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The problem of the gain of dichroic devices is analyzed in a framework based on a vectorial pure operatorial (non-matrix) Pauli algebraic approach to polarization optics. We show that the partially polarized light and the partial polarizers can be described in this framework by absolutely similar, symmetric quantities. In this sense, a physically essential device parameter, the degree of dichroism, is defined. This symmetry between the description of the polarized light and of the polarization devices leads to an expressive form of the generalized Malus' law, the consequences of which are analyzed in detail. Most importantly, one can for the first time describe and graphically illustrate the generalized structure of the gain of a dichroic device.

Keywords: Pauli algebra, Malus' law, dichroic polarization device

1. Introduction

The generalized Malus' law is a classical subject in the theory of light polarization, e.g. (1) p. 110 and (2, 3), which is intimately connected, in fact equivalent, with that of determining the gain given by a partial polarizer for partially polarized incident light (1, 4, 5).

Letting aside the use of some theorems and procedures of linear algebra expressed in their pure operational form (e.g. polar decomposition, singular value decomposition), all the approaches to these problems are, in the last instance, matrix ones (Jones or Mueller). The matrix approaches have the advantage to "follow a fixed routine in which little thought is required beyond looking up the vectors and matrices in a table and performing the standard multiplication operations", (6) p. 122, but the drawback of keeping us far away from the intuition about the physical phenomenon. When one wants, finally, to grasp the physical signification of the results, a considerable effort is required for coming back from the mathematics to the physics involved.

In what concerns the gain, the Jones matrix approach, for example, uses the well-known formula:

$$g = \frac{Tr(\mathsf{T}\mathsf{J}_i\mathsf{T}^\dagger)}{Tr\mathsf{J}_i} \ . \tag{1}$$

Let us quote from Azzam and Bashara (1) p. 145: "[This is] an elegant form of the intensity transmittance in terms of the coherency matrix of the incident light, J_i, and the Jones matrix of the optical system, T. However this equation is difficult to interpret because of the implicate way the coherence matrix J carries the information about the intensity *I*, degree of polarization *p* and polarization form ϑ, ε of partially coherent light." By consequence, the efforts of interpreting such a simple and elegant formula often involve cumbersome calculi and lead to very complicated expressions of the gain in terms of the physical parameters of the device and of the incident light (e.g. (1), pp.103-110).

The Mueller calculus, in its standard matrix form, uses also 4×4, base dependent, collections of numbers for characterizing the devices/media. Nevertheless, in the last time, a semi-vectorial language (7-10) was developed in the general frame of Mueller calculus, which highlights some characteristic vectorial quantities in the structure of the Mueller matrices: the retardance, the diattenuation and the polarizance vectors. In this language, the Mueller matrices are written in a partitioned form, containing two vectors (generally specified in terms of their components) plus a 3×3 matrix. This way, Mueller calculus has very much gained in physical expressivity and efficiency. Particularly, it can express the gain of an orthogonal dichroic device in a compact form in terms of the diattenuation vector — equal in this case to the polarizance vector (see Appendix). A mathematical discomfort still resides in the hybrid nature of this language.

In this paper, we will analyze the problem of the gain in a vectorial pure operatorial Pauli algebraic approach to the interaction of light with the polarization devices. Firstly, this approach is mathematically self-consistent: pure operatorial and coordinate-free. Secondly, it is intimately connected with the geometrical intuitive handling of the interaction light-devices/media on the Poincaré sphere, because the Pauli operators, well-known, are the basic rotors on the sphere.

As we have shown in detail in (11) and (12), this approach leads straightforwardly, in the most direct manner and only in few lines of calculus to the whole group of three quantities which characterize the action of the system on the state of optical polarization (SOP): the gain g, the Poincaré unit vector $\mathbf{s}_o$ of the polarization state and the degree of polarization p_o of the emergent light. All the relevant quantities, $\mathbf{s}_o$, p_o, g appear in block in the expression of the polarization density operator of the output state. It is a unique expression which contains all the information about the interaction which occurred.

A remarkable aspect of this method, which will be apparent in the analysis of the action of the dichroic devices on partially polarized light, is that the results have a high degree of symmetry. This symmetry, as well as the compactness of the results, is due to the fact that our approach is parameterized in a manner well adapted to the symmetries of both the polarization states space and of the devices.

The main aims of this paper are:

1. To prove the feasibility of the vectorial pure operatorial Pauli algebraic approach for an already classical problem in polarization theory — the calculation of the gain of a polarization device.

2. To introduce a symmetrical description — which is straightforward and quite natural in this approach — of the polarized light and of the polarization devices. In the operatorial Pauli algebraic framework, this symmetric description is the direct manifestation of the fact that the state of optical polarization and the dichroic devices are similarly represented by Hermitian operators.

Pointing out this symmetry, a simple and expressive form of the generalized Malus' law will be obtained.

For the sake of clarity we restrict our analysis only to the dichroic devices. The extension to any linear deterministic (13) device is straightforward on the basis of the polar decomposition (14) of the device operators: The gain of the device is determined exclusively by the Hermitian component of the device, in other words by its modulus. The unitary factor of the device operator contributes only to the alteration of the state of polarization. For a general linear deterministic operator all the analysis and its results apply to the Hermitian factor of the operator and are unaffected by the unitary factor.

The layout of the paper is the following. In Section 2 we briefly review some basic results of the vectorial and pure operatorial Pauli algebra's applications in polarization optics (11, 12) that will be of use in the present work. In Section 3, starting from the expression of the gain, we show that one can define for the dichroic device a similar quantity as the degree of polarization for the light. This quantity, which we denominate degree of dichroism of the device, represents the natural symmetric partner of the light degree of polarization in the expressions which characterize the interaction polarized light – dichroic device/media. In Section 4, this symmetry is emphasized by obtaining the most compact and insightful form of the generalized Malus' law, completely specifying the manner in which the characteristic quantities of the partial polarizer and of the partially polarized light couple to give the output observables of their interaction.

2. Action of an orthogonal dichroic device on partially polarized light

The orthogonal dichroic devices (homogeneous partial polarizers) (15) are represented in polarization optics by Hermitian operators, which have the following general Pauli algebraic expression:

$$\mathsf{H}_{\mathbf{n}}(\rho,\eta) = e^{\rho} e^{\frac{\eta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}} = e^{\rho}\left(\sigma_0 \cosh\frac{\eta}{2} + \mathbf{n}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2}\right), \tag{2}$$

where:

$$e^{\rho} = e^{\frac{\eta_1+\eta_2}{2}}, \tag{3}$$

$$e^{\eta} = e^{\eta_1-\eta_2}, \tag{4}$$

are the isotropic and the relative amplitude transmittances of the device, respectively, while e^{η_1} and e^{η_2} are its two amplitude eigentransmittances, major and minor, respectively, (6) p. 36. In the following we shall label by $\tau_M = e^{2\eta_1}$ and $\tau_m = e^{2\eta_2}$ the major and minor intensity transmittances, respectively. In the most widespread case of dichroic device, namely in the case of diattenuators, the two coefficients η_1 and η_2 are both negative and the eigentransmittances are lower than unity. Other types of dichroic devices have been reported in the literature, (1) p.111, (16), consisting of active elements (with optically pumped crystals), which act as diamplifiers or as squeeze devices (amplifier on one channel, attenuator on the other). In the case of diamplifiers both transmittance coefficients η_1 and η_2 are positive, while in the case of squeeze devices one of the coefficients is positive, while the other is negative. The major eigenstate will be defined as the state of maximum transmittance, irrespective of the fact that the device is a diattenuator or a diamplifier. That is, in out notations, $\eta_1 > \eta_2$.

In equation (2), $\boldsymbol{\sigma}$ denotes a vector operator whose components are the three Pauli scalar operators, σ_i (i = 1, 2, 3). Together with σ_0, the "two-dimensional unit operator", they define the Pauli basis, in which any "2 x 2 operator" (linear operator defined on a space of two dimensions over the field of complex numbers **C**) can be expanded. The vector **n** is the Poincaré unit vector of the device (1). For a Hermitian operator it is a real vector. In a Poincaré sphere representation (1), the unit vector **n** points towards the major eigenstate of the dichroic device. In the Pauli algebraic formalism, the state of optical polarization is described by means of its density operator (corresponding in the matrix representation to the coherency, or polarization matrix).

For the most general case, of partially polarized light, the density operator of the SOP has the Pauli algebraic form:

$$\mathsf{J} = \frac{I}{2}(\sigma_0 + p\mathbf{s}\cdot\boldsymbol{\sigma}), \tag{5}$$

where I is the light intensity, p is the degree of polarization and **s** is a real unit vector. The product p**s** is called the Poincaré vector of the polarization state and it gives the representation of the SOPs in the Poincaré states space. In the case of completely polarized light (p = 1), the top of this vector lies on the Poincaré sphere (of radius 1) Σ_2^1, while in the case of partially polarized light it lies in the Poincaré ball Σ_3^1, on the sphere of radius p, Σ_2^p.

In the following we will take the intensity of the input polarization state equal to unity, case in which the intensity of the output polarization state gives directly the value of the gain of the interaction with the dichroic device. The density operators of the input and output SOPs may be written in this case:

$$\mathsf{J}_i = \frac{1}{2}(\sigma_0 + p_i \mathbf{s}_i \cdot \boldsymbol{\sigma}), \tag{6}$$

$$\mathsf{J}_o = \frac{g}{2}(\sigma_0 + p_o \mathbf{s}_o \cdot \boldsymbol{\sigma}), \tag{7}$$

where g labels the intensity gain, and the indices i and o stand for the input and output polarization states, respectively.

The action of the Hermitian operator, equation (2), corresponding to the dichroic device on the density operator of the input state, equation (6), is given by (1, 9):

$$\mathsf{J}_o = \mathsf{H}_{\mathbf{n}}(\rho,\eta)\mathsf{J}_i \mathsf{H}_{\mathbf{n}}(\rho,\eta), \tag{8}$$

By using equations (2) and (6) in equation (8) one obtains for the density operator of the output SOP the following expression:

$$\begin{aligned}\mathsf{J}_o &= \frac{1}{2}e^{2\rho}(\sigma_0 \cosh\frac{\eta}{2} + \mathbf{n}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2})(\sigma_0 + p_i\,\mathbf{s}_i\cdot\boldsymbol{\sigma})(\sigma_0\cosh\frac{\eta}{2} + \mathbf{n}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2}) \\ &= \frac{1}{2}e^{2\rho}\left\{\sigma_0(\cosh\eta + p_i\,\mathbf{s}_i\cdot\mathbf{n}\sinh\eta) + \left[p_i\,\mathbf{s}_i + \mathbf{n}\sinh\eta + 2p_i\,(\mathbf{n}\cdot\mathbf{s}_i)\mathbf{n}\sinh^2\frac{\eta}{2}\right]\cdot\boldsymbol{\sigma}\right\},\end{aligned} \tag{9}$$

which, by comparison to equation (7), gives directly the expressions of the fundamental quantities for the interaction of the light with the dichroic device, namely the gain g and the Poincaré vector of the output polarization state, $p_o\mathbf{s}_o$, whose modulus is equal with the degree of polarization:

$$g = e^{2\rho}\left(\cosh\eta + p_i\mathbf{s}_i\cdot\mathbf{n}\sinh\eta\right) = e^{2\eta_1}\,\frac{1+p_i\cos\alpha}{2} + e^{2\eta_2}\,\frac{1-p_i\cos\alpha}{2}, \tag{10}$$

$$p_o\mathbf{s}_o = \frac{p_i\,\mathbf{s}_i + \mathbf{n}\sinh\eta + 2p_i\left(\mathbf{n}\cdot\mathbf{s}_i\right)\mathbf{n}\sinh^2\frac{\eta}{2}}{\cosh\eta + p_i\left(\mathbf{n}\cdot\mathbf{s}_i\right)\sinh\eta} = \frac{p_i\,\mathbf{s}_i + \mathbf{n}\sinh\eta + \mathbf{n}p_i\cos\alpha\left(\cosh\eta - 1\right)}{\cosh\eta + p_i\cos\alpha\sinh\eta}, \tag{11}$$

where we have denoted by α the angle between the Poincaré unit vectors of the incident light and of the device, $\mathbf{s}_i$ and $\mathbf{n}$, respectively (figure 1).

