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# Partitioning Harary graphs into connected subgraphs containing prescribed vertices

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## Abstract

A graph  $G$  is *arbitrarily partitionable* (AP for short) if for every partition  $(\tau_1, \dots, \tau_p)$  of  $|V(G)|$  there exists a partition  $(V_1, \dots, V_p)$  of  $V(G)$  such that each  $V_i$  induces a connected subgraph of  $G$  with order  $\tau_i$ . If, additionally,  $k$  of these subgraphs ( $k \leq p$ ) each contains an arbitrary vertex of  $G$  prescribed beforehand, then  $G$  is *arbitrarily partitionable under  $k$  prescriptions* (AP+ $k$  for short). Every AP+ $k$ -graph on  $n$  vertices must be  $(k+1)$ -connected, and have thus at least  $\lceil \frac{n(k+1)}{2} \rceil$  edges. We show that there exist AP+ $k$ -graphs on  $n$  vertices and  $\lceil \frac{n(k+1)}{2} \rceil$  edges for every  $k \geq 1$  and  $n \geq k$ .

**Keywords:** arbitrarily partitionable graph, partition under prescriptions, Harary graph.

## 1 Introduction

We denote by  $V(G)$  and  $E(G)$  the sets of vertices and edges, respectively, of a graph  $G$ . The *order* (resp. *size*) of  $G$  is the cardinality of the set  $V(G)$  (resp.  $E(G)$ ). If  $X$  is a subset of  $V(G)$ , then  $G[X]$  denotes the subgraph of  $G$  induced by  $X$ .

In the late 1970s, Györi and Lovász independently proved the following well-known result.

**Theorem 1 (Györi [4] and Lovász [6], independently)** *If  $G$  is a  $k$ -connected graph, then, given  $k$  pairwise distinct vertices  $(v_1, \dots, v_k)$  of  $G$  and  $k$  positive integers  $(\tau_1, \dots, \tau_k)$  adding up to  $|V(G)|$ , there exists a partition  $(V_1, \dots, V_k)$  of  $V(G)$  such that  $v_i \in V_i$ , the subgraph  $G[V_i]$  is connected and  $|V_i| = \tau_i$  for every  $i \in [1, k]$ .*

In this paper, we consider a more general partition problem resulting from the combination of the notion of *arbitrarily partitionable* graphs [1] with the constraint of prescribing a set of vertices from Theorem 1. Let  $G$  be a connected graph of order  $n$ . A sequence  $\tau = (\tau_1, \dots, \tau_p)$  of positive integers is *admissible for  $G$*  if it performs a partition of  $n$ , that is if  $\sum_{i=1}^p \tau_i = n$ . If, additionally, we can partition  $V(G)$  into  $p$  parts  $(V_1, \dots, V_p)$  such that each  $V_i$  induces a connected subgraph of  $G$  with order  $\tau_i$ , then  $\tau$  is *realizable in  $G$* , and the partition  $(V_1, \dots, V_p)$  is a *realization of  $\tau$  in  $G$* . If every admissible sequence for  $G$  is also realizable in  $G$ , then  $G$  is *arbitrarily partitionable* (AP for short). The interested reader is referred to [1, 2, 7] for a review of some results on AP-graphs.

Now suppose that we still want to partition  $G$  into an arbitrary number, say  $p$ , of connected subgraphs  $G_1, \dots, G_p$  of prescribed orders, but in such a way that each connected subgraph

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$G_i \in \{G_1, \dots, G_k\}$ , with  $k \leq p$ , contains some vertex  $v_i$  of  $G$  arbitrarily chosen. To model this additional requirement, the definition of AP-graphs can be strengthened as follows [3]. A  $k$ -prescription of  $G$  is a subset  $P = (v_1, \dots, v_k)$  of  $k$  distinct vertices of  $G$ . We say that a sequence  $(\tau_1, \dots, \tau_p)$  with  $p \geq k$  elements is *realizable in  $G$  under  $P$*  if there exists a realization  $(V_1, \dots, V_p)$  of  $\tau$  in  $G$  such that for every  $i \in [1, k]$ , the vertex  $v_i$  belongs to  $V_i$ . Notice that we have adopted the convention that the sizes associated to the prescribed vertices are located at the beginning of the sequence. We now say that  $G$  is  $(p, k)$ -partitionable if every sequence admissible for  $G$  consisting of exactly  $p$  elements is realizable in  $G$  under every  $k$ -prescription. Finally,  $G$  is *arbitrarily partitionable under  $k$  prescriptions* (AP+ $k$  for short) if  $G$  is  $(p, k)$ -partitionable for every  $p \in [k, n]$ .

According to these definitions, an AP+0-graph is an AP-graph. Stated differently, Theorem 1 says that every  $k$ -connected graph is  $(k, k)$ -partitionable. Hence, throughout this paper, we mainly consider  $(p, k)$ -partitions with  $p > k$ . For any  $k \geq 1$ , the set of complete graphs on at least  $k$  vertices is a trivial class of AP+ $k$ -graphs, each of them having the largest possible number of edges. In this work, we will focus on *optimal AP+ $k$ -graphs*, that is AP+ $k$ -graphs having the least possible number of edges.

The study of AP+ $k$ -graphs can be motivated by the following computer science problem [3]. Suppose that we have a network of  $n$  computing resources we want to share between an arbitrary number of users, say  $p$ , where the  $i^{\text{th}}$  user needs exactly  $\tau_i$  resources. For the sake of performance, we also suppose that the sharing must not be done arbitrarily but in such a way that the following two conditions are fulfilled:

1. each resource is attributed to exactly one user,
2. two resources attributed to a user must be able to communicate within his subnetwork.

Clearly, the problem of satisfying our users with this specific demand is equivalent to the problem of seeking a realization of the sequence  $(\tau_1, \dots, \tau_p)$  in the graph modelling our network. Hence, if we want our network to be shareable for every demand, that is no matter what  $p$  and the  $\tau_i$ 's are, then it must have an AP-graph topology. If, additionally, each of  $k$  of our users is allowed to request one specific resource to belong to his subnetwork, then our network must have an AP+ $k$ -graph topology.

This paper is organized as follows. In Section 2, we introduce some notations and preliminary results which are useful to prove, in Sections 3 and 4, respectively, some results on the partitioning of powers of paths and cycles under many prescriptions. In Section 5, these results are then used to show that, for every  $k \neq 2$ , every  $(k+1)$ -connected Harary graph is an optimal AP+ $k$ -graph. We finally show, in Section 6, that 3-connected Harary graphs are not necessarily AP+2, and provide another class of optimal AP+2-graphs instead.

