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## MONOIDS OF *O*-TYPE, SUBWORD REVERSING, AND ORDERED GROUPS

PATRICK DEHORNOY

ABSTRACT. We describe a simple scheme for constructing finitely generated monoids in which left-divisibility is a linear ordering and for practically investigating these monoids. The approach is based on subword reversing, a general method of combinatorial group theory, and connected with Garside theory, here in a non-Noetherian context. As an application we describe several families of ordered groups whose space of left-invariant orderings has isolated points.

A group  $G$  is left-orderable if there exists a linear ordering on  $G$  that is left-invariant, *i.e.*,  $g < g'$  implies  $hg < hg'$  for every  $h$  in  $G$ . Viewing an ordering on  $G$  as a subset of  $G \times G$ , one equips the family  $\text{LO}(G)$  of all left-invariant orderings of  $G$  with a topology induced by the product topology of  $\mathfrak{P}(G \times G)$ . Then  $\text{LO}(G)$  is a compact space and, in many cases, in particular when  $G$  is a countable non-abelian free group,  $\text{LO}(G)$  has no isolated points and it is a Cantor set [20, 3]. By contrast, apart from the cases when  $\text{LO}(G)$  is finite and therefore discrete, as is the case for the Klein bottle group and, more generally, for the Tararin groups [21, 14], not so many examples are known when  $\text{LO}(G)$  contains isolated points. By the results of [11], this happens when  $G$  is an Artin braid group (see also [15]), and, by those of [16, 12], when  $G$  is a torus knot group, *i.e.*, a group of presentation  $\langle x, y \mid x^m = y^n \rangle$  with  $m, n \geq 2$ . These results, as well as the further results of [13], use non-elementary techniques.

The aim of this paper is to observe that a number of ordered groups with similar properties, including the above ones, can be constructed easily using a monoid approach. A necessary and sufficient condition for a submonoid  $M$  of a group  $G$  to be, when 1 is removed, the positive cone of a left-invariant ordering on  $G$  is that  $M$  is what will be called *of O-type*, namely it is cancellative, has no nontrivial invertible element, and its left- and right-divisibility relations (see Definition 1.1) are linear orderings. Moreover, the involved ordering is isolated in the corresponding space  $\text{LO}(G)$  whenever  $M$  is finitely generated. We are thus naturally led to the question of recognizing which (finite) presentations define monoids of *O*-type.

Here we focus on presentations of a certain syntactical type called triangular. Although no complete decidability result can probably be expected, the situation is that, in practice, many cases can be successfully addressed, actually almost all cases in the samples we tried. The main tool we use here is subword reversing [4, 5, 6, 7], a general method of combinatorial group theory that is especially suitable for investigating divisibility in a presented monoid and provides efficient algorithms that make experiments easy. Both in the positive case (when the defined monoid is of *O*-type) and in the negative one (when it is not), the approach leads to sufficient  $\Sigma_1^1$ -conditions, *i.e.*, provides effective procedures returning a result when the conditions are met but possibly running forever otherwise. At a technical level, the main new observation is that subword reversing can be useful even in a context where all traditional Noetherianity assumptions fail.

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The outcome is the construction of families of finitely generated monoids of  $O$ -type, hence of ordered groups with isolated points in the space of left-orderings, together with algorithmic tools for analysing these structures. There is a close connection with Garside theory [8] in that all examples we obtain admit a Garside element. The scheme is summarized in the following result (see Propositions 5.4, 5.5, and 5.9 for more general versions).

**Theorem 1.** *Assume that  $G$  (resp.  $M$ ) is the group (resp. monoid) defined by a (finite or infinite) presentation  $(\mathbf{a}_1, \mathbf{a}_2, \dots; \mathbf{a}_1 = \mathbf{a}_2 w_2 \mathbf{a}_2, \mathbf{a}_2 = \mathbf{a}_3 w_3 \mathbf{a}_3, \dots)$  with  $w_2, w_3, \dots$  words in  $\{\mathbf{a}_1, \mathbf{a}_2, \dots\}$  (no  $\mathbf{a}_i^{-1}$ ). Assume moreover that there exists in  $M$  a power  $\Delta$  of  $\mathbf{a}_1$  such that, for every  $i \geq 2$ , there exist  $g, g'$  satisfying  $\mathbf{a}_i \Delta = \Delta g$  and  $\Delta \mathbf{a}_i = g' \Delta$ . Then (i)  $M \setminus \{1\}$  is the positive cone of a left-invariant ordering  $\leq$  on  $G$ ; (ii) If  $\{\mathbf{a}_1, \mathbf{a}_2, \dots\}$  is finite,  $\leq$  is isolated in the space  $\text{LO}(G)$ , and the word problem of  $G$  and the decision problem for  $\leq$  are decidable.*

Among others, the approach applies to the above mentioned torus knot groups, providing a short construction of an isolated ordering, and to the group  $B_3$  of 3-strand braids, providing one more proof of its orderability. More examples are listed in Table 1.

1:	$\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b}(\mathbf{a}^p \mathbf{b})^q \rangle$	$\Delta = \mathbf{a}^{p+1}$ central (Proposition 6.4); is also $\langle x, y \mid x^{p+1} = y^{q+1} \rangle$ (torus knot group)
2:	$\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b}(\mathbf{a} \mathbf{b}^p)^q \mathbf{a} \mathbf{b} \rangle$	$\Delta = (\mathbf{a} \mathbf{b}^{p-1})^2$ central (Proposition 6.5); is also $\langle x, y \mid x^{q+2} = y^2 \rangle$ (torus knot group)
3:	$\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b} \mathbf{a}^{pq-1} \mathbf{b} \mathbf{a}^{p-1} \mathbf{b} \mathbf{a}^{pq-1} \mathbf{b} \rangle$	$\Delta = \mathbf{a}^{2pq-p}$ central; is also $\langle x, y \mid x^p = (y x^{pq-p} y)^2 \rangle$ , and $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{b} \mathbf{a}^{p-1} \mathbf{b}, \mathbf{b} = \mathbf{c} \mathbf{a}^{pq-1} \mathbf{c} \rangle$
4:	$\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b} \mathbf{a}^{pq} \mathbf{b} \mathbf{a}^{p-1} \mathbf{b} \mathbf{a}^{pq} \mathbf{b} \rangle$	$\Delta = (\mathbf{a}^{pq} \mathbf{b})^2$ quasi-central (Propos. 6.7); is also $\langle x, y \mid x^p = (y x^{pq-p+1} y)^2 \rangle$ , and $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{b} \mathbf{a}^{p-1} \mathbf{b}, \mathbf{b} = \mathbf{c} \mathbf{a}^{pq} \mathbf{c} \rangle$
5:	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{b} \mathbf{a}^{p-1} \mathbf{b}, \mathbf{b} = \mathbf{c} \mathbf{a}^{pq-1} \mathbf{b} \mathbf{c} \rangle$	$\Delta = \mathbf{a}^{2pq-p(p-1)}$ central
6:	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{b} \mathbf{a}^{p-1} \mathbf{b}, \mathbf{b} = \mathbf{c} \mathbf{a}^{pq} \mathbf{b} \mathbf{c} \rangle$	$\Delta = \mathbf{a}^{pq} \mathbf{b}^2$ quasi-central (central for $p = 1$ )
7:	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{b}(\mathbf{a} \mathbf{b})^p, \mathbf{b} = \mathbf{c} \mathbf{b}(\mathbf{a}^{2q-1} \mathbf{b})^p \mathbf{c} \rangle$	$\Delta = \mathbf{a}^{2p(p+1)(q-1)}$ central
8:	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{b}(\mathbf{a}^p \mathbf{b})^q, \mathbf{b} = \mathbf{c}((\mathbf{a}^p \mathbf{b})^r \mathbf{a}^p \mathbf{c})^s \rangle$	$\Delta = \mathbf{a}^{(p+1)(r+1)}$ central (Proposition 8.2); is also $\langle x, y, z \mid x^{p+1} = y^{q+1}, y^{r+1} = z^{s+1} \rangle$

TABLE 1. Some groups eligible for the current approach, hence ordered with an isolated point in the space of orderings: all values  $p, q, r, s \geq 1$  are permitted.

The paper is organized as follows. In Section 1, we introduce the notion of a monoid of  $O$ -type and describe its connection with ordered groups. In Section 2, we define triangular presentations, raise the central question, namely recognizing when a (right)-triangular presentation defines a monoid of (right)- $O$ -type (and therefore leads to an ordered group), and state without proof the main technical result (“Main Lemma”), which reduces the central question to the existence of common right-multiples. Section 3 contains a brief introduction to subword reversing, with observations about the particular form it takes in the context of right-triangular presentations. In Section 4, we establish that every right-triangular presentation is what we call complete for right-reversing, and deduce a proof of the Main Lemma. Next, we investigate in Section 5 the notion of a quasi-central element in a monoid and, putting things together, we obtain the expected sufficient conditions for a presentation to define a monoid of  $O$ -type, completing in particular the proof of Theorem 1. Various examples are then exhibited in Sections 6 and 8, with an intermission in Section 7 where we establish some negative results about the existence of triangular presentations. Finally, some questions are gathered in Section 9.

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## 1. MONOIDS OF $O$ -TYPE

If  $G$  is an orderable group and  $<$  is a left-invariant ordering of  $G$ , the positive cone  $\mathcal{P}_<$  of  $<$ , *i.e.*, the set  $\{g \in G \mid g > 1\}$ , is a subsemigroup of  $G$  satisfying  $G = \mathcal{P}_< \amalg \mathcal{P}_<^{-1} \amalg \{1\}$ . Conversely, if  $P$  is a subsemigroup of  $G$  satisfying  $G = P \amalg P^{-1} \amalg \{1\}$ , then the relation  $g^{-1}h \in P$  defines a left-invariant ordering on  $G$  and  $P$  is the associated positive cone.

In the sequel, the notions of divisors and multiples will play a central role. It is convenient to consider them in the context of monoids, *i.e.*, semigroups with a unit.

**Definition 1.1.** Assume that  $M$  is a monoid. For  $g, h$  in  $M$ , we say that  $g$  is a *left-divisor* of  $h$ , or, equivalently,  $h$  is a *right-multiple* of  $g$ , denoted  $g \preceq h$ , if there exists  $h'$  in  $M$  satisfying  $gh' = h$ . Symmetrically, we say that  $g$  is a *right-divisor* of  $h$ , or, equivalently,  $h$  is a *left-multiple* of  $g$ , denoted  $h \succcurlyeq g$ , if there exists  $h'$  in  $M$  satisfying  $h = h'g$ .

For every monoid  $M$ , left- and right-divisibility are partial preorders on  $M$ , and they are partial orders whenever 1 is the only invertible element of  $M$ . Note that the right-divisibility relation of a monoid  $M$  is the left-divisibility relation of the opposite monoid  $\widetilde{M}$ , *i.e.*, the monoid with the same domain equipped with the operation defined by  $g \widetilde{\cdot} h = hg$ .

It is easy to translate the existence of an invariant ordering in a group into the language of monoids and divisibility. We recall that a monoid is called left-cancellative (*resp.* right-cancellative), for all  $g, g', h$  in the monoid,  $hg = hg'$  (*resp.*  $gh = g'h$ ) implies  $g = g'$ . A monoid is cancellative if it is both left- and right-cancellative. The monoids we shall investigate are as follows.

**Definition 1.2.** A monoid  $M$  is said to be of *right- $O$ -type* (*resp.* *left- $O$ -type*) if  $M$  is left-cancellative (*resp.* right-cancellative), 1 is the only invertible element in  $M$ , and, for all  $g, h$  in  $M$ , at least one of  $g \preceq h$ ,  $h \preceq g$  (*resp.*  $g \succcurlyeq h$ ,  $h \succcurlyeq g$ ) holds. A monoid is of  *$O$ -type* if it is both of right- and left- $O$ -type.

In other words, a monoid  $M$  is of right- $O$ -type if it is left-cancellative and left-divisibility is a linear ordering on  $M$ , and it is of  $O$ -type if it is cancellative and left- and right-divisibility are linear orderings on  $M$ . The letter  $O$  stands for “order”; it may seem strange that the notion connected with left-divisibility is called “right- $O$ -type”, but this option is natural when one thinks in terms of multiples and it is more coherent with the forthcoming terminology. The connection with ordered groups is easy.

**Lemma 1.3.** For  $G$  a group and  $M$  a submonoid of  $G$ , the following are equivalent:

- (i) The group  $G$  admits a left-invariant ordering whose positive cone is  $M \setminus \{1\}$ ;
- (ii) The monoid  $M$  is of  $O$ -type.

*Proof.* Assume (i). Put  $P = M \setminus \{1\}$ . First, by assumption,  $M$  is included in a group, hence it must be cancellative. Next, assume that  $g$  is an invertible element of  $M$ , *i.e.*, there exists  $h$  in  $M$  satisfying  $gh = 1$ . If  $g$  belongs to  $P$ , then so does  $h$  and, therefore,  $g$  belongs to  $P \cap P^{-1}$ , contradicting the assumption that  $P$  is a positive cone. So 1 must be the only invertible element of  $M$ . Now, let  $g, h$  be distinct elements of  $M$ . Then one of  $g^{-1}h$ ,  $h^{-1}g$  belongs to  $P$ , hence to  $M$ : in the first case,  $g \preceq h$  holds, in the second,  $h \preceq g$ . Symmetrically, one of  $gh^{-1}$ ,  $hg^{-1}$  belongs to  $P$ , hence to  $M$ , now implying  $g \succcurlyeq h$  or  $h \succcurlyeq g$ . So any two elements of  $M$  are comparable with respect to  $\preceq$  and  $\succcurlyeq$ . Hence  $M$  is of  $O$ -type, and (i) implies (ii).

Conversely, assume that  $M$  is of  $O$ -type. Put  $P = M \setminus \{1\}$  again. Then  $P$  is a subsemigroup of  $G$ . The assumption that 1 is the only invertible element in  $M$  implies  $P \cap P^{-1} = \emptyset$ . Next, the assumption that any two elements of  $M$  are comparable with respect to  $\preceq$  implies

*a fortiori* that any two of its elements admit a common right-multiple. By Ore's theorem [2], this implies that  $G$  is a group of right-fractions for  $M$ , *i.e.*, every element of  $G$  admits an expression of the form  $gh^{-1}$  with  $g, h$  in  $M$ . Now, let  $f$  be an element of  $G$ . As said above, there exist  $g, h$  in  $M$  satisfying  $f = gh^{-1}$ . By assumption, at least one of  $g \succsim h$ ,  $h \succsim g$  holds in  $M$ . This means that at least one of  $f \in M$ ,  $f \in M^{-1}$  holds. Therefore, we have  $G = M \cup M^{-1}$ , which is also  $G = P \cup P^{-1} \cup \{1\}$ . So  $P$  is a positive cone on  $G$ , and (ii) implies (i).  $\square$

It will be convenient to restate Lemma 1.3 in terms of presentations. A *semigroup presentation* (or *positive group presentation* [6]) is a pair  $(S; R)$  with  $S$  a (nonempty) set and  $R$  a family of relations of the form  $u = v$ , where  $u, v$  are nonempty words in the alphabet  $S$ . Every semigroup presentation  $(S; R)$  naturally defines three structures, namely a semigroup, a monoid, and a group; we use  $\langle S | R \rangle^+$  and  $\langle S | R \rangle$  for the monoid and the group so defined.

**Lemma 1.4.** *For every semigroup presentation  $(S; R)$ , the following are equivalent:*

- (i) *The group  $\langle S | R \rangle$  admits a left-invariant ordering whose positive cone admits, as a semigroup, the presentation  $(S; R)$ ;*
- (ii) *The monoid  $\langle S | R \rangle^+$  is of  $O$ -type.*

*Proof.* The proof that (i) implies (ii) is the same as for Lemma 1.3: the only additional point is that, if a semigroup  $P$  admits, as a semigroup, the presentation  $(S; R)$ , then the monoid  $P \cup \{1\}$  admits, as a monoid, the same presentation.

As for (ii) implying (i), let  $G$  be a group with presentation  $(S; R)$  and let  $M$  be a monoid with that presentation. By Ore's theorem, as in the proof of Lemma 1.3, the hypothesis that  $M$  is of right- $O$ -type implies that  $M$  embeds in a group of right-fractions. Moreover (this is the additional point) the latter group admits the presentation  $(S; R)$ . So there exists an embedding  $\iota$  of  $M$  into  $G$ . Then the rest of the verification is the same, and  $\iota M \setminus \{1\}$  is a positive cone on  $G$ .  $\square$

## 2. TRIANGULAR PRESENTATIONS

We are thus led to looking for monoids of  $O$ -type and, more specifically, for recognizing which presentations define monoids of  $O$ -type. Owing to the symmetry of the definition, we shall mainly focus on recognizing monoids of right- $O$ -type and then use the criteria for the opposite presentation. Now, if a monoid  $M$  is of right- $O$ -type and it is generated by some subset  $S$ , then, for all  $s, s'$  in  $S$ , the elements  $s$  and  $s'$  are comparable with respect to  $\preceq$ , *i.e.*,  $s' = sg$  holds for some  $g$ , or *vice versa*. In other words, some relation of the particular form  $s' = sw$  must be satisfied in  $M$ . We shall consider presentations in which all relations have this form (see Section 7 for a discussion about the relevance of this approach).

**Definition 2.1.** A semigroup relation  $u = v$  is called *triangular* if either  $u$  or  $v$  consists of a single letter.

So, a triangular relation has the generic form  $s' = sw$ , where  $s, s'$  belong to the reference alphabet. For instance,  $\mathbf{a} = \mathbf{bab}$  and  $\mathbf{b} = \mathbf{c}^2\mathbf{ba}$  are typical triangular relations in the alphabet  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ . The problem we shall address now is

**Question 2.2.** *Assume that  $(S; R)$  is a presentation consisting of triangular relations. Is the associated monoid necessarily of right- $O$ -type?*

The following counter-example shows that a uniform positive answer is impossible.

**Example 2.3.** The presentation  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{c} = \mathbf{ab}, \mathbf{c} = \mathbf{ba})$  consists of two triangular relations. The associated monoid  $M$  is a rank 2 free Abelian monoid based on  $\mathbf{a}$  and  $\mathbf{b}$ , and neither of  $\mathbf{a}, \mathbf{b}$  is a right-multiple of the other. So  $M$  is not of right- $O$ -type.

Clearly, the problem in Example 2.3 is the existence of several relations  $c = \dots$  simultaneously. We are thus led to restricting to particular families of triangular relations. If  $S$  is a nonempty set, we denote by  $S^*$  the free monoid of all words in the alphabet  $S$ . We use  $\varepsilon$  for the empty word.

**Definition 2.4.** A semigroup presentation  $(S; R)$  is called *right-triangular* if there exist  $S' \subseteq S$  and maps  $N$  (“next”) :  $S' \rightarrow S$  and  $C$  (“complement”) :  $S' \rightarrow S^* \setminus \{\varepsilon\}$  such that  $N$  is injective with no fixpoint and  $R$  consists of the relations  $N(s) = sC(s)$  for  $s$  in  $S'$ . We write  $C^i(s)$  for  $C(s)C(N(s))\dots C(N^{i-1}(s))$  when  $N^i(s)$  is defined, and  $\widehat{R}$  for  $R \cup \{N^i(s) = sC^i(s) \mid i \geq 2\}$ . A *left-triangular* presentation is defined symmetrically by relations  $\widetilde{N}(s) = \widetilde{C}(s)s$ . A presentation is *triangular* if it is both right- and left-triangular.

**Example 2.5.** Assume  $S = \{a, b, c\}$  and  $R = \{a = bac, b = cba\}$ . Then  $(S; R)$  is a right-triangular presentation. The associated maps  $N$  and  $C$  are given by  $N(b) = a$ ,  $C(b) = ac$ ,  $N(c) = b$ ,  $C(c) = ba$ , and we have  $\widehat{R} = R \cup \{a = cba^2c\}$ . The presentation  $(S; R)$  is also left-triangular, with  $\widetilde{N}$  and  $\widetilde{C}$  defined by  $\widetilde{N}(a) = b$ ,  $\widetilde{C}(a) = cb$ ,  $\widetilde{N}(c) = a$ ,  $\widetilde{C}(c) = ba$ .

If  $(S; R)$  is a right-triangular presentation, the family  $\widehat{R}$  is a sort of transitive closure of  $R$ , and the presentations  $(S; R)$  and  $(S; \widehat{R})$  define the same monoid and the same group. Triangular presentations can be described in terms of the left- and right-graphs [1, 18]. The *left-graph* (*resp. right-graph*) of  $(S; R)$  is the unoriented graph with vertex set  $S$  such that  $\{s, s'\}$  is an edge if and only if there exists a relation  $s\dots = s'\dots$  (*resp. ...s = ...s'*) in  $R$ . Then a presentation  $(S; R)$  is right-triangular if it consists of triangular relations and, in addition, the left-graph of  $(S; R)$  is a union of discrete chains. In practice, we shall be mostly interested in the case when there is only one (countable) chain, in which case there exists a (finite or infinite) subset  $I$  of  $\mathbb{Z}$  such that  $S$  is  $\{a_i \mid i \in I\}$  and  $R$  consists of one relation  $a_{i-1} = a_i w_i$  for each  $i$  in  $I$  that is not minimal.

Our main technical result will be a criterion for recognizing which right-triangular presentations give rise to a monoid of right- $O$ -type.

**Proposition 2.6** (Main Lemma). *Assume that  $(S; R)$  is a right-triangular presentation. Then the following are equivalent:*

- (i) *The monoid  $\langle S \mid R \rangle^+$  is of right- $O$ -type;*
- (ii) *Any two elements of  $\langle S \mid R \rangle^+$  admit a common right-multiple.*

The proof of the Main Lemma will be completed at the end of Section 4 below. In terms of ordered groups, this result merged with Lemma 1.4 provides a characterization for ordered groups with an explicitly presented positive cone.

**Corollary 2.7.** *For every triangular presentation  $(S; R)$ , the following are equivalent:*

- (i) *The group  $\langle S \mid R \rangle$  admits a left-invariant ordering whose positive cone admits, as a semigroup, the presentation  $(S; R)$ ;*
- (ii) *Any two elements of the monoid  $\langle S \mid R \rangle^+$  admit a common right-multiple and a common left-multiple.*

*Proof.* If (i) is satisfied, then, by Lemma 1.4, the monoid  $\langle S \mid R \rangle^+$  is of  $O$ -type, and Proposition 2.6 applied to  $(S; R)$  and to the opposite presentation  $(S, \widetilde{R})$  implies (ii). Conversely, if  $\langle S \mid R \rangle^+$  satisfies (ii), Proposition 2.6 applied to  $(S; R)$  and  $(S, \widetilde{R})$  implies that  $\langle S \mid R \rangle^+$  is both of right- and left- $O$ -type. Then Lemma 1.4 implies (i).  $\square$

### 3. SUBWORD REVERSING

We shall prove the Main Lemma by using subword reversing. In essence, subword reversing is a strategy for constructing van Kampen diagrams in a context of monoids, *i.e.*, equivalently, for finding derivations between words, and we shall see that it is especially

relevant for investigating triangular presentations (due to the special form of triangular presentations, it might well be that alternative arguments using rewrite systems or other approaches also exist, but this is not clear).

The description given below is sketchy, as we only mention the definition and the needed technical results. We refer to [6, 7] for additional motivation and explanation.

As is usual with presented groups, if  $S$  is an alphabet, we introduce a formal copy  $S^{-1}$  of  $S$  consisting of one letter  $s^{-1}$  for each letter of  $S$ . The letters of  $S$  are then called *positive*, whereas those of  $S^{-1}$  are called *negative*. Accordingly, a word in the alphabet  $S \cup S^{-1}$  will be called a *signed  $S$ -word*, whereas a word in the alphabet  $S$  is called an  $S$ -word, or a *positive  $S$ -word* if we wish to insist that there is no negative letter. If  $w$  is a signed  $S$ -word,  $w^{-1}$  denotes the word obtained from  $w$  by exchanging  $s$  and  $s^{-1}$  everywhere and reversing the order of the letters. A word of the form  $u^{-1}v$  with  $u, v$  positive is called *negative-positive*.

**Definition 3.1.** Assume that  $(S; R)$  a semigroup presentation and  $w, w'$  are signed  $S$ -words. We say that  $w$  is *right- $R$ -reversible to  $w'$  in one step*, denoted  $w \curvearrowright_R^{(1)} w'$ , if either there exist  $s, s'$  in  $S$ , a relation  $sv' = s'v$  of  $R$ , and signed words  $w_1, w_2$  satisfying

$$(3.1) \quad w = w_1 s^{-1} s' w_2 \quad \text{and} \quad w' = w_1 v' v^{-1} w_2,$$

or there exist  $s$  in  $S$  and signed  $S$ -words  $w_1, w_2$  satisfying

$$(3.2) \quad w = w_1 s^{-1} s w_2 \quad \text{and} \quad w' = w_1 w_2.$$

We say that  $w$  is *right- $R$ -reversible to  $w'$  in  $n$  steps*, denoted  $w \curvearrowright_R^{(n)} w'$ , if there exist  $w_0, \dots, w_n$  satisfying  $w_0 = w$ ,  $w_n = w'$  and  $w_i \curvearrowright_R^{(1)} w_{i+1}$  for each  $i$ . We write  $w \curvearrowright_R w'$  if  $w \curvearrowright_R^{(n)} w'$  holds for some  $n$ .

Note that (3.2) becomes an instance of (3.1) if, for every  $s$  in  $S$ , the trivial relation  $s = s$  is considered to belong to  $R$ . Right-reversing consists in replacing a negative-positive length two subword with a positive-negative word, hence somehow reversing the signs, whence the terminology. We shall often write “reversing” for “right-reversing” (except at the end of Section 5 where left-reversing, the symmetric counterpart of right-reversing, occurs).

**Example 3.2.** Assume  $S = \{a, b, c\}$  and  $R = \{a = bab, b = cbc, a = cbcab\}$ . Starting for instance with  $w = a^{-1}c^{-1}a$ , we find

$$w = a^{-1} \underline{c^{-1}a} \curvearrowright_R^{(1)} \underline{a^{-1}bcab} \curvearrowright_R^{(1)} b^{-1}a^{-1}cab,$$

where, at each step, the reversed subword is underlined. Observe that the word obtained after two reversing steps is  $b^{-1}w^{-1}b$ , so that  $w \curvearrowright_R^{(4n)} b^{-2n}w b^{2n}$  holds for every  $n$ .

It is useful to associate with every sequence of reversing steps a rectangular grid diagram that illustrates it (see [7] for full details). Assume that  $w_0, w_1, \dots$  is an  $R$ -reversing sequence, *i.e.*,  $w_i \curvearrowright_R^{(1)} w_{i+1}$  holds for every  $i$ . The diagram is analogous to a van Kampen diagram, and it is constructed inductively. First we associate with  $w_0$  a path shaped like an ascending staircase by reading  $w_0$  from left to right and iteratively appending a horizontal right-oriented edge labeled  $s$  for each letter  $s$ , and a vertical down-oriented edge labeled  $s$  for each letter  $s^{-1}$ . Then, assume that the diagram for  $w_0, \dots, w_i$  has been constructed, and  $w_{i+1}$  is obtained from  $w_i$  by reversing some subword  $s^{-1}s'$  into  $v'v^{-1}$ . Inductively, the

subword  $s^{-1}s'$  corresponds to an open pattern  $\begin{array}{c} \xrightarrow{s'} \\ s \downarrow \end{array}$  in the diagram, and we complete it by appending new arrows forming the closed pattern  $\begin{array}{c} \xrightarrow{s'} \\ s \downarrow \curvearrowright_R v \\ \xrightarrow{v'} \end{array}$ . If the length  $\ell$  of  $v$  is

more than one, the arrow labeled  $v$  consists of  $\ell$  concatenated arrows. If  $v$  is empty, we

append a equality sign, as in  $s \begin{array}{c} \xrightarrow{s'} \\ \Downarrow \text{\scriptsize } \curvearrowright_R \\ \xrightarrow{v'} \end{array} \parallel$ . It then follows from the inductive definition that

all words  $w_i$  can be read in the diagram by following the paths that connect the bottom-left corner to the top-right corner, see Figure 1.

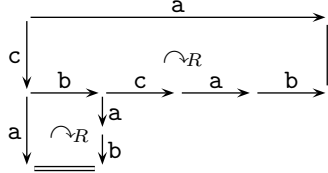


FIGURE 1. Reversing diagram associated with the reversing sequence of Example 3.2: starting from the signed word  $a^{-1}c^{-1}a$ , which corresponds to the left and top arrows, we successively reverse  $c^{-1}a$  into  $bcab$ , and  $a^{-1}b$  into  $b^{-1}a^{-1}$ , thus obtaining the final word  $b^{-1}a^{-1}cab$ .

If  $(S; R)$  is a semigroup presentation, and  $u, v$  are  $S$ -words, applying iterated subword reversing to the signed word  $u^{-1}v$  may lead to three different behaviours:

- either the process continues for ever (as in the case of Example 3.2),
- or one gets stuck with a factor  $s^{-1}s'$  such that  $R$  contains no relation  $s\dots = s'\dots$ ,
- or the process leads in finitely many steps to a positive–negative word  $v'u'^{-1}$  where  $u'$  and  $v'$  are  $S$ -words (no letter  $s^{-1}$ ): then the sequence cannot be extended since the last word contains no subword of the form  $s^{-1}s'$ ; this case corresponds to a reversing diagram

of the form  $u \begin{array}{c} \xrightarrow{v} \\ \Downarrow \text{\scriptsize } \curvearrowright_R \\ \xrightarrow{v'} \end{array} u'$ , and we shall then say that the reversing of  $u^{-1}v$  is *terminating*.

We shall use without proof two (elementary) results about reversing. The first one connects  $R$ -reversing with  $R$ -equivalence and it expresses that a reversing diagram projects to a van Kampen diagram when the vertices connected with equality signs are identified.

**Notation 3.3.** For  $(S; R)$  a semigroup presentation, we denote by  $\equiv_R^+$  the smallest congruence on  $S^*$  that includes  $R$ , so that  $\langle S | R \rangle^+ \equiv S^* / \equiv_R^+$ . For  $w$  an  $S$ -word, we denote by  $[w]^+$  the  $\equiv_R^+$ -class of  $w$ , *i.e.*, the element of the monoid  $\langle S | R \rangle^+$  represented by  $w$ .

**Lemma 3.4.** [6, Proposition 1.9] *Assume that  $(S; R)$  is a positive presentation, and  $u, v, u', v'$  are  $S$ -words satisfying  $u^{-1}v \curvearrowright_R v'u'^{-1}$ . Then  $uv' \equiv_R^+ vu'$  holds. In particular,  $u^{-1}v \curvearrowright_R \varepsilon$  implies  $u \equiv_R^+ v$ .*

In other words, the existence of a reversing diagram  $u \begin{array}{c} \xrightarrow{v} \\ \Downarrow \text{\scriptsize } \curvearrowright_R \\ \xrightarrow{v'} \end{array} u'$  implies  $uv' \equiv_R^+ vu'$ ,

which also reads  $[u]^+[v']^+ = [v]^+[u']^+$ : the element represented by  $uv'$  and  $vu'$  is a common right-multiple of  $[u]^+$  and  $[v]^+$  in the associated monoid. Thus subword reversing can be seen as a tool for constructing common right-multiples in presented monoids.

The second basic result says that, when the reversing of a compound word  $u^{-1}v_1v_2$  terminates, the reversing steps involving  $v_1$  and  $v_2$  can be separated.

**Lemma 3.5.** [6, Lemma 1.8] *Assume that  $(S; R)$  is a semigroup presentation and  $u, v, u', v'$  are  $S$ -words satisfying  $u^{-1}v \curvearrowright_R^{(n)} v'u'^{-1}$ . Then, for every decomposition  $v = v_1v_2$ , there exist*



an  $S$ -word  $u_0$  and decompositions  $v' = v'_1 v'_2$  and  $n = n_1 + n_2$  such that  $u^{-1} v_1 \curvearrowright_R^{(n_1)} v'_1 u_0^{-1}$  and  $u_0^{-1} v_2 \curvearrowright_R^{(n_2)} v'_2 u'^{-1}$  hold.

In other words, every diagram  $u \begin{array}{ccc} \xrightarrow{v_1} & & \xrightarrow{v_2} \\ \curvearrowright_R^{(n)} & & \\ \xrightarrow{v'} & & \end{array} u'$  splits into  $u \begin{array}{ccc} \xrightarrow{v_1} & & \xrightarrow{v_2} \\ \curvearrowright_R^{(n_1)} & & \\ \xrightarrow{v'_1} & & \end{array} u_0 \begin{array}{ccc} \xrightarrow{v_2} & & \\ \curvearrowright_R^{(n_2)} & & \\ \xrightarrow{v'_2} & & \end{array} u'$ .

Here comes the first specific observation about reversing with triangular relations.

**Lemma 3.6.** *If  $(S; R)$  is a semigroup presentation consisting of triangular relations, and  $u, v, u', v'$  are  $S$ -words satisfying  $u^{-1} v \curvearrowright_R v' u'^{-1}$ , then at least one of  $u', v'$  is empty.*

*Proof.* We use induction on the number of reversing steps, *i.e.*, the number  $n$  such that  $u^{-1} v \curvearrowright_R^{(n)} v' u'^{-1}$  holds. For  $n = 0$ , the only possibility is that  $u, v, u', v'$  all are empty and the result is trivial. For  $n = 1$ , the only possibility is that  $u$  or  $v$  consists of one letter, say for instance  $u = s \in S$ . Write  $v = s'w$  with  $s' \in S$ . If the (unique) reversing step is of the type  $s^{-1}s \curvearrowright \varepsilon$ , we obtain  $u' = \varepsilon$  (and  $v' = w$ ). Otherwise, the reversing step is either of the type  $s^{-1}s' \curvearrowright w'$  with  $sw' = s'$  a relation of  $R$ , or of the type  $s^{-1}s' \curvearrowright w'^{-1}$  with  $s'w' = s$  a relation of  $R$ . In the first case, we obtain  $u' = \varepsilon$  (and  $v' = w'w$ ); in the second case, the final word  $w'^{-1}w$  is positive–negative only if  $w$  is empty, and, in this case, we have  $v' = \varepsilon$  (and  $u' = w'$ ). The argument is similar if  $v$ , instead of  $u$ , has length one.

Assume now  $n \geq 2$ . Then at least one of the words  $u, v$  has length two or more. Assume that  $v$  does, and write it as  $v_1 v_2$  with  $v_1, v_2$  nonempty. By Lemma 3.5, the assumption that  $u^{-1} v_1 v_2$  reverses to  $v' u'^{-1}$  in  $n$  steps implies the existence of  $S$ -words  $v'_1, v'_2, u_0$  and of numbers  $n_1, n_2$  satisfying

$$v' = v'_1 v'_2, \quad n = n_1 + n_2, \quad u^{-1} v_1 \curvearrowright_R^{(n_1)} v'_1 u_0^{-1}, \quad \text{and} \quad u_0^{-1} v_2 \curvearrowright_R^{(n_2)} v'_2 u'^{-1}.$$

Two cases are possible. Assume first  $n_1 = n$ , whence  $n_2 = 0$ . As, by assumption,  $v_2$  is nonempty, the hypothesis that  $u_0^{-1} v_2$  is a positive–negative word implies that  $u_0$  is empty, and so is  $u'$ . Assume now  $n_1 < n$ . The value  $n_1 = 0$  is impossible as it would imply that  $u$  or  $v_1$  is empty, contrary to the assumption. Hence we also have  $n_2 < n$ . Now assume that  $u'$  is nonempty. Then, as we have  $n_2 < n$ , the induction hypothesis implies that  $v'_2$  is empty. Next,  $u'$  can be nonempty only if  $u_0$  is nonempty. Then, as we have  $n_1 < n$ , the induction hypothesis implies that  $v'_1$  is empty as well, and we conclude that  $v'$ , which is  $v'_1 v'_2$ , is empty. See Figure 2.  $\square$

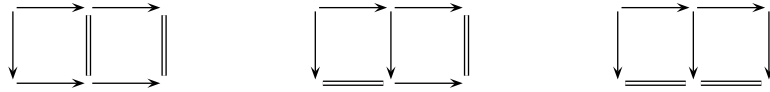


FIGURE 2. The three possible ways of concatenating two reversing diagrams in which one of the output words is empty: in each case, one of the final output words has to be empty.

So, in the context of triangular relations, when reversing is terminating, it shows not only that the elements of the monoid represented by the initial words admit a common right-multiple, but also that these elements are comparable with respect to left-divisibility. Indeed, by Lemma 3.4, if we have  $u^{-1} v \curvearrowright_R v'$  with  $v'$  a (positive)  $S$ -word, we deduce  $[u]^+ [v']^+ = [v]^+$ , whence  $[u]^+ \preceq [v]^+$  in the monoid  $\langle S \mid R \rangle^+$  and, symmetrically, if we have  $u^{-1} v \curvearrowright_R u'^{-1}$  with  $u'$  an  $S$ -word, we deduce  $[v]^+ [u']^+ = [u]^+$ , whence  $[v]^+ \preceq [u]^+$ .

## 4. COMPLETENESS OF SUBWORD REVERSING

Owing to Lemma 3.6, if  $(S; R)$  is a semigroup presentation that consists of triangular relations, in order to prove that any two elements of the monoid  $\langle S \mid R \rangle^+$  are comparable with respect to left-divisibility, it is enough to show that, for all  $S$ -words  $u, v$ , there exists at least one reversing sequence from  $u^{-1}v$  that is terminating, *i.e.*, is finite and finishes with a positive–negative word.

A natural situation in which reversing is guaranteed to be terminating is the case when, for every pair of letters  $s, s'$ , there exists at least one relation  $s\dots = s'\dots$  in  $R$  (so that one never gets stuck) and all relations  $u = v$  of  $R$  involve words  $u, v$  of length at most two, so that reversing does not increase the length of words. More generally, termination is guaranteed when one can identify a set of  $S$ -words  $\widehat{S}$  including  $S$  so that, for all  $u, v$  in  $\widehat{S}$ , there exist  $u', v'$  in  $\widehat{S}$  satisfying  $u^{-1}v \curvearrowright_R v'u'^{-1}$  (which amounts to meet the above conditions with respect to the extended alphabet  $\widehat{S}$ ). However, except in a few trivial examples, this approach fails when applied to presentations with triangular relations: usually, the closure  $\widehat{S}$  of  $S$  under reversing is infinite and difficult to work with. Therefore, we must use a more subtle approach in two steps, namely showing that

(i) if two elements represented by words  $u, v$  admit a common right-multiple, then the reversing of  $u^{-1}v$  terminates, and

(ii) any two elements of the considered monoids admit a common right-multiple.

When this is done, Lemma 3.6 can be applied and one is close to concluding that the considered monoid is of right- $O$ -type. In this section, we address point (i). Here comes the second, more important observation of the paper, namely that (i) is *always* true for a right-triangular presentation. Technically, the proof relies on what is known as the completeness condition.

If  $(S; R)$  is any semigroup presentation, then, by Lemma 3.4,  $u^{-1}v \curvearrowright_R \varepsilon$ , *i.e.*, the existence of a diagram

$u \begin{array}{c} \xrightarrow{v} \\ \curvearrowright_R \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \parallel$ , implies  $u \equiv_R^+ v$ . We consider now the converse implication.

**Definition 4.1 (complete).** A semigroup presentation  $(S; R)$  is called *complete for right-reversing* if, for all  $S$ -words  $u, v$ ,

$$(4.1) \quad u \equiv_R^+ v \quad \text{implies} \quad u^{-1}v \curvearrowright_R \varepsilon.$$

As the converse of (4.1) is always true, if  $(S; R)$  is complete, (4.1) is an equivalence.

**Remark 4.2.** If there exist two letters  $s, s'$  of  $S$  such that, in the presentation  $R$ , there is more than one relation of the type  $s\dots = s'\dots$  (including the case when there exists a relation  $s\dots = s\dots$  different from the implicit trivial relation  $s = s$ ), then  $R$ -reversing need not be a deterministic process and, starting from some words  $u, v$ , there may exist several pairs  $u', v'$  satisfying  $u^{-1}v \curvearrowright_R v'u'^{-1}$ . According to our definitions, the condition  $u^{-1}v \curvearrowright_R \varepsilon$  involved in (4.1) means that there exists *at least one way* of obtaining the empty word starting from  $u^{-1}v$ . However, this type of non-determinism never occurs with a right-triangular presentation  $(S; R)$  or its completion  $(S, \widehat{R})$ : by definition,  $\widehat{R}$  contains at most one relation  $s\dots = s'\dots$  for each pair of letters, so  $R$ - and  $\widehat{R}$ -reversings are deterministic.

The intuition behind completeness is that, when a presentation is complete for right-reversing, the *a priori* complicated relation  $\equiv_R^+$  can be replaced with the more simple relation  $\curvearrowright_R$ . As explained in [7], this makes recognizing some properties of the associated monoid and group easy. In our current context, in order to address point (i) above, we are interested in connecting the existence of common multiples and termination of reversing. When the completeness condition is satisfied, this is easy.

**Lemma 4.3.** *Assume that  $(S ; R)$  is a positive presentation that is complete for right-reversing. Then, for all  $g, h$  in  $\langle S | R \rangle^+$ , the following are equivalent:*

- (i) *The elements  $g$  and  $h$  admit a common right-multiple;*
- (ii) *For some  $S$ -words  $u, v$  representing  $g$  and  $h$ , the reversing of  $u^{-1}v$  is terminating;*
- (iii) *For all  $S$ -words  $u, v$  representing  $g$  and  $h$ , the reversing of  $u^{-1}v$  is terminating.*

*Proof.* Assume that  $g$  and  $h$  admit a common right-multiple  $f$ . By definition, there exist  $g', h'$  satisfying  $f = gh' = hg'$ . Let  $u, v, u', v'$  be arbitrary  $S$ -words representing  $g, h, g',$  and  $h'$ . Then we have  $uv' \equiv_R^+ vu'$ , whence  $(uv')^{-1}(vu') \curvearrowright_R \varepsilon$ , i.e.,  $v'^{-1}u^{-1}vu' \curvearrowright_R \varepsilon$  since  $(S ; R)$  is complete for right-reversing. Applying Lemma 3.5 twice, we split the reversing diagram of  $(uv')^{-1}(vu')$  into four diagrams:

$$\begin{array}{ccc} & \xrightarrow{v} & \xrightarrow{u'} \\ \begin{array}{c} u \\ \downarrow \\ v' \end{array} & \begin{array}{c} \curvearrowright_R \\ \curvearrowright_R \end{array} & \begin{array}{c} \curvearrowright_R \\ \curvearrowright_R \end{array} \\ & \xrightarrow{\quad} & \xrightarrow{\quad} \end{array} \quad \parallel \quad \parallel$$

Each of the four diagrams above necessarily corresponds to a terminating reversing and, in particular, the reversing of  $u^{-1}v$  must terminate. So (i) implies (iii).

On the other hand, it is obvious that (iii) implies (ii). Finally, by Lemma 3.4, any relation  $u^{-1}v \curvearrowright_R v'u'^{-1}$  implies that the element of  $\langle S | R \rangle^+$  both represented by  $uv'$  and  $vu'$  is a common right-multiple of the elements represented by  $u$  and  $v$ , so (ii) implies (i).  $\square$

Owing to Lemmas 3.6 and 4.3, we are led to wondering whether a presentation consisting of triangular relations is necessarily complete for right-reversing. A priori, the question may seem hopeless as the only method known so far for establishing that a presentation  $(S ; R)$  is complete for right-reversing [5, 6] consists in establishing a certain combinatorial condition (the ‘‘cube condition’’) using an induction that is possible only when the associated monoid  $M$  satisfies some Noetherianity condition, namely that, for every  $g$  in  $M$ ,

$$(4.2) \quad \text{there is no infinite sequence } g_0, g_1, \dots \text{ satisfying } g_0 \prec g_1 \prec g_2 \prec \dots \preceq g,$$

where  $g \prec h$  means  $g \preceq h$  with  $g \neq h$ . Now, (4.2) turns out to fail whenever  $R$  contains a relation of the form  $s = \dots s \dots$ , hence in most of the cases we are interested in. However, right-triangular presentations turn out to be eligible for an alternative completeness argument.

**Proposition 4.4.** *For every right-triangular presentation  $(S ; R)$ , the associated presentation  $(S, \hat{R})$  is complete for right-reversing.*

The proof will be split into several steps. Until the end of the proof, we assume that  $(S ; R)$  is a fixed right-triangular presentation, with associated functions  $N$  and  $C$ . We recall that this means that  $R$  consists of the relations  $N(s) = sC(s)$  with  $s$  in  $S$ . We recall also that  $C^i(s)$  stands for  $C(s)C(N(s))\dots C(N^{i-1}(s))$  whenever  $N^i(s)$  is defined.

By definition, if  $u, v$  are  $S$ -words, then  $u \equiv_R^+ v$  holds, i.e.,  $u$  and  $v$  represent the same element in  $\langle S | R \rangle^+$ , if and only if there exists an  $R$ -derivation from  $u$  to  $v$ , i.e., a sequence  $w_0 = u, w_1, \dots, w_n = v$  such that each word  $w_k$  is obtained from  $w_{k-1}$  by applying exactly one relation of  $R$ . We write  $u \equiv_R^{+(n)} v$  when there exists a length  $n$  derivation from  $u$  to  $v$ .

**Definition 4.5.** Assume that  $w$  is a nonempty  $S$ -word. We denote by  $I(w)$  the initial letter of  $w$ , and by  $T(w)$  (like ‘‘tail’’) the subword satisfying  $w = I(w)T(w)$ . We say that a letter  $s$  of  $S$  *underlies*  $w$  if  $I(w) = N^i(s)$  holds for some  $i \geq 0$ . In this case, we put  $E_s(w) = sC^i(s)T(w)$ ; otherwise, we put  $E_s(w) = w$ .

A straightforward induction on  $i$  gives  $N^i(s) \equiv_R^+ sC^i(s)$  whenever defined, and we deduce  $w = N^i(s)T(w) \equiv_R^+ sC^i(s)T(w) = E_s(w)$  whenever  $I(w) = N^i(s)$  holds.

We begin with a direct consequence of the definition of a right-triangular presentation.

**Lemma 4.6.** *Assume that  $(w_0, \dots, w_n)$  is a sequence of  $S$ -words such that, for every  $k$ , some letter of  $S$  underlies  $w_k$  and  $w_{k+1}$ . Then some letter underlies all of  $w_0, \dots, w_n$ .*

*Proof.* We use induction on  $n$ . The result is obvious for  $n \leq 1$ . Assume  $n \geq 2$ . By induction hypothesis, there exists  $s$  underlying  $w_0$  and  $w_1$ , and  $s'$  underlying  $w_1, \dots, w_n$ . So there exist  $i, j$  such that  $I(w_1)$  is both  $N^i(s)$  and  $N^j(s')$ . Assume first  $i \leq j$ . The injectivity of the map  $N$  implies  $s = N^{j-i}(s')$ . Hence  $s'$  underlies  $w_0$  as well, and, therefore,  $s'$  underlies all of  $w_0, \dots, w_n$ . The argument is similar in the case  $i \geq j$ , with now  $s' = N^{i-j}(s)$  and  $s$  underlying all of  $w_0, \dots, w_n$ .  $\square$

**Lemma 4.7.** *Assume that  $u, v$  are nonempty  $S$ -words satisfying  $u \equiv_R^{+(1)} v$ . Then there exists  $s$  underlying  $u$  and  $v$  and, for every  $s$  in  $S$ , we have  $E_s(u) \equiv_R^{+(\leq 1)} E_s(v)$ . Moreover, exactly one of the following holds:*

- (i) *we have  $I(u) = I(v)$  and  $T(u) \equiv_R^{+(1)} T(v)$ ;*
- (ii) *we have  $I(u) \neq I(v)$  and  $E_s(u) = E_s(v)$  for all  $s$  underlying  $u$  and  $v$ .*

*Proof.* The assumption that  $u \equiv_R^{+(1)} v$  holds means that there exist a number  $p \geq 1$  and a relation of  $R$  such that  $v$  is obtained from  $u$  by applying that relation to its subword starting at position  $p$ . Assume first  $p \geq 2$ . In this case, the initial letter is not changed, *i.e.*, we have  $I(u) = I(v)$ , whereas  $T(v)$  is obtained from  $T(u)$  by applying a relation of  $R$  at position  $p-1$ , and  $T(u) \equiv_R^{+(1)} T(v)$  holds. Next, underlying a word  $w$  depends on the initial letter of  $w$  only, hence, as  $u$  and  $v$  have the same initial letter, the letters underlying  $u$  and  $v$  coincide. Finally, let  $s$  belong to  $S$ . Assume first that  $I(u)$ , which is also  $I(v)$ , is  $N^i(s)$ . Then, by definition, we have  $E_s(u) = sC^i(s)T(u)$  and  $E_s(v) = sC^i(s)T(v)$ , so that  $T(u) \equiv_R^{+(1)} T(v)$  implies  $E_s(u) \equiv_R^{+(1)} E_s(v)$ . Otherwise, we have  $E_s(u) = u$  and  $E_s(v) = v$ , whence  $E_s(u) \equiv_R^{+(1)} E_s(v)$  again. Hence  $E_s(u) \equiv_R^{+(1)} E_s(v)$  holds for every  $s$  in this case.

Assume now  $p = 1$ . This means that there exists  $s$  and  $w$  satisfying  $u = sC(s)w$  and  $v = N(s)w$ , or *vice versa*. In this case, we have  $I(u) = s \neq N(s) = I(v)$  and, by definition,  $s$  underlies both  $u$  and  $v$ . Now, assume that  $s'$  is any element of  $S$  that underlies  $u$ . This means that we have  $I(u) = s = N^i(s')$  for some  $i \geq 0$ , and we then have  $I(v) = N(s) = N^{i+1}(s')$ , so  $s'$  underlies  $v$  as well. Then we find

$$E_{s'}(u) = s'C^i(s')T(u) = s'C^i(s')C(s)w = s'C^{i+1}(s')w = E_{s'}(v).$$

On the other hand, assume that  $s'$  is an element of  $S$  that does not underlie  $u$ . Then we have  $E_{s'}(u) = u$ . If  $s'$  underlies  $v$ , owing to the fact that  $I(v)$  is  $N(I(u))$  and  $N$  is injective, the only possibility is  $s' = N(s)$  and, in this case, we have  $E_{s'}(v) = v$ . If  $s'$  does not underlie  $v$ , by definition we have  $E_{s'}(v) = v$  as well. So, in every such case, we find  $E_{s'}(u) = u \equiv_R^{+(1)} v = E_{s'}(v)$ .

The case  $u = N(s)w$ ,  $v = sC(s)w$  is of course similar. Then the proof is complete since the relation  $E_s(u) \equiv_R^{+(\leq 1)} E_s(v)$  has been established for every  $s$  in every case.  $\square$

**Lemma 4.8.** *Assume that  $u, v$  are nonempty  $S$ -words satisfying  $u \equiv_R^{+(n)} v$ . Then at least one of the following holds:*

- *we have  $I(u) = I(v)$  and  $T(u) \equiv_R^{+(n)} T(v)$ ;*
- *there exists  $s$  underlying  $u$  and  $v$  and satisfying  $E_s(u) \equiv_R^{+(\leq n)} E_s(v)$ .*

*Proof.* Let  $(w_0, \dots, w_n)$  be an  $R$ -derivation from  $u$  to  $v$ . Two cases are possible. Assume first that the initial letter never changes in the considered derivation, *i.e.*,  $I(w_k) = I(u)$  holds for every  $k$ . Then all one step derivations  $(w_k, w_{k+1})$  correspond to case (i) in Lemma 4.7. The

latter implies  $I(w_k) = I(w_{k+1})$  and  $T(w_k) \equiv_R^{+(1)} T(w_{k+1})$  for every  $k$ , whence  $I(u) = I(v)$  and  $T(u) \equiv_R^{+(n)} T(v)$ .

Assume now that the initial letter changes at least once in  $(w_0, \dots, w_n)$ , say  $I(w_i) \neq I(w_{i+1})$ . First, Lemma 4.7 together with Lemma 4.6 implies the existence of  $s$  in  $S$  that underlies  $w_k$  for every  $k$ . Next, each one step derivation  $(w_i, w_{i+1})$  corresponds to case (ii) in Lemma 4.7. So, as  $s$  underlies  $w_i$  and  $w_{i+1}$ , we have  $E_s(w_i) = E_s(w_{i+1})$ . On the other hand, by Lemma 4.7 again, we have  $E_s(w_k) \equiv_R^{+(\leq 1)} E_s(w_{k+1})$  for  $k \neq i$ , so, summing up, we obtain  $E_s(u) \equiv_R^{+(\leq n)} E_s(v)$  for this particular choice of  $s$ .  $\square$

We can now complete the argument establishing that the presentation  $(S, \widehat{R})$  is complete for right-reversing. We denote by  $|w|$  the length (number of letters) of a word  $w$ .

*Proof of Proposition 4.4.* We show using induction on  $n \geq 0$  and, for a given value of  $n$ , on  $\max(|u|, |v|)$ , that  $u \equiv_R^{+(n)} v$  implies  $u^{-1}v \curvearrowright_{\widehat{R}} \varepsilon$ .

Assume first  $n = 0$ . Then the assumption implies  $u = v$ , in which case  $u^{-1}v$  reverses to the empty word by  $|u|$  successive deletions of subwords  $s^{-1}s$ .

Assume now  $n \geq 1$ . Then  $u$  and  $v$  must be nonempty. Assume first that  $I(u) = I(v)$  and  $T(u) \equiv_R^{+(n)} T(v)$  hold. By definition, we have

$$\max(|T(u)|, |T(v)|) = \max(|u|, |v|) - 1,$$

so the induction hypothesis implies  $T(u)^{-1}T(v) \curvearrowright_{\widehat{R}} \varepsilon$ . On the other hand, as  $I(u)$  and  $I(v)$  are equal, we have

$$u^{-1}v = T(u)^{-1}I(u)^{-1}I(v)T(v) \curvearrowright_{\widehat{R}}^{(1)} T(u)^{-1}T(v).$$

By transitivity of reversing, we deduce  $u^{-1}v \curvearrowright_{\widehat{R}} \varepsilon$ .

Assume now that  $I(u) = I(v)$  and  $T(u) \equiv_R^{+(n)} T(v)$  do not hold. Then, by Lemma 4.3, there must exist  $s$  such that  $s$  underlies  $u$  and  $v$  and  $E_s(u) \equiv_R^{+(n')} E_s(v)$  holds for some  $n' < n$ . Then the induction hypothesis implies  $E_s(u)^{-1}E_s(v) \curvearrowright_{\widehat{R}} \varepsilon$ . Write  $s_k$  for  $N^k(s)$ . As  $s$  underlies  $u$  and  $v$ , there exist  $i, j$  satisfying  $I(u) = s_i$  and  $I(v) = s_j$ . Assume for instance  $i \leq j$ . By definition, we have  $E_s(u) = sC^i(s)T(u)$  and  $E_s(v) = sC^j(s)T(v) = sC^i(s)C^{j-i}(s_i)T(v)$ , so that, if  $\ell$  is the length of the word  $sC^i(s)$ , the first  $\ell$  steps in any reversing sequence starting from  $E_s(u)^{-1}E_s(v)$  must be

$$\begin{aligned} E_s(u)^{-1}E_s(v) &= T(u)^{-1}C^i(s)^{-1}s^{-1}sC^i(s)C^{j-i}(s_i)T(v) \\ &\curvearrowright_{\widehat{R}}^{(\ell)} T(u)^{-1}C^{j-i}(s_i)T(v). \end{aligned}$$

It follows that the relation  $E_s(u)^{-1}E_s(v) \curvearrowright_{\widehat{R}} \varepsilon$  deduced above from the induction hypothesis implies

$$(4.3) \quad T(u)^{-1}C^{j-i}(s_i)T(v) \curvearrowright_{\widehat{R}} \varepsilon.$$

Now, let us consider the  $\widehat{R}$ -reversing of  $u^{-1}v$ , i.e., of  $T(u)^{-1}s_i^{-1}s_jT(v)$ . By definition, the only relation of  $\widehat{R}$  of the form  $s_i \dots = s_j \dots$  is  $s_i C^{j-i}(s_i) = s_j$ , so the first step in the reversing must be  $T(u)^{-1}s_i^{-1}s_jT(v) \curvearrowright_{\widehat{R}} T(u)^{-1}C^{j-i}(s_i)T(v)$ . Concatenating this with (4.3), we deduce  $u^{-1}v \curvearrowright_{\widehat{R}} \varepsilon$  again, which completes the induction.  $\square$

With the above completeness at hand, we are now ready for assembling pieces and establishing the Main Lemma (Proposition 2.6).

*Proof of the Main Lemma.* Put  $M = \langle S | R \rangle^+$ . If  $M$  is of right- $O$ -type, any two elements of  $M$  are comparable with respect to left-divisibility, hence they certainly admit a common right-multiple, namely the larger of them. So (i) trivially implies (ii).

Conversely, assume that any two elements of  $M$  admit a common right-multiple. First, as  $M$  admits a right-triangular presentation, hence a semigroup presentation, 1 is the only invertible element in  $M$ .

Next,  $M$  must be left-cancellative. Indeed, the point is to prove that, if  $s$  belongs to  $S$  and  $u, v$  are  $S$ -words satisfying  $su \equiv_R^+ sv$ , then  $u \equiv_R^+ v$  holds. By Proposition 4.4, the presentation  $(S, \widehat{R})$  is complete for right-reversing. Hence  $su \equiv_R^+ sv$  implies  $(su)^{-1}(sv) \curvearrowright_{\widehat{R}} \varepsilon$ , *i.e.*,  $u^{-1}s^{-1}sv \curvearrowright_{\widehat{R}} \varepsilon$ . Now, the first step in any reversing sequence from  $u^{-1}s^{-1}sv$  is  $u^{-1}s^{-1}sv \curvearrowright_{\widehat{R}} u^{-1}v$ , so the assumption implies  $u^{-1}v \curvearrowright_{\widehat{R}} \varepsilon$ , whence  $u \equiv_R^+ v$ .

Finally, let  $g, h$  be two elements of  $M$ . Hence, by Lemma 4.3, which is relevant as  $(S, \widehat{R})$  is complete for right-reversing, there exist  $S$ -words  $u, v$  representing  $g$  and  $h$  and such that the  $\widehat{R}$ -reversing of  $u^{-1}v$  is terminating, *i.e.*, there exist  $S$ -words  $u', v'$  satisfying  $u^{-1}v \curvearrowright_{\widehat{R}} v'u'^{-1}$ . By construction, the family  $\widehat{R}$  consists of triangular relations so, by Lemma 3.6, at least one of the words  $u', v'$  is empty. This means that at least one of  $g \preceq h$  or  $h \preceq g$  holds in  $M$ , *i.e.*,  $g$  and  $h$  are comparable with respect to left-divisibility. So  $M$  is a monoid of right- $O$ -type, and (ii) implies (i).  $\square$

To conclude this section, we observe in view of future examples that the triangular presentations defining monoids admitting common right-multiples must be of some simple type.

**Lemma 4.9.** *Assume that  $(S; R)$  is a right-triangular presentation defining a monoid in which any two elements admit a common right-multiple. Then there exists a (finite or infinite) interval  $I$  of  $\mathbb{Z}$  such that  $S$  is  $\{\mathbf{a}_i \mid i \in I\}$  and  $R$  consists of one relation  $\mathbf{a}_{i-1} = \mathbf{a}_i C(\mathbf{a}_i)$  for each non-minimal  $i$  in  $I$ .*

*Proof.* Let  $s, s'$  belong to  $S$ . As  $s$  and  $s'$  admit a common right-multiple, Lemma 4.3 implies that the  $\widehat{R}$ -reversing of  $s^{-1}s'$  terminates, which in turn requires that  $\widehat{R}$  contains at least one relation of the form  $s\dots = s'\dots$ . Hence the left-graph of  $(S; R)$  consists of a unique chain, which, by definition of a right-triangular presentation, means that  $(S; R)$  has the form stated in the lemma.  $\square$

## 5. QUASI-CENTRAL ELEMENTS

Owing to the Main Lemma (Proposition 2.6), the point in order to decide whether a monoid specified by a right-triangular presentation is of right- $O$ -type is to recognize whether any two elements admit a common right-multiple. We shall establish and use a sufficient condition that involves the existence of a *right-quasi-central* element.

**Definition 5.1.** An element  $\Delta$  of a monoid  $M$  is called *right-quasi-central* (*resp.* *left-quasi-central*) if there exists an endomorphism  $\phi$  of  $M$  such that, for every  $g$  in  $M$ , we have

$$(5.1) \quad g\Delta = \Delta\phi(g) \quad (\text{resp. } \phi(g)\Delta = \Delta g).$$

When  $\phi$  is the identity, we recover the standard notion of a central element, *i.e.*, one that commutes with every element. Note that a right-quasi-central element such that the associated endomorphism is an automorphism is necessarily left-quasi-central as well. The interest of considering quasi-central elements here is given by the following result.

**Lemma 5.2.** *Assume that  $M$  is a left-cancellative monoid that is generated by a set  $S$  and admits a right-quasi-central element  $\Delta$  such that  $s \preceq \Delta$  holds for every  $s$  in  $S$ . Then any two elements of  $M$  admit a common right-multiple.*

*Proof.* Let  $\phi$  be the (necessarily unique) endomorphism of  $M$  witnessing that  $\Delta$  is right-quasi-central. First, (5.1) applied with  $g = \Delta$  gives  $\Delta^2 = \Delta\phi(\Delta)$ , whence  $\phi(\Delta) = \Delta$  since  $M$  is left-cancellative.

Next, we claim that  $g \preceq h$  implies  $\phi(g) \preceq \phi(h)$ . Indeed, by definition,  $g \preceq h$  implies the existence of  $h'$  satisfying  $gh' = h$ , whence  $\phi(g)\phi(h') = \phi(h)$  since  $\phi$  is an endomorphism. This shows that  $\phi(g) \preceq \phi(h)$  is satisfied.

Now, we prove using induction on  $n$  that  $g \in S^n$  implies  $g \preceq \Delta^n$ . For  $n = 0$ , the property is obvious and, for  $n = 1$ , it is one of the assumptions. Assume  $n \geq 2$  and  $g \in S^n$ . Write  $g = sh$  with  $s \in S$  and  $h \in S^{n-1}$ . By induction hypothesis, we have  $h \preceq \Delta^{n-1}$ , whence  $g = sh \preceq s\Delta^{n-1}$ . Applying (5.1)  $n - 1$  times, we obtain  $s\Delta^{n-1} = \Delta^{n-1}\phi^{n-1}(s)$ . Next, by the above remarks, the assumption  $s \preceq \Delta$  implies  $\phi^{n-1}(s) \preceq \phi^{n-1}(\Delta) = \Delta$ , and we deduce  $g \preceq \Delta^{n-1}\phi^{n-1}(s) \preceq \Delta^{n-1}\Delta = \Delta^n$ .

Finally, if  $g, h$  are elements of  $M$ , we have  $g \preceq \Delta^n$  and  $h \preceq \Delta^p$  for some  $n, p$ . Then  $\Delta^{\max(n,p)}$  is a common right-multiple of  $g$  and  $h$ . So any two elements of  $M$  admit a common right-multiple.  $\square$

On the other hand, establishing that an element of a presented monoid is eligible for Lemma 5.2 is easy.

**Lemma 5.3.** *Assume that  $M$  is a left-cancellative monoid generated by a set  $S$ . Then, for every  $\Delta$  in  $M$ , the following are equivalent:*

- (i)  $\Delta$  is right-quasi-central and  $s \preceq \Delta$  holds for every  $s$  in  $S$ ,
- (ii) The relations  $s \preceq \Delta \preceq s\Delta$  hold for every  $s$  in  $S$ .

*Proof.* Assume (i) and let  $\phi$  be the witnessing endomorphism. Let  $s$  belong to  $S$ . By assumption,  $s \preceq \Delta$  is true. Let  $g$  be the element satisfying  $sg = \Delta$ . Then (5.1) implies  $s\Delta = \Delta\phi(s)$ , whence  $s\Delta = sg\phi(s)$ , and  $\Delta = g\phi(s)$  since  $M$  is left-cancellative. We deduce  $\Delta = sg \preceq sg\phi(s) = s\Delta$ , and (i) implies (ii).

Conversely, assume (ii). We shall define an endomorphism  $\phi$  witnessing that  $\Delta$  is right-quasi-central in  $M$ . First, for  $s$  in  $S$ , we define  $\phi(s)$  to be the unique element satisfying

$$(5.2) \quad s\Delta = \Delta\phi(s),$$

which exists since, by assumption,  $\Delta \preceq s\Delta$  holds. Now, assume that  $s_1, \dots, s_n, s'_1, \dots, s'_p$  are elements of  $S$  and  $s_1 \cdots s_n = s'_1 \cdots s'_p$  holds in  $M$ . By applying (5.2) repeatedly, we obtain

$$\Delta\phi(s_1) \cdots \phi(s_n) = s_1 \cdots s_n \Delta = s'_1 \cdots s'_p \Delta = \Delta\phi(s'_1) \cdots \phi(s'_p),$$

whence  $\phi(s_1) \cdots \phi(s_n) = \phi(s'_1) \cdots \phi(s'_p)$  since  $M$  is left-cancellative. It follows that, for every  $g$  in  $M \setminus \{1\}$ , we can define  $\phi(g)$  to be the common value of  $\phi(s_1) \cdots \phi(s_n)$  for all expressions of  $g$  as a product of elements of  $S$ . We complete with  $\phi(1) = 1$ . Then, by construction,  $\phi$  is an endomorphism of  $M$  and (5.1) is satisfied for every  $g$  in  $M$ .  $\square$

Putting things together, we immediately deduce the expected condition for a presented monoid to be of right- $O$ -type.

**Proposition 5.4.** *Assume that  $(S; R)$  is a right-triangular presentation and there exists  $\Delta$  in  $\langle S | R \rangle^+$  satisfying  $s \preceq \Delta \preceq s\Delta$  for every  $s$  in  $S$ . Then  $\langle S | R \rangle^+$  is a monoid of right- $O$ -type.*

*Proof.* By Lemma 5.3, the element  $\Delta$  is right-quasi-central in  $M$  so, by Lemma 5.2, any two elements of  $M$  admit a common right-multiple. As the presentation  $(S; R)$  is right-triangular, it is eligible for the Main Lemma (Proposition 2.6), which implies that  $\langle S | R \rangle^+$  is of right- $O$ -type.  $\square$

It is now easy to establish Theorem 1 of the introduction. We state the results in a slightly more general form avoiding unnecessary symmetries. We begin with point (i).

**Proposition 5.5.** *Assume that  $(S; R)$  is a triangular presentation and there exist  $\Delta$  and  $\tilde{\Delta}$  in  $\langle S | R \rangle^+$  satisfying  $s \preceq \Delta \preceq s\Delta$  and  $\tilde{\Delta}s \preceq \tilde{\Delta} \preceq \tilde{\Delta}s$  for every  $s$  in  $S$ . Then  $\langle S | R \rangle^+$  is a monoid of  $O$ -type, and  $\langle S | R \rangle^+ \setminus \{1\}$  is the positive cone of a left-invariant ordering on  $\langle S | R \rangle$ .*

*Proof.* Proposition 5.4 applied to  $(S; R)$  and  $\Delta$  implies that  $\langle S | R \rangle^+$  is of right- $O$ -type, and, applied to the opposite presentation  $(S, \tilde{R})$  and  $\tilde{\Delta}$ , it implies that  $\langle S | R \rangle^+$  is of left- $O$ -type. Hence  $\langle S | R \rangle^+$  is of  $O$ -type. Then, by Corollary 2.7, the group  $\langle S | R \rangle$  admits a left-invariant ordering whose positive cone is  $\langle S | R \rangle^+ \setminus \{1\}$ .  $\square$

As for point (ii) in Theorem 1, the definition of the topology on the space  $\text{LO}(G)$  [20] implies that, if the positive cone of a left-ordering on the group  $G$  is generated, as a semi-group, by a finite set  $S$ , then the ordering is an isolated point in the space  $\text{LO}(G)$  because this ordering is the only one in which  $S$  is positive and the set of all such orderings is open.

As for the solvability of the decision problem of the ordering (which immediately implies that of the word problem), we shall establish the correctness of Algorithm 5.6 below. The latter simultaneously appeals to right-reversing as defined in Section 3 and to *left-reversing*  $\tilde{\curvearrowright}$ , the symmetric procedure that replaces  $s's^{-1}$  with  $v^{-1}v'$  such that  $vs' = v's$  is a relation. The properties of left-reversing are of course symmetric to those of right-reversing: formally, using  $\tilde{w}$  for the mirror-image of  $w$  (same letters in reserved order),  $w \tilde{\curvearrowright}_R w'$  is equivalent to  $\tilde{w} \curvearrowright_{\tilde{R}} \tilde{w}'$  where, as usual,  $\tilde{R}$  refers to the opposite presentation, *i.e.*, the family of all relations  $\tilde{u} = \tilde{v}$  for  $u = v$  in  $R$ . We denote by  $\tilde{R}$  the family obtained by adding to  $R$  the relations  $\tilde{N}^i(s) = \tilde{C}^i(s)s$  with  $i \geq 2$  (the “left-completion” of  $R$ , symmetric to  $\tilde{R}$ ). If  $w$  is a signed  $S$ -word, we denote by  $[w]$  the element of the group  $\langle S | R \rangle$  represented by  $w$ .

**Algorithm 5.6** (decision problem of the ordering).

- **Data:** A finite (or recursive) triangular presentation  $(S; R)$ ;
- **Input:** A signed  $S$ -word  $w$ ;
- **Procedure:**
  - Right- $\tilde{R}$ -reverse  $w$  into  $vu^{-1}$  with  $u, v$  in  $S^*$ ;
  - Left- $\tilde{R}$ -reverse  $vu^{-1}$  into  $u'^{-1}v'$  with  $u', v'$  in  $S^*$ ;
- **Output:**
  - For  $u' \neq \varepsilon$  and  $v' = \varepsilon$ , return “ $[w] < 1$ ”;
  - For  $u' = v' = \varepsilon$ , return “ $[w] = 1$ ”;
  - For  $u' = \varepsilon$  and  $v' \neq \varepsilon$ , return “ $[w] > 1$ ”.

**Proposition 5.7.** Assume that  $(S; R)$  is a triangular presentation and there exist  $\Delta$  and  $\tilde{\Delta}$  in  $\langle S | R \rangle^+$  satisfying  $s \preceq \Delta \preceq s\Delta$  and  $\tilde{\Delta}s \succcurlyeq \tilde{\Delta} \succcurlyeq s$  for every  $s$  in  $S$ . Then Algorithm 5.6 solves the decision problem for the ordering of Proposition 5.5, as well as the word problem of the group  $\langle S | R \rangle$ .

*Proof.* Put  $M = \langle S | R \rangle^+$ . First, as  $(S; R)$  is finite or, at least, recursive, the relations  $\curvearrowright_{\tilde{R}}$  and  $\tilde{\curvearrowright}_{\tilde{R}}$  are recursive, so Algorithm 5.6 is indeed effective. Next, as the presentation  $(S, \tilde{R})$  is complete for right-reversing and, by Proposition 5.5, any two elements of the monoid  $M$  admit a common right-multiple, every right- $\tilde{R}$ -reversing sequence is terminating: for every signed  $S$ -word  $w$ , there exist positive  $S$ -words  $u, v$  satisfying  $w \curvearrowright_{\tilde{R}} uv^{-1}$ . Similarly, as  $(S, \tilde{R})$  is complete for left-reversing and any two elements of  $M$  admit a common left-multiple, every  $\tilde{R}$ -reversing sequence is terminating and, therefore, there exist positive  $S$ -words  $u', v'$  satisfying  $uv^{-1} \tilde{\curvearrowright}_{\tilde{R}} u'^{-1}v'$ . Hence Algorithm 5.6 always terminates. Moreover, by (the counterpart of) Lemma 3.6, at least one of the words  $u', v'$  is empty.

By construction,  $w \curvearrowright_{\tilde{R}} uv^{-1} \tilde{\curvearrowright}_{\tilde{R}} u'^{-1}v'$  implies  $[w] = [u'^{-1}v']$  in  $\langle S | R \rangle$ . If  $u'$  is nonempty and  $v'$  is empty, we deduce  $[w] = [u'^{-1}] \in M^{-1} \setminus \{1\}$ , whence  $[w] < 1$  for the ordering whose positive cone is  $M^{-1} \setminus \{1\}$ . If  $u'$  and  $v'$  are empty, we deduce  $[w] = [\varepsilon] = 1$ . Finally, if  $u'$  is empty and  $v'$  is nonempty, we obtain  $[w] = [v'] \in M \setminus \{1\}$ , whence  $[w] > 1$ . So Algorithm 5.6 decides the relation  $<$ . As  $<$  is a strict linear ordering, the algorithm also solves the word problem as  $[w] \neq 1$  is equivalent to the disjunction of  $[w] < 1$  and  $[w] > 1$ .  $\square$



Thus the proof of Theorem 1 is complete. We add two more observations. The first one involves monoids that are of right- $O$ -type but not necessarily of  $O$ -type. In this case, the termination of left-reversing is not guaranteed, and the monoid need not be connected with a left-invariant ordering in the group. However, the group is still a group of right-fractions for the monoid, and we can solve its word problem by appealing to right-reversing only.

**Algorithm 5.8** (word problem).

- **Data:** *A finite (or recursive) right-triangular presentation  $(S; R)$ ;*
- **Input:** *A signed  $S$ -word  $w$ ;*
- **Procedure:**
  - *Right- $\widehat{R}$ -reverse  $w$  into  $vu^{-1}$  with  $u, v$  in  $S^*$ ;*
  - *Right- $\widehat{R}$ -reverse  $u^{-1}v$  into  $v'u'^{-1}$  with  $u', v'$  in  $S^*$ ;*
- **Output:**
  - *For  $u' = v' = \varepsilon$ , return “ $[w] = 1$ ”;*
  - *For  $u' \neq \varepsilon$  or  $v' \neq \varepsilon$ , return “ $[w] \neq 1$ ”.*

**Proposition 5.9.** *Assume that  $(S; R)$  is a finite (or recursive) right-triangular presentation and there exists  $\Delta$  in  $\langle S | R \rangle^+$  satisfying  $s \preceq \Delta \preceq s\Delta$  for every  $s$  in  $S$ . Then Algorithm 5.8 solves the word problem of the group  $\langle S | R \rangle$ .*

*Proof.* By Proposition 4.4, the presentation  $(S, \widehat{R})$  is complete for right-reversing and, by Proposition 5.5, any two elements of the monoid  $M$  admit a common right-multiple, hence every  $\widehat{R}$ -reversing sequence is terminating: for every signed  $S$ -word  $w$ , there exist positive  $S$ -words  $u, v$  satisfying  $w \curvearrowright_{\widehat{R}} uv^{-1}$ . Hence Algorithm 5.8, which consists of two concatenated reversings, always terminates.

Then, by construction,  $w \curvearrowright_{\widehat{R}} uv^{-1}$  implies  $[w] = [uv^{-1}]$  in  $\langle S | R \rangle$ . Hence  $[w] = 1$  holds if and only if we have  $[uv^{-1}] = 1$ , or, equivalently,  $[u] = [v]$ . By Ores’s theorem, the monoid  $\langle S | R \rangle^+$  embeds in the group  $\langle S | R \rangle$ , so the latter condition is equivalent to  $[u]^+ = [v]^+$ , *i.e.*, to  $u \equiv_R^+ v$ . As  $(S, \widehat{R})$  is complete for right-reversing, the latter condition is equivalent to  $u^{-1}v \curvearrowright_{\widehat{R}} \varepsilon$ , *i.e.*, with the notation of Algorithm 5.8, to  $u' = v' = \varepsilon$ .  $\square$

The second observation is a connection with Garside theory [8].

**Proposition 5.10.** *Assume that  $M$  is a monoid of right- $O$ -type and  $\Delta$  is a right-quasi-central (resp. simultaneously right- and left-quasi-central) element of  $M$  whose left-divisors generate  $M$ . Then  $\Delta$  is a right-Garside (resp. Garside) element in  $M$  in the sense of [8, Definitions VI.1.36 and 2.29].*

*Proof.* By assumption, the monoid  $M$  is left-cancellative and the left-divisors of  $\Delta$  generate  $M$ . As  $\Delta$  is right-quasi-central, every right-divisor of  $\Delta$  is left-divides  $\Delta$  since, as noted in the proof of Lemma 5.2,  $\Delta = g'g$  implies  $g\phi(g') = \Delta$ . Finally, for every  $g$  in  $M$ , the elements  $g$  and  $\Delta$  admit a greatest common left-divisor (left-gcd), namely the smaller of them with respect to  $\preceq$ . Hence, by definition,  $\Delta$  is a right-Garside element in  $M$ .

If  $\Delta$  is also left-quasi-central, then, by symmetry, the left-divisors of  $\Delta$  must be included in its right-divisors, and, therefore, the left- and right-divisors of  $\Delta$  coincide. Then  $\Delta$  is a Garside element in  $M$ .  $\square$

It follows that, under the hypotheses of Proposition 5.10, the left-divisors of  $\Delta$  in  $M$  form what is called a Garside family [8, Definition I.1.34] and every element of  $M$  admits a distinguished decomposition in terms of these elements. However, as left-divisibility is a linear ordering here, this decomposition is rather trivial: every element is left-divisible by some maximal power of  $\Delta$ , and the normal decompositions all have the simple form  $(\Delta, \dots, \Delta, g)$  with  $g \preceq \Delta$ .

## 6. EXAMPLES

We now apply the approach of Proposition 5.4 to construct examples of monoids of right- $O$ -type specified by a right-triangular presentation. Owing to Lemma 4.9, we consider presentations of form

$$(6.1) \quad (\mathbf{a}_1, \dots, \mathbf{a}_n ; \mathbf{a}_1 = \mathbf{a}_2 C(\mathbf{a}_2), \dots, \mathbf{a}_{n-1} = \mathbf{a}_n C(\mathbf{a}_n))$$

where  $C(\mathbf{a}_2), \dots, C(\mathbf{a}_n)$  are words in the alphabet  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ . Computer experiments are then easy. On the positive side, Proposition 5.4 leads to the following sufficient condition.

**Lemma 6.1.** *Assume that  $(S; R)$  is a semigroup presentation of the form (9.1), and there exists  $m \geq 1$  such that, for  $2 \leq i \leq n$ , there exists a positive  $S$ -word  $w_i$  such that*

$$(6.2) \quad \mathbf{a}_1^{-m} \mathbf{a}_i \mathbf{a}_1^m \curvearrowright_{\widehat{R}} w_i.$$

*Then  $\mathbf{a}^m$  is right-quasi-central in  $\langle S | R \rangle^+$  and the latter is a monoid of right- $O$ -type.*

*Proof.* By Lemma 5.3, the element  $\mathbf{a}_1^m$  is right-quasi-central in  $\langle S | R \rangle^+$ . Moreover, by construction, we have  $\mathbf{a}_n \preceq \mathbf{a}_{n-1} \preceq \dots \preceq \mathbf{a}_1 \preceq \mathbf{a}_1^m$ , whence  $\mathbf{a}_i \preceq \mathbf{a}_1^m$  for every  $i$ . By Proposition 5.4, the monoid  $\langle S | R \rangle^+$  is of right- $O$ -type.  $\square$

On the negative side, recognizing that a presentation gives rise to a monoid that is not of right- $O$ -type seems difficult, since the non-existence of a common right-multiple corresponds to a non-terminating reversing, which *a priori* escapes experiment. However, some cases may be detected using purely syntactic criteria.

**Lemma 6.2.** *Assume that  $(S; R)$  is a triangular presentation.*

(i) *If a relation of  $\widehat{R}$  has the form  $s = w$  with  $|w| > 1$  and  $w$  finishing with  $s$ , then  $\langle S | R \rangle^+$  is not right-cancellative and, therefore,  $\langle S | R \rangle^+$  is not of right- $O$ -type.*

(ii) *If a relation of  $\widehat{R}$  has the form  $s = w$  with  $w$  beginning with  $(uv)^r us$  with  $r \geq 1$ ,  $u$  nonempty, and  $v$  such that  $v^{-1}s$  reverses to a word beginning with  $s$ , hence in particular if  $v$  is empty or it can be decomposed as  $u_1, \dots, u_m$  where  $u_k s$  is a prefix of  $w$  for every  $k$ , then the elements  $s$  and  $[u]^+ s$  have no common right-multiple in  $\langle S | R \rangle^+$  and, therefore,  $\langle S | R \rangle^+$  is not of right- $O$ -type.*

*Proof.* (i) If  $R$  contains a relation  $s = us$  with  $u$  nonempty,  $s = [u]^+ s$  holds in  $\langle S | R \rangle^+$ , whereas  $1 = [u]^+$  fails. So  $\langle S | R \rangle^+$  is not right-cancellative.

(ii) We claim that the right- $\widehat{R}$ -reversing of  $s^{-1}us$  cannot be terminating, see Figure 3. Indeed, writing the involved relation  $s = (uv)^r usw_1$  with  $v^{-1}s \curvearrowright_{\widehat{R}} sw_2$ , we find

$$\begin{aligned} s^{-1}us &\curvearrowright_{\widehat{R}} w_1^{-1}s^{-1}(vu)^{-(r-1)}u^{-1}v^{-1}s \\ &\curvearrowright_{\widehat{R}} w_1^{-1}s^{-1}(vu)^{-(r-1)}u^{-1}sw_2 \\ &\curvearrowright_{\widehat{R}} w_1^{-1}s^{-1}(vu)^{-(r-1)}(vu)^{r-1}v^{-1}usw_1w_2 \\ &\curvearrowright_{\widehat{R}} w_1^{-1}s^{-1}v^{-1}usw_1w_2 \\ &\curvearrowright_{\widehat{R}} w_1^{-1}w_2^{-1} \cdot s^{-1}us \cdot w_1w_2. \end{aligned}$$

We deduce that  $s^{-1}us \curvearrowright_{\widehat{R}} (w_1^{-1}w_2^{-1})^n \cdot s^{-1}us \cdot (w_1w_2)^n$  holds for every  $n$  and, therefore, it is impossible that  $s^{-1}us$  leads in finitely many steps to a positive-negative word. Then, by Lemma 4.3, which is relevant since, by Proposition 4.4,  $(S, \widehat{R})$  is complete for right-reversing,  $s$  and  $[u]^+ s$  admit no common right-multiple in  $\langle S | R \rangle^+$ .  $\square$

For instance, a relation  $\mathbf{a} = \mathbf{babab}^3\mathbf{a}^2\dots$  is impossible in a right-triangular presentation for a monoid of right- $O$ -type: indeed, the right-hand side of the relation can be written as  $(\mathbf{ba})\mathbf{bab}^2(\mathbf{ba})\mathbf{a}\dots$ , which is eligible for Lemma 6.2(ii) with  $u = \mathbf{ba}$  and  $v = \mathbf{bab} \cdot \mathbf{b}$ , a product of two words  $u_1, u_2$  such that  $u_i \mathbf{a}$  is a prefix of the right-hand term of the relation.

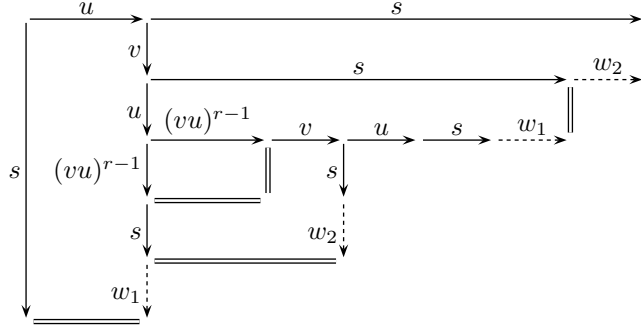


FIGURE 3. Proof of Lemma 6.2(ii): in a positive number of steps, the word  $s^{-1}us$  reverses to a word that includes it and, therefore, the reversing cannot be terminating.

In this section, we focus on the case of two generators, namely presentations of the form  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}w)$  where  $w$  is an  $\{\mathbf{a}, \mathbf{b}\}$ -word. It turns out that Lemmas 6.1 and 6.2 cover almost all cases, *i.e.*, for almost all presentations that are not *a priori* discarded by Lemma 6.2, one can find some power  $\mathbf{a}^m$  of the generator  $\mathbf{a}$  satisfying (6.2), which reduces here to the unique condition that  $\mathbf{a}^{-m}\mathbf{b}\mathbf{a}^m$  reverses to a positive word.

**Fact 6.3.** *Among the 1,023 presentations  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}w)$  with  $w$  an  $\{\mathbf{a}, \mathbf{b}\}$ -word of length  $\leq 9$ ,*

- 157 are eligible for Lemma 6.1, yielding a monoid of right-O-type (28 are of O-type);
- 511 + 343 are eligible for Lemma 6.2(i) or (ii), yielding a monoid not of right-O-type;
- 12 are eligible neither for Lemma 6.1 nor for Lemma 6.2.

See Table 2 for some typical examples. The presentations left aside by Lemmas 6.1, 6.2, and 6.2 can be addressed one by one; it turns out that, in all cases involved in Fact 6.3, either there exists a quasi-central element that is not a power of  $\mathbf{a}$ , or, although the relation does not obey the syntactical conditions of Lemma 6.2, some explicit reversing can be proved to be non-terminating because of some relation  $u \curvearrowright^{(n)} \dots u \dots$  with  $n > 0$ , see Table 3.

Besides systematic experiments, we now describe infinite families of monoids of (right)-O-type. We begin with a family that includes the torus knot groups (Row 1 in Table 1).

**Proposition 6.4.** *For  $p, q, r \geq 1$ , let  $G$  (resp.  $M$ ) be the group (resp. monoid) defined by*

$$(6.3) \quad (\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}(\mathbf{a}^p\mathbf{b}^r)^q).$$

- (i) *The element  $\mathbf{a}^{p+1}$  is right-quasi-central and  $M$  is a monoid of right-O-type.*
- (ii) *For  $r = 1$ , the group  $G$  is the torus knot group  $\langle x, y \mid x^{p+1} = y^{q+1} \rangle$ ; the element  $\mathbf{a}^{p+1}$  is central,  $M$  is of O-type, and  $M \setminus \{1\}$  is the positive cone of an ordering that is isolated in  $\text{LO}(G)$ .*

*Proof.* (i) We argue in  $M$ . Applying (6.3), we first find

$$\mathbf{a} = \mathbf{b}(\mathbf{a}^p\mathbf{b}^r)^q = \mathbf{b} \cdot \mathbf{a} \cdot (\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1},$$

whence, repeating the operation  $r$  times and moving the brackets,

$$\mathbf{a} = \mathbf{b}^r \cdot \mathbf{a} \cdot ((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^r = \mathbf{b}^r(\mathbf{a}^p\mathbf{b}^r)^q \cdot ((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^{r-1}.$$

Let  $\Delta = \mathbf{a}^{p+1}$ . Substituting the above value of  $\mathbf{a}$  at the underlined position, moving the brackets, and applying the relation once in the contracting direction, we find

$$\begin{aligned} \mathbf{b} \cdot \Delta &= \mathbf{b}\mathbf{a}^p\mathbf{a} = \mathbf{b}\mathbf{a}^p \cdot \mathbf{b}^r(\mathbf{a}^p\mathbf{b}^r)^q \cdot ((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^{r-1} \\ &= \mathbf{b}(\mathbf{a}^p\mathbf{b}^r)^q \cdot \mathbf{a}^p\mathbf{b}^r((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^{r-1} \\ &= \mathbf{a} \cdot \mathbf{a}^p \cdot \mathbf{b}^r((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^{r-1} = \Delta \cdot \mathbf{b}^r((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^{r-1}. \end{aligned}$$

	right- $O$	left- $O$	
$a = \text{bababab}$	YES	YES	$\Delta = a^2$ central; (6.3) with $p = 1, q = 2, r = 1$
$a = \text{ba}^2\text{babab}$	YES		$\Delta = a^3$ right-quasi-central, $\phi(a) = a, \phi(b) = (\text{babab})^3$
		NO	$a = \dots a(\text{ab})(\text{ab})(\text{ab})$ : Lemma 6.2 with $u = (\text{ab}), v = \varepsilon$
$a = \text{baba}^2\text{bab}$	NO	NO	$a = (\text{ba})(\text{ba})a\dots$ : Lemma 6.2 with $u = \text{ba}$ and $v = \varepsilon$
$a = \text{ba}^2\text{ba}^2\text{bab}$	YES		$\Delta = a^3$ right-quasi-central, $\phi(a) = a, \phi(b) = \text{ba}^2(\text{bab})^2$
		NO	$a = \dots a(\text{ab})(\text{ab})$ : Lemma 6.2 with $u = (\text{ab})$ and $v = \varepsilon$
$a = \text{ba}^2\text{baba}^2\text{b}$	??	??	not covered by Lemma 6.1 and 6.2, see Table 3
$a = \text{ba}^3\text{babab}$	YES		$\Delta = a^4$ right-quasi-central, $\phi(a) = a, \phi(b) = \text{b}(\text{ab})^8$
		NO	$a = \dots a(\text{ab})(\text{ab})$ : Lemma 6.2 with $u = (\text{ab})$ and $v = \varepsilon$
$a = \text{baba}^3\text{bab}$	NO	NO	$a = (\text{ba})(\text{ba})a\dots$ : Lemma 6.2 with $u = \text{ba}$ and $v = \varepsilon$
$a = \text{ba}^2\text{ba}^2\text{ba}^2\text{b}$	YES	YES	$\Delta = a^4$ central; (6.3) with $p = 2, q = 2, r = 1$
$a = \text{ba}^3\text{ba}^2\text{bab}$	YES		$\Delta = a^5$ right-quasi-central, $\phi(a) = a, \phi(b) = \text{ba}^2\text{baba}(\text{ba}^2\text{bab})^2$
		NO	$a = \dots a(\text{ab})(\text{ab})$ : Lemma 6.2 with $u = (\text{ab})$ and $v = \varepsilon$
$a = \text{ba}^3\text{baba}^2\text{b}$	YES		$\Delta = a^4$ right-quasi-central, $\phi(a) = a, \phi(b) = (\text{baba}^2\text{b})^2\text{ababa}^2\text{b}$
		NO	$a = \dots a(\text{a}^2\text{b})\text{ab}(\text{a}^2\text{b})$ : Lemma 6.2 with $u = (\text{a}^2\text{b})$ and $v = \text{ab}$
$a = \text{ba}^2\text{ba}^3\text{bab}$	NO		$a = (\text{ba}^2)(\text{ba}^2)a\dots$ : Lemma 6.2 with $u = \text{ba}^2$ and $v = \varepsilon$
		NO	$a = \dots a(\text{ab})(\text{ab})$ : Lemma 6.2 with $u = (\text{ab})$ and $v = \varepsilon$
$a = \text{ba}^4\text{babab}$	YES		$\Delta = a^5$ right-quasi-central, $\phi(a) = a, \phi(b) = (\text{babab})(\text{a}^2(\text{babab}))^2$
		NO	$a = \dots a(\text{ab})(\text{ab})(\text{ab})$ : Lemma 6.2 with $u = (\text{ab})$ and $v = \varepsilon$
$a = \text{baba}^4\text{bab}$	NO	NO	$a = (\text{ba})(\text{ba})a\dots$ : Lemma 6.2 with $u = \text{ba}$ and $v = \varepsilon$

TABLE 2. Examples of two-generator monoids with a triangular presentation, here all relations of the form  $a = \text{ba}^p\text{ba}^q\text{ba}^r\text{b}$  with  $p, q, r > 0$  and  $p + q + r \leq 6$  (not repeating symmetric cases): all but one turn out to be eligible either for Lemma 6.1 (there exists a right-quasi-central power of  $a$ ) or for Lemma 6.2 (the syntactic form of the relation results in a non-terminating reversing) or its symmetric counterpart, here denoted by Lemma 6.2.

Now, everything is clear. First, we have  $\mathbf{b} \preceq \mathbf{a} \preceq \mathbf{a}^{p+1} = \Delta$  in  $M$ . Next, as  $\Delta$  is a power of  $\mathbf{a}$ , it commutes with  $\mathbf{a}$ . So, by Lemma 5.3,  $\Delta$  is right-quasi-central in  $M$ , with associated endomorphism defined by

$$\phi(\mathbf{a}) = \mathbf{a}, \quad \phi(\mathbf{b}) = \mathbf{b}^r((\mathbf{a}^{p-1}\mathbf{b}^r)(\mathbf{a}^p\mathbf{b}^r)^{q-1})^{r-1}.$$

By Proposition 5.4,  $M$  is of right- $O$ -type.

(ii) Assume now  $r = 1$ , so that (6.3) reduces to  $\mathbf{a} = \mathbf{b}(\mathbf{a}^p\mathbf{b})^q$ . Put  $x = \mathbf{a}$  and  $y = \mathbf{a}^p\mathbf{b}$  in  $G$ . Then  $x^{p+1} = y^{q+1}$  holds in  $G$ . Conversely, if we define  $\mathbf{a} = x$  and  $\mathbf{b} = x^{-p}y$  in the group  $\langle x, y \mid x^{p+1} = y^{q+1} \rangle$ , then we obtain  $\mathbf{a} = \mathbf{b}(\mathbf{a}^p\mathbf{b})^q$ . An isomorphism between  $G$  and the torus knot group  $\langle x, y \mid x^{p+1} = y^{q+1} \rangle$  follows.

Next, by (i),  $M$  is of right- $O$ -type. As the defining relation is symmetric, the opposite monoid  $\bar{M}$  is also of right- $O$ -type, so  $M$  is of left- $O$ -type, hence of  $O$ -type. Note that, in this case, the endomorphism  $\phi$  is the identity, and  $\Delta$ , *i.e.*,  $\mathbf{a}^{p+1}$ , is central. By Lemma 1.3,  $M \setminus \{1\}$  is the positive cone of an ordering that is isolated in the space  $\text{LO}(G)$ .  $\square$

Some of the groups of Proposition 6.4 are well known. For instance, for  $p = q = r = 1$ , the group  $G$  is the Klein bottle group  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \text{bab} \rangle$ . For  $p = 2$  and  $q = r = 1$ , the group  $G$ , *i.e.*,  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \text{ba}^2\mathbf{b} \rangle$ , is Artin's braid group  $B_3$ . In terms of the standard Artin generators  $\sigma_i$ , the elements  $\mathbf{a}$  and  $\mathbf{b}$  can be realized as  $\sigma_1\sigma_2$  and  $\sigma_2^{-1}$ , and the associated ordering is the isolated ordering described by Dubrovina–Dubrovin in [11] (see also [17]). The braid group  $B_3$  is also obtained for  $p = r = 1$  and  $q = 2$ , *i.e.*, for  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \text{babab} \rangle$ , with  $\mathbf{a}$  and  $\mathbf{b}$  now realizable as  $\sigma_1\sigma_2\sigma_1$  and  $\sigma_2^{-1}$ . Note that, when realized as above, the associated

	right- $O$	left- $O$	
$\mathbf{a} = \mathbf{bab}^3\mathbf{ab}$	YES	YES	$\Delta = (\mathbf{ab})^3 = (\mathbf{ba})^3$ central
$\mathbf{a} = \mathbf{bab}^4\mathbf{ab}$	YES	YES	$\Delta = (\mathbf{ab}^2)^3 = (\mathbf{b}^2\mathbf{a})^3$ central
$\mathbf{a} = \mathbf{ba}^2\mathbf{baba}^2\mathbf{b}$	YES	YES	$\Delta = (\mathbf{a}^2\mathbf{b})^2 = (\mathbf{ba}^2)^2$ right- and left-quasi-central, $\phi(\mathbf{a}) = \mathbf{a}(\mathbf{ba}^2\mathbf{b})^2$ , $\tilde{\phi}(\mathbf{a}) = (\mathbf{ba}^2\mathbf{b})^2\mathbf{a}$ , $\phi(\mathbf{b}) = \tilde{\phi}(\mathbf{b}) = \mathbf{b}$
$\mathbf{a} = \mathbf{ba}^2\mathbf{b}^3\mathbf{a}^2\mathbf{b}$	NO	NO	non-terminating right-reversing: $u \curvearrowright^{(10)} v^{-1}uv$ for $u = \mathbf{a}^{-2}\mathbf{ba}^2\mathbf{ba}$ and $v = \mathbf{ba}^2\mathbf{b}^3$
$\mathbf{a} = \mathbf{bab}^4\mathbf{ab}^2$	YES		$\Delta = (\mathbf{ab}^2)^2$ right-quasi-central, $\phi(\mathbf{a}) = (\mathbf{b}^2\mathbf{ab}^4)^2\mathbf{ab}^2$ , $\phi(\mathbf{b}) = \mathbf{b}^4\mathbf{ab}^4$
$\mathbf{a} = \mathbf{bab}^5\mathbf{ab}$	YES	NO	$\mathbf{a} = \dots\mathbf{a}(\mathbf{b})(\mathbf{b})$ : Lemma 6.2 with $u = \mathbf{b}$ and $v = \varepsilon$
$\mathbf{a} = \mathbf{ba}^2\mathbf{bab}^2\mathbf{a}^2\mathbf{b}$	YES	YES	$\Delta = (\mathbf{ab}^3)^3 = (\mathbf{b}^3\mathbf{a})$ central
		NO	$\Delta = (\mathbf{a}^2\mathbf{b})^2$ right-quasi-central, $\phi(\mathbf{a}) = \mathbf{ab}(\mathbf{ba}^2\mathbf{b})^2$ , $\phi(\mathbf{b}) = (\mathbf{ba}^2\mathbf{b}^2)^2$
			non-terminating left-reversing: $u \curvearrowleft^{(26)} vuv^{-1}$ for $u = \mathbf{a}^2\mathbf{b}^2\mathbf{a}^2\mathbf{bab}^3\mathbf{a}^2\mathbf{ba}^{-1}$ and $v = \mathbf{b}$
$\mathbf{a} = \mathbf{ba}^2\mathbf{b}^4\mathbf{a}^2\mathbf{b}$	NO	NO	non-terminating right-reversing: $u \curvearrowright^{(12)} v^{-1}uv$ for $u = \mathbf{b}^{-1}\mathbf{a}^{-2}\mathbf{ba}^2\mathbf{ba}$ and $v = \mathbf{b}^4\mathbf{a}^2\mathbf{bab}^4\mathbf{a}^2\mathbf{b}$
$\mathbf{a} = \mathbf{bab}^2\mathbf{ab}^3\mathbf{ab}$	YES		$\Delta = (\mathbf{ab})^2$ right-quasi-central, $\phi(\mathbf{a}) = (\mathbf{bab}^2)^2\mathbf{ab}$ , $\phi(\mathbf{b}) = \mathbf{b}(\mathbf{b}^2\mathbf{ab}^2)^2$
		NO	non-terminating left-reversing: $u \curvearrowleft^{(8)} vuv^{-1}$ for $u = \mathbf{ab}^2\mathbf{aba}^{-1}$ and $v = \mathbf{b}$
$\mathbf{a} = \mathbf{bab}^5\mathbf{ab}^2$	YES		$\Delta = (\mathbf{ab}^3)^2$ right-quasi-central, $\phi(\mathbf{a}) = (\mathbf{b}^2\mathbf{ab}^5)^3\mathbf{ab}^3$ , $\phi(\mathbf{b}) = \mathbf{b}^4\mathbf{ab}^5$
		NO	$\mathbf{a} = \dots\mathbf{a}(\mathbf{b})(\mathbf{b})$ : Lemma 6.2 with $u = \mathbf{b}$ and $v = \varepsilon$
$\mathbf{a} = \mathbf{bab}^6\mathbf{ab}$	YES	YES	$\Delta = (\mathbf{ab}^4)^3 = (\mathbf{b}^4\mathbf{a})^3$ central

TABLE 3. Presentations  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{bw})$  with  $|w| \leq 9$  that escape Lemmas 6.1, 6.2 and/or  $\widetilde{6.2}$  (up to a symmetry): in every case, one can either find a (quasi)-central element or identify a non-terminating reversing, hence decide whether the associated monoid is of right- or left- $O$ -type; note that one can have both  $\mathbf{a} \preceq \phi(\mathbf{a})$  (for instance for  $\mathbf{a} = \mathbf{ba}^2\mathbf{baba}^2\mathbf{b}$ ) and  $\phi(\mathbf{a}) \preceq \mathbf{a}$  (for instance for  $\mathbf{a} = \mathbf{bab}^4\mathbf{ab}$ , in which case  $\mathbf{a} = \phi(\mathbf{a}) \cdot \mathbf{b}^6\mathbf{ab}^2$  holds).

submonoids of  $B_3$  coincide as, using  $\mathbf{a}$  in the case  $p = 2, q = 1$  and  $\mathbf{a}'$  in the case  $p = 1, q = 2$ , we find  $\mathbf{a} = \mathbf{a}'\mathbf{b}$  and  $\mathbf{a}' = \mathbf{ba}^2$ . Therefore the associated (isolated) orderings of  $B_3$  coincide.

For  $r \geq 2$ , the group  $G$  is not isomorphic to the opposite group. The left counterpart of Lemma 6.2 (“Lemma  $\widetilde{6.2}$ ”) implies that  $\mathbf{a}$  and  $\mathbf{ab}$  have no common left-multiple in  $M$ . Hence  $M$  is not of left- $O$ -type, and the group  $G$ , which is a group of right-fractions for  $M$ , is not a group of left-fractions for that monoid: the right-fraction  $\mathbf{aba}^{-1}$  is an element of  $G$  that cannot be expressed as a left-fraction. As a consequence, the semigroup  $M \setminus \{1\}$  defines a *partial* left-invariant ordering on  $G$  only: for instance, the elements  $\mathbf{b}^{-1}\mathbf{a}^{-1}$  and  $\mathbf{a}^{-1}$  are not comparable as their quotient  $\mathbf{aba}^{-1}$  belongs neither to  $M$  nor to  $M^{-1}$ . Note that, for  $p = q = 1$ , the group  $G$ , *i.e.*,  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{bab}^{r+1} \rangle$ , is the Baumslag–Solitar group  $\text{BS}(r+1, -1)$ , whereas the opposite group  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b}^{r+1}\mathbf{ab} \rangle$  is  $\text{BS}(-1, q+1)$ .

Besides the above examples, the case  $p = r = 2, q = 1$ , *i.e.*,  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{ba}^2\mathbf{b}^2 \rangle$ , is the first non-classical example in the family. In this case,  $\mathbf{a}^3$  is a right-quasi-central element that is not central, and the associated endomorphism is given by  $\phi(\mathbf{a}) = \mathbf{a}$  and  $\phi(\mathbf{b}) = \mathbf{b}^2\mathbf{ab}^2$ .

We now consider the family of Row 2 in Table 1.

**Proposition 6.5.** *For  $p, q \geq 1$ , let  $G$  (resp.  $M$ ) be the group (resp. monoid) defined by*

$$(6.4) \quad (\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}(\mathbf{ab}^p)^q\mathbf{ab}).$$

*Then  $G$  is  $\langle x, y \mid x^{q+2} = y^2 \rangle$ , the element  $(\mathbf{ab}^p)^{q+2}$ , which is also  $(\mathbf{ab}^{p-1})^2$ , is central,  $M$  is of  $O$ -type, and  $M \setminus \{1\}$  is the positive cone of an ordering that is in  $\text{LO}(G)$ .*

*Proof.* Put  $x = \mathbf{ab}^p$  and  $y = \mathbf{bx}^{q+1}$  in  $G$ . Then  $G$  is generated by  $x$  and  $y$  and (6.4) implies  $\mathbf{ab}^p = \mathbf{b}(\mathbf{ab}^p)^q\mathbf{ab}^p\mathbf{b}$ , *i.e.*,  $x = \mathbf{bx}^{q+1}\mathbf{b}$ , whence in turn  $x^{q+2} = (\mathbf{bx}^{q+1})^2$ , *i.e.*,  $x^{q+2} = y^2$ .

Conversely, define  $\mathbf{b} = yx^{-(q+1)}$  and  $\mathbf{a} = xb^{-p}$  in  $\langle x, y \mid x^{q+2} = y^2 \rangle$ . Then the group is generated both by  $x$  and  $\mathbf{b}$ , and by  $\mathbf{a}$  and  $\mathbf{b}$ , and the relations  $x = \mathbf{b}x^{q+1}\mathbf{b}$  and  $\mathbf{a} = \mathbf{b}(\mathbf{ab}^p)^q\mathbf{ab}$  are satisfied. It follows that  $G$  admits the presentations  $\langle x, \mathbf{b} \mid x = \mathbf{b}x^{q+1}\mathbf{b} \rangle$  and  $\langle x, y \mid x^{q+2} = y^2 \rangle$ .

Now, in  $M$ , let us define  $\dot{x} = \mathbf{ab}^p$  and  $\Delta = \dot{x}^{q+2}$  (at this point, we do not know that  $M$  embeds in  $G$ , so using  $x$  here would be confusing). First, we find

$$\Delta = (\mathbf{ab}^p)^{q+2} = \mathbf{ab}^{p-1}(\mathbf{b}(\mathbf{ab}^p)^q\mathbf{ab})\mathbf{b}^{p-1} = \mathbf{ab}^{p-1}\mathbf{ab}^{p-1} = (\mathbf{ab}^{p-1})^2.$$

Next, by the same computation as above, the relation  $\dot{x} = \mathbf{b}\dot{x}^{q+1}\mathbf{b}$  holds in  $M$ . We deduce

$$\mathbf{b} \cdot \Delta = \mathbf{b}\dot{x}^{q+1}(\dot{x}) = \mathbf{b}\dot{x}^{q+1}(\mathbf{b}\dot{x}^{q+1}\mathbf{b}) = (\mathbf{b}\dot{x}^{q+1}\mathbf{b})\dot{x}^{q+1}\mathbf{b} = (\dot{x})\dot{x}^{q+1}\mathbf{b} = \Delta \cdot \mathbf{b},$$

so  $\Delta$  commutes with  $\mathbf{b}$  in  $M$ . On the other hand,  $\Delta$  commutes with  $\dot{x}$ , *i.e.*, with  $\mathbf{ab}^p$ . So, always in  $M$ , we have  $\mathbf{ab}^p\Delta = \Delta\mathbf{ab}^p$ , whence  $\mathbf{a}\Delta\mathbf{b}^p = \Delta\mathbf{ab}^p$  as  $\Delta$  commutes with  $\mathbf{b}$ , and  $\mathbf{a}\Delta = \Delta\mathbf{a}$  as  $M$  is right-cancellative since it admits a left-triangular presentation. So  $\Delta$  is central in  $M$ , and Proposition 5.5 gives the expected results.  $\square$

**Remark 6.6.** In the above proof, once noted that  $G$  is  $\langle x, y \mid x^{q+2} = y^2 \rangle$ , it is obvious that  $x^{q+2}$  is central in  $G$ . However, this is not *a priori* sufficient to deduce that  $\dot{x}^{q+2}$  is central in  $M$  as long as  $M$  is not known to embed in  $G$ . That is why we carefully check that the expected commutation relations can be established inside  $M$ , *i.e.*, without using inverses except those provided by cancellativity.

We skip the verification for the family of Row 3 in Table 1, which is similar, and now consider Row 4, which is more interesting in that it involves elements that are both right- and left-quasi-central, but not central for  $p \geq 2$  since the associated endomorphism is not the identity. For  $p = 2, q = 1$ , the presentation occurs in Row 3 of Table 3.

**Proposition 6.7.** For  $p, q \geq 1$ , let  $G$  (*resp.*  $M$ ) be the group (*resp.* monoid) defined by

$$(6.5) \quad (\mathbf{a}, \mathbf{b} ; \mathbf{a} = \mathbf{ba}^{pq}\mathbf{ba}^{p-1}\mathbf{ba}^{pq}\mathbf{b}).$$

Then  $G$  is  $\langle x, y \mid x^p = (yx^{pq-p+1}y)^2 \rangle$ , the element  $(\mathbf{a}^{pq}\mathbf{b})^2$  is right-quasi-central with  $\phi(\mathbf{a}) = \mathbf{a}^{p-1}(\mathbf{ba}^{pq}\mathbf{b})^2$ ,  $\phi(\mathbf{b}) = \mathbf{b}$ , the monoid  $M$  is of *O*-type and  $M \setminus \{1\}$  is the positive cone of an ordering that is isolated in  $\text{LO}(G)$ .

*Proof.* We argue in  $M$ . First, we claim that  $\mathbf{a}^p$  and  $\mathbf{ba}^{pq}\mathbf{b}$  commute. Indeed, we find

$$\begin{aligned} \mathbf{a}^p \cdot \mathbf{ba}^{pq}\mathbf{b} &= (\mathbf{a})\mathbf{a}^{p-1}\mathbf{ba}^{pq}\mathbf{b} = (\mathbf{ba}^{pq}\mathbf{ba}^{p-1}\mathbf{ba}^{pq}\mathbf{b})\mathbf{a}^{p-1}\mathbf{ba}^{pq}\mathbf{b} \\ &= \mathbf{ba}^{pq}\mathbf{ba}^{p-1}(\mathbf{ba}^{pq}\mathbf{ba}^{p-1}\mathbf{ba}^{pq}\mathbf{b}) = \mathbf{ba}^{pq}\mathbf{ba}^{p-1}(\mathbf{a}) = \mathbf{ba}^{pq}\mathbf{b} \cdot \mathbf{a}^p. \end{aligned}$$

Now put  $\Delta = (\mathbf{a}^{pq}\mathbf{b})^2$ . We claim that  $\Delta = (\mathbf{ba}^{pq})^2$  also holds. Indeed, using the above commutation relation  $q$  times, we find

$$\Delta = (\mathbf{a}^{pq}\mathbf{b})^2 = (\mathbf{a}^p)^q \cdot \mathbf{ba}^{pq}\mathbf{b} = \mathbf{ba}^{pq}\mathbf{b} \cdot (\mathbf{a}^p)^q = (\mathbf{ba}^{pq})^2.$$

We immediately deduce

$$\mathbf{b} \cdot \Delta = \mathbf{b}(\mathbf{a}^{pq}\mathbf{ba}^{pq}\mathbf{b}) = (\mathbf{ba}^{pq}\mathbf{ba}^{pq})\mathbf{b} = \Delta \cdot \mathbf{b}.$$

On the other hand, using the above commutation relation to push the underlined factor  $\mathbf{a}^{pq}$  to the left through  $\mathbf{ba}^{pq}\mathbf{b}$  and  $\mathbf{a}^{p-1}$ , we find

$$\begin{aligned} \mathbf{a} \cdot \Delta &= (\mathbf{a})\mathbf{a}^{pq}\mathbf{ba}^{pq}\mathbf{b} = (\mathbf{ba}^{pq}\mathbf{ba}^{p-1}\mathbf{ba}^{pq}\mathbf{b})\underline{\mathbf{a}^{pq}}\mathbf{ba}^{pq}\mathbf{b} \\ &= \mathbf{ba}^{pq}\mathbf{ba}^{pq}\mathbf{a}^{p-1}\mathbf{ba}^{pq}\mathbf{ba}^{pq}\mathbf{b} \\ &= (\mathbf{ba}^{pq})^2 \cdot \mathbf{a}^{p-1}(\mathbf{ba}^{pq}\mathbf{b})^2 = \Delta \cdot \mathbf{a}^{p-1}(\mathbf{ba}^{pq}\mathbf{b})^2. \end{aligned}$$

It follows that  $\Delta$  is right-quasi-central, with  $\phi(\mathbf{a}) = \mathbf{a}^{p-1}(\mathbf{ba}^{pq}\mathbf{b})^2$  and  $\phi(\mathbf{b}) = \mathbf{b}$ . For  $p = 1$ , we find  $\phi(\mathbf{a}) = (\mathbf{bab})^2 = \mathbf{a}$ , so  $\Delta$  is central, but, for  $p \geq 2$ , we have  $\mathbf{a}^{p-2}(\mathbf{ba}^{pq}\mathbf{b})^2 \neq 1$  and  $\phi$  is not the identity. Owing to the symmetry of the presentation and of the equalities

$\Delta = (\mathbf{a}^{p^q}\mathbf{b})^2 = (\mathbf{b}\mathbf{a}^{p^q})^2$ , the element  $\Delta$  is also left-quasi-central. By Proposition 5.5, the monoid  $M$  is of  $O$ -type and the rest follows. Note that, by symmetry, the endomorphism associated with  $\Delta$  as a left-quasi-central element has to be  $\phi^{-1}$  so, in particular,  $\phi$  is an automorphism of  $M$  and we must have  $\phi((\mathbf{b}\mathbf{a}^{p^q}\mathbf{a})^2\mathbf{a}^{p-1}) = \mathbf{a}$ .  $\square$

The previous results fail to cover all presentations  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}\mathbf{a}^q\mathbf{b}\mathbf{a}^p\mathbf{b}\mathbf{a}^q\mathbf{b})$ . Actually, some cases remain unclear, typically  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}\mathbf{a}^4\mathbf{b}\mathbf{a}^2\mathbf{b}\mathbf{a}^4\mathbf{b})$  and  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}\mathbf{a}^5\mathbf{b}\mathbf{a}^3\mathbf{b}\mathbf{a}^5\mathbf{b})$ .

## 7. NON-EXAMPLES

So far, we did not discuss the range of our approach as expressed by the converse of Question 2.2, namely whether all monoids of (right)- $O$ -type admit (right)-triangular presentations. The positive results of Section 6, which provide a number of monoids of right- $O$ -type with a right-triangular presentation, might suggest a positive answer. In this section, we show that this is not the case, and give a simple criterion discarding a number of such monoids, in particular the  $n$ -strand Dubrovina–Dubrovin braid monoids for  $n \geq 4$ .

So our starting point is

**Question 7.1.** *Assume that  $M$  is a monoid of right- $O$ -type and  $S$  is a generating subfamily of  $M$ . Does  $M$  admit a right-triangular presentation in terms of  $S$ ?*

What is significant in a right-triangular presentation is not the fact that it consists of triangular relations, but the condition that there is at most one letter  $N(s)$  and one relation  $N(s) = sC(s)$  for every  $s$ : every semigroup presentation can be trivially transformed into a presentation of the same monoid consisting of triangular relations by introducing, for every relation  $u = v$ , a new letter  $s$  and replacing  $u = v$  with the triangular relations  $s = u$ ,  $s = v$ .

The following result, which is a special case of a result of [10] for monoids in which any two elements admit a least common right-multiple, may appear relevant for Question 7.1.

**Fact 7.2.** *Assume that  $M$  is a monoid of right- $O$ -type that satisfies Condition (4.2), and  $S$  is any generating subfamily of  $M$ . For all  $s, s'$  in  $S$  with  $s \preccurlyeq s'$ , choose an  $S$ -word  $w$  such that  $sw$  represents  $s'$ . Let  $R$  be the family of all relations  $sw = s'$  so obtained. Then  $(S; R)$  is a presentation of  $M$ .*

*Proof (sketch).* We wish to prove for all  $S$ -words  $u, v$  that  $[u]^+ = [v]^+$  is equivalent to  $u \equiv_R^+ v$ . By construction,  $R$  consists of relations that are valid in  $M$ , hence  $u \equiv_R^+ v$  always implies  $[u]^+ = [v]^+$ , and the problem is the converse implication. Standard arguments show that (4.2) is equivalent to the existence of a map  $\lambda$  from  $M$  to the ordinals such that  $s \neq 1$  implies  $\lambda(sg) > \lambda(g)$ . Then one proves that  $[u]^+ = [v]^+$  with  $\lambda([u]^+) = \alpha$  implies  $u \equiv_R^+ v$  using induction on  $\alpha$ . For  $\alpha = 0$ , we have  $\lambda([u]^+) = \lambda([v]^+) = 0$ , hence  $[u]^+$  is minimum with respect to proper right-divisibility in  $M$ , implying  $[u]^+ = [v]^+ = 1$ , whence  $u = v = \varepsilon$ . Assume now  $\alpha > 0$ . Then  $u$  and  $v$  cannot be empty. Write  $u = su_0$ ,  $v = s'v_0$  with  $s, s'$  in  $S$ . Then, by definition, we have (\*)  $\lambda([u_0]^+) < \lambda([u]^+)$  and  $\lambda([v_0]^+) < \lambda([v]^+)$ . Assume first  $s' = s$ . By assumption, we have  $[u]^+ = [v]^+$ , i.e.,  $s[u_0]^+ = s[v_0]^+$ . As  $M$  is left-cancellative, we deduce  $[u_0]^+ = [v_0]^+$ . By (\*) and the induction hypothesis, this implies  $u_0 \equiv_R^+ v_0$ , whence *a fortiori*  $u = su_0 \equiv_R^+ sv_0 = v$ . Finally, assume  $s' \neq s$ . In  $M$ , the elements  $s$  and  $s'$  are comparable for  $\preccurlyeq$ , say for instance  $s \preccurlyeq s'$ . Then, by construction, there exists in  $R$  one relation  $sw = s'$  such that  $s[w]^+ = s'$  holds in  $M$ . We deduce  $s[u_0]^+ = s'[v_0]^+ = s[w]^+[v_0]^+ = s[wv_0]^+$ , whence  $[u_0]^+ = [wv_0]^+$  since  $M$  is left-cancellative. By (\*) and the induction hypothesis, this implies  $u_0 \equiv_R^+ wv_0$ , whence  $u = su_0 \equiv_R^+ swv_0 \equiv_R^+ s'v_0 = v$ . So the induction is complete.  $\square$

The above positive result is misleading. The range of Fact 7.2 is nonempty since it applies at least to the monoid  $(\mathbb{N}, +)$ , but, as already mentioned, the Noetherianity condition (4.2)

fails in almost all monoids that admit triangular presentations, and the following example shows that, when (4.2) fails, we cannot hope for a result similar to Fact 7.2.

**Example 7.3.** Let  $M$  be the Klein bottle monoid  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{bab} \rangle^+$ . Then  $M$  is of right- $O$ -type, and it is generated by  $\mathbf{a}$  and  $\mathbf{b}$ . Now, in  $M$ , we have  $\mathbf{a} = \mathbf{b}^2 \mathbf{ab}^2$ , so, if Fact 7.2 were valid here,  $(\mathbf{a}, \mathbf{b} ; \mathbf{a} = \mathbf{b}^2 \mathbf{ab}^2)$  would be an alternative presentation of  $M$ . This is not the case: by Lemma 6.2 applied with  $u = \mathbf{b}$  and  $v = \varepsilon$ , monoid  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{b}^2 \mathbf{ab}^2 \rangle^+$  is not of right- $O$ -type and, therefore, it is not isomorphic to  $M$ .

Actually, we shall establish a rather general negative answer to Question 7.1 in the case of generating families with at least three elements.

**Definition 7.4.** Assume that  $M$  is a monoid and  $S$  is included in  $M$ . An element  $s$  of  $S$  is called *preponderant in  $S$*  if  $g \preceq hs$  holds for all  $g, h$  in the submonoid generated by  $S \setminus \{s\}$ .

**Proposition 7.5.** Assume that  $M$  is a monoid of right- $O$ -type and  $S$  is a generating subfamily of  $M$  that contains a preponderant element and has at least three elements. Then  $M$  admits no right-triangular presentation in terms of  $S$ .

*Proof.* We assume that  $M$  admits a right-triangular presentation  $(S ; R)$  and shall derive a contradiction by exhibiting two elements of  $M$  that cannot admit a common right-multiple.

As  $M$  is of right- $O$ -type, owing to Lemma 4.9, we can enumerate  $S$  as  $\{\mathbf{a}_i \mid i \in I\}$  so that all relations in  $R$  have the form  $\mathbf{a}_{i-1} = \mathbf{a}_i C(\mathbf{a}_i)$ . Assume that  $\mathbf{a}_i$  is preponderant in  $S$ . Then  $i$  must be minimal in  $I$  as  $\mathbf{a}_j \prec \mathbf{a}_i$  holds for every  $j \neq i$ . So we may assume  $I = \{1, 2, \dots\}$  (finite or infinite), and that  $\mathbf{a}_1$  is preponderant in  $S$ .

As  $\mathbf{a}_1$  is preponderant in  $S$ , it may occur in no word  $C(\mathbf{a}_i)$  with  $i \geq 3$  for, otherwise, writing  $C(\mathbf{a}_i) = u\mathbf{a}_1v$  with no  $\mathbf{a}_1$  in  $u$ , applying the definition of preponderance with  $g = \mathbf{a}_{i-1}^2$  and  $h = [\mathbf{a}_i u]^+$  would lead to the contradiction

$$\mathbf{a}_{i-1} \prec \mathbf{a}_{i-1}^2 \preceq [\mathbf{a}_i u \mathbf{a}_1]^+ \preceq [\mathbf{a}_i C(\mathbf{a}_i)]^+ = \mathbf{a}_{i-1}.$$

On the other hand,  $\mathbf{a}_1$  must occur in  $C(\mathbf{a}_2)$  for, otherwise, we would obtain similarly the contradiction  $[\mathbf{a}_2 C(\mathbf{a}_2)]^+ \prec [\mathbf{a}_2 C(\mathbf{a}_2) \mathbf{a}_2]^+ \preceq \mathbf{a}_1 = [\mathbf{a}_2 C(\mathbf{a}_2)]^+$ . Write  $\mathbf{a}_2 C(\mathbf{a}_2) = u_0 \mathbf{a}_1 v_0$  with no  $\mathbf{a}_1$  in  $u_0$ .

*Claim.* Assume that  $w$  is an  $S$ -word that is  $\equiv_R^+$ -equivalent to a word beginning with  $\mathbf{a}_1$ . Then  $w$  contains at least one letter  $\mathbf{a}_1$  and, if  $u$  is the initial fragment of  $w$  that goes up to the first letter  $\mathbf{a}_1$ , there exists  $r \geq 0$  satisfying  $u \equiv_R^+ u_0^r$ .

We prove the claim using induction on the combinatorial distance  $n$  of  $w$  to a word beginning with  $\mathbf{a}_1$ , i.e., on the length of an  $R$ -derivation from  $w$  to such a word. For  $n = 0$ , i.e., if  $w$  begins with  $\mathbf{a}_1$ , the word  $u$  is empty, and we have  $u = \varepsilon = u_0^0$ . Assume  $n > 0$ . Let  $w'$  be a word obtained from  $w$  by applying one relation of  $R$  that lies at distance  $n - 1$  from a word beginning with  $\mathbf{a}_1$ . By induction hypothesis,  $w'$  contains at least one letter  $\mathbf{a}_1$ , and we have  $w' = u' \mathbf{a}_1 v'$  with no  $\mathbf{a}_1$  in  $u'$  and  $u' \equiv_R^+ u_0^{r'}$  for some  $r'$ . We consider the various ways  $w$  can be obtained from  $w'$ . First, if one relation of  $R$  is applied inside  $v'$ , we have  $w = u' \mathbf{a}_1 v$  with  $v \equiv_R^+ v'$  and the result is clear with  $u = u'$  and  $r = r'$ . Next, assume that the distinguished letter  $\mathbf{a}_1$  is involved. By hypothesis,  $N(\mathbf{a}_1)$  is not defined, so there is no relation  $s = \mathbf{a}_1 C(\mathbf{a}_1)$  in  $R$ . On the other hand,  $u'$  contains no  $\mathbf{a}_1$  and, therefore,  $\mathbf{a}_1$  occurs in no relation  $s = \dots$  for  $s$  occurring in  $u'$ . So the only ways  $\mathbf{a}_1$  may be involved is either  $\mathbf{a}_1$  being replaced with  $\mathbf{a}_2 C(\mathbf{a}_2)$ , or  $\mathbf{a}_2 C(\mathbf{a}_2)$  (which contains at least one  $\mathbf{a}_1$ ) being replaced with  $\mathbf{a}_1$ . In the first case, we obtain  $u = u' u_0 \mathbf{a}_1 v_0 v'$ , which shows that  $w$  contains a letter  $\mathbf{a}_1$  and gives  $u = u' u_0$ , whence  $u \equiv_R^+ u_0^{r'+1}$ , the expected result with  $r = r' + 1$ . In the second case, there must exist decompositions  $u' = uu_0$  and  $v' = v_0 v$  so that we have  $w' = uu_0 \mathbf{a}_1 v_0 v$  and  $w = u \mathbf{a}_1 v$ . Again  $w$  contains  $\mathbf{a}_1$ , and we find now  $uu_0 \equiv_R^+ u' \equiv_R^+ u_0^{r'}$ , whence  $u \equiv_R^+ u_0^{r'-1}$  because, by assumption,  $M$  is right-cancellative. This is again the expected result, this time



with  $r = r' - 1$ . Finally, it remains the case when one relation of  $R$  is applied inside  $u'$ . In this case, we obtain  $w = u\mathbf{a}_1v'$  with  $u \equiv_R^+ u'$ , whence  $u \equiv_R^+ u' \equiv_R^+ u_0'$ , and the result is clear with  $r = r'$ . So the proof of the claim is complete.

We shall now easily obtain a contradiction. Indeed, by construction, the word  $u_0$  begins with the letter  $\mathbf{a}_2$ , so  $\mathbf{a}_2 \preceq [u_0]^+$  holds. By assumption,  $\mathbf{a}_2$  and  $\mathbf{a}_3$  are distinct, so  $C(\mathbf{a}_3)$  is nonempty, and we obtain  $1 \prec \mathbf{a}_3 \prec \mathbf{a}_3[C(\mathbf{a}_3)]^+ = \mathbf{a}_2 \preceq [u_0]^+$ , so that  $\mathbf{a}_3 = [u_0']^+$  fails for every  $r$ . Then the above claim implies that no  $S$ -word beginning with  $\mathbf{a}_3\mathbf{a}_1$  may be  $\equiv_R^+$ -equivalent to an  $S$ -word beginning with  $\mathbf{a}_1$ . In other words, the elements  $\mathbf{a}_1$  and  $\mathbf{a}_3\mathbf{a}_1$  cannot admit a common right-multiple in  $M$ , contrary to the assumption that  $M$  is of right- $O$ -type.  $\square$

Proposition 7.5 prevents a number of monoids of right- $O$ -type from admitting a right-triangular presentation.

**Corollary 7.6.** *Assume that  $M$  is a monoid of right- $O$ -type that is generated by  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  with  $\mathbf{a} \succ \mathbf{b} \succ \mathbf{c}$  and  $\mathbf{b}, \mathbf{c}$  satisfying some relation  $\mathbf{b} = \mathbf{c}v$  with no  $\mathbf{a}$  in  $v$ . Then, unless  $M$  is generated by  $\mathbf{b}$  and  $\mathbf{c}$ , there is no way to complete  $\mathbf{b} = \mathbf{c}v$  with a relation  $\mathbf{a} = \mathbf{b}u$  so as to obtain a presentation of  $M$ .*

*Proof.* For a contradiction, assume that  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{b}u, \mathbf{b} = \mathbf{c}v)$  is a presentation of  $M$ . If there is no  $\mathbf{a}$  in  $u$ , the assumption that  $\mathbf{a} = \mathbf{b}u$  is valid in  $M$  implies that  $\mathbf{a}$  belongs to the submonoid generated by  $\mathbf{b}$  and  $\mathbf{c}$ , so  $M$  must be generated by  $\mathbf{b}$  and  $\mathbf{c}$ .

Assume now that there is at least one  $\mathbf{a}$  in  $u$ . As  $\mathbf{a}$  does not occur in  $\mathbf{b} = \mathbf{c}v$ , a word containing  $\mathbf{a}$  cannot be equivalent to a word not containing  $\mathbf{a}$ . This implies that  $\mathbf{a}$  is preponderant in  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ . Indeed, assume that  $g, h$  belong to the submonoid of  $M$  generated by  $\mathbf{b}$  and  $\mathbf{c}$ . By the above remark,  $h\mathbf{a}g' = g$  is impossible, hence so is  $h\mathbf{a} \preceq g$ . As, by assumption,  $M$  is of right- $O$ -type, we deduce  $g \preceq h\mathbf{a}$ . Then Proposition 7.5 gives the result.  $\square$

So, for instance, no right-triangular presentation made of  $\mathbf{b} = \mathbf{c}\mathbf{b}\mathbf{c}$  (Klein bottle relation) or  $\mathbf{b} = \mathbf{c}\mathbf{b}^2\mathbf{c}$  (Dubrovina–Dubrovin braid relation) plus a relation of the form  $\mathbf{a} = \mathbf{b}...$  may define a monoid of right- $O$ -type. In the case of braids, we obtain the following general result.

**Corollary 7.7.** *Let  $B_n^\oplus$  be the submonoid of the braid group  $B_n$  generated by  $s_1 = \sigma_1 \cdots \sigma_{n-1}$ ,  $s_2 = (\sigma_2 \cdots \sigma_{n-1})^{-1}$ ,  $s_3 = \sigma_3 \cdots \sigma_{n-1}$ , ...,  $s_{n-1} = \sigma_{n-1}^{(-1)^n}$ . Then  $B_n^\oplus$  is a monoid of  $O$ -type that admits no right-triangular presentation in terms of  $s_1, \dots, s_{n-1}$  for  $n \geq 4$ .*

*Proof.* That  $B_n^\oplus$  is of  $O$ -type was established by Dubrovina–Dubrovin in [11]. Now, as a braid that admits an expression containing at least one  $\sigma_1$  and no  $\sigma_1^{-1}$  cannot admit an expression with no  $\sigma_1^{\pm 1}$  [9], the generator  $s_1$  is preponderant in  $\{s_1, \dots, s_{n-1}\}$ . Proposition 7.5 implies that  $B_n^\oplus$  admits no triangular presentation in terms of  $s_1, \dots, s_{n-1}$  for  $n \geq 4$ .  $\square$

One can indeed convert the standard presentation of the braid group  $B_n$  into a presentation in terms of the generators  $s_1, \dots, s_{n-1}$  of Corollary 7.7. For instance, writing  $\mathbf{a}, \mathbf{b}, \dots$  for  $s_1, s_2, \dots$ , one can check that  $B_4^\oplus$  admits the presentation

$$(7.1) \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{b}^2\mathbf{a}^2\mathbf{b}\mathbf{a}\mathbf{b}^2\mathbf{b}^2, \mathbf{b} = \mathbf{c}\mathbf{b}^2\mathbf{c}, \mathbf{a}\mathbf{b}\mathbf{c} = \mathbf{c}\mathbf{a}\mathbf{b}),$$

a triangular presentation augmented with a third, additional relation. But the triangular presentation made of the first two relations in (7.1) is not a presentation of  $B_4^\oplus$ , nor of any monoid of  $O$ -type either.

## 8. MORE EXAMPLES

In view of the negative results of Section 7, we might now wonder whether triangular presentations can be relevant at all in the case of more than two generators. Actually they can, and computer experiments again provide a number of right-triangular presentations defining

		right- $O$	left- $O$	
$\mathbf{a} = \mathbf{bab}$	$\mathbf{b} = \mathbf{caaaaac}$	YES	YES	$\Delta = \mathbf{a}^{10}$ central
$\mathbf{a} = \mathbf{bab}$	$\mathbf{b} = \mathbf{caaabac}$	YES	YES	$\Delta = \mathbf{a}^4$ central
$\mathbf{a} = \mathbf{bab}$	$\mathbf{b} = \mathbf{caaacac}$	YES		$\Delta = \mathbf{a}^6$ right-quasi-central, $\phi(\mathbf{a}) = \mathbf{a}$ , $\phi(\mathbf{b}) = \mathbf{b}$ , $\phi(\mathbf{c}) = \mathbf{cacab}((\mathbf{ac})^2\mathbf{ab})^2((\mathbf{ac})^2\mathbf{ab}(\mathbf{ac})^2)^2$
			NO	$\mathbf{a} = \dots\mathbf{a}(\mathbf{ac})(\mathbf{ac})$ : Lemma 6.2 with $u = \mathbf{ca}$ , $v = \varepsilon$
$\mathbf{a} = \mathbf{bab}$	$\mathbf{b} = \mathbf{cababac}$	YES	YES	$\Delta = \mathbf{a}^6$ central
$\mathbf{a} = \mathbf{bab}$	$\mathbf{b} = \mathbf{cabacac}$	YES		$\Delta = \mathbf{a}^2$ right-quasi-central, $\phi(\mathbf{a}) = \mathbf{a}$ , $\phi(\mathbf{b}) = \mathbf{b}$ , $\phi(\mathbf{c}) = \mathbf{c}(\mathbf{ac})^3$
			NO	$\mathbf{b} = \dots\mathbf{b}(\mathbf{ac})(\mathbf{ac})$ : Lemma 6.2 with $u = \mathbf{ca}$ , $v = \varepsilon$
$\mathbf{a} = \mathbf{bab}$	$\mathbf{b} = \mathbf{cacacac}$	YES	YES	$\Delta = \mathbf{a}^2$ central

TABLE 4. Three-generator monoids with a triangular presentation: examples with  $\mathbf{a} = \mathbf{bab}$  plus a relation of the form  $\mathbf{b} = \mathbf{ca}s_1\mathbf{a}s_2\mathbf{ac}$  with  $s_1, s_2$  in  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ .

monoids of (right)- $O$ -type. We shall focus below on triangular presentations consisting of the Klein bottle relation  $\mathbf{a} = \mathbf{bab}$  completed with another relation  $\mathbf{b} = \mathbf{cw}$ . Qualitative results are similar in the general case. As in the case of two generators, it turns out that Lemmas 6.1 and 6.2 cover a lot of presentations.

**Fact 8.1.** *Among the 1,090 presentations  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{bab}, \mathbf{b} = \mathbf{cw})$  with  $|w| \leq 6$ ,*  
 - 48 are eligible for Lemma 6.1, yielding a monoid of right- $O$ -type (42 are of  $O$ -type);  
 - 193 + 726 are eligible for Lemma 6.2(i) or (ii), yielding a monoid not of right- $O$ -type;

See Table 4 for some examples. It is natural that more presentations (123 in the context of Fact 8.1) escape Lemmas 6.1 and 6.2 than in the case of two generators because there exist analogs of Lemma 6.2 simultaneously involving two or more letters and leading to non-terminating reversings. We shall not go into detail here, and refer to Table 5, which displays the examples with shortest length.

As in Section 6, we conclude with the description of some infinite families. We skip those corresponding to Rows 5 to 7 in Table 1 (in the case of Row 6, the element  $\mathbf{a}^p\mathbf{b}^2$  is quasi-central for  $p \geq 2$  with  $\phi(\mathbf{a}) = \mathbf{a}^{p-1}\mathbf{ba}^{p-2}\mathbf{b}^3$ ,  $\phi(\mathbf{b}) = \mathbf{b}$ , and  $\phi(\mathbf{c}) = \mathbf{c}$ ), but consider the family of Row 8, which extends the torus knot groups of Row 1 and Proposition 6.4. Actually, we establish a general result for all “multi-toric” groups.

**Proposition 8.2.** *Let  $G$  be a group defined by a presentation of the form*

$$(8.1) \quad (x_1, x_2, \dots, x_\ell; x_1^{m_2+1} = x_2^{n_2+1}, x_2^{m_3+1} = x_3^{n_3+1}, \dots, x_{\ell-1}^{m_\ell+1} = x_\ell^{n_\ell+1}).$$

*Then  $G$  is orderable and there exists an isolated point in the space  $\text{LO}(G)$ .*

*Proof.* Owing to Proposition 5.5, it is sufficient to prove that admits a triangular presentation  $(S; R)$  such that there exists a central element  $\Delta$  in the monoid  $\langle S | R \rangle^+$ .

Starting from  $x_1, \dots, x_\ell$ , we inductively define elements  $\mathbf{a}_1, \dots, \mathbf{a}_\ell$  of  $G$  by

$$(8.2) \quad x_1 = \mathbf{a}_1, \quad x_2 = x_1^{m_2}\mathbf{a}_2, \quad x_3 = x_2^{m_3}x_1^{m_2}\mathbf{a}_3, \quad \dots, \quad x_\ell = x_{\ell-1}^{m_\ell} \dots x_2^{m_3}x_1^{m_2}\mathbf{a}_\ell.$$

Inductively, (8.2) implies for each  $i$  an equality  $x_i = W_i(\mathbf{a}_1, \dots, \mathbf{a}_i)$ , where  $W_i$  is some expression in  $\mathbf{a}_1, \dots, \mathbf{a}_i$ , namely  $W_1(\mathbf{a}_1) = \mathbf{a}_1$  and  $W_i(\mathbf{a}_1, \dots, \mathbf{a}_i) = F_{m_2} \circ \dots \circ F_{m_i}(\mathbf{a}_1)$  for  $i \geq 2$ , where  $F_m$  is the alphabetical homomorphism that maps  $\mathbf{a}_k$  to  $\mathbf{a}_1^m \mathbf{a}_{k+1}$  for each  $k$ . For instance, we find  $W_2(\mathbf{a}_1, \mathbf{a}_2) = \alpha_{m_2}(\mathbf{a}_1) = \mathbf{a}_1^{m_2}\mathbf{a}_2$ , and

$$W_3(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = F_{m_2}(F_{m_3}(\mathbf{a}_1)) = F_{m_2}(\mathbf{a}_1^{m_3}\mathbf{a}_2) = (\mathbf{a}_1^{m_2}\mathbf{a}_2)^{m_3}\mathbf{a}_1^{m_2}\mathbf{a}_3,$$

*etc.* Now we compute

$$x_{i-1}^{m_i+1} = x_{i-1}^{m_i} x_{i-1} = x_{i-1}^{m_i} x_{i-2}^{m_{i-1}} \cdots x_1^{m_2} \mathbf{a}_{i-1},$$

$$x_i^{n_i+1} = x_i x_i^{n_i} = x_{i-1}^{m_i} \cdots x_1^{m_2} \mathbf{a}_i x_i^{n_i} = x_{i-1}^{m_i} x_{i-2}^{m_{i-1}} \cdots x_1^{m_2} \mathbf{a}_i W_i(\mathbf{a}_1, \dots, \mathbf{a}_i)^{n_i}.$$

For every  $i$ , the relation  $x_{i-1}^{m_i+1} = x_i^{n_i+1}$  is valid in  $G$ , hence so is  $\mathbf{a}_{i-1} = \mathbf{a}_i W_i(\mathbf{a}_1, \dots, \mathbf{a}_i)^{n_i}$ .

Conversely, let  $(S, R)$  be defined by

$$(8.3) \quad S = \{\mathbf{a}_1, \dots, \mathbf{a}_\ell\}, \quad R = \{\mathbf{a}_{i-1} = \mathbf{a}_i W_i(\mathbf{a}_1, \dots, \mathbf{a}_i)^{n_i} \mid i = 2, \dots, \ell\}.$$

In the group  $\langle S \mid R \rangle$ , (8.2) defines elements  $x_1, \dots, x_\ell$ , and the above computation shows that these elements satisfy the relations  $x_{i-1}^{m_i+1} = x_i^{n_i+1}$  for every  $i$ . One deduces that the groups  $G$  and  $\langle S \mid R \rangle$  are isomorphic, *i.e.*, that  $(S; R)$  is a presentation of  $G$ .

Let  $M$  be the monoid  $\langle S \mid R \rangle^+$ . We shall now show the existence of a central element in  $M$ . Let  $m$  be a common multiple of  $m_2 + 1, \dots, m_\ell + 1$  (or, simply, a number such that  $m(n_2 + 1) \cdots (n_i + 1)$  is a multiple of  $(m_2 + 1) \cdots (m_{i+1} + 1)$  for each  $i$ ). Then, it is almost obvious that  $x_1^m$  is central in  $G$ . As already noted in Remark 6.6, this is not enough to directly claim that  $x_1^m$  is central in  $M$ .

Now, (8.2) non-ambiguously defines elements  $\dot{x}_1, \dots, \dot{x}_\ell$  in  $M$  (no inverse occurs), and, once again but now in  $M$ , the above computation shows that these elements satisfy, in  $M$ , the relations  $\dot{x}_{i-1}^{m_i+1} = \dot{x}_i^{n_i+1}$ . Then, by construction, the chosen power  $\mathbf{a}_1^m$  commutes with  $\dot{x}_1, \dots, \dot{x}_\ell$  in  $M$  and, as  $M$  is cancellative since the presentation is triangular, hence complete for right- and left-reversing, this implies that  $\mathbf{a}_1^m$  commutes with  $\mathbf{a}_1, \dots, \mathbf{a}_\ell$ , hence it is central in  $M$ .

Now, we are done: the presentation  $(S; R)$  is triangular, and, by construction, some power of  $\mathbf{a}_1$  is central in the monoid  $\langle S \mid R \rangle^+$ , *i.e.*, in  $M$ . Then Proposition 5.5 gives the expected results about the ordering of  $G$  associated with  $M$ .  $\square$

	right- $O$	left- $O$	
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{caac}$	YES	YES	$\Delta = (\mathbf{a}^2 \mathbf{c})^2$ right-quasi-central, $\phi(\mathbf{a}) = \mathbf{ab}^2$ , $\phi(\mathbf{b}) = \mathbf{b}$ , $\phi(\mathbf{c}) = \mathbf{c}$
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{cacc}$	NO		non-terminating reversing: $u \curvearrowright^{(4)} v^{-1} uv$ for $u = \mathbf{a}^{-1} \mathbf{cab}$ and $v = \mathbf{b}^2$
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{caabc}$	YES	NO	$\mathbf{a} = \dots \mathbf{a}(\mathbf{c})(\mathbf{c})$ : Lemma 6.2 with $u = \mathbf{c}$ , $v = \varepsilon$
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{caacc}$	YES	YES	$\Delta = \mathbf{a}^2 \mathbf{b}^2$ right- and left-quasi-central, $\phi(\mathbf{a}) = \mathbf{ab}^4$ , $\tilde{\phi}(\mathbf{a}) = \mathbf{b}^4 \mathbf{a}$ , $\phi(\mathbf{b}) = \tilde{\phi}(\mathbf{b}) = \mathbf{b}$ , $\phi(\mathbf{c}) = \tilde{\phi}(\mathbf{c}) = \mathbf{c}$
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{cabcc}$	NO	NO	$\Delta = (\mathbf{a}^2 \mathbf{c}^2)^2$ right-quasi-central, $\phi(\mathbf{a}) = \mathbf{abac}^2 \mathbf{abc}^2 \mathbf{a}^2 \mathbf{c}^2$ , $\phi(\mathbf{b}) = \mathbf{abac}^4 \mathbf{a}^2 \mathbf{c}^2$ , $\phi(\mathbf{c}) = \mathbf{abac}^4$
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{cacbc}$	NO	NO	$\mathbf{a} = \dots \mathbf{a}(\mathbf{c})(\mathbf{c})$ : Lemma 6.2 with $u = \mathbf{c}$ , $v = \varepsilon$
$\mathbf{a} = \mathbf{bab}$ $\mathbf{b} = \mathbf{cacc}$	NO	NO	non-terminating reversing: $u \curvearrowright^{(8)} v^{-1} uv$ for $u = \mathbf{c}^{-1} \mathbf{b}^{-1} \mathbf{c}^2 \mathbf{ab}^2$ and $v = \mathbf{c}^2$
		NO	$\mathbf{b} = \dots \mathbf{b}(\mathbf{c})(\mathbf{c})$ : Lemma 6.2 with $u = \mathbf{c}$ , $v = \varepsilon$
		NO	non-terminating reversing: $u \curvearrowright^{(8)} v^{-1} uv$ for $u = \mathbf{b}^{-1} \mathbf{cbcab}$ and $v = (\mathbf{bc})^2$
		NO	$\mathbf{a} = \dots \mathbf{a}(\mathbf{c})\mathbf{b}(\mathbf{c})$ : Lemma 6.2 with $u = \mathbf{c}$ , $v = \mathbf{b}$
		NO	non-terminating reversing: $u \curvearrowright^{(8)} v^{-1} uv$ for $u = \mathbf{b}^{-1} \mathbf{c}^3 \mathbf{ab}$ and $v = \mathbf{bc}^3 \mathbf{abc}^3$
		NO	$\mathbf{a} = \dots \mathbf{a}(\mathbf{c})(\mathbf{c})$ : Lemma 6.2 with $u = \mathbf{c}$ , $v = \varepsilon$

TABLE 5. The nine presentations of the form  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{bab}, \mathbf{b} = \mathbf{cw})$  with  $|w| \leq 4$  that are not covered by Lemma 6.1 or Lemma 6.2: all cases have been successfully addressed. In the right- $O$ -type cases, the right-quasi-central element turns out to be  $[w^2]^+$  where the relation is  $\mathbf{b} = \mathbf{cw}$  but this need not be true in general.

**Example 8.3.** If we assume  $m_2 = n_2 = \dots = m_\ell = n_\ell = p$ , the group  $G$  admits the presentation  $(x_1, \dots, x_\ell; x_1^{p+1} = x_2^{p+1} = \dots = x_\ell^{p+1})$ , and the result applies with  $\Delta = x_1^{p+1}$ . The positive cone of the associated isolated ordering is then defined by the presentation (8.3), whose relations, in the current case, take the form (as usual, we write  $\mathbf{a}, \mathbf{b}, \dots$  for  $\mathbf{a}_1, \mathbf{a}_2, \dots$ )

$$\mathbf{a} = \mathbf{b}(\mathbf{a}^p\mathbf{b})^p, \quad \mathbf{b} = \mathbf{c}((\mathbf{a}^p\mathbf{b})^p\mathbf{a}^p\mathbf{c})^p, \quad \mathbf{c} = \mathbf{d}(((\mathbf{a}^p\mathbf{b})^p\mathbf{a}^p\mathbf{c})^p(\mathbf{a}^p\mathbf{b})^p\mathbf{a}^p\mathbf{d})^p, \text{ etc.}$$

See Table 1 for other particular cases.

## 9. FURTHER QUESTIONS

Apart from the generic question of better understanding all the above mentioned monoids of  $O$ -type as well as the associated ordered groups, we mention a few more specific problems.

**Existence of a quasi-central element.** The striking point in Facts 6.3 and 8.1 is that all identified monoids of right- $O$ -type involve a right-quasi-central element.

**Question 9.1.** *Does every monoid of right- $O$ -type with a finite right-triangular presentation necessarily admit a right-quasi-central element whose left-divisors generate the monoid (hence a right-Garside element)?*

(The precision about divisors generating the monoid is necessary in Proposition 5.5: in every case, 1 is a central element.) As the involved right-quasi-central elements seem to be of several different types (often a power of the largest generator, but not always), and as so are the various possible obstructions to being of right- $O$ -type, a general answer seems not obvious. Note that a given monoid admits in general many right-Garside elements: unless in a Noetherian case, there need not exist a smallest such element. For instance, in the case of the Klein bottle monoid  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{bab} \rangle^+$ , the element  $\mathbf{a}^2$  is right-Garside (hence right-quasi-central), but so are all elements  $\mathbf{a}^2\mathbf{b}^i$  with  $i \geq 1$ , which are smaller w.r.t. left-divisibility.

Candidates for a negative answer to Question 9.1 could be the monoid

$$\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{ba}^4\mathbf{ba}^2\mathbf{ba}^4\mathbf{b} \rangle^+ \quad (\text{or, equivalently, } \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{ba}^2\mathbf{b}, \mathbf{b} = \mathbf{ca}^4\mathbf{c} \rangle^+),$$

for which neither a quasi-central element nor a non-terminating reversing was found so far.

**Existence of a triangular presentation.** We saw in Section 7 that there exist monoids of right- $O$ -type that admit no right-triangular presentation. However, the argument of Proposition 7.5 requires the existence of at least three generators.

**Question 9.2.** *Does every two-generator monoid of right- $O$ -type admit a right-triangular presentation?*

If  $M$  is a monoid of right- $O$ -type generated by two elements  $\mathbf{a}, \mathbf{b}$  with  $\mathbf{a} \succ \mathbf{b}$ , some triangular relation  $\mathbf{a} = \mathbf{bw}$  must be satisfied in  $M$ . However, as observed in Example 7.3, the choice of  $w$  is not unique and, more generally, nothing excludes that other, possibly non-triangular relations connect  $\mathbf{a}$  and  $\mathbf{b}$  in  $M$ : in the Klein bottle monoid, the relation  $\mathbf{a} = \mathbf{b}(\mathbf{bab}^2)$  holds, but so does for instance the relation  $\mathbf{ab} = \mathbf{bab}^2$ , which is not a consequence of the former. Question 9.2 asks in particular whether there exists a preferred choice for the above word  $w$ .

**Symmetry of presentation.** In the case of two generators, all identified monoids of  $O$ -type admit presentations in which the right-hand term of the relation is a palindrome, *i.e.*, is invariant under reversing the order of letters.

**Question 9.3.** *Is every two-generator triangular presentation defining a monoid of  $O$ -type necessarily palindromic?*

In the case of three generators and more, non-palindromic presentations may define monoids of  $O$ -type with a central element: for instance,  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{bab}, \mathbf{b} = \mathbf{ca}^p\mathbf{bc})$ , which is not palindromic, defines for  $p$  odd a monoid of  $O$ -type in which  $\mathbf{a}^{2p}$  is central. The same result holds for  $p$  even, now with  $\mathbf{a}^p\mathbf{b}^2$  quasi-central, but the case is less convincing as the presentation is then equivalent to the palindromic presentation  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{bab}, \mathbf{b} = \mathbf{c}(\mathbf{ba})^p\mathbf{bc})$ . On the other hand, although the latter presentation is palindromic, it gives rise to a quasi-central element only, with  $\phi(\mathbf{a}) = \mathbf{ab}^4$ . A similar phenomenon occurs with the palindromic presentation  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{bcacb}, \mathbf{b} = \mathbf{cabac})$ , where  $\mathbf{a}^2$  is quasi-central with  $\phi(\mathbf{b}) = (\mathbf{ca})^2\mathbf{b}$ .

Returning to the case of two generators, we can also raise the converse of Question 9.3.

**Question 9.4.** *Does every two-generator triangular presentation not discarded by Lemma 6.2 and its counterpart necessarily define a monoid of  $O$ -type?*

Already mentioned for Question 9.1,  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{ba}^4\mathbf{ba}^2\mathbf{ba}^4\mathbf{b})$  seems critical.

**Remark 9.5.** All the examples mentioned so far are weakly symmetric in that the left- and right-graphs of the considered presentations coincide: all relations have the form  $s' = s\dots s$ . This is not necessary. For instance, the monoid  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a} = \mathbf{bac}, \mathbf{b} = \mathbf{cba} \rangle^+$  is of  $O$ -type, with  $(\mathbf{ab})^3$  central, but we have  $\mathbf{c} \preccurlyeq \mathbf{b} \preccurlyeq \mathbf{a}$  and  $\mathbf{b} \succcurlyeq \mathbf{a} \succcurlyeq \mathbf{c}$ , *i.e.*, the left and the right ordering of the generators is not the same. The above monoid  $M$  turns out to be the 3-strand braid monoid  $B_3^\oplus$ . Indeed,  $M$  can be realized inside  $B_3^\oplus$  by putting  $\mathbf{a} = s_1^2s_2^3 (= \sigma_2\sigma_1\sigma_2^{-1})$ ,  $\mathbf{b} = s_1s_2 (= \sigma_1)$ ,  $\mathbf{c} = s_2 (= \sigma_2^{-1})$ , whereas  $B_3^\oplus$  can be realized inside  $M$  by putting  $s_1 = \mathbf{cb}^2\mathbf{a}$  and  $s_2 = \mathbf{c}$ . Thus the associated ordering on  $B_3$  is, once again, the Dubrovina–Dubrovin order.

**Complexity of reversing.** The existence of a right-quasi-central element implies the existence of common right-multiples, hence, in the context of a presentation that is complete for right-reversing, the termination of every right-reversing. However, the argument gives no complexity upper bound, at least no polynomial bound.

**Example 9.6.** Consider the presentation  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{bab}^{r+1})$  of the Baumslag–Solitar group  $\text{BS}(r+1, -1)$ . For every  $n$ , the equality  $\mathbf{ba}^n = \mathbf{ab}^{(r+1)^n}$  holds in  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a} = \mathbf{bab}^{r+1} \rangle^+$  and the signed word  $\mathbf{a}^{-n}\mathbf{ba}^n$  reverses to the word  $\mathbf{b}^{(r+1)^n}$ , whose length is exponential in  $n$ . As every reversing step adds at most  $r$  letters, the number of steps needed to reverse the length  $2n+1$  word  $\mathbf{a}^{-n}\mathbf{ba}^n$  must be exponential in  $n$  (in this case, the exact number is  $((r+1)^n + rn - 1)/r$ ).

A similar exponential complexity may occur whenever there exists a right-quasi-central element such that the associated endomorphism duplicates some letter. It turns out that, in all involved examples, the opposite presentation does not define a monoid of right- $O$ -type: typically, the opposite of  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{bab}^{r+1})$  is the Baumslag–Solitar presentation  $\text{BS}(-1, r+1)$ , *i.e.*,  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{b}^{r+1}\mathbf{ab})$ , which defines a monoid that, by Lemma 6.2, is not of right- $O$ -type. Similarly, in the case of  $(\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{a} = \mathbf{bacb}, \mathbf{b} = \mathbf{cac})$ , the element  $\mathbf{a}^2$  is right-quasi-central with  $\phi(\mathbf{c}) = (\mathbf{cb})^2$ , leading to an exponential complexity for reversing, but, again, the opposite presentation defines a monoid that is not of right- $O$ -type as one finds  $u \curvearrowright^{(12)} v^{-1}uv$  for  $u = \mathbf{b}^{-1}\mathbf{c}^2\mathbf{ab}$  and  $v = \mathbf{c}^2$ . By contrast, such behaviours could not be found for monoids of  $O$ -type.

**Question 9.7.** *If a triangular presentation defines a monoid of  $O$ -type, does the associated reversing necessarily have a polynomial (quadratic) complexity?*

Note that the existence of a quasi-central element that is not central need not imply an exponential complexity. For instance, for the presentation  $(\mathbf{a}, \mathbf{b}; \mathbf{a} = \mathbf{ba}^2\mathbf{baba}^2\mathbf{b})$  of Proposition 6.7 with  $\Delta = (\mathbf{a}^2\mathbf{b})^2$ , we have  $\phi(\mathbf{a}) = \mathbf{a}(\mathbf{ba}^2\mathbf{b})^2$ , and the shortest expression of  $\phi(\mathbf{a})$  is longer by 8 letters than that of  $\mathbf{a}$ . However,  $\phi(\mathbf{a}^2) = \mathbf{a}^2$  holds in the monoid, and the right-reversing of  $\Delta^{-n}\mathbf{a}\Delta^n$  leads to a word of length linear in  $n$  (precisely  $8n+9$ ) in a

quadratic number of steps  $(8n^2 + 14n + 6)$ , supporting the conjecture that the associated reversing has a quadratic complexity in every case.

**Isolated points in the space of left-orderings, case of a direct limit.** Theorem 1, as well as Propositions 5.4–5.5, are valid both in the case of a finite and infinite presentation, thus leading to orderable groups with an explicitly presented positive cone. But the argument showing that the involved ordering is isolated in its space of orderings is valid only when the presentation is finite. However, as observed by C. Rivas [19], a non-finitely generated monoid may give rise to an isolated ordering, so it makes sense to raise

**Question 9.8.** *If  $(S, R)$  is an infinite triangular presentation defining a monoid of O-type, may the associated ordering be isolated in the space  $\text{LO}(\langle S | R \rangle)$ ?*

In the direction of a positive answer, it would be natural to address Question 9.8 in the context when the considered monoid is a direct limit of finitely generated monoids. The properties of subword reversing make this situation easy to recognize.

**Proposition 9.9.** *Assume that  $(S; R)$  is an infinite triangular presentation*

$$(9.1) \quad (\mathbf{a}_1, \mathbf{a}_2, \dots; \mathbf{a}_1 = \mathbf{a}_2 w_2 \mathbf{a}_2, \mathbf{a}_2 = \mathbf{a}_3 w_3 \mathbf{a}_3, \dots)$$

with  $w_i$  in  $\{\mathbf{a}_1, \dots, \mathbf{a}_i\}^*$  and, putting  $S_n = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  and  $R_n = \{\mathbf{a}_{i-1} = \mathbf{a}_i w_i \mathbf{a}_i \mid i \leq n\}$ , that the monoid  $\langle S_n | R_n \rangle^+$  is of O-type for every  $n$  (or, at least, for unboundedly many  $n$ ). Then  $\langle S | R \rangle^+$  is a direct limit of the monoids  $\langle S_n | R_n \rangle^+$ , it is of O-type, and  $\langle S | R \rangle^+ \setminus \{1\}$  is the positive cone of a left-invariant ordering on the group  $\langle S | R \rangle$ .

*Proof.* Assume  $n < m$ . Owing to the assumption about  $w_i$ , the presentations  $(S_n; R_n)$  and  $(S_m; R_m)$  are well defined and, by definition, they are right-triangular, so that, by Proposition 4.4,  $(S_n; \widehat{R}_n)$  and  $(S_m; \widehat{R}_m)$  are complete for right-reversing. Now assume that  $u, v$  are  $S_n$ -words. Then  $u$  and  $v$  represent the same element in  $\langle S_n | R_n \rangle^+$  (resp. in  $\langle S_m | R_m \rangle^+$ ) if and only if  $u^{-1}v$  is  $\widehat{R}_n$ -reversible (resp.  $\widehat{R}_m$ -reversible) to the empty word. By definition of reversing, the relations in  $R_m \setminus R_n$  are never involved in the reversing of  $u^{-1}v$ , so the latter two relations are both equivalent to  $u^{-1}v$  being  $R$ -reversible to  $\varepsilon$ . It follows that the identity on  $S_n$  induces an embedding of  $\langle S_n | R_n \rangle^+$  into  $\langle S_m | R_m \rangle^+$ . So  $\langle S_n | R_n \rangle^+$  identifies with the submonoid of  $\langle S_m | R_m \rangle^+$  generated by  $S_n$  and  $\langle S | R \rangle^+$  is then the direct limit, here the union, of all monoids  $\langle S_n | R_n \rangle^+$ .

It follows that  $\langle S | R \rangle^+$  is of O-type. Indeed, a direct limit of monoids of right-O-type is of right-O-type: any two elements of the limit belong to some monoid of the considered direct system, hence are comparable with respect to left-divisibility in that monoid, and therefore in the limit.  $\square$

The interest of Proposition 9.9 is to provide *local* conditions for recognizing a monoid of O-type: in order to show that the monoid  $\langle S | R \rangle^+$  is, say, of right-O-type, it is sufficient to exhibit for every  $n$  an element  $\Delta_n$  that is right-quasi-central in  $\langle S_n | R_n \rangle^+$ , which is less demanding than exhibiting a unique element  $\Delta$  that is right-quasi-central in the whole of  $\langle S | R \rangle^+$ . A typical example is the group

$$(9.2) \quad \langle x_1, x_2, \dots \mid x_1^2 = x_2^q, x_2^2 = x_3^q, \dots \rangle$$

considered in Proposition 8.2. For  $q = 2$ , the element  $\Delta = x_1^2$  is central in  $\langle S | R \rangle^+$ , and Theorem 1 implies that  $\langle S | R \rangle^+$  is of O-type. By contrast, for  $q$  odd, the element  $\Delta_n = x_1^{2^{n-2}}$  is central in  $\langle S_n | R_n \rangle^+$ , but not in  $\langle S_{n+1} | R_{n+1} \rangle^+$ . It follows that  $\langle S_n | R_n \rangle^+$  is of O-type for every  $n$ , and  $\langle S | R \rangle^+$  is of O-type by Proposition 9.9, but, in this case, no power of  $x_1$  is central in  $\langle S | R \rangle^+$ . It seems that the infinite “multi-toric” groups (9.2) are natural first candidates for addressing Question 9.8.

## REFERENCES

- [1] S.I. Adyan, *Defining relations and algorithmic problems for groups and semigroups*, Proc. Steklov Inst. Math., **85** (1966).
- [2] A.H. Clifford & G.B. Preston, *The algebraic Theory of Semigroups, vol. 1*, Amer. Math. Soc. Surveys **7**, (1961).
- [3] M. Dabkowska, M. Dabkowski, V. Harizanov, J. Przytycki, and M. Veve, *Compactness of the space of left orders*, J. Knot Theory Ramifications **16** (2007) 267–256.
- [4] P. Dehornoy, *Deux propriétés des groupes de tresses*, C. R. Acad. Sci. Paris **315** (1992) 633–638.
- [5] P. Dehornoy, *Groups with a complemented presentation*, J. Pure Appl. Algebra **116** (1997) 115–137.
- [6] P. Dehornoy, *Complete positive group presentations*, J. Algebra **268** (2003) 156–197.
- [7] P. Dehornoy, *The subword reversing method*, Intern. J. Alg. and Comput. **21** (2011) 71–118
- [8] P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, J. Michel, *Garside Theory*, in progress; <http://www.math.unicaen.fr/~garside/Garside.pdf>.
- [9] P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, *Ordering Braids*, Mathematical Surveys and Monographs vol. 148, Amer. Math. Soc. (2008).
- [10] P. Dehornoy & L. Paris, *Gaussian groups and Garside groups, two generalizations of Artin groups*, Proc. London Math. Soc. **79-3** (1999) 569–604.
- [11] T. Dubrovina and N. Dubrovin, *On braid groups*, Sb. Math. **192** (2001) 693–703.
- [12] T. Ito, *Dehornoy-like left-orderings and isolated left-orderings*, arXiv:1103.4669.
- [13] T. Ito, *Construction of isolated left-orderings via partially central cyclic amalgamation*, arXiv:1107.0545.
- [14] A.I. Kokorin, V.M. Kopyutov, and N.Ya. Medvedev, *Right-Ordered Groups*, Plenum Publishing Corporation (1996).
- [15] A. Navas, *On the dynamics of (left) orderable groups*, Ann. Inst. Fourier **60** (2010) 1685–1740.
- [16] A. Navas, *A remarkable family of left-ordered groups: central extensions of Hecke groups*, J. Algebra **328** (2011) 31–42.
- [17] M. Picantin, *Petits groupes gaussiens*, Thèse de doctorat, Université de Caen (2000).
- [18] J.H. Remmers, *On the geometry of semigroup presentations*, Advances in Math. **36** (1980) 283–296.
- [19] C. Rivas, *Left-orderings on free products of groups*, J. Algebra; 350; 2012; 318–329.
- [20] A.S. Sikora, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. **36** (2004) 519–526.
- [21] V. Tararin, *On the theory of right orderable groups*, Matem. Zametki **54** (1993) 96–98. Translation to english in Math. Notes 54 (1994), 833–834.

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