All the information about the action of the dichroic device on the partially polarized incident light is given directly and compactly, in block, by equation (9), or equivalently by equations (10) and (11). The expression (10) of the gain constitutes *the largest generalization of the Malus' law*, valid for partially polarized light passed through any canonical dichroic device. Equation (11) provides the state of polarization of the emergent light. It is quite remarkable that in the final results, equations (10) and (11), the information concerning the gain of the device, g, and that concerning the SOP of the outgoing light, $p_o\mathbf{s}_o$, are completely separated: both the gain and the polarization state of the emerging light are expressed exclusively on the basis of the characteristics of the device (ρ, η, $\mathbf{n}$) or (η_1, η_2, $\mathbf{n}$) and of that of the incident light (p_i, $\mathbf{s}_i$).

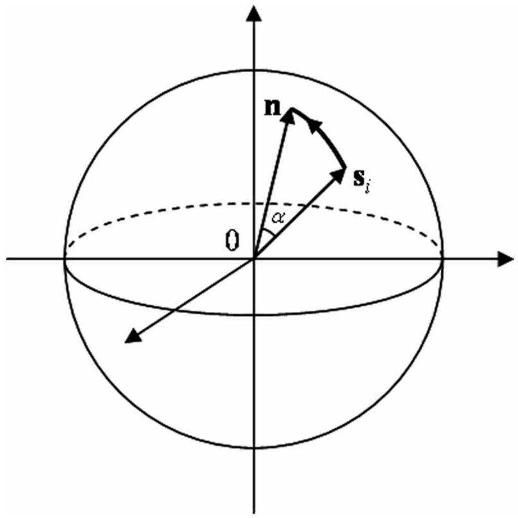

Figure 1. The Poincaré unit vectors of the dichroic device and of the incident light. Notations

In the following we will be concerned with some aspects of the gain given by a dichroic device for, generally, partially polarized incident light and, on this basis, we will give a physically expressive form of the generalized Malus' law.

3. The gain, transmittance and degree of dichroism of the dichroic device

Equation (10) is the general expression of the gain given by any dichroic device for any SOP of the incident light in function of the primary parameters of the device ($\mathbf{n}$, η_1, η_2) and of the incident light SOP ($\mathbf{s}_i$, p_i). Let us analyze now some essential aspects of the problem of the gain. In equation (10) it is apparent that the gain depends on the principal intensity transmittances of the device, on the degree of polarization of the incident light and on the angle between the Poincaré unit vector of the device and of the incident light. We shall analyze in these terms the conditions in which the gain takes its maximum and minimum values.

The only quantity which can take positive as well as negative values in the expression (10) of the gain is the scalar product $\mathbf{n}\cdot\mathbf{s}_i$. Corresponding to its extreme values ±1, some relative maximum and minimum values of the gain are obtained when the Poincaré unit vector of the dichroic device is aligned parallel, $\mathbf{n}\uparrow\uparrow\mathbf{s}_i$, and antiparallel, $\mathbf{n}\uparrow\downarrow\mathbf{s}_i$, respectively, with the Poincaré unit vector of the incident light:

$$g_M = e^{2\rho}(\cosh\eta + p_i \sinh\eta)\,, \tag{12}$$

$$g_m = e^{2\rho}(\cosh\eta - p_i \sinh\eta)\,. \tag{13}$$

They are easily interpretable: The gain reaches a maximum when the major eigenstate of the density operator of the incident light is transmitted with the major intensity transmittance of the device and the minor eigenstate with the minor transmittance. When, on the contrary, the major eigenstate of the density operator of the incident light passes through the device with a minimum intensity transmittance and it is the minor eigenstate that benefits of the maximum transmittance, the gain is minimum. These (relative) extreme values of the gain, equations (12), (13), depend on the dichroic device, but also on the degree of polarization of the incident light.

Physically, both conditions $\mathbf{n}\uparrow\uparrow\mathbf{s}_i$ and $\mathbf{n}\uparrow\downarrow\mathbf{s}_i$ mean that the *dichroic device is matched with the polarization state of the incident light*, hence its major eigenstate has the same (generally elliptical) form as the SOP of the incident light. The former condition requires that the major eigenstate of the device coincides with the SOP of the incident light, the latter that the minor eigenstate of the device coincides with this SOP. It is worth mentioning that the condition $\mathbf{n}\uparrow\uparrow\mathbf{s}_i$ also requires that the handedness of the corresponding polarization forms (of the incident light and of the major eigenstate of the device) is the same. On the other hand, for the condition $\mathbf{n}\uparrow\downarrow\mathbf{s}_i$, the same polarization forms must have the major axes orthogonal and opposite handedness.

When the angle α between $\mathbf{n}$ and $\mathbf{s}_i$ is continuously varied, the gain changes between g_M and g_m according to equation (10). A *gain contrast* obtained at the variation of the angle α can be defined:

$$\gamma = \frac{g_M - g_m}{g_M + g_m} \tag{14}$$

and, with equations (12) and (13), its value is given by:

$$\gamma = p_i \tanh\eta \, . \tag{15}$$

In the expression (15) of the gain contrast, the incident light and the device are similarly represented by two parameters, p_i — the degree of polarization of the incident light — and $\tanh\eta$, respectively. Let us exploit further this similarity for getting a precise physical interpretation of the device parameter $\tanh\eta$.

On the one hand, when $\tanh\eta = 1$, i.e. when the dichroic device becomes an ideal (i.e. total) polarizer, the gain contrast takes the value of the degree of polarization of the incident light:

$$\gamma = p_i \, . \tag{16}$$

Thus the *degree of polarization* of a beam of light could be measured by means of a total polarizer matched with its SOP (having its unique eigenstate identical with the major eigenstate of the density operator of the incident light), namely as the gain contrast in such a measurement. This would come to a direct separation of the completely polarized and unpolarized components of the light.

On the other hand, when $p_i = 1$, i.e. for completely polarized incident light, the gain contrast is:

$$\gamma = \tanh\eta \, . \tag{17}$$

The quantity $\tanh\eta$ is the most relevant parameter of the dichroic device, the most suitable measure of its anisotropy. It varies from 0, for an isotropic device, to 1, for a device of highest anisotropy. We will call it the *degree of dichroism* of the device and will label it by p_d.

As well as the degree of polarization of a beam of light could be measured, as the gain contrast, by means of a dichroic device matched on the SOP of the incident light, the degree of dichroism of a dichroic device could be measured, as a gain contrast too, by means of a completely polarized beam of light whose SOP is matched on the state of the device. In conclusion, the expressions (15) of the gain contrast may be written in a symmetrical form with respect to the device and to the incident light as:

$$\boxed{\gamma = \frac{g_M - g_m}{g_M + g_m} = p_i p_d} \, . \tag{18}$$

There is some confusion, some overlapping, in the literature concerning the term transmittance and gain, e.g. reference (17). We believe that things would be well clarified by attributing the term *gain* (g) to the interaction light – device (it depends on the device as well as on the incident SOP), and the term *transmittance* (τ) solely to the device (it depends exclusively on the device structure).

Well known, the dichroic device is one of the building blocks in any decomposition of polarization elements (1, 10, 17-19) and the gain of any polarization element is determined exclusively by the characteristics of this building block. Thus, the previous results are applicable in the analysis of any deterministic (13) polarization device.

Equation (18) gives us a first insight into the fact that the gain of a dichroic device is determined by two completely similar quantities, describing the partially polarized light and the partial polarizers. The full mathematical expression of this dependence, as well as its structure, will be explored in the following section.

4. The generalized Malus' law

As we have already mentioned, equation (10) gives the generalized form of the Malus' law for any dichroic device and any SOP of the incident light. For a linear ideal polarizer ($\eta_1 = 0,\ \eta_2 \to -\infty$) acting on totally linearly polarized light ($p_i = 1$), the familiar squared-cosine Malus' law is obtained:

$$g = \cos^2 \frac{\alpha}{2}. \tag{19}$$

For a total elliptic polarizer acting on totally elliptical polarized light, Malus' law keeps the same form, but with α the angle between the Poincaré vectors of the polarizer and of the incident light, $\mathbf{n}$ and $\mathbf{s}_i$ respectively (figure 1). This result, of a certain generality, was first established in reference (2) by a calculus of spherical trigonometry, considered tedious even by its authors (20).

Having in mind the symmetry between the descriptions of the incident light SOP and of the device, introduced in the previous section, we can lead the expression (10) of the gain to a very symmetric form:

$$\begin{aligned} g &= e^{2\eta_1} \frac{1 + p_i \cos\alpha}{2} + e^{2\eta_2} \frac{1 - p_i \cos\alpha}{2} \\ &= \frac{\tau_M + \tau_m}{2} + p_i \mathbf{n} \cdot \mathbf{s}_i \frac{\tau_M - \tau_m}{2} \\ &\boxed{g = \bar{\tau}(1 + p_d \mathbf{n} \cdot p_i \mathbf{s}_i)} \end{aligned} \tag{20}$$

Equation (20) is a compact and expressive formula, showing that the variation of the gain is completely determined, up to an isotropic factor, $\bar{\tau}$, by the scalar product of two vectors, $p_i\mathbf{s}_i$, the Poincaré vector of the incident light, and $p_d\,\mathbf{n}$, which can be denominated as Poincaré vector of the dichroic device. One can notice, in addition to what was already apparent from equation (18), that the symmetry between the quantities which describe the polarization state of the light, on the one hand, and the dichroic device, one the other hand, does not only manifest in the gain contrast, equation (18), but is present in the very expression of the gain. This symmetry has its roots in the fact that the SOPs and the dichroic devices are similarly represented in the Pauli algebra by Hermitian operators.

For *unpolarized incident light* the gain given by the dichroic device reduces to:

$$g_{up} = \frac{\tau_M + \tau_m}{2} = \bar{\tau}\ . \tag{21}$$

Well known, the intensity of an unpolarized beam of light incident on a dichroic device is transmitted in a ratio given by the mean intensity transmittance of the device (the arithmetic mean of its principal intensity transmittances). Half of the incident light is transmitted with a gain τ_M, half with the gain τ_m, and the results are *incoherently superposed* in the output.

For *totally polarized light* the gain given by a dichroic device may be put successively in the forms:

$$g_{tp} = \bar{\tau}(1 + p_d \mathbf{s}_i \cdot \mathbf{n})$$
$$= \tau_M \frac{1 + \mathbf{s}_i \cdot \mathbf{n}}{2} + \tau_m \frac{1 - \mathbf{s}_i \cdot \mathbf{n}}{2} = \tau_M \cos^2 \frac{\alpha}{2} + \tau_m \sin^2 \frac{\alpha}{2}, \tag{22}$$

where, as we have already mentioned, α is the angle between the Poincaré unit vectors of the incident light's SOP and of the device, $\mathbf{s}_i$ and $\mathbf{n}$, respectively (figure 1). In the (very particular) case of linearly polarized light and a linear partial polarizer, $\alpha/2$ represents the angle, in the real physical space, between the azimuth of the incident SOP and that of the polarizer. In the more general case of an elliptical polarization state of the incident light and an elliptical polarizer, the connection between the angle on the Poincaré sphere and the real physical angle (for example, between the major axes of the corresponding ellipses) is less straightforward.

The last form of the gain, equation (22), gives a good physical insight in what concerns the transfer of completely polarized light through a dichroic device. The light is (*coherently*) decomposed on the principal directions of the dichroic device (in the eigenstates of the device), transmitted with the corresponding principal intensity transmittances of the device, τ_M and τ_m, and (*coherently*) recomposed at the exit. The additivity of the intensities of these components in the output is due to the fact that they are orthogonally polarized.

For getting a good physical insight in the general case of *partially polarized light*, we may use the decomposition corresponding to the so-called "polarized-unpolarized dichotomy", (6) p. 11. The SOP of the partially polarized incident light is decomposed into a completely polarized state and an unpolarized one. This decomposition is unique:

$$\mathsf{J}_i(I, p_i) = I(1 - p_i)\mathsf{J}_{i.up} + I\, p_i \mathsf{J}_{i.tp}\,. \tag{23}$$

By using equations (20-22) it is straightforward to verify the additivity (weighted by p_i and $1 - p_i$ respectively) of the gains for these components:

$$g = (1 - p_i)g_{up} + p_i g_{tp}\,. \tag{24}$$

Making use now of equations (21) and (22) this decomposition of the gain may be put in the insightful form:

$$\boxed{g = (1 - p_i)\bar{\tau} + p_i\left(\tau_M \cos^2 \frac{\alpha}{2} + \tau_m \sin^2 \frac{\alpha}{2}\right)}. \tag{25}$$

Maybe this is the most expressive form of the generalized Malus' law. The partially polarized incident light is (mentally) decomposed incoherently into an unpolarized component and a completely polarized one. The gain given by the dichroic device (e.g. a partial polarizer) for the unpolarized component is the mean intensity transmittance of the device $\bar{\tau}$. The totally polarized component, in its turn, is coherently decomposed in two orthogonally polarized components along the eigenstates of the device. These last components are transmitted through the device with its intensity transmittances, τ_M and τ_m, and all the components are added in intensity at the output.

In the following we shall present some graphical illustrations of the generalized Malus' law. We will refer to the most widespread case of dichroic devices, namely that of diattenuators. The graphs represent the gain as a function of the angle α between the Poincaré unit vectors of the incident light SOP, $\mathbf{s}_i$, and of the device, $\mathbf{n}$. We have to note that the variation of α means a change of the structure (characteristics) of the dichroic device or of the polarization of the incident light: for $\alpha = 0$, the device is matched parallel ($\mathbf{n} \uparrow\uparrow \mathbf{s}_i$) and for $\alpha = \pi$ antiparallel ($\mathbf{n} \uparrow\downarrow \mathbf{s}_i$) with the incident light SOP. Only in the case of linear polarizers acting on linearly partially polarized light the variation of α has a straightforward correspondent in the real physical space.

In figure 2 the variation of the gain with the angle α is illustrated for the action of a partial polarizer on partially polarized incident light. The first term of equation (25), corresponding to the unpolarized component of the incident light, does not depend on the angle α. For a given degree of polarization of the incident light, this term depends solely on the mean intensity transmittance of the device, $\bar{\tau}$, and constitutes an invariable contribution to the total gain. In figure 2 the term is represented by the dashed line. The second term of equation (25), corresponding to the completely polarized component of the incident light, is represented in figure 2 by the part of the graph situated above the dashed line. It is a variable contribution with respect to the angle α. Its extreme values, equal to $p_i \tau_M$ and $p_i \tau_m$, are obtained when the SOP is aligned with the major and minor axes of the dichroic device (α is 0 and π, respectively).

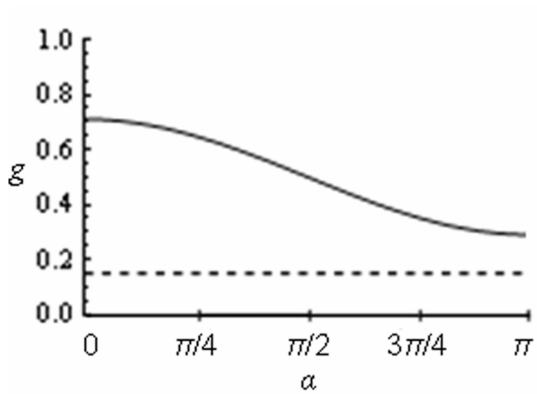

Figure 2. Variation of the gain with the angle α for partially polarized incident light, when p_i and $\bar{\tau}$ are constant. The dashed line represents the contribution of the unpolarized component
$p_i = 0.7$, $\tau_M = 0.8$, $\tau_m = 0.2$

The variation of the gain with the angle α emerges entirely from the completely polarized component of the incident light, which is thus the source of the gain contrast γ. The unpolarized component of the incident light limits the gain contrast to a value p_i. There are two reasons for which the gain contrast, γ, is, in general, lower than unity: the partial polarization of the light, $p_i < 1$, and the imperfection of the polarizer, $p_d < 1$. They manifest apparently in figure 2: the first in the constant gain term (dashed line) corresponding to the unpolarized component, the second in the fact that the gain does not decrease all the way to the dashed line (for $\alpha = \pi$).

In figure 3 the variation of the gain with the angle α is illustrated for the action of a perfect polarizer on partially polarized incident light. In the case of a perfect polarizer, the unpolarized component of the incident light is half transmitted and half absorbed by the device. From equation (25), the invariable contribution of this component to the total gain is equal to $(1 - p_i)/2$. It follows that the extreme values of the gain for partially polarized light incident on a total polarizer are equal to $(1 - p_i)/2$ and $(1 + p_i)/2$, respectively. They are represented in figure 3 by the two dashed lines, which are symmetric with respect to a line situated at g = $\bar{\tau} = 0.5$.

In the limit case of completely polarized light incident on a total polarizer — even not a linear polarizer — the elementary square cosine form of the Malus' law is obtained. In this case the gain contrast is completely maximized.

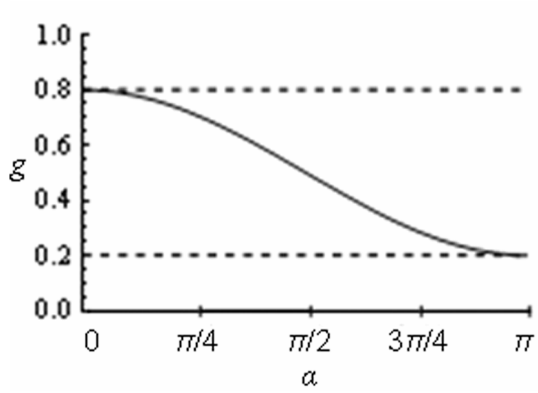

Figure 3. Variation of the gain with the angle α for partially polarized incident light, when p_d and $\bar{\tau}$ are equal to 1. The dashed lines represent the extreme values of the gain $p_i = 0.6$, $\tau_M = 1$, $\tau_m = 0$

5. Conclusions

In this paper we have applied a vectorial pure operatorial Pauli algebraic approach in analyzing the gain given by a dichroic device for partially polarized incident light.

The parameterization of the problem specific to this approach leads to simple, expressive and symmetric results — e.g. the expression (20) of the gain, or the form (25) of the generalized Malus' law. Our approach is parameterized in a manner well adapted to the symmetries of both the polarization states and of the devices. On the one hand, this approach is performed in the Hilbert space of the density operators of the polarization states and of the devices, which have an identical geometric representation on the Poincaré sphere. On the other hand, concerning the devices, it addresses to their eigenstates and eigenvalues, which both reflect the symmetry of the device.

An essential aspect of this parameterization is the fact that the incident light and the device are described by similar, complementary parameters: the degree of polarization, p_i , and the degree of dichroism, p_d, respectively, and the corresponding Poincaré vectors, $p_i\mathbf{s}_i$ and $p_d\,\mathbf{s}_d$.

In these terms, the gain contrast is simply the product of the degree of polarization of the incident light and the degree of dichroism of the device, equation (18). The expression of the gain takes a simple and symmetric form, equation (20): the gain is essentially determined by the scalar product of the Poincaré vectors of the light and of the device. A remarkable aspect of the complementarity light/device revealed by this approach is that both the degree of polarization of the incident light and the degree of dichroism of the device are equal with the gain contrast in suitable complementary arrangements, equations (16) and (17) respectively.

Finally, by using the polarized/unpolarized dichotomy a simple and physically expressive form of the Malus' law, equation (25), is obtained.

In fact, this approach sets in the simplest and most suitable mathematical terms our intuitive representation of the interaction polarized light – dichroic devices.

These results could be applied straightforwardly in polarization metrology (21, 22).

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Appendix

One of the reviewers has signaled an important aspect connected with our paper, namely that the partitioned-matrix, semi-vectorialized form of the Mueller calculus leads to a compact vectorial form of the gain expression, similar to our Eq. (20).

Because, to the best of our knowledge, the deduction of the gain equation (A4) in the frame of the partitioned Muller matrix language (7-10) isn't formulated in the literature, we present it here in the form given by our reviewer:

The partitioned form of the Mueller matrix of a pure dichroic system is:

$$\mathbf{M}_D = m_{00}\begin{bmatrix} 1 & \mathbf{D}^T \\ \mathbf{P} & \mathbf{m}_D \end{bmatrix}, \tag{A1}$$

where

- **D** is the “diattenuation vector” whose modulus D is the diattenuation;
- **P** is the “polarizance vector” whose modulus P is the polarizance (in the considered case of orthogonal (or homogeneous) diattenuators, **P** = **D** and

$$m_D = (1-D^2)^{1/2}\mathbf{I} + \frac{1}{D^2}[1-(1-D^2)^{1/2}]\mathbf{D}\times\mathbf{D}^T, \tag{A2}$$

where **I** is the 3×3 identity matrix;

- m_{00} is the gain for unpolarized light.

Using the partitioned form of the Stokes vector:

$$\mathbf{s}^T = I(1, p\mathbf{s})^T, \tag{A3}$$

where p is the degree of polarization, **s** is the normalized vectorial part of the Stokes vector and I is the intensity of the incident light, then

$$g = m_{00}(1+\mathbf{Ds}) = m_{00}(1+Dp\cos\alpha), \tag{A4}$$

where α is the angle between the vectorial part of the incident light **s** and the diattenuation vector **D** of the system.