## 2 Definitions, notation, and preliminary results

A subgraph  $H$  of a graph  $G$  is a *spanning subgraph* of  $G$  if  $V(H) = V(G)$ . We will also say that  $G$  is *spanned by  $H$* . Given a graph  $G$  and an integer  $k \geq 1$ , the  $k^{\text{th}}$  *power of  $G$* , denoted by  $G^k$ , is the graph with the same vertex set as  $G$ , two of its vertices being adjacent if they are at distance at most  $k$  in  $G$ . Let  $P_n$  (resp.  $C_n$ ) denote the path (resp. the cycle) on  $n$  vertices, and  $\{v_0, \dots, v_{n-1}\}$  its set of vertices. For convenience, the vertices  $v_0$  and  $v_{n-1}$  of  $P_n$  are called the *first vertex* and the *last vertex* of  $P_n$ , respectively. We use the same terminology to deal with the vertices of  $P_n^k$  (resp.  $C_n^k$ ) according to its natural spanning path  $P_n$  (resp. spanning cycle  $C_n$ ).

Let  $k \geq 1$  and  $n \geq k$  be any two integers. The  $k$ -connected Harary graph on  $n$  vertices, denoted by  $H_{k,n}$ , has vertex set  $\{v_0, \dots, v_{n-1}\}$  and the following edges:

- if  $k = 2r$  is even, then two vertices  $v_i$  and  $v_j$  are linked if and only if  $i - r \leq j \leq i + r$ ;

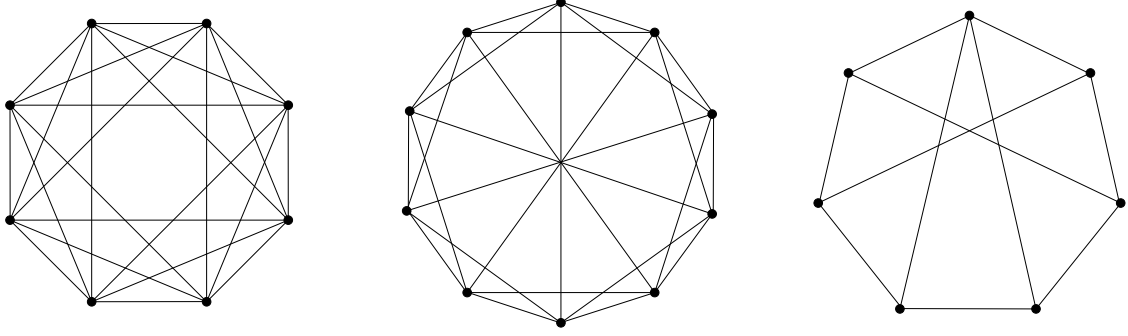


Figure 1: The Harary graphs  $H_{6,8}$ ,  $H_{5,10}$  and  $H_{3,7}$

- if  $k = 2r + 1$  is odd and  $n$  is even, then  $H_{k,n}$  is obtained by joining  $v_i$  and  $v_{i+\frac{n}{2}}$  in  $H_{2r,n}$  for every  $i \in [0, \frac{n}{2} - 1]$ ;
- if  $k = 2r + 1$  and  $n$  are odd, then  $H_{k,n}$  is obtained from  $H_{2r,n}$  by first linking  $v_0$  to both  $v_{\lfloor \frac{n}{2} \rfloor}$  and  $v_{\lceil \frac{n}{2} \rceil}$ , and then each vertex  $v_i$  to  $v_{i+\lceil \frac{n}{2} \rceil}$  for every  $i \in [1, \lfloor \frac{n}{2} \rfloor - 1]$ ;

where the subscripts are taken modulo  $n$ . Three examples of Harary graphs are given in Figure 1. When  $k$  is odd, the neighbours of a vertex  $v$  of  $H_{k,n}$  which are at distance strictly more than  $k$  from  $v$  in its spanning  $C_n$  are called the *antipodal neighbours* of  $v$ . Clearly,  $v$  has at most two antipodal neighbours in  $H_{k,n}$ . In particular,  $v$  has exactly two antipodal neighbours if and only if  $i = 0$ , and  $k$  and  $n$  are both odd. A *diagonal edge* of  $H_{k,n}$  is an edge linking two antipodal neighbours of  $H_{k,n}$ .

If  $G$  is a graph with a natural ordering of its vertices (like powers of paths and cycles, or Harary graphs), then, for any vertex  $v$  of  $G$ , we denote by  $v^+$  (resp.  $v^-$ ) the neighbour of  $v$  following  $v$  (resp. preceding  $v$ ) in this ordering.

One of the first important property of AP-graphs is the following one.

**Proposition 2** *If a graph  $G$  admits a spanning AP-subgraph (resp. a spanning AP+k-subgraph for some  $k \geq 1$ ), then  $G$  is AP (resp. AP+k).*

Recall that a graph is *traceable* if it admits a Hamiltonian path. Since every path is obviously AP, the previous proposition implies that every traceable graph is AP.

We now point out the following property of AP+k-graphs, from which we deduce the size of an optimal AP+k-graph.

**Observation 3** *Every AP+k-graph is  $(k + 1)$ -connected. Therefore, an optimal AP+k-graph has  $\lceil \frac{n(k+1)}{2} \rceil$  edges.*

Indeed, if there exist  $k$  vertices  $v_1, \dots, v_k$  such that  $G - \{v_1, \dots, v_k\}$  is disconnected into  $q$  connected components on  $n_1, \dots, n_q$  vertices, respectively, with  $n_1 \geq \dots \geq n_q$ , then the sequence  $(1, \dots, 1, n_1 + 1, (\sum_{i=2}^q n_i) - 1)$  cannot be realized in  $G$  under  $(v_1, \dots, v_k)$ . Since an edge-minimal  $(k + 1)$ -connected graph has size  $\lceil \frac{n(k+1)}{2} \rceil$ , this is the size of an optimal AP+k-graph.

Baudon *et al.* [3] proved that, for every  $k \geq 1$ , there exist non complete AP+k-graphs, namely the powers of paths and the powers of cycles:

**Theorem 4 (Baudon et al. [3])** *The graph  $P_n^{k+1}$  is AP+k for every  $k \geq 0$  and  $n \geq k + 1$ . The graph  $C_n^k$  is AP+(2k - 1) for every  $k \geq 1$  and  $n \geq 2k$ .*

Since the number of edges of  $P_n^{k+1}$  is  $(k + 1)(n - (k + 1)) + \sum_{i=1}^k i$ , this graph has more edges than an optimal AP+k-graph having the same number of vertices. However,  $C_n^k$  is  $2k$ -regular and hence is an edge-minimal  $2k$ -connected graph. According to Observation 3, it follows that the set of  $k^{\text{th}}$  powers of cycles is a set of optimal AP+(2k - 1)-graphs.

### 3 Partitioning $P_n^k$ under strictly more than $k - 1$ prescriptions

In this section, we point out, in terms of prescribed vertices location, some situations in which it is possible to partition the graph  $P_n^k$  under strictly more than  $k - 1$  prescriptions.

**Lemma 5 ([3])** *Let  $P = (v_{i_1}, \dots, v_{i_k})$  be a  $k$ -prescription of  $P_n^k$  with  $k \geq 1$ ,  $n \geq k$  and  $0 \leq i_1 < \dots < i_k \leq n - 1$ . If  $i_1 = 0$  or  $i_k = n - 1$ , then every partition  $\tau = (\tau_1, \dots, \tau_p)$  of  $n$  with  $p \geq k$  elements is realizable in  $P_n^k$  under  $P$ .*

**Lemma 6** *Let  $P = (v_{i_1}, \dots, v_{i_{k+1}})$  be a  $(k + 1)$ -prescription of  $G = P_n^k$  with  $k \geq 1$ ,  $n \geq k$  and  $0 \leq i_1 < \dots < i_{k+1} \leq n - 1$ . If  $i_1 = 0$  and  $i_{k+1} = n - 1$ , then every partition  $\tau = (\tau_1, \dots, \tau_p)$  of  $n$  with  $p \geq k + 1$  elements is realizable in  $G$  under  $P$ .*

**Proof.** We prove this claim by induction on  $k$ . For  $k = 1$ , the result is obvious; thus, we now suppose that  $k \geq 2$  and that the claim holds for every  $k'$  such that  $k' < k$ . If  $\tau_1 \leq i_2$ , then a correct realization of  $\tau$  in  $G$  under  $P$  is  $(V_1, \dots, V_p)$  where  $V_1 = \{v_0, \dots, v_{\tau_1-1}\}$  and  $(V_2, \dots, V_p)$  is a realization of  $(\tau_2, \dots, \tau_p)$  in  $G - V_1$  under the prescription  $(v_{i_2}, \dots, v_{i_{k+1}})$ . This realization necessarily exists according to Lemma 5 since  $v_{i_{k+1}}$  is the last vertex of  $G - V_1$ .

Suppose now that  $\tau_1 > i_2$ . Observe that  $[0, k - 1] - (\bigcup_{j=2}^k i_j \pmod k)$  is not empty, and let us denote by  $r$  one of these values. The subset  $V_1$  of the realization is constructed as follows. It first contains all the vertices between  $v_0$  and  $v_{i_2-1}$ , that is  $V_1 \supseteq \{v_0, \dots, v_{i_2-1}\}$ . We then add the vertex  $v_j$  to  $V_1$ , where  $j \in [i_2 + 1, i_2 + k - 1]$  is such that  $j \equiv r \pmod k$ . Finally, as long as  $|V_1| < \tau_1$  and we do not reach  $v_{n-1}$ , we repeatedly add to  $V_1$  the vertex at distance  $k$  on the right from the last one added to  $V_1$  ( $v_{j+k}, v_{j+2k}$ , etc.). According to our choice of  $r$ , these vertices are not prescribed ones and, at any moment of this procedure, the subgraph  $G - V_1$  is spanned by the  $(k - 1)^{th}$  power of a path and the subgraph  $G[V_1]$  is connected.

Thus, on the one hand, if  $V_1 = \tau_1$  holds after the procedure, then  $(V_1, \dots, V_p)$  is a correct realization of  $\tau$  under  $P$ , where  $(V_2, \dots, V_p)$  is a realization of  $(\tau_2, \dots, \tau_p)$  in  $G - V_1$  under the prescription  $(v_{i_2}, \dots, v_{i_{k+1}})$  which necessarily exists by the induction hypothesis since  $v_{i_2}$  and  $v_{i_{k+1}}$  are the first and last vertices of  $G - V_1$ .

On the other hand, if  $|V_1| < \tau_1$  holds once the procedure is achieved, then each vertex from  $V(G) - V_1$  has a neighbour in  $V_1$ . Hence, we can obtain a correct realization  $(V_1 \cup V_1', V_2, \dots, V_p)$  of  $\tau$  in  $G$  under  $P$ , where  $(V_2, \dots, V_p, V_1')$  is a realization of  $(\tau_2, \dots, \tau_p, \tau_1 - |V_1|)$  in  $G - V_1$  under the prescription  $(v_{i_2}, \dots, v_{i_{k+1}})$ . Once again, such a realization necessarily exists in this subgraph according to the induction hypothesis. ■

**Lemma 7** *Let  $P = (v_{i_1}, \dots, v_{i_k})$  be a  $k$ -prescription of  $G = P_n^k$  with  $k \geq 1$ ,  $n \geq k$  and  $0 \leq i_1 < \dots < i_k \leq n - 1$ . If  $i_k \neq i_1 + k - 1$ , then every partition  $\tau = (\tau_1, \dots, \tau_p)$  of  $n$  with  $p \geq k$  elements is realizable in  $P_n^k$  under  $P$ .*

**Proof.** If  $x = \sum_{j=k+1}^p \tau_j \leq i_1$ , then a correct realization of  $\tau$  in  $G$  under  $P$  is  $(V_1, \dots, V_p)$  where  $(V_{k+1}, \dots, V_p)$  is an arbitrary realization of  $(\tau_{k+1}, \dots, \tau_p)$  in the traceable subgraph  $G[\{v_0, \dots, v_{x-1}\}]$ , and  $(V_1, \dots, V_k)$  is a realization of  $(\tau_1, \dots, \tau_k)$  in  $G - \{v_0, \dots, v_{x-1}\}$  under  $P$  obtained using Theorem 1.

Suppose now that  $x > i_1$ . On the one hand, if  $\tau_1 > i_1$ , then a correct realization of  $\tau$  in  $G$  under  $P$  is  $(V_1' \cup V_1'', V_2, \dots, V_p)$ , where  $V_1' = \{v_0, \dots, v_{i_1-1}\}$  and  $(V_1'', V_2, \dots, V_p)$  is a realization of  $(\tau_1 - i_1, \tau_2, \dots, \tau_p)$  in  $G - V_1'$  under  $P$  obtained via Lemma 5. On the other hand, if  $\tau_1 \leq i_1$ , then let  $V_1$  be a subset of  $\{v_0, \dots, v_{i_1}\}$  obtained as follows. First, we let  $V_1 = \{v_{i_1}\}$  and we then repeatedly add to  $V_1$  the vertex located at distance 2 on the left of the last vertex added to  $V_1$  as long as  $|V_1| < \tau_1$  and  $v_0$  is not reached. If there is no vertex at distance 2 on the left of the last vertex added to  $V_1$ , then we add to  $V_1$  every remaining vertex from  $\{v_0, \dots, v_{i_1-1}\} - V_1$  from left to right until  $V_1$  has size  $\tau_1$ . Let  $X = \{v_0, \dots, v_{i_1-1}\} - V_1$ . Notice that, at the end of the previous procedure,  $G[V_1]$  is connected,  $G[X]$  is traceable, and  $v_{i_1-1} \in X$ . Now, if there exists  $y \in [k+1, p]$

such that  $\sum_{j=k+1}^y \tau_j = |X|$ , then a correct realization of  $\tau$  in  $G$  under  $P$  is  $(V_1, \dots, V_p)$  where  $(V_{k+1}, \dots, V_y)$  is a realization of  $(\tau_{k+1}, \dots, \tau_y)$  in  $G[X]$  and  $(V_2, \dots, V_k, V_{y+1}, \dots, V_p)$  is a realization of  $(\tau_2, \dots, \tau_k, \tau_{y+1}, \dots, \tau_p)$  in  $G - \{v_1, \dots, v_{i_1}\}$  under  $\{v_{i_2}, \dots, v_{i_k}\}$  obtained using Lemma 5.

Theorem 1. On the contrary, if for some  $y$  we have  $\sum_{j=k+1}^{y-1} \tau_j < |X|$  and  $\sum_{j=k+1}^y \tau_j > |X|$ , then let  $\tau'_y = |X| - \sum_{j=k+1}^{y-1} \tau_j$ ,  $\tau''_y = \tau_y - \tau'_y$ , and  $v_\alpha \notin P$  be the nearest neighbour of  $v_{i_{y-1}}$  located on the right of  $v_{i_1}$ . Such a vertex necessarily exists since otherwise this would imply that our  $k$  prescribed vertices are located consecutively along  $G$ . Moreover,  $v_\alpha$  is the first vertex of  $G - \{v_0, \dots, v_{i_1}\}$  if  $v_{i_1}^+ \neq v_{i_2}$ . We then obtain a desired realization  $(V_1, \dots, V_{y-1}, V'_y \cup V''_y, V_{y+1}, \dots, V_p)$  of  $\tau$  in  $G$  under  $P$ , where  $(V'_y, V_{k+1}, \dots, V_{y-1})$  is a realization of  $(\tau'_y, \tau_{k+1}, \dots, \tau_{y-1})$  in  $G[X]$  under  $(v_{i_{y-1}})$ , and  $(V_2, \dots, V_k, V''_y, V_{y+1}, \dots, V_p)$  is a realization of  $(\tau_2, \dots, \tau_k, \tau''_y, \tau_{y+1}, \dots, \tau_p)$  in  $G[\{v_{i_{y+1}}, \dots, v_{i_{n-1}}\}]$  under  $(v_{i_2}, \dots, v_{i_k}, v_\alpha)$ . Those two realizations exist according to Lemma 5.  $\blacksquare$

## 4 Partitioning $C_n^k$ under $2k$ prescriptions

We now prove, for any  $k \geq 1$  and  $n \geq 2k$ , that  $C_n^k$  can be partitioned under  $2k$  prescriptions when the prescribed vertices are "conveniently" located in  $C_n^k$ . Since the Harary graph  $H_{2k+1, n}$  is spanned by  $C_n^k$ , these results imply that, in some situations, we can deduce a partitioning of  $H_{2k+1, n}$  under  $2k$  prescriptions using a spanning subgraph argument.

We first introduce some notations. Suppose that  $G = C_n^k$  for some  $k \geq 1$  and  $n \geq 2k$ , and that we want to realize the sequence  $\tau = (\tau_0, \dots, \tau_{p-1})$  in  $G$  under  $P = (v_{i_0}, \dots, v_{i_{2k-1}})$ , with  $p \geq 2k$  and  $0 \leq i_0 < \dots < i_{2k-1} \leq n-1$ . For every  $j \in [0, 2k-1]$ , we denote by  $d_j$  the value  $i_j - i_{j-1} - 1$ . In particular, we have  $n = 2k + \sum_{j=0}^{2k-1} d_j$ . Given two distinct integers  $x, y \in [0, 2k-1]$  such that  $x < y$ , the *garden of the prescribed vertices*  $v_{i_x}, \dots, v_{i_y}$  in  $G$  is defined as  $G_{x,y} = \{v_{i_x}^+, \dots, v_{i_y}^-\}$  and is the maximum subset of consecutive vertices of  $G$  containing no other prescribed vertices than  $v_{i_x}, \dots, v_{i_y}$ . In particular,  $|G_{x,y}| = (y - x + 1) + \sum_{j=x}^{y-1} d_j$ . We say that the prescribed vertices  $v_{i_x}, \dots, v_{i_y}$  are *saturated for  $\tau$  in their garden* if  $\sum_{j=x}^y \tau_j > |G_{x,y}|$ , that is if the parts associated to the prescribed vertices  $v_{i_x}, \dots, v_{i_y}$  of a realization of  $\tau$  in  $G$  under  $P$  must contain some vertices of  $V(G) - G_{x,y}$ .

**Lemma 8** *Let  $P = (v_{i_0}, \dots, v_{i_{2k-1}})$  be a  $2k$ -prescription of  $G = C_n^k$  with  $k \geq 1$ ,  $n \geq 2k$  and  $0 \leq i_0 < \dots < i_{2k-1} \leq n-1$ , and  $\tau = (\tau_0, \dots, \tau_{p-1})$  be an admissible sequence for  $G$  with  $p \geq 2k$ . For any  $x \in [0, 2k-1]$ , if the prescribed vertices  $v_{i_x}, \dots, v_{i_{x+k-1}}$  are saturated for  $\tau$  in their garden, then neither the prescribed vertices  $v_{i_{x+1}}, \dots, v_{i_{x+k}}$  nor  $v_{i_{x+k+1}}, \dots, v_{i_x}$  are saturated in their respective garden.*

**Proof.** Let us consider, without loss of generality, that  $x = 0$ . If the prescribed vertices  $v_{i_0}, \dots, v_{i_{k-1}}$  are saturated for  $\tau$  in  $G_{0, k-1}$ , then we have:

$$\sum_{j=0}^{k-1} \tau_j > k + \sum_{j=0}^{k-1} d_j,$$

and

$$\sum_{j=k}^{2k-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j.$$

If the prescribed vertices  $v_{i_1}, \dots, v_{i_k}$  are also saturated for  $\tau$  in  $G_{1, k}$ , then

$$\sum_{j=1}^k \tau_j > k + \sum_{j=1}^{k+1} d_j,$$

and

$$\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j < k + d_0 + \sum_{j=k+2}^{2k-1} d_j,$$

which implies that both  $n_1 - n_{k+1} > d_1 - d_{k+2}$  and  $n_1 - n_{k+1} < d_1 - d_{k+2}$  hold, a contradiction.

If we now suppose that  $v_{i_{k+1}}, \dots, v_0$  are saturated for  $\tau$  in  $G_{k+1,0}$ , then

$$\sum_{j=1}^k \tau_j < k + \sum_{j=2}^k d_j,$$

and

$$\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j > k + d_0 + d_1 + \sum_{j=k+1}^{2k-1} d_j,$$

and we thus get that both

$$\left( \sum_{j=1}^{k-1} \tau_j \right) - \left( \sum_{j=k+1}^{2k-1} \tau_j \right) > \left( \sum_{j=2}^k d_j \right) - \left( \sum_{j=k-1}^{2k-1} d_j \right)$$

and

$$\left( \sum_{j=1}^{k-1} \tau_j \right) - \left( \sum_{j=k+1}^{2k-1} \tau_j \right) < \left( \sum_{j=2}^k d_j \right) - \left( \sum_{j=k-1}^{2k-1} d_j \right)$$

hold, again a contradiction. ■

In what follows, a *maximal block*  $B$  of  $P$  in  $C_n^k$  is a maximal subset  $B \subseteq P$  of consecutive vertices of  $C_n^k$ . In particular, if  $\{v_{i_q}, \dots, v_{i_q+\alpha}\}$  is a maximal block of  $P$  in  $C_n^k$ , then neither  $v_{i_q}^-$  nor  $v_{i_q+\alpha}^+$  are prescribed vertices.

**Lemma 9** *Let  $P = (v_{i_0}, \dots, v_{i_{2k-1}})$  be a  $2k$ -prescription of  $G = C_n^k$  with  $k \geq 1$ ,  $n \geq 2k$  and  $0 \leq i_1 < \dots < i_k \leq n - 1$ . If there exists exactly one maximal block  $B$  of  $P$  in  $G$  of size at least  $k$ , then every partition  $\tau = (\tau_0, \dots, \tau_{p-1})$  of  $n$  with  $p \geq 2k$  elements is realizable in  $G$  under  $P$ .*

**Proof.** Let us suppose, without loss of generality, that  $v_{i_0}, \dots, v_{i_{k-1}}$  belong to  $B$ . We distinguish two subcases depending on whether  $v_{i_0}, \dots, v_{i_{k-1}}$  are saturated for  $\tau$  in their garden or not.

*Case 1: The prescribed vertices  $v_{i_0}, \dots, v_{i_{k-1}}$  are not saturated for  $\tau$  in  $G_{0,k-1}$ .*

Let  $X$  be a subset of  $\sum_{j=0}^{k-1} \tau_j$  consecutive vertices of  $G_{0,k-1}$  such that  $X \cap P = \{v_{i_0}, \dots, v_{i_{k-1}}\}$ . According to Theorem 1, there exists a realization  $(V_0, \dots, V_{k-1})$  of  $(\tau_0, \dots, \tau_{k-1})$  in  $G[X]$  under the prescription  $(v_{i_0}, \dots, v_{i_{k-1}})$  since this subgraph is  $k$ -connected. Now observe that a realization  $(V_k, \dots, V_{p-1})$  of  $(\tau_k, \dots, \tau_{p-1})$  in  $G - X$  under the prescription  $(v_{i_k}, \dots, v_{i_{2k-1}})$  necessarily exists since  $G - X$  is isomorphic to the  $k^{\text{th}}$  power of a path and we supposed that the prescribed vertices perform only one maximal block of size at least  $k$  in  $G$ . This implies that either  $v_{i_k}$  is the first vertex of  $G - X$ ,  $v_{i_{2k-1}}$  is the last vertex of  $G - X$ , or the prescribed vertices  $v_{i_k}, \dots, v_{i_{2k-1}}$  are not consecutive in  $G - X$ . In the first two cases, the realization is obtained thanks to Lemma 5, and it follows from Lemma 7 in the third case. Finally, observe that  $(V_0, \dots, V_{p-1})$  is a correct realization of  $\tau$  in  $G$  under  $P$ .

*Case 2: The prescribed vertices  $v_{i_0}, \dots, v_{i_{k-1}}$  are saturated for  $\tau$  in  $G_{0,k-1}$ .*

By Lemma 8, we know that neither  $v_{i_1}, \dots, v_{i_k}$  nor  $v_{i_{k+1}}, \dots, v_{i_0}$  are saturated for  $\tau$  in  $G_{1,k}$  and  $G_{k+1,0}$ , respectively. Hence, only two cases may occur.

*Case 2.a.* If there exists a subset  $X'$  of  $\sum_{j=1}^k \tau_j$  consecutive vertices of  $G_{1,k}$  such that  $P \cap X' = \{v_{i_1}, \dots, v_{i_k}\}$ , then, since  $v_{i_0}$  is the last vertex of  $G - X'$ , all the conditions are met to apply

the procedure used in *Case 1* with  $v_{i_1}, \dots, v_{i_k}$  and  $X'$  to get the desired realization. A similar realization can be obtained analogously if the assumption above holds for  $v_{i_{k+1}}, \dots, v_{i_0}$  and their garden  $G_{k+1,0}$ .

*Case 2.b.* Suppose now that both  $\sum_{j=1}^k \tau_j < k + \sum_{j=2}^k d_j$  and  $\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j < k + d_0 + \sum_{j=k+2}^{2k-1} d_j$  hold. Let  $X = \{v_{i_1}, \dots, v_{i_k}\}$ . If there exists a  $x \in [2k, p-1]$  such that  $(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^x \tau_j) = |X|$  then we can partition  $G$  into two graphs  $G[X]$  and  $G - X$  and obtain a whole realization of  $\tau$  in  $G$  under  $P$  by considering realizations of  $(\tau_1, \dots, \tau_k, \tau_{2k}, \dots, \tau_x)$  in  $G[X]$  under  $(v_{i_1}, \dots, v_{i_k})$  and of  $(\tau_0, \tau_{k+1}, \dots, \tau_{2k-1}, \tau_{x+1}, \dots, \tau_{p-1})$  in  $G - X$  under  $(v_{i_0}, v_{i_{k+1}}, \dots, v_{i_{2k-1}})$  which necessarily exist since the conditions of Lemma 5 are satisfied in both cases ( $v_{i_0}$  is the last vertex of  $G - X$  and  $v_{i_1}$  is the first vertex of  $G[X]$ ). Finally,  $(V_0, \dots, V_{p-1})$  is a realization of  $\tau$  in  $G$  under  $P$ .

If such a  $x$  does not exist, then let  $x \in [2k, p-1]$  be such that  $(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^{x-1} \tau_j) < |X|$  and  $(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^x \tau_j) > |X|$ , and let  $\tau'_x = |X| - [(\sum_{j=1}^k \tau_j) + (\sum_{j=2k}^{x-1} \tau_j)]$  and  $\tau''_x = \tau_x - \tau'_x$ . Since we supposed that there is only one maximal block of  $P$  in  $G$  with size at least  $k$  and  $v_{i_0}$  and  $v_{i_1}$  are adjacent according to the natural ordering of  $V(G)$ , there exist  $\alpha \in [i_1, i_k]$  and  $\beta \in [i_k + 1, i_k + k - 1]$  such that  $v_\alpha$  and  $v_\beta$  are adjacent and are not prescribed vertices (otherwise there would exist a second maximal block of  $P$  in  $G$  with size at least  $k$  since  $v_{i_1}, \dots, v_{i_k}$  are not saturated for  $\tau$  in their garden). In particular, let us choose  $v_\beta$  in such a way that the distance between  $v_\alpha$  and  $v_\beta$  is the smallest possible according to the natural ordering of  $V(G)$ . We get that either  $v_\beta$  or  $v_{i_{k+1}}$  is the first vertex of  $G - X$ .

Finally, we obtain the desired realization of  $\tau$  in  $G$  under  $P$  as follows. First, let  $(V_1, \dots, V_k, V'_x, V_{2k}, \dots, V_{x-1})$  be a realization of  $(\tau_1, \dots, \tau_k, \tau'_x, \tau_{2k}, \dots, \tau_{x-1})$  in  $G[X]$  under  $(v_{i_1}, \dots, v_{i_k}, v_\alpha)$ . Then, let  $(V_0, V_{k+1}, \dots, V_{2k-1}, V''_x, V_{x+1}, \dots, V_{p-1})$  be a realization of  $(\tau_0, \tau_{k+1}, \dots, \tau_{2k-1}, \tau''_x, \tau_{x+1}, \dots, \tau_{p-1})$  in  $G - X$  under  $(v_{i_0}, v_{i_{k+1}}, \dots, v_{i_{2k-1}}, v_\beta)$ . These two realizations necessarily exist according to Lemma 6. Moreover, the subgraph  $G[V'_x \cup V''_x]$  is connected since  $v_\alpha v_\beta \in E(G)$ ; therefore, we eventually get that  $(V_0, \dots, V_{x-1}, V'_x \cup V''_x, V_{x+1}, \dots, V_{p-1})$  is a correct realization of  $\tau$  in  $G$  under  $P$ .  $\blacksquare$

**Lemma 10** *Let  $P = (v_{i_0}, \dots, v_{i_{2k-1}})$  be a  $2k$ -prescription of  $G = C_n^k$  with  $k \geq 1$ ,  $n \geq 2k$  and  $0 \leq i_1 < \dots < i_k \leq n - 1$ . If there does not exist a maximal block of  $P$  in  $G$  of size at least  $k$ , then every partition  $\tau = (\tau_0, \dots, \tau_{p-1})$  of  $n$  with  $p \geq 2k$  elements is realizable in  $G$  under  $P$ .*

**Proof.** We can suppose, without loss of generality, that neither  $v_{i_0}, \dots, v_{i_{k-1}}$  nor  $v_{i_k}, \dots, v_{i_{2k-1}}$  are saturated for  $\tau$  in  $G_{0,k-1}$  and  $G_{k,2k-1}$ , respectively. Indeed, if one of these conditions holds, then this cannot be simultaneously the case for  $v_{i_1}, \dots, v_{i_k}$  nor  $v_{i_{k+1}}, \dots, v_{i_{2k-1}}, v_{i_0}$  in their respective garden according to Lemma 8; we could thus easily relabel our prescribed vertices and part sizes so that the above assertion holds.

We distinguish the following two subcases.

*Case 1.* If we can find a subset  $X$  of  $\sum_{j=0}^{k-1} \tau_j$  consecutive vertices of  $G_{0,k-1}$  such that  $P \cap X = \{v_{i_0}, \dots, v_{i_{k-1}}\}$ , then we can obtain a correct realization of  $\tau$  in  $G$  under  $P$  in a same way as what we did in *Case 2.a* of Lemma 9. A similar argument holds if there exists a satisfying subset  $X$  included in the garden of  $v_{i_k}, \dots, v_{i_{2k-1}}$ .

*Case 2.* The last case which may happen is the one where both  $\sum_{j=0}^{k-1} \tau_j < k + \sum_{j=1}^{k-1} d_j$  and  $\sum_{j=k}^{2k-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j$  hold, and neither  $v_{i_0}$  and  $v_{i_{2k-1}}$  nor  $v_{i_{k-1}}$  and  $v_{i_k}$  are consecutive vertices of  $G$  (otherwise, we could use the arguments of *Case 2.b* of Lemma 9 to deduce a realization of  $\tau$  in  $G$  under  $P$ ). We can suppose that there does not exist  $x \in [2k, p-1]$  such that  $\sum_{j=0}^{k-1} \tau_j + \sum_{j=2k}^x \tau_j = k + \sum_{j=1}^{k-1} d_j$  or  $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=2k}^x \tau_j = k + \sum_{j=k+1}^{2k-1} d_j$  for otherwise we could find a realization of  $\tau$  in  $G$  under  $P$  using Lemma 7 since there is no maximal block of  $P$  in  $G$  with size at least  $k$ . We explain below how to deduce a realization of  $\tau$  in  $G$  under  $P$ .

If  $\tau_0 + \tau_{2k-1} \geq d_0 + 2$ , then we can obtain a correct realization of  $\tau$  in  $G$  under  $P$  as follows. Let  $V'_0 = \{v_{i_0-1}, v_{i_0-2}, \dots, v_\alpha\}$  and  $V'_{2k-1} = \{v_{i_{2k-1}+1}, v_{i_{2k-1}+2}, \dots, v_\alpha^-\}$



where  $|V_0'| < \tau_0$ ,  $|V_{2k-1}'| < \tau_{2k-1}$ , and  $\alpha \in [i_{2k-1} + 1, i_0 - 1]$ . Then, consider a realization  $(V_0'', V_1, \dots, V_{2k-2}, V_{2k-1}'', V_{2k}, \dots, V_{p-1})$  of the sequence  $(\tau_0 - |V_0'|, \tau_1, \dots, \tau_{2k-2}, \tau_{2k-1} - |V_{2k-1}'|, \tau_{2k}, \dots, \tau_{p-1})$  in the subgraph  $G[\{v_{i_0}, \dots, v_{i_{2k-1}}\}]$  under  $P$  obtained in the same way as what we did for *Case 2.b* (that is by doing as if  $v_{i_0}$  and  $v_{i_{2k-1}}$  were consecutive vertices of  $G$ ). Such a realization necessarily exists since we supposed that there is no maximal block of  $P$  in  $G$  with size at least  $k$ . Finally, observe that the realization  $(V_0' \cup V_0'', V_1, \dots, V_{2k-2}, V_{2k-1}' \cup V_{2k-1}'', V_{2k}, \dots, V_{p-1})$  is a correct realization of  $\tau$  in  $G$  under  $P$ .

Now, if  $\tau_0 + \tau_{2k-1} < d_0 + 2$ , then  $\tau_0 - 1 \geq \lfloor \frac{d_0(k-1)}{k} \rfloor$  and  $\tau_{2k-1} - 1 \geq \lfloor \frac{d_0(k-1)}{k} \rfloor$  cannot hold simultaneously since otherwise we would get a contradiction. Let  $Z = \{v_{i_{2k-1}}^+, \dots, v_{i_0}^-\}$  and suppose that  $\tau_0 - 1 < \lfloor \frac{d_0(k-1)}{k} \rfloor$ ; thus, there exists a subset  $V_0$  of  $Z \cup \{v_{i_0}\}$  such that  $G[V_0]$  is connected on  $\tau_0$  vertices,  $V_0$  contains the vertex  $v_{i_0}$  but does not contain neither  $v_{i_{2k-1}}^+$  nor  $v_{i_0}^-$ , and the subgraph  $G[Z - V_0]$  is traceable.

Once again, we can suppose that there does not exist  $x \in [2k, p-1]$  such that  $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^x \tau_j = |G_{1,k-1}|$  since otherwise we could easily deduce a realization of  $\tau$  in  $G$  under  $P$  using Lemmas 5 and 6 (recall that  $v_{i_k}, \dots, v_{i_{2k-1}}$  are not saturated for  $\tau$  in their garden). Thus, let  $x \in [2k, p-1]$  be such that  $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^{x-1} \tau_j < |G_{1,k-1}|$  and  $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^x \tau_j > |G_{1,k-1}|$ , and let  $\tau'_x = |G_{1,k-1}| - \sum_{j=1}^{k-1} \tau_j - \sum_{j=2k}^{x-1} \tau_j$  and  $\tau''_x = \tau_x - \tau'_x$ . For the sake of the proof, let us also denote  $\tau_p$  instead of  $\tau''_x$ . Since  $G_{1,k-1}$  only includes  $k-1$  prescribed vertices, then we can obtain a realization  $(V_1, \dots, V_{k-1}, V'_x, V_{2k}, \dots, V_{x-1})$  of  $(\tau_1, \dots, \tau_{k-1}, \tau'_x, \tau_{2k}, \dots, \tau_{x-1})$  in  $G[G_{1,k-1}]$  under  $(v_{i_1}, \dots, v_{i_{k-1}}, v_\alpha)$  where  $v_\alpha$  is a non-prescribed vertex of  $G_{1,k-1}$  neighbouring  $v_{i_0}^-$ . Such a vertex necessarily exists since there are no maximal block of  $P$  of size at least  $k$  in  $G$ ; in particular, choose  $v_\alpha = v_{i_0}^+$  if  $v_{i_1} \neq v_{i_0}^+$ . The previous realization exists by Lemma 5.

For the same reasons as above, let us suppose that there does not exist  $y \in [x+1, p]$  such that  $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^y \tau_j = k + \sum_{j=k+1}^{2k-1} d_j$ . Then denote by  $y \in [x+1, p]$  the least integer such that  $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^{y-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j$  and  $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^y \tau_j > k + \sum_{j=k+1}^{2k-1} d_j$ . Next, let us split  $\tau_y$  into two elements  $\tau'_y$  and  $\tau''_y$  as above, and use the arguments above once again to obtain a realization  $(V_k, \dots, V_{2k-1}, V'_y, V_{x+1}, \dots, V_{y-1})$  of  $(\tau_k, \dots, \tau_{2k-1}, \tau'_y, \tau_{x+1}, \dots, \tau_{y-1})$  in  $G[\{v_{i_k}, \dots, v_{i_{2k-1}}\}]$  under  $(v_{i_k}, \dots, v_{i_{2k-1}}, v_\beta)$  where  $v_\beta$  is a non-prescribed vertex of  $\{v_{i_k}, \dots, v_{i_{2k-1}}\}$  neighbouring  $v_{i_{2k-1}}^+$ . This realization exists by Lemma 6 since  $v_{i_k}$  and  $v_{i_{2k-1}}$  are the first and last vertices of  $G[\{v_{i_k}, \dots, v_{i_{2k-1}}\}]$ , respectively.

On the one hand, if  $y = p$ , then it means that the element  $\tau_x$  was split into three elements  $\tau'_x$ ,  $\tau'_y$  and  $\tau''_y$ . In this case, the realization  $(V_0, \dots, V_{x-1}, V'_x \cup V'_y \cup Z, V_{y+1}, \dots, V_{p-1})$  is a correct realization of  $\tau$  in  $G$  under  $P$ . On the other hand, if  $y < p$ , then, since  $G[Z - V_0]$  is spanned by a path whose first and last vertices are  $v_{i_{2k-1}}^+$  and  $v_{i_0}^-$ , respectively, by Lemma 6 there exists a realization  $(V''_x, V''_y, V_{y+1}, \dots, V_{p-1})$  of  $(\tau''_x, \tau''_y, \tau_{y+1}, \dots, \tau_{p-1})$  in  $G[Z - V_0]$  under  $(v_{i_0}^-, v_{i_{2k-1}}^+)$ . By construction,  $G[V'_x \cup V''_x]$  and  $G[V'_y \cup V''_y]$  both induce a connected subgraph of  $G$ . It follows that  $(V_0, \dots, V_{x-1}, V'_x \cup V''_x, V_{x+1}, \dots, V_{y-1}, V'_y \cup V''_y, V_{y+1}, \dots, V_{p-1})$  is a realization of  $\tau$  in  $G$  under  $P$ .  $\blacksquare$

## 5 Partitioning Harary graphs under prescriptions

Harary graphs are known to be graphs having the smallest possible size regarding their connectivity [5]. They trivially have the property of being AP since they are Hamiltonian; thus, one could wonder about the maximum number of prescribed vertices we can allow while partitioning them. In particular, can we always prescribe  $k$  vertices when partitioning a  $(k+1)$ -connected Harary graph? If this is the case, then such a graph will also be an optimal AP+ $k$ -graph.

We investigate, in this section, the question for the three different constructions of Harary graphs, according to the parities of both  $n$  and  $k$ .

### 5.1 Construction 1: $k$ is even

For every  $k \geq 2$  and  $n \geq 2k$  such that  $k$  is even,  $H_{k,n}$  is isomorphic to  $C_n^{\frac{k}{2}}$  which is  $AP+(k-1)$  according to Theorem 4. Thus, the following result derives directly.

**Corollary 11** *For every even  $k \geq 2$  and  $n \geq 2k$ , the Harary graph  $H_{k,n}$  is  $AP+(k-1)$ .*

### 5.2 Construction 2: $k$ is odd and $n$ is even

Let  $k \geq 2$  and  $n \geq 2$  be two integers such that  $k$  is odd and  $n$  is even. By construction,  $H_{2k+1,n}$  is spanned by  $H_{2k,n}$  and thus is  $AP+(2k-1)$  according to Corollary 11. Although this number of prescribed vertices is good since  $H_{2k+1,n}$  is  $(2k+1)$ -connected, the connectivity of this graph suggests that we could up to one more prescribed vertex while partitioning it (see Observation 3).

Before proving that  $H_{2k+1,n}$  is actually  $AP+2k$ , we first introduce the following lemma which deals with the traceability of a graph composed by two linked squares of paths.

**Lemma 12** *If  $G$  is a graph such that  $V(G) = V_1 \cup V_2$ , the subgraphs  $G[V_1]$  and  $G[V_2]$  are both spanned by the square of a path, and there exists an edge joining one vertex of  $V_1$  and one of  $V_2$ , then  $G$  is traceable.*

**Proof.** Let  $v_1, \dots, v_{n_1}$  and  $u_1, \dots, u_{n_2}$  denote the vertices of  $G[V_1]$  and  $G[V_2]$ , from left to right, and  $v_\alpha$  and  $u_\beta$  be two vertices of  $G$  such that  $v_\alpha u_\beta \in E(G)$ . Consider the following subpaths of  $G$ :

- $P = v_1 v_2 \dots v_{\alpha-1}$ ;
- $Q = \begin{cases} v_{\alpha+1} v_{\alpha+3} \dots v_{n_1-1} v_{n_1} v_{n_1-2} v_{n_1-4} \dots v_{\alpha+2} & \text{if } n_1 - \alpha \text{ is even,} \\ v_{\alpha+1} v_{\alpha+3} \dots v_{n_1} v_{n_1-1} v_{n_1-3} \dots v_{\alpha+2} & \text{otherwise;} \end{cases}$
- $R = \begin{cases} u_{\beta+2} u_{\beta+4} \dots u_{n_2} u_{n_2-1} u_{n_2-3} \dots u_{\beta+1} & \text{if } n_2 - \beta \text{ is even,} \\ u_{\beta+2} u_{\beta+4} \dots u_{n_2-1} u_{n_2} u_{n_2-2} u_{n_2-4} \dots u_{\beta+1} & \text{otherwise;} \end{cases}$
- $S = u_{\beta-1} u_{\beta-2} \dots u_1$ .

It is then easy to check that  $PQv_\alpha u_\beta RS$  is a Hamiltonian path of  $G$ . ■

We now prove the main result of this section.

**Theorem 13** *For every odd  $k \geq 2$  and even  $n \geq 2k+1$ , the Harary graph  $H_{2k+1,n}$  is  $AP+2k$ .*

**Proof.** Let  $G = H_{2k+1,n}$  be the  $(2k+1)$ -connected Harary graph on  $n$  vertices. We prove that every partition  $\tau = (\tau_0, \dots, \tau_{p-1})$  of  $n$  with  $p \geq 2k+1$  elements is realizable in  $G$  under every prescription  $P = (v_{i_0}, \dots, v_{i_{2k-1}})$  with  $0 \leq i_0 < \dots < i_{2k-1} \leq n-1$ . We distinguish three cases depending on whether there exists a realization of  $\tau$  in  $C_n^k$  under  $P$  or not. Since  $C_n^k$  is a spanning subgraph of  $G$ , such a realization would also be a realization of  $\tau$  in  $G$  under  $P$ .

*Case 1.* First observe that if there exists exactly one maximal block of  $P$  in  $G$  with size at least  $k$ , then we can obtain a correct realization of  $\tau$  in  $G$  under  $P$  by considering a realization of  $\tau$  in  $C_n^k$  under  $P$ . Such a realization necessarily exists according to Lemma 9.

*Case 2.* Similarly, if there does not exist a maximal block of  $P$  in  $G$  with size at least  $k$ , then we can use Lemma 10 to deduce a realization of  $\tau$  in  $C_n^k$  under  $P$ . This realization is also a realization of  $\tau$  in  $G$  under  $P$ .

*Case 3.* The last case we have to consider is the one where there exist two maximal blocks  $B_1$  and  $B_2$  of  $P$  in  $G$  with size exactly  $k$ . In this case, the prescribed vertices disconnect the graph into two components linked by some diagonal edges. Indeed, without loss of generality,

let us suppose that  $B_1 = \{v_{i_0}, \dots, v_{i_{k-1}}\}$  and  $B_2 = \{v_{i_k}, \dots, v_{i_{2k-1}}\}$ ; necessarily, the antipodal neighbours of  $v_{i_0}^-$  and  $v_{i_{k-1}}^+$  do not both belong to  $P$  since otherwise there would exist a maximal block of  $P$  in  $G$  with size at least  $k + 2$ . Let us denote by  $v_\alpha$  and  $v_\beta$  two antipodal neighbours of  $G$  such that  $v_\alpha, v_\beta \notin B_1 \cup B_2$ ,  $\alpha \in [i_{k-1} + 1, i_k - 1]$  and  $\beta \in [i_{2k-1} + 1, i_0 - 1]$ . Besides, let  $a_1 = \alpha - i_{k-1} - 1$ ,  $a_2 = i_k - \alpha - 1$ ,  $a_3 = i_0 - \beta - 1$  and  $a_4 = \beta - i_{2k-1} - 1$  denote the number of consecutive vertices between  $B_1$ ,  $B_2$  and the two vertices  $v_\alpha$  and  $v_\beta$  according to the natural ordering of  $V(G)$ .

If  $\sum_{j=0}^{k-1} \tau_j \leq a_1 + a_3 + k$  and  $\sum_{j=k}^{2k-1} \tau_j \leq a_2 + a_4 + k$ , then we can find two subsets  $X$  and  $Y$  of consecutive vertices of  $G$  such that  $|X| = \sum_{j=0}^{k-1} \tau_j$ ,  $|Y| = \sum_{j=k}^{2k-1} \tau_j$ ,  $\{v_{i_0}, \dots, v_{i_{k-1}}\} \subseteq X$ ,  $\{v_{i_k}, \dots, v_{i_{2k-1}}\} \subseteq Y$ , and  $v_\alpha, v_\beta \notin X \cup Y$ . Since  $G[X]$  and  $G[Y]$  are both isomorphic to the  $k^{\text{th}}$  power of a path, then using Theorem 1 we can deduce two realizations  $(V_0, \dots, V_{k-1})$  and  $(V_k, \dots, V_{2k-1})$  of  $(\tau_0, \dots, \tau_{k-1})$  and  $(\tau_k, \dots, \tau_{2k-1})$  in  $G[X]$  and  $G[Y]$  under the prescriptions  $(v_{i_0}, \dots, v_{i_{k-1}})$  and  $(v_{i_k}, \dots, