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Partitioning Harary graphs into connected subgraphs containing prescribed vertices

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Abstract

A graph G is *arbitrarily partitionable* (AP for short) if for every partition (τ_1, \dots, τ_p) of $|V(G)|$ there exists a partition (V_1, \dots, V_p) of $V(G)$ such that each V_i induces a connected subgraph of G with order τ_i . If, additionally, k of these subgraphs ($k \leq p$) each contains an arbitrary vertex of G prescribed beforehand, then G is *arbitrarily partitionable under k prescriptions* (AP+ k for short). Every AP+ k -graph on n vertices must be $(k+1)$ -connected, and have thus at least $\lceil \frac{n(k+1)}{2} \rceil$ edges. We show that there exist AP+ k -graphs on n vertices and $\lceil \frac{n(k+1)}{2} \rceil$ edges for every $k \geq 1$ and $n \geq k$.

Keywords: arbitrarily partitionable graph, partition under prescriptions, Harary graph.

1 Introduction

We denote by $V(G)$ and $E(G)$ the sets of vertices and edges, respectively, of a graph G . The *order* (resp. *size*) of G is the cardinality of the set $V(G)$ (resp. $E(G)$). If X is a subset of $V(G)$, then $G[X]$ denotes the subgraph of G induced by X .

In the late 1970s, Györi and Lovász independently proved the following well-known result.

Theorem 1 (Györi [4] and Lovász [6], independently) *If G is a k -connected graph, then, given k pairwise distinct vertices (v_1, \dots, v_k) of G and k positive integers (τ_1, \dots, τ_k) adding up to $|V(G)|$, there exists a partition (V_1, \dots, V_k) of $V(G)$ such that $v_i \in V_i$, the subgraph $G[V_i]$ is connected and $|V_i| = \tau_i$ for every $i \in [1, k]$.*

In this paper, we consider a more general partition problem resulting from the combination of the notion of *arbitrarily partitionable* graphs [1] with the constraint of prescribing a set of vertices from Theorem 1. Let G be a connected graph of order n . A sequence $\tau = (\tau_1, \dots, \tau_p)$ of positive integers is *admissible for G* if it performs a partition of n , that is if $\sum_{i=1}^p \tau_i = n$. If, additionally, we can partition $V(G)$ into p parts (V_1, \dots, V_p) such that each V_i induces a connected subgraph of G with order τ_i , then τ is *realizable in G* , and the partition (V_1, \dots, V_p) is a *realization of τ in G* . If every admissible sequence for G is also realizable in G , then G is *arbitrarily partitionable* (AP for short). The interested reader is referred to [1, 2, 7] for a review of some results on AP-graphs.

Now suppose that we still want to partition G into an arbitrary number, say p , of connected subgraphs G_1, \dots, G_p of prescribed orders, but in such a way that each connected subgraph

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$G_i \in \{G_1, \dots, G_k\}$, with $k \leq p$, contains some vertex v_i of G arbitrarily chosen. To model this additional requirement, the definition of AP-graphs can be strengthened as follows [3]. A k -prescription of G is a subset $P = (v_1, \dots, v_k)$ of k distinct vertices of G . We say that a sequence (τ_1, \dots, τ_p) with $p \geq k$ elements is *realizable in G under P* if there exists a realization (V_1, \dots, V_p) of τ in G such that for every $i \in [1, k]$, the vertex v_i belongs to V_i . Notice that we have adopted the convention that the sizes associated to the prescribed vertices are located at the beginning of the sequence. We now say that G is (p, k) -partitionable if every sequence admissible for G consisting of exactly p elements is realizable in G under every k -prescription. Finally, G is *arbitrarily partitionable under k prescriptions* (AP+ k for short) if G is (p, k) -partitionable for every $p \in [k, n]$.

According to these definitions, an AP+0-graph is an AP-graph. Stated differently, Theorem 1 says that every k -connected graph is (k, k) -partitionable. Hence, throughout this paper, we mainly consider (p, k) -partitions with $p > k$. For any $k \geq 1$, the set of complete graphs on at least k vertices is a trivial class of AP+ k -graphs, each of them having the largest possible number of edges. In this work, we will focus on *optimal AP+ k -graphs*, that is AP+ k -graphs having the least possible number of edges.

The study of AP+ k -graphs can be motivated by the following computer science problem [3]. Suppose that we have a network of n computing resources we want to share between an arbitrary number of users, say p , where the i^{th} user needs exactly τ_i resources. For the sake of performance, we also suppose that the sharing must not be done arbitrarily but in such a way that the following two conditions are fulfilled:

1. each resource is attributed to exactly one user,
2. two resources attributed to a user must be able to communicate within his subnetwork.

Clearly, the problem of satisfying our users with this specific demand is equivalent to the problem of seeking a realization of the sequence (τ_1, \dots, τ_p) in the graph modelling our network. Hence, if we want our network to be shareable for every demand, that is no matter what p and the τ_i 's are, then it must have an AP-graph topology. If, additionally, each of k of our users is allowed to request one specific resource to belong to his subnetwork, then our network must have an AP+ k -graph topology.

This paper is organized as follows. In Section 2, we introduce some notations and preliminary results which are useful to prove, in Sections 3 and 4, respectively, some results on the partitioning of powers of paths and cycles under many prescriptions. In Section 5, these results are then used to show that, for every $k \neq 2$, every $(k+1)$ -connected Harary graph is an optimal AP+ k -graph. We finally show, in Section 6, that 3-connected Harary graphs are not necessarily AP+2, and provide another class of optimal AP+2-graphs instead.

2 Definitions, notation, and preliminary results

A subgraph H of a graph G is a *spanning subgraph* of G if $V(H) = V(G)$. We will also say that G is *spanned by H* . Given a graph G and an integer $k \geq 1$, the k^{th} *power of G* , denoted by G^k , is the graph with the same vertex set as G , two of its vertices being adjacent if they are at distance at most k in G . Let P_n (resp. C_n) denote the path (resp. the cycle) on n vertices, and $\{v_0, \dots, v_{n-1}\}$ its set of vertices. For convenience, the vertices v_0 and v_{n-1} of P_n are called the *first vertex* and the *last vertex* of P_n , respectively. We use the same terminology to deal with the vertices of P_n^k (resp. C_n^k) according to its natural spanning path P_n (resp. spanning cycle C_n).

Let $k \geq 1$ and $n \geq k$ be any two integers. The k -connected Harary graph on n vertices, denoted by $H_{k,n}$, has vertex set $\{v_0, \dots, v_{n-1}\}$ and the following edges:

- if $k = 2r$ is even, then two vertices v_i and v_j are linked if and only if $i - r \leq j \leq i + r$;

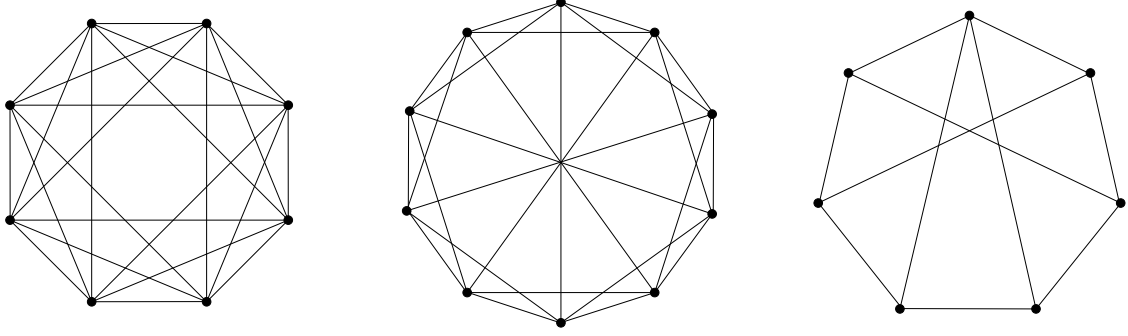


Figure 1: The Harary graphs $H_{6,8}$, $H_{5,10}$ and $H_{3,7}$

- if $k = 2r + 1$ is odd and n is even, then $H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in [0, \frac{n}{2} - 1]$;
- if $k = 2r + 1$ and n are odd, then $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lceil \frac{n}{2} \rceil}$, and then each vertex v_i to $v_{i+\lceil \frac{n}{2} \rceil}$ for every $i \in [1, \lfloor \frac{n}{2} \rfloor - 1]$;

where the subscripts are taken modulo n . Three examples of Harary graphs are given in Figure 1. When k is odd, the neighbours of a vertex v of $H_{k,n}$ which are at distance strictly more than k from v in its spanning C_n are called the *antipodal neighbours* of v . Clearly, v has at most two antipodal neighbours in $H_{k,n}$. In particular, v has exactly two antipodal neighbours if and only if $i = 0$, and k and n are both odd. A *diagonal edge* of $H_{k,n}$ is an edge linking two antipodal neighbours of $H_{k,n}$.

If G is a graph with a natural ordering of its vertices (like powers of paths and cycles, or Harary graphs), then, for any vertex v of G , we denote by v^+ (resp. v^-) the neighbour of v following v (resp. preceding v) in this ordering.

One of the first important property of AP-graphs is the following one.

Proposition 2 *If a graph G admits a spanning AP-subgraph (resp. a spanning AP+k-subgraph for some $k \geq 1$), then G is AP (resp. AP+k).*

Recall that a graph is *traceable* if it admits a Hamiltonian path. Since every path is obviously AP, the previous proposition implies that every traceable graph is AP.

We now point out the following property of AP+k-graphs, from which we deduce the size of an optimal AP+k-graph.

Observation 3 *Every AP+k-graph is $(k + 1)$ -connected. Therefore, an optimal AP+k-graph has $\lceil \frac{n(k+1)}{2} \rceil$ edges.*

Indeed, if there exist k vertices v_1, \dots, v_k such that $G - \{v_1, \dots, v_k\}$ is disconnected into q connected components on n_1, \dots, n_q vertices, respectively, with $n_1 \geq \dots \geq n_q$, then the sequence $(1, \dots, 1, n_1 + 1, (\sum_{i=2}^q n_i) - 1)$ cannot be realized in G under (v_1, \dots, v_k) . Since an edge-minimal $(k + 1)$ -connected graph has size $\lceil \frac{n(k+1)}{2} \rceil$, this is the size of an optimal AP+k-graph.

Baudon *et al.* [3] proved that, for every $k \geq 1$, there exist non complete AP+k-graphs, namely the powers of paths and the powers of cycles:

Theorem 4 (Baudon et al. [3]) *The graph P_n^{k+1} is AP+k for every $k \geq 0$ and $n \geq k + 1$. The graph C_n^k is AP+(2k - 1) for every $k \geq 1$ and $n \geq 2k$.*

Since the number of edges of P_n^{k+1} is $(k + 1)(n - (k + 1)) + \sum_{i=1}^k i$, this graph has more edges than an optimal AP+k-graph having the same number of vertices. However, C_n^k is $2k$ -regular and hence is an edge-minimal $2k$ -connected graph. According to Observation 3, it follows that the set of k^{th} powers of cycles is a set of optimal AP+(2k - 1)-graphs.

3 Partitioning P_n^k under strictly more than $k - 1$ prescriptions

In this section, we point out, in terms of prescribed vertices location, some situations in which it is possible to partition the graph P_n^k under strictly more than $k - 1$ prescriptions.

Lemma 5 ([3]) *Let $P = (v_{i_1}, \dots, v_{i_k})$ be a k -prescription of P_n^k with $k \geq 1$, $n \geq k$ and $0 \leq i_1 < \dots < i_k \leq n - 1$. If $i_1 = 0$ or $i_k = n - 1$, then every partition $\tau = (\tau_1, \dots, \tau_p)$ of n with $p \geq k$ elements is realizable in P_n^k under P .*

Lemma 6 *Let $P = (v_{i_1}, \dots, v_{i_{k+1}})$ be a $(k + 1)$ -prescription of $G = P_n^k$ with $k \geq 1$, $n \geq k$ and $0 \leq i_1 < \dots < i_{k+1} \leq n - 1$. If $i_1 = 0$ and $i_{k+1} = n - 1$, then every partition $\tau = (\tau_1, \dots, \tau_p)$ of n with $p \geq k + 1$ elements is realizable in G under P .*

Proof. We prove this claim by induction on k . For $k = 1$, the result is obvious; thus, we now suppose that $k \geq 2$ and that the claim holds for every k' such that $k' < k$. If $\tau_1 \leq i_2$, then a correct realization of τ in G under P is (V_1, \dots, V_p) where $V_1 = \{v_0, \dots, v_{\tau_1-1}\}$ and (V_2, \dots, V_p) is a realization of (τ_2, \dots, τ_p) in $G - V_1$ under the prescription $(v_{i_2}, \dots, v_{i_{k+1}})$. This realization necessarily exists according to Lemma 5 since $v_{i_{k+1}}$ is the last vertex of $G - V_1$.

Suppose now that $\tau_1 > i_2$. Observe that $[0, k - 1] - (\bigcup_{j=2}^k i_j \bmod k)$ is not empty, and let us denote by r one of these values. The subset V_1 of the realization is constructed as follows. It first contains all the vertices between v_0 and v_{i_2-1} , that is $V_1 \supseteq \{v_0, \dots, v_{i_2-1}\}$. We then add the vertex v_j to V_1 , where $j \in [i_2 + 1, i_2 + k - 1]$ is such that $j \equiv r \pmod k$. Finally, as long as $|V_1| < \tau_1$ and we do not reach v_{n-1} , we repeatedly add to V_1 the vertex at distance k on the right from the last one added to V_1 (v_{j+k}, v_{j+2k} , etc.). According to our choice of r , these vertices are not prescribed ones and, at any moment of this procedure, the subgraph $G - V_1$ is spanned by the $(k - 1)^{th}$ power of a path and the subgraph $G[V_1]$ is connected.

Thus, on the one hand, if $|V_1| = \tau_1$ holds after the procedure, then (V_1, \dots, V_p) is a correct realization of τ under P , where (V_2, \dots, V_p) is a realization of (τ_2, \dots, τ_p) in $G - V_1$ under the prescription $(v_{i_2}, \dots, v_{i_{k+1}})$ which necessarily exists by the induction hypothesis since v_{i_2} and $v_{i_{k+1}}$ are the first and last vertices of $G - V_1$.

On the other hand, if $|V_1| < \tau_1$ holds once the procedure is achieved, then each vertex from $V(G) - V_1$ has a neighbour in V_1 . Hence, we can obtain a correct realization $(V_1 \cup V_1', V_2, \dots, V_p)$ of τ in G under P , where (V_2, \dots, V_p, V_1') is a realization of $(\tau_2, \dots, \tau_p, \tau_1 - |V_1|)$ in $G - V_1$ under the prescription $(v_{i_2}, \dots, v_{i_{k+1}})$. Once again, such a realization necessarily exists in this subgraph according to the induction hypothesis. ■

Lemma 7 *Let $P = (v_{i_1}, \dots, v_{i_k})$ be a k -prescription of $G = P_n^k$ with $k \geq 1$, $n \geq k$ and $0 \leq i_1 < \dots < i_k \leq n - 1$. If $i_k \neq i_1 + k - 1$, then every partition $\tau = (\tau_1, \dots, \tau_p)$ of n with $p \geq k$ elements is realizable in P_n^k under P .*

Proof. If $x = \sum_{j=k+1}^p \tau_j \leq i_1$, then a correct realization of τ in G under P is (V_1, \dots, V_p) where (V_{k+1}, \dots, V_p) is an arbitrary realization of $(\tau_{k+1}, \dots, \tau_p)$ in the traceable subgraph $G[\{v_0, \dots, v_{x-1}\}]$, and (V_1, \dots, V_k) is a realization of (τ_1, \dots, τ_k) in $G - \{v_0, \dots, v_{x-1}\}$ under P obtained using Theorem 1.

Suppose now that $x > i_1$. On the one hand, if $\tau_1 > i_1$, then a correct realization of τ in G under P is $(V_1' \cup V_1'', V_2, \dots, V_p)$, where $V_1' = \{v_0, \dots, v_{i_1-1}\}$ and (V_1'', V_2, \dots, V_p) is a realization of $(\tau_1 - i_1, \tau_2, \dots, \tau_p)$ in $G - V_1'$ under P obtained via Lemma 5. On the other hand, if $\tau_1 \leq i_1$, then let V_1 be a subset of $\{v_0, \dots, v_{i_1}\}$ obtained as follows. First, we let $V_1 = \{v_{i_1}\}$ and we then repeatedly add to V_1 the vertex located at distance 2 on the left of the last vertex added to V_1 as long as $|V_1| < \tau_1$ and v_0 is not reached. If there is no vertex at distance 2 on the left of the last vertex added to V_1 , then we add to V_1 every remaining vertex from $\{v_0, \dots, v_{i_1-1}\} - V_1$ from left to right until V_1 has size τ_1 . Let $X = \{v_0, \dots, v_{i_1-1}\} - V_1$. Notice that, at the end of the previous procedure, $G[V_1]$ is connected, $G[X]$ is traceable, and $v_{i_1-1} \in X$. Now, if there exists $y \in [k+1, p]$

such that $\sum_{j=k+1}^y \tau_j = |X|$, then a correct realization of τ in G under P is (V_1, \dots, V_p) where (V_{k+1}, \dots, V_y) is a realization of $(\tau_{k+1}, \dots, \tau_y)$ in $G[X]$ and $(V_2, \dots, V_k, V_{y+1}, \dots, V_p)$ is a realization of $(\tau_2, \dots, \tau_k, \tau_{y+1}, \dots, \tau_p)$ in $G - \{v_1, \dots, v_{i_1}\}$ under $\{v_{i_2}, \dots, v_{i_k}\}$ obtained using Lemma 5.

Theorem 1. On the contrary, if for some y we have $\sum_{j=k+1}^{y-1} \tau_j < |X|$ and $\sum_{j=k+1}^y \tau_j > |X|$, then let $\tau'_y = |X| - \sum_{j=k+1}^{y-1} \tau_j$, $\tau''_y = \tau_y - \tau'_y$, and $v_\alpha \notin P$ be the nearest neighbour of $v_{i_{y-1}}$ located on the right of v_{i_1} . Such a vertex necessarily exists since otherwise this would imply that our k prescribed vertices are located consecutively along G . Moreover, v_α is the first vertex of $G - \{v_0, \dots, v_{i_1}\}$ if $v_{i_1}^+ \neq v_{i_2}$. We then obtain a desired realization $(V_1, \dots, V_{y-1}, V'_y \cup V''_y, V_{y+1}, \dots, V_p)$ of τ in G under P , where $(V'_y, V_{k+1}, \dots, V_{y-1})$ is a realization of $(\tau'_y, \tau_{k+1}, \dots, \tau_{y-1})$ in $G[X]$ under $(v_{i_{y-1}})$, and $(V_2, \dots, V_k, V''_y, V_{y+1}, \dots, V_p)$ is a realization of $(\tau_2, \dots, \tau_k, \tau''_y, \tau_{y+1}, \dots, \tau_p)$ in $G[\{v_{i_{y+1}}, \dots, v_{i_{n-1}}\}]$ under $(v_{i_2}, \dots, v_{i_k}, v_\alpha)$. Those two realizations exist according to Lemma 5. \blacksquare

4 Partitioning C_n^k under $2k$ prescriptions

We now prove, for any $k \geq 1$ and $n \geq 2k$, that C_n^k can be partitioned under $2k$ prescriptions when the prescribed vertices are "conveniently" located in C_n^k . Since the Harary graph $H_{2k+1, n}$ is spanned by C_n^k , these results imply that, in some situations, we can deduce a partitioning of $H_{2k+1, n}$ under $2k$ prescriptions using a spanning subgraph argument.

We first introduce some notations. Suppose that $G = C_n^k$ for some $k \geq 1$ and $n \geq 2k$, and that we want to realize the sequence $\tau = (\tau_0, \dots, \tau_{p-1})$ in G under $P = (v_{i_0}, \dots, v_{i_{2k-1}})$, with $p \geq 2k$ and $0 \leq i_0 < \dots < i_{2k-1} \leq n-1$. For every $j \in [0, 2k-1]$, we denote by d_j the value $i_j - i_{j-1} - 1$. In particular, we have $n = 2k + \sum_{j=0}^{2k-1} d_j$. Given two distinct integers $x, y \in [0, 2k-1]$ such that $x < y$, the *garden of the prescribed vertices* v_{i_x}, \dots, v_{i_y} in G is defined as $G_{x,y} = \{v_{i_x}^+, \dots, v_{i_y}^-\}$ and is the maximum subset of consecutive vertices of G containing no other prescribed vertices than v_{i_x}, \dots, v_{i_y} . In particular, $|G_{x,y}| = (y - x + 1) + \sum_{j=x}^{y-1} d_j$. We say that the prescribed vertices v_{i_x}, \dots, v_{i_y} are *saturated for τ in their garden* if $\sum_{j=x}^y \tau_j > |G_{x,y}|$, that is if the parts associated to the prescribed vertices v_{i_x}, \dots, v_{i_y} of a realization of τ in G under P must contain some vertices of $V(G) - G_{x,y}$.

Lemma 8 *Let $P = (v_{i_0}, \dots, v_{i_{2k-1}})$ be a $2k$ -prescription of $G = C_n^k$ with $k \geq 1$, $n \geq 2k$ and $0 \leq i_0 < \dots < i_{2k-1} \leq n-1$, and $\tau = (\tau_0, \dots, \tau_{p-1})$ be an admissible sequence for G with $p \geq 2k$. For any $x \in [0, 2k-1]$, if the prescribed vertices $v_{i_x}, \dots, v_{i_{x+k-1}}$ are saturated for τ in their garden, then neither the prescribed vertices $v_{i_{x+1}}, \dots, v_{i_{x+k}}$ nor $v_{i_{x+k+1}}, \dots, v_{i_x}$ are saturated in their respective garden.*

Proof. Let us consider, without loss of generality, that $x = 0$. If the prescribed vertices $v_{i_0}, \dots, v_{i_{k-1}}$ are saturated for τ in $G_{0, k-1}$, then we have:

$$\sum_{j=0}^{k-1} \tau_j > k + \sum_{j=0}^{k-1} d_j,$$

and

$$\sum_{j=k}^{2k-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j.$$

If the prescribed vertices v_{i_1}, \dots, v_{i_k} are also saturated for τ in $G_{1, k}$, then

$$\sum_{j=1}^k \tau_j > k + \sum_{j=1}^{k+1} d_j,$$

and

$$\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j < k + d_0 + \sum_{j=k+2}^{2k-1} d_j,$$

which implies that both $n_1 - n_{k+1} > d_1 - d_{k+2}$ and $n_1 - n_{k+1} < d_1 - d_{k+2}$ hold, a contradiction.

If we now suppose that $v_{i_{k+1}}, \dots, v_0$ are saturated for τ in $G_{k+1,0}$, then

$$\sum_{j=1}^k \tau_j < k + \sum_{j=2}^k d_j,$$

and

$$\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j > k + d_0 + d_1 + \sum_{j=k+1}^{2k-1} d_j,$$

and we thus get that both

$$\left(\sum_{j=1}^{k-1} \tau_j \right) - \left(\sum_{j=k+1}^{2k-1} \tau_j \right) > \left(\sum_{j=2}^k d_j \right) - \left(\sum_{j=k-1}^{2k-1} d_j \right)$$

and

$$\left(\sum_{j=1}^{k-1} \tau_j \right) - \left(\sum_{j=k+1}^{2k-1} \tau_j \right) < \left(\sum_{j=2}^k d_j \right) - \left(\sum_{j=k-1}^{2k-1} d_j \right)$$

hold, again a contradiction. ■

In what follows, a *maximal block* B of P in C_n^k is a maximal subset $B \subseteq P$ of consecutive vertices of C_n^k . In particular, if $\{v_{i_q}, \dots, v_{i_q+\alpha}\}$ is a maximal block of P in C_n^k , then neither $v_{i_q}^-$ nor $v_{i_q+\alpha}^+$ are prescribed vertices.

Lemma 9 *Let $P = (v_{i_0}, \dots, v_{i_{2k-1}})$ be a $2k$ -prescription of $G = C_n^k$ with $k \geq 1$, $n \geq 2k$ and $0 \leq i_1 < \dots < i_k \leq n - 1$. If there exists exactly one maximal block B of P in G of size at least k , then every partition $\tau = (\tau_0, \dots, \tau_{p-1})$ of n with $p \geq 2k$ elements is realizable in G under P .*

Proof. Let us suppose, without loss of generality, that $v_{i_0}, \dots, v_{i_{k-1}}$ belong to B . We distinguish two subcases depending on whether $v_{i_0}, \dots, v_{i_{k-1}}$ are saturated for τ in their garden or not.

Case 1: The prescribed vertices $v_{i_0}, \dots, v_{i_{k-1}}$ are not saturated for τ in $G_{0,k-1}$.

Let X be a subset of $\sum_{j=0}^{k-1} \tau_j$ consecutive vertices of $G_{0,k-1}$ such that $X \cap P = \{v_{i_0}, \dots, v_{i_{k-1}}\}$. According to Theorem 1, there exists a realization (V_0, \dots, V_{k-1}) of $(\tau_0, \dots, \tau_{k-1})$ in $G[X]$ under the prescription $(v_{i_0}, \dots, v_{i_{k-1}})$ since this subgraph is k -connected. Now observe that a realization (V_k, \dots, V_{p-1}) of $(\tau_k, \dots, \tau_{p-1})$ in $G - X$ under the prescription $(v_{i_k}, \dots, v_{i_{2k-1}})$ necessarily exists since $G - X$ is isomorphic to the k^{th} power of a path and we supposed that the prescribed vertices perform only one maximal block of size at least k in G . This implies that either v_{i_k} is the first vertex of $G - X$, $v_{i_{2k-1}}$ is the last vertex of $G - X$, or the prescribed vertices $v_{i_k}, \dots, v_{i_{2k-1}}$ are not consecutive in $G - X$. In the first two cases, the realization is obtained thanks to Lemma 5, and it follows from Lemma 7 in the third case. Finally, observe that (V_0, \dots, V_{p-1}) is a correct realization of τ in G under P .

Case 2: The prescribed vertices $v_{i_0}, \dots, v_{i_{k-1}}$ are saturated for τ in $G_{0,k-1}$.

By Lemma 8, we know that neither v_{i_1}, \dots, v_{i_k} nor $v_{i_{k+1}}, \dots, v_{i_0}$ are saturated for τ in $G_{1,k}$ and $G_{k+1,0}$, respectively. Hence, only two cases may occur.

Case 2.a. If there exists a subset X' of $\sum_{j=1}^k \tau_j$ consecutive vertices of $G_{1,k}$ such that $P \cap X' = \{v_{i_1}, \dots, v_{i_k}\}$, then, since v_{i_0} is the last vertex of $G - X'$, all the conditions are met to apply

the procedure used in *Case 1* with v_{i_1}, \dots, v_{i_k} and X' to get the desired realization. A similar realization can be obtained analogously if the assumption above holds for $v_{i_{k+1}}, \dots, v_{i_0}$ and their garden $G_{k+1,0}$.

Case 2.b. Suppose now that both $\sum_{j=1}^k \tau_j < k + \sum_{j=2}^k d_j$ and $\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j < k + d_0 + \sum_{j=k+2}^{2k-1} d_j$ hold. Let $X = \{v_{i_1}, \dots, v_{i_k}\}$. If there exists a $x \in [2k, p-1]$ such that $(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^x \tau_j) = |X|$ then we can partition G into two graphs $G[X]$ and $G - X$ and obtain a whole realization of τ in G under P by considering realizations of $(\tau_1, \dots, \tau_k, \tau_{2k}, \dots, \tau_x)$ in $G[X]$ under $(v_{i_1}, \dots, v_{i_k})$ and of $(\tau_0, \tau_{k+1}, \dots, \tau_{2k-1}, \tau_{x+1}, \dots, \tau_{p-1})$ in $G - X$ under $(v_{i_0}, v_{i_{k+1}}, \dots, v_{i_{2k-1}})$ which necessarily exist since the conditions of Lemma 5 are satisfied in both cases (v_{i_0} is the last vertex of $G - X$ and v_{i_1} is the first vertex of $G[X]$). Finally, (V_0, \dots, V_{p-1}) is a realization of τ in G under P .

If such a x does not exist, then let $x \in [2k, p-1]$ be such that $(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^{x-1} \tau_j) < |X|$ and $(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^x \tau_j) > |X|$, and let $\tau'_x = |X| - [(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^{x-1} \tau_j)]$ and $\tau''_x = \tau_x - \tau'_x$. Since we supposed that there is only one maximal block of P in G with size at least k and v_{i_0} and v_{i_1} are adjacent according to the natural ordering of $V(G)$, there exist $\alpha \in [i_1, i_k]$ and $\beta \in [i_k + 1, i_k + k - 1]$ such that v_α and v_β are adjacent and are not prescribed vertices (otherwise there would exist a second maximal block of P in G with size at least k since v_{i_1}, \dots, v_{i_k} are not saturated for τ in their garden). In particular, let us choose v_β in such a way that the distance between v_α and v_β is the smallest possible according to the natural ordering of $V(G)$. We get that either v_β or $v_{i_{k+1}}$ is the first vertex of $G - X$.

Finally, we obtain the desired realization of τ in G under P as follows. First, let $(V_1, \dots, V_k, V'_x, V_{2k}, \dots, V_{x-1})$ be a realization of $(\tau_1, \dots, \tau_k, \tau'_x, \tau_{2k}, \dots, \tau_{x-1})$ in $G[X]$ under $(v_{i_1}, \dots, v_{i_k}, v_\alpha)$. Then, let $(V_0, V_{k+1}, \dots, V_{2k-1}, V''_x, V_{x+1}, \dots, V_{p-1})$ be a realization of $(\tau_0, \tau_{k+1}, \dots, \tau_{2k-1}, \tau''_x, \tau_{x+1}, \dots, \tau_{p-1})$ in $G - X$ under $(v_{i_0}, v_{i_{k+1}}, \dots, v_{i_{2k-1}}, v_\beta)$. These two realizations necessarily exist according to Lemma 6. Moreover, the subgraph $G[V'_x \cup V''_x]$ is connected since $v_\alpha v_\beta \in E(G)$; therefore, we eventually get that $(V_0, \dots, V_{x-1}, V'_x \cup V''_x, V_{x+1}, \dots, V_{p-1})$ is a correct realization of τ in G under P . \blacksquare

Lemma 10 *Let $P = (v_{i_0}, \dots, v_{i_{2k-1}})$ be a $2k$ -prescription of $G = C_n^k$ with $k \geq 1$, $n \geq 2k$ and $0 \leq i_1 < \dots < i_k \leq n - 1$. If there does not exist a maximal block of P in G of size at least k , then every partition $\tau = (\tau_0, \dots, \tau_{p-1})$ of n with $p \geq 2k$ elements is realizable in G under P .*

Proof. We can suppose, without loss of generality, that neither $v_{i_0}, \dots, v_{i_{k-1}}$ nor $v_{i_k}, \dots, v_{i_{2k-1}}$ are saturated for τ in $G_{0,k-1}$ and $G_{k,2k-1}$, respectively. Indeed, if one of these conditions holds, then this cannot be simultaneously the case for v_{i_1}, \dots, v_{i_k} nor $v_{i_{k+1}}, \dots, v_{i_{2k-1}}, v_{i_0}$ in their respective garden according to Lemma 8; we could thus easily relabel our prescribed vertices and part sizes so that the above assertion holds.

We distinguish the following two subcases.

Case 1. If we can find a subset X of $\sum_{j=0}^{k-1} \tau_j$ consecutive vertices of $G_{0,k-1}$ such that $P \cap X = \{v_{i_0}, \dots, v_{i_{k-1}}\}$, then we can obtain a correct realization of τ in G under P in a same way as what we did in *Case 2.a* of Lemma 9. A similar argument holds if there exists a satisfying subset X included in the garden of $v_{i_k}, \dots, v_{i_{2k-1}}$.

Case 2. The last case which may happen is the one where both $\sum_{j=0}^{k-1} \tau_j < k + \sum_{j=1}^{k-1} d_j$ and $\sum_{j=k}^{2k-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j$ hold, and neither v_{i_0} and $v_{i_{2k-1}}$ nor $v_{i_{k-1}}$ and v_{i_k} are consecutive vertices of G (otherwise, we could use the arguments of *Case 2.b* of Lemma 9 to deduce a realization of τ in G under P). We can suppose that there does not exist $x \in [2k, p-1]$ such that $\sum_{j=0}^{k-1} \tau_j + \sum_{j=2k}^x \tau_j = k + \sum_{j=1}^{k-1} d_j$ or $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=2k}^x \tau_j = k + \sum_{j=k+1}^{2k-1} d_j$ for otherwise we could find a realization of τ in G under P using Lemma 7 since there is no maximal block of P in G with size at least k . We explain below how to deduce a realization of τ in G under P .

If $\tau_0 + \tau_{2k-1} \geq d_0 + 2$, then we can obtain a correct realization of τ in G under P as follows. Let $V'_0 = \{v_{i_0-1}, v_{i_0-2}, \dots, v_\alpha\}$ and $V'_{2k-1} = \{v_{i_{2k-1}+1}, v_{i_{2k-1}+2}, \dots, v_\alpha^-\}$

where $|V_0'| < \tau_0$, $|V_{2k-1}'| < \tau_{2k-1}$, and $\alpha \in [i_{2k-1} + 1, i_0 - 1]$. Then, consider a realization $(V_0'', V_1, \dots, V_{2k-2}, V_{2k-1}'', V_{2k}, \dots, V_{p-1})$ of the sequence $(\tau_0 - |V_0'|, \tau_1, \dots, \tau_{2k-2}, \tau_{2k-1} - |V_{2k-1}'|, \tau_{2k}, \dots, \tau_{p-1})$ in the subgraph $G[\{v_{i_0}, \dots, v_{i_{2k-1}}\}]$ under P obtained in the same way as what we did for *Case 2.b* (that is by doing as if v_{i_0} and $v_{i_{2k-1}}$ were consecutive vertices of G). Such a realization necessarily exists since we supposed that there is no maximal block of P in G with size at least k . Finally, observe that the realization $(V_0' \cup V_0'', V_1, \dots, V_{2k-2}, V_{2k-1}' \cup V_{2k-1}'', V_{2k}, \dots, V_{p-1})$ is a correct realization of τ in G under P .

Now, if $\tau_0 + \tau_{2k-1} < d_0 + 2$, then $\tau_0 - 1 \geq \lfloor \frac{d_0(k-1)}{k} \rfloor$ and $\tau_{2k-1} - 1 \geq \lfloor \frac{d_0(k-1)}{k} \rfloor$ cannot hold simultaneously since otherwise we would get a contradiction. Let $Z = \{v_{i_{2k-1}}^+, \dots, v_{i_0}^-\}$ and suppose that $\tau_0 - 1 < \lfloor \frac{d_0(k-1)}{k} \rfloor$; thus, there exists a subset V_0 of $Z \cup \{v_{i_0}\}$ such that $G[V_0]$ is connected on τ_0 vertices, V_0 contains the vertex v_{i_0} but does not contain neither $v_{i_{2k-1}}^+$ nor $v_{i_0}^-$, and the subgraph $G[Z - V_0]$ is traceable.

Once again, we can suppose that there does not exist $x \in [2k, p-1]$ such that $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^x \tau_j = |G_{1,k-1}|$ since otherwise we could easily deduce a realization of τ in G under P using Lemmas 5 and 6 (recall that $v_{i_k}, \dots, v_{i_{2k-1}}$ are not saturated for τ in their garden). Thus, let $x \in [2k, p-1]$ be such that $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^{x-1} \tau_j < |G_{1,k-1}|$ and $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^x \tau_j > |G_{1,k-1}|$, and let $\tau'_x = |G_{1,k-1}| - \sum_{j=1}^{k-1} \tau_j - \sum_{j=2k}^{x-1} \tau_j$ and $\tau''_x = \tau_x - \tau'_x$. For the sake of the proof, let us also denote τ_p instead of τ''_x . Since $G_{1,k-1}$ only includes $k-1$ prescribed vertices, then we can obtain a realization $(V_1, \dots, V_{k-1}, V'_x, V_{2k}, \dots, V_{x-1})$ of $(\tau_1, \dots, \tau_{k-1}, \tau'_x, \tau_{2k}, \dots, \tau_{x-1})$ in $G[G_{1,k-1}]$ under $(v_{i_1}, \dots, v_{i_{k-1}}, v_\alpha)$ where v_α is a non-prescribed vertex of $G_{1,k-1}$ neighbouring $v_{i_0}^-$. Such a vertex necessarily exists since there are no maximal block of P of size at least k in G ; in particular, choose $v_\alpha = v_{i_0}^+$ if $v_{i_1} \neq v_{i_0}^+$. The previous realization exists by Lemma 5.

For the same reasons as above, let us suppose that there does not exist $y \in [x+1, p]$ such that $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^y \tau_j = k + \sum_{j=k+1}^{2k-1} d_j$. Then denote by $y \in [x+1, p]$ the least integer such that $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^{y-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j$ and $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^y \tau_j > k + \sum_{j=k+1}^{2k-1} d_j$. Next, let us split τ_y into two elements τ'_y and τ''_y as above, and use the arguments above once again to obtain a realization $(V_k, \dots, V_{2k-1}, V'_y, V_{x+1}, \dots, V_{y-1})$ of $(\tau_k, \dots, \tau_{2k-1}, \tau'_y, \tau_{x+1}, \dots, \tau_{y-1})$ in $G[\{v_{i_k}, \dots, v_{i_{2k-1}}\}]$ under $(v_{i_k}, \dots, v_{i_{2k-1}}, v_\beta)$ where v_β is a non-prescribed vertex of $\{v_{i_k}, \dots, v_{i_{2k-1}}\}$ neighbouring $v_{i_{2k-1}}^+$. This realization exists by Lemma 6 since v_{i_k} and $v_{i_{2k-1}}$ are the first and last vertices of $G[\{v_{i_k}, \dots, v_{i_{2k-1}}\}]$, respectively.

On the one hand, if $y = p$, then it means that the element τ_x was split into three elements τ'_x , τ'_y and τ''_y . In this case, the realization $(V_0, \dots, V_{x-1}, V'_x \cup V'_y \cup Z, V_{y+1}, \dots, V_{p-1})$ is a correct realization of τ in G under P . On the other hand, if $y < p$, then, since $G[Z - V_0]$ is spanned by a path whose first and last vertices are $v_{i_{2k-1}}^+$ and $v_{i_0}^-$, respectively, by Lemma 6 there exists a realization $(V''_x, V''_y, V_{y+1}, \dots, V_{p-1})$ of $(\tau''_x, \tau''_y, \tau_{y+1}, \dots, \tau_{p-1})$ in $G[Z - V_0]$ under $(v_{i_0}^-, v_{i_{2k-1}}^+)$. By construction, $G[V'_x \cup V''_x]$ and $G[V'_y \cup V''_y]$ both induce a connected subgraph of G . It follows that $(V_0, \dots, V_{x-1}, V'_x \cup V''_x, V_{x+1}, \dots, V_{y-1}, V'_y \cup V''_y, V_{y+1}, \dots, V_{p-1})$ is a realization of τ in G under P . ■

5 Partitioning Harary graphs under prescriptions

Harary graphs are known to be graphs having the smallest possible size regarding their connectivity [5]. They trivially have the property of being AP since they are Hamiltonian; thus, one could wonder about the maximum number of prescribed vertices we can allow while partitioning them. In particular, can we always prescribe k vertices when partitioning a $(k+1)$ -connected Harary graph? If this is the case, then such a graph will also be an optimal AP+ k -graph.

We investigate, in this section, the question for the three different constructions of Harary graphs, according to the parities of both n and k .

5.1 Construction 1: k is even

For every $k \geq 2$ and $n \geq 2k$ such that k is even, $H_{k,n}$ is isomorphic to $C_n^{\frac{k}{2}}$ which is $AP+(k-1)$ according to Theorem 4. Thus, the following result derives directly.

Corollary 11 *For every even $k \geq 2$ and $n \geq 2k$, the Harary graph $H_{k,n}$ is $AP+(k-1)$.*

5.2 Construction 2: k is odd and n is even

Let $k \geq 2$ and $n \geq 2$ be two integers such that k is odd and n is even. By construction, $H_{2k+1,n}$ is spanned by $H_{2k,n}$ and thus is $AP+(2k-1)$ according to Corollary 11. Although this number of prescribed vertices is good since $H_{2k+1,n}$ is $(2k+1)$ -connected, the connectivity of this graph suggests that we could up to one more prescribed vertex while partitioning it (see Observation 3).

Before proving that $H_{2k+1,n}$ is actually $AP+2k$, we first introduce the following lemma which deals with the traceability of a graph composed by two linked squares of paths.

Lemma 12 *If G is a graph such that $V(G) = V_1 \cup V_2$, the subgraphs $G[V_1]$ and $G[V_2]$ are both spanned by the square of a path, and there exists an edge joining one vertex of V_1 and one of V_2 , then G is traceable.*

Proof. Let v_1, \dots, v_{n_1} and u_1, \dots, u_{n_2} denote the vertices of $G[V_1]$ and $G[V_2]$, from left to right, and v_α and u_β be two vertices of G such that $v_\alpha u_\beta \in E(G)$. Consider the following subpaths of G :

- $P = v_1 v_2 \dots v_{\alpha-1}$;
- $Q = \begin{cases} v_{\alpha+1} v_{\alpha+3} \dots v_{n_1-1} v_{n_1} v_{n_1-2} v_{n_1-4} \dots v_{\alpha+2} & \text{if } n_1 - \alpha \text{ is even,} \\ v_{\alpha+1} v_{\alpha+3} \dots v_{n_1} v_{n_1-1} v_{n_1-3} \dots v_{\alpha+2} & \text{otherwise;} \end{cases}$
- $R = \begin{cases} u_{\beta+2} u_{\beta+4} \dots u_{n_2} u_{n_2-1} u_{n_2-3} \dots u_{\beta+1} & \text{if } n_2 - \beta \text{ is even,} \\ u_{\beta+2} u_{\beta+4} \dots u_{n_2-1} u_{n_2} u_{n_2-2} u_{n_2-4} \dots u_{\beta+1} & \text{otherwise;} \end{cases}$
- $S = u_{\beta-1} u_{\beta-2} \dots u_1$.

It is then easy to check that $PQv_\alpha u_\beta RS$ is a Hamiltonian path of G . ■

We now prove the main result of this section.

Theorem 13 *For every odd $k \geq 2$ and even $n \geq 2k+1$, the Harary graph $H_{2k+1,n}$ is $AP+2k$.*

Proof. Let $G = H_{2k+1,n}$ be the $(2k+1)$ -connected Harary graph on n vertices. We prove that every partition $\tau = (\tau_0, \dots, \tau_{p-1})$ of n with $p \geq 2k+1$ elements is realizable in G under every prescription $P = (v_{i_0}, \dots, v_{i_{2k-1}})$ with $0 \leq i_0 < \dots < i_{2k-1} \leq n-1$. We distinguish three cases depending on whether there exists a realization of τ in C_n^k under P or not. Since C_n^k is a spanning subgraph of G , such a realization would also be a realization of τ in G under P .

Case 1. First observe that if there exists exactly one maximal block of P in G with size at least k , then we can obtain a correct realization of τ in G under P by considering a realization of τ in C_n^k under P . Such a realization necessarily exists according to Lemma 9.

Case 2. Similarly, if there does not exist a maximal block of P in G with size at least k , then we can use Lemma 10 to deduce a realization of τ in C_n^k under P . This realization is also a realization of τ in G under P .

Case 3. The last case we have to consider is the one where there exist two maximal blocks B_1 and B_2 of P in G with size exactly k . In this case, the prescribed vertices disconnect the graph into two components linked by some diagonal edges. Indeed, without loss of generality,

let us suppose that $B_1 = \{v_{i_0}, \dots, v_{i_{k-1}}\}$ and $B_2 = \{v_{i_k}, \dots, v_{i_{2k-1}}\}$; necessarily, the antipodal neighbours of $v_{i_0}^-$ and $v_{i_{k-1}}^+$ do not both belong to P since otherwise there would exist a maximal block of P in G with size at least $k + 2$. Let us denote by v_α and v_β two antipodal neighbours of G such that $v_\alpha, v_\beta \notin B_1 \cup B_2$, $\alpha \in [i_{k-1} + 1, i_k - 1]$ and $\beta \in [i_{2k-1} + 1, i_0 - 1]$. Besides, let $a_1 = \alpha - i_{k-1} - 1$, $a_2 = i_k - \alpha - 1$, $a_3 = i_0 - \beta - 1$ and $a_4 = \beta - i_{2k-1} - 1$ denote the number of consecutive vertices between B_1 , B_2 and the two vertices v_α and v_β according to the natural ordering of $V(G)$.

If $\sum_{j=0}^{k-1} \tau_j \leq a_1 + a_3 + k$ and $\sum_{j=k}^{2k-1} \tau_j \leq a_2 + a_4 + k$, then we can find two subsets X and Y of consecutive vertices of G such that $|X| = \sum_{j=0}^{k-1} \tau_j$, $|Y| = \sum_{j=k}^{2k-1} \tau_j$, $\{v_{i_0}, \dots, v_{i_{k-1}}\} \subseteq X$, $\{v_{i_k}, \dots, v_{i_{2k-1}}\} \subseteq Y$, and $v_\alpha, v_\beta \notin X \cup Y$. Since $G[X]$ and $G[Y]$ are both isomorphic to the k^{th} power of a path, then using Theorem 1 we can deduce two realizations (V_0, \dots, V_{k-1}) and (V_k, \dots, V_{2k-1}) of $(\tau_0, \dots, \tau_{k-1})$ and $(\tau_k, \dots, \tau_{2k-1})$ in $G[X]$ and $G[Y]$ under the prescriptions $(v_{i_0}, \dots, v_{i_{k-1}})$ and $(v_{i_k}, \dots, v_{i_{2k-1}})$. Moreover, since $k \geq 2$, $G - (X, Y)$ is traceable according to Lemma 12, there exists a realization (V_{2k}, \dots, V_{p-1}) of $(\tau_{2k}, \dots, \tau_{p-1})$ in $G - (X, Y)$. Finally, (V_0, \dots, V_{p-1}) is a correct realization of τ in G under P .

Let us now suppose that, without loss of generality, $\sum_{j=0}^{k-1} \tau_j > a_1 + a_3 + k$ and $\sum_{j=k}^{2k-1} \tau_j < a_2 + a_4 + k$. If $\sum_{j=0}^{2k-1} \tau_j \geq a_1 + a_2 + 2k + 1$, then we can find two subsets of consecutive vertices $X, Y \subseteq V(G)$ such that, for some $\gamma \in [i_{k-1} + 1, i_k - 1]$, $v_\gamma \in X$, $v_\gamma^+ \in Y$, $\{v_{i_0}, \dots, v_{i_{k-1}}\} \subseteq X$, $\{v_{i_k}, \dots, v_{i_{2k-1}}\} \subseteq Y$, $|X| = \sum_{j=0}^{k-1} \tau_j$ and $|Y| = \sum_{j=k}^{2k-1} \tau_j$. By Theorem 1, we know that there exists a realization (V_0, \dots, V_{k-1}) of $(\tau_0, \dots, \tau_{k-1})$ in $G[X]$ under the prescription $(v_{i_0}, \dots, v_{i_{k-1}})$, and a realization (V_k, \dots, V_{2k-1}) of $(\tau_k, \dots, \tau_{2k-1})$ in $G[Y]$ under $(v_{i_k}, \dots, v_{i_{2k-1}})$. Finally, $G - (X, Y)$ is isomorphic to the k^{th} power of a path, and thus there exists a realization (V_{2k}, \dots, V_p) of the remaining sequence $(\tau_{2k}, \dots, \tau_{p-1})$ in it. We get that (V_0, \dots, V_{p-1}) is a realization of τ in G under P .

If it is not possible to choose the subsets X and Y in such a way that they have two neighbouring consecutive vertices along the arc of size $a_1 + a_2 + 1$ of G , then we can necessarily find two such subsets along the arc of size $a_3 + a_4 + 1$ of G . Indeed, in such a situation we have $\sum_{j=0}^{k-1} \tau_j > a_1 + a_3 + k$ by hypothesis, but $\sum_{j=0}^{k-1} \tau_j < a_1 + a_2 + k + 1$. This implies that $a_2 \geq a_3$, and, since $a_1 + a_3 = a_2 + a_4$, that $a_1 \geq a_4$. Hence, we get $\sum_{j=0}^{k-1} \tau_j > a_4 + a_3 + k$ and it is therefore possible to choose the two subsets X and Y along the arc of size $a_3 + a_4 + 1$ of G to deduce a realization of τ in G under P . \blacksquare

5.3 Construction 3: k and n are odd

The Harary graph $G = H_{2k+1, n+1}$, where $k \geq 2$ and $n + 1 \geq 2k + 1$ are odd, is spanned by C_n^k . Thus, it follows, according to Lemmas 9 and 10, that there always exists a realization of τ in G under a $2k$ -prescription which admits at most one maximal block in G with size at least k . Moreover, $H_{2k+1, n}$ and $H_{2k+1, n+1}$ differ by their diagonal edges, but the arguments we used to prove *Case 3* of Theorem 13 also hold when considering $H_{2k+1, n+1}$ instead of $H_{2k+1, n}$.

Thus, we can directly derive the following result.

Theorem 14 *For every odd $k \geq 2$ and odd $n \geq 2k + 1$, the Harary graph $H_{2k+1, n}$ is AP+2k.*

6 On the existence of optimal AP+2-graphs

The proof of Theorems 13 and 14 relies on the fact that some particular subgraphs of a Harary graph $H_{k, n}$ necessarily satisfy the conditions of Lemma 12 when $k \neq 3$. Although, one can easily check that this lemma cannot be used when $H_{k, n}$ is not connected enough, in particular when $k = 3$. Hence, our proof cannot be used directly to prove that a Harary graph $H_{3, n}$ is AP+2 for some n .

Besides, these graphs are not all AP+2 anyway:

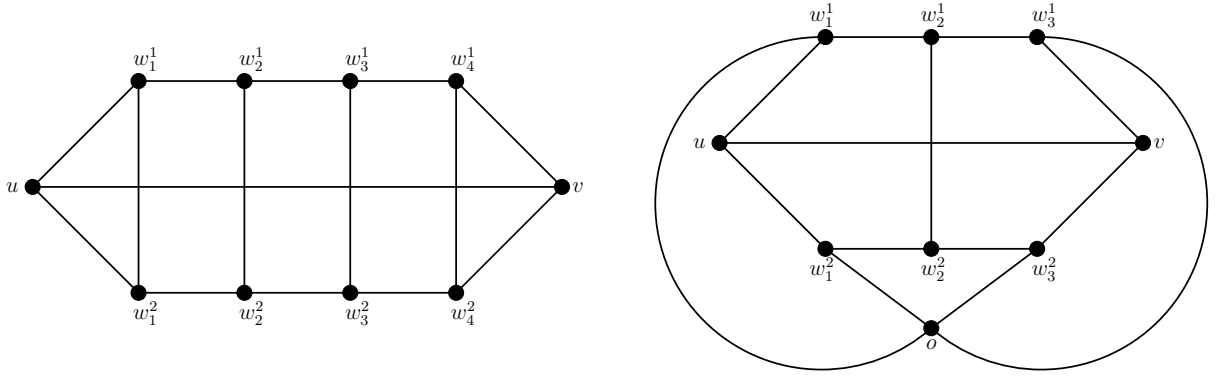


Figure 2: The graphs Pr_{10} and Pr_9

Observation 15 For every $n \equiv 2 \pmod{4}$, the Harary graph $H_{3,n}$ is not AP+2.

Indeed, let $n \equiv 2 \pmod{4}$ and $G = H_{3,n}$. One can check that the subgraph $G - \{v_0, v_2\}$ does not admit a perfect matching. Therefore, the sequence $(1, 1, 2, \dots, 2)$ is not realizable in G under (v_0, v_2) .

In order to prove that there actually exist optimal AP+2-graphs, we introduce a new class of 3-connected graphs.

Definition 16 Let $n \geq 4$. The graph Pr_n is constructed as follows:

- If n is even, Pr_n is obtained from the cycle C_n , whose vertices are successively denoted by $u, w_1^1, \dots, w_{\frac{n-2}{2}}^1, v, w_{\frac{n-2}{2}}^2, \dots, w_1^2$, by adding it the edge uv and all edges $w_i^1 w_i^2$, for every $i \in [1, \frac{n-2}{2}]$.
- If n is odd, Pr_n is obtained by first removing the edges $w_1^1 w_1^2$ and $w_{\frac{n-3}{2}}^1 w_{\frac{n-3}{2}}^2$ from Pr_{n-1} , and then adding it a new vertex o linked to $w_1^1, w_1^2, w_{\frac{n-3}{2}}^1$, and $w_{\frac{n-3}{2}}^2$.

The two graphs Pr_9 and Pr_{10} are drawn in Figure 2.

For every $n \geq 4$, the graph Pr_n is 3-connected and has the least possible number of edges, that is $\lceil \frac{3n}{2} \rceil$. To now prove that such a graph is AP+2, we consider the following sufficient condition for a graph to be AP+2 which is easier to determine for Pr graphs. Recall that a graph G is *Hamiltonian-connected* if, for every two distinct vertices u and v of G , there exists a Hamiltonian path in G with endvertices u and v .

Lemma 17 If a graph G is Hamiltonian-connected, then G is AP+2.

This lemma obviously holds since, by Lemma 6, we can partition P_n under every 2-prescription (u, v) as long as u and v are the endvertices of P_n .

Before showing that $G = Pr_n$ is Hamiltonian-connected for every $n \geq 4$, we first introduce some notations to deal with the vertices of such a graph. Let $q = \frac{n-2}{2}$ if n is even (resp. $q = \frac{n-3}{2}$ if n is odd); given two distinct integers x and y in $[1, q]$ (resp. in $[2, q-1]$) such that $x < y$, we denote by $P_{x,y}^{\nearrow}(G)$ and $P_{x,y}^{\searrow}(G)$ the paths of G defined as follows.

$$P_{x,y}^{\nearrow}(G) = \begin{cases} w_x^2 w_x^1 & \text{if } x = y, \\ w_x^2 w_x^1 P_{x+1,y}^{\searrow}(G) & \text{otherwise.} \end{cases}$$

$$P_{x,y}^{\searrow}(G) = \begin{cases} w_x^1 w_x^2 & \text{if } x = y, \\ w_x^1 w_x^2 P_{x+1,y}^{\nearrow}(G) & \text{otherwise.} \end{cases}$$

s	t	P
u	v	$uP_{1,q}^{\nearrow}(G)v$
u	w_i^1	$uP_{1,i-1}^{\nearrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ if $i-1$ is even $uP_{1,i-1}^{\searrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ otherwise
w_i^1	w_j^1	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\searrow}(G)uvP_{q,j}^{\nwarrow}(G)$ if $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\swarrow}(G)$ otherwise
w_i^1	w_j^2	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\searrow}(G)uvP_{q,j}^{\swarrow}(G)$ if $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\nwarrow}(G)$ otherwise

Table 1: Proof that Pr_n is Hamiltonian-connected for every $n \geq 4$ even

The paths $P_{x,y}^{\nwarrow}(G)$ and $P_{x,y}^{\swarrow}(G)$ of G are defined analogously from right to left when $x > y$. For every $\alpha \in \{1, 2\}$, we additionally define $P_{x,y}^{\alpha,\rightarrow}(G)$ for $x < y$ (resp. $P_{x,y}^{\alpha,\leftarrow}(G)$ for $x > y$) to be the path $w_x^\alpha w_{x+1}^\alpha \dots w_y^\alpha$ of G (resp. the path $w_x^\alpha w_{x-1}^\alpha \dots w_y^\alpha$ of G). For convenience, let us assume that $P_{x,y}^{\nearrow}(G) = \emptyset$, $P_{x,y}^{\searrow}(G) = \emptyset$ and $P_{x,y}^{\alpha,\rightarrow}(G) = \emptyset$ (resp. $P_{x,y}^{\nwarrow}(G) = \emptyset$, $P_{x,y}^{\swarrow}(G) = \emptyset$ and $P_{x,y}^{\alpha,\leftarrow}(G) = \emptyset$) whenever x or y do not belong to the interval above or when $x > y$ (resp. when $x < y$). Using these notations, we get, for instance, that $uP_{1,4}^{1,\rightarrow}(Pr_{10})vP_{4,1}^{2,\leftarrow}(Pr_{10})$ and $uP_{1,4}^{\nearrow}(Pr_{10})v$ are Hamiltonian paths of Pr_{10} .

We are now ready to prove that every Pr_n graph is Hamiltonian-connected, and thus AP+2 according to Lemma 17.

Proposition 18 *For every $n \geq 4$, the graph Pr_n is Hamiltonian-connected.*

Proof. Let $G = Pr_n$, and $q = \frac{n-2}{2}$ if n is even, $q = \frac{n-3}{2}$ otherwise. Table 1 (resp. Table 2) exhibits, given two distinct vertices s and t of G , a Hamiltonian path P of G whose endvertices are s and t when n is even (resp. n is odd). In Table 1 (resp. Table 2), it is assumed that $1 \leq i \leq q$ when j is not defined (resp. $1 < i < q$), and $0 \leq i < j \leq q$ otherwise (resp. $1 < i < j < q$). All the Hamiltonian paths which do not appear in these two tables can be deduced from other Hamiltonian paths using the symmetries of G . ■

Corollary 19 *For every $n \geq 4$, the graph Pr_n is AP+2.*

7 Conclusions

We summarize Corollaries 11 and 19 and Theorems 13 and 14 in this concluding theorem.

Theorem 20 *For every $k \geq 1$ and $n \geq k$, there exist optimal AP+k-graphs on n vertices.*

According to Observation 3, we know that the number of optimal AP+k-graphs is upper bounded by the number of edge-minimal $(k+1)$ -connected graphs. However, it seems difficult to make an estimation on the number of optimal AP+k-graphs in general.

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s	t	P
o	u	$oP_{1,q}^{1,\rightarrow}(G)vP_{q,1}^{2,\leftarrow}(G)u$
o	w_1^1	$ow_q^1vw_q^2P_{q-1,2}^{\nwarrow}w_1^2uw_1^1$ if q is even $ow_q^2vw_q^1P_{q-1,2}^{\swarrow}w_1^2uw_1^1$ otherwise
o	w_i^1	$ow_1^1uw_1^2P_{2,i-1}^{\swarrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ if i is even $ow_1^2uw_1^1P_{2,i-1}^{\nwarrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ otherwise
u	v	$uP_{1,q}^{2,\rightarrow}(G)oP_{1,q}^{1,\rightarrow}(G)v$
u	w_1^1	$uvP_{q,1}^{2,\leftarrow}(G)oP_{q,1}^{1,\leftarrow}(G)$
u	w_q^1	$uvP_{q,1}^{2,\leftarrow}(G)oP_{1,q}^{1,\rightarrow}(G)$
u	w_i^1	$uP_{1,i-1}^{1,\rightarrow}(G)P_{i-1,1}^{2,\leftarrow}(G)ow_q^2vw_q^1P_{q-1,i}^{\swarrow}(G)$ if $q-i$ is even $uP_{1,i-1}^{1,\rightarrow}(G)P_{i-1,1}^{2,\leftarrow}(G)ow_q^1vw_q^2P_{q-1,i}^{\nwarrow}(G)$ otherwise
w_1^1	w_1^2	$w_1^1uvP_{q,2}^{1,\leftarrow}(G)P_{2,q}^{2,\rightarrow}(G)ow_1^2$
w_1^1	w_q^1	$P_{1,q-1}^{1,\rightarrow}(G)P_{q-1,1}^{1,\leftarrow}(G)uvw_q^2ow_q^1$
w_1^1	w_q^2	$w_1^1ow_1^2uvP_{q,2}^{1,\leftarrow}(G)P_{2,q}^{2,\rightarrow}(G)$
w_i^1	w_j^1	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i and $q-j$ are even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i is odd and $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i is even and $q-j$ is odd $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i and $q-j$ are odd
w_i^1	w_j^2	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i and $q-j$ are even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^2vw_q^1P_{q-1,j}^{\nwarrow}(G)$ if i is odd and $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i is even and $q-j$ is odd $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^1vw_q^2P_{q-1,j}^{\swarrow}(G)$ if i and $n-j$ are odd

Table 2: Proof that Pr_n is Hamiltonian-connected for every $n \geq 5$ odd

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