}}^{-1} \circ (\text{Id}_V^{n-1} \otimes \lambda) \circ (\Theta_{-\sigma}^{n-1, 1} \otimes \text{Id}_M) \circ c_{M, V^n}, \end{aligned}$$

defines a bidegree  $-1$  tensor bidifferential for  $V$  with coefficient in  $M$  on the left.

By definition,  $(d^\lambda)_n$  is a signed sum of terms of the form



*Proof.* Relations  $(\rho d)_{n-1} \circ (\rho d)_n = 0$  and

$$\begin{aligned} (d^\lambda)_{n-1} \circ (d^\lambda)_n &= c_{M, V^{n-2}}^{-1} \circ (d^\lambda)_{n-1} \circ c_{M, V^{n-1}} \circ c_{M, V^{n-1}}^{-1} \circ (d^\lambda)_n \circ c_{M, V^n} \\ &= c_{M, V^{n-2}}^{-1} \circ (d^\lambda)_{n-1} \circ (d^\lambda)_n \circ c_{M, V^n} = 0, \end{aligned}$$

with  $(d^\lambda)_n := (-1)^{n-1} (\text{Id}_V^{n-1} \otimes \lambda) \circ (\Theta_{-\sigma}^{n-1, 1} \otimes \text{Id}_M)$ , follow directly from the corresponding identities in theorem 1<sup>coeffs</sup>.

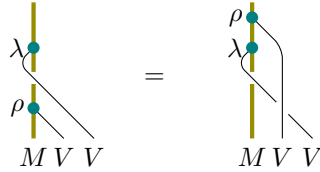
To prove the compatibility between  $(\rho d)_n$  and  $(d^\lambda)_n$ , observe that

$$(d^\lambda)_n = (-1)^{n-1} ((\lambda \circ c_{M, V}) \otimes \text{Id}_V^{n-1}) \circ (\text{Id}_M \otimes c_{V, V^{n-1}}^{-1}) \circ (\text{Id}_M \otimes \Theta_{-\sigma}^{n-1, 1}),$$

then use the defining property of a bimodule, the naturality of the braiding  $c$  and the Yang-Baxter relation for  $\sigma$ .  $\square$

*Remark 7.8.* We have kept the notation  $c^{-1}$ , redundant for symmetric  $c$ , to be able to treat the non symmetric situation. In this case, on the picture showing  $(d^\lambda)_n$  the thick line (corresponding to  $M$ ) should go behind all normal lines, in order to distinguish  $c$  from  $c^{-1}$ . One should be careful to differentiate two braidings,  $c$  and  $\sigma$ , which is difficult to do pictorially. For the above theorem to be still valid, one should change the compatibility condition defining a bimodule to the following one, different from the old one in general:

$$\lambda \circ (\text{Id}_V \otimes \rho) \circ c_{M \otimes V, V} = \rho \circ (\lambda \otimes \text{Id}_V) \circ c_{M, V} \circ c_{V, V}^{-1} : M \otimes V \otimes V \rightarrow M.$$





All the crossings correspond to the braiding  $c$  here.

A more elegant solution for the non symmetric case would be welcome.

Let us terminate this section with examples, as usual very familiar.

1. Taking a vector space  $V$  with a simple flip as braiding and, for instance, its symmetric algebra  $S(V)$  as a module over  $V$  (the action coming from concatenation, as usual), one obtains more complicated versions of the Koszul complex.
2. In the case of shelves one recovers the rack homology with coefficients, hinted at in [5].
3. For Leibniz algebras our machinery gives Leibniz homology with coefficients, generalizing Chevalley-Eilenberg homology (cf. [15]).

In these three cases one generally puts the coefficients only on the left (cf. remark 7.5).

4. Coefficients on both sides turn out to be particularly useful for associative algebras in a symmetric category. Proposition 7.7 gives in this setting a differential for any bimodule:

$$\begin{aligned}
 ({}^{\rho}d - d^{\lambda})_n &:= \rho \otimes \text{Id}_V^{n-1} + \sum_{i=1}^{n-1} (-1)^i \mu_i + (-1)^n (\lambda \otimes \text{Id}_V^{n-1}) \circ c_{M \otimes V^{n-1}, V} \\
 &\quad + \text{some terms involving } \nu.
 \end{aligned}$$

For  $\mathcal{C} = \mathbf{Vect}_{\mathbb{k}}$ , one can get rid of the terms with  $\nu$  as it was done in the proof of proposition 5.5, getting the *Hochschild differential*.

5. The co-version of the previous differential is the *Cartier differential* for coalgebras (c.f. [4], where it was first introduced). It is easily obtained by duality.

## 7.2 Structure mixing techniques

Another approach to studying (bi)modules over an associative or a Leibniz algebra consists in interpreting these structures as an associative / Leibniz multiplication on a larger vector space, mixing the module structure and the multiplication on the acting algebra. It resembles to what is often done when studying Hochschild or Leibniz extensions (see [1] and [17] for example).

We have chosen not to raise this section to the categoric level for simplicity, but this can be done in the setting of an additive category.

We study only the example of a **bimodule**  $V \in {}_A\text{Mod}_B$  over associative algebras here.

Take three vector spaces  $A, B, V$  with four bilinear operations

$$\begin{array}{ll}
 \mu_A : A \otimes A & \longrightarrow A & \mu_B : B \otimes B & \longrightarrow B \\
 \rho_A : A \otimes V & \longrightarrow V & \rho_B : V \otimes B & \longrightarrow V.
 \end{array}$$

These operations are denoted by a dot, e.g.  $a \cdot v = \rho_A(a \otimes v)$ , when it doesn't lead to confusion.

Now mix these structures, posing  $V' = A \oplus B \oplus V$  and defining a bilinear operation  $\mu$  on  $V'$  by

$$\begin{aligned} \mu|_{A \otimes A} &= \mu_A, & \mu|_{B \otimes B} &= \mu_B, \\ \mu|_{A \otimes V} &= \rho_A, & \mu|_{V \otimes B} &= \rho_B, \end{aligned}$$

extended by zero for other couples of spaces. One easily checks the following

**Lemma 7.9.** The associativity of  $\mu$  is equivalent to a set of conditions:

- ⊗  $A$  and  $B$  are associative algebras;
- ⊗  $\rho_A$  is a left action of  $A$  on  $V$ ;
- ⊗  $\rho_B$  is a right action of  $B$  on  $V$ ;
- ⊗ these actions are compatible, in the sense that

$$(a \cdot v) \cdot b = a \cdot (v \cdot b), \quad \forall a \in A, v \in V, b \in B.$$

Add a formal unit  $\tilde{V} := V' \oplus \mathbb{k}\mathbf{1}$  and consider the application  $\sigma_\mu$  from section 5.3. Combining the preceding lemma with lemma 5.3, one gets:

**Corollary 7.10.** *The application  $\sigma_\mu$  is a braiding on  $\tilde{V}$  if and only if the maps  $\mu_A, \mu_B, \rho_A, \rho_B$  define a structure of two associative algebras  $A$  and  $B$  and a bimodule  $V \in {}_A\text{Mod}_B$ .*

Thus our braiding encodes the structure of a bimodule.

Still proceeding as in section 5.3, consider the left-stable upper cut  $\epsilon_0(V \oplus A \oplus B) \equiv 0, \epsilon_0(\mathbf{1}) = 1$ . The subspace  $T(\text{Ker } \epsilon_0)$  is then  ${}^{\epsilon_0}d$ -stable. Now we try to restrict  ${}^{\epsilon_0}d$  even further. First, the linear span of pure tensors in  $T(\tilde{V})$  containing an element of  $V$  exactly once is stable under the action of the positive braid monoid given by  $\sigma_\mu$  (cf. section 2) and is  $\epsilon_0$ -stable (since  $\epsilon_0(V) \equiv 0$ ), hence  ${}^{\epsilon_0}d$ -stable. The same holds for its "ordered" part  $T(A \oplus \mathbb{k}\mathbf{1}) \otimes V \otimes T(B \oplus \mathbb{k}\mathbf{1})$  (i.e. one forces the element of  $V$  to follow all elements of  $A$  and to precede those of  $B$ ). The intersection of the last space with  $T(\text{Ker } \epsilon_0)$  is then a  ${}^{\epsilon_0}d$ -stable space

$$T(A; V; B) := T(A) \otimes V \otimes T(B) \subseteq T(\tilde{V}),$$

thus proving

**Proposition 7.11.** *Take a bimodule  $V \in {}_A\text{Mod}_B$ . The restriction of  ${}^{\epsilon_0}d$  described above to  $T(A; V; B)$  gives a differential*

$$\begin{aligned} {}^{\epsilon_0}d(a_1 \dots a_n v b_1 \dots b_m) &= \\ &= \sum_{i=1}^{n-1} (-1)^i a_1 \dots a_{i-1} (a_i \cdot a_{i+1}) a_{i+2} \dots a_n v b_1 \dots b_m \\ &+ (-1)^n a_1 \dots a_{n-1} (a_n \cdot v) v b_1 \dots b_m + (-1)^{n+1} a_1 \dots a_n (v \cdot b_1) b_2 \dots b_m \\ &+ \sum_{i=1}^{m-1} (-1)^{n+1+i} a_1 \dots a_n v b_1 \dots b_{i-1} (b_i \cdot b_{i+1}) b_{i+2} \dots b_m \end{aligned}$$

The differential from the proposition can be upgraded to the Hochschild differential, giving an approach alternative to that presented in the previous section. This will be done and analyzed in detail in a subsequent paper.

## 8 What next?

This paper is the first one in a series of publications devoted to a braided interpretation of algebraic and homological phenomena. The tools developed here will be further expanded and applied to new examples. Two follow-up papers are in preparation. The first one treats the "bi-world": bialgebras and Hopf algebras, Hopf and Yetter-Drinfeld modules. We will present quite elaborate braidings for these structures and will then easily derive formulas for corresponding differentials which seem otherwise quite heavy and obscure. In the center of the second one is the "cyclic world", with cyclic homologies of associative and Hopf algebras understood in our new braided settings. The "shuffle world" and notably Harrison homology for associative algebras can be treated with the same ideas as guidelines.

Many other questions wait for a braided interpretation. For instance, a braiding giving Poisson algebra homology would be interesting to obtain. Gerstenhaber structure on Hochschild homology is also likely to have a quantum shuffle explanation. Further, a homology theory for Zinbiel algebra and, in the same vein, a braided version of the operad duality do not seem impossible.

The author's dream is to get new homologies for certain algebraic structures with the help of the "braided" tools presented here.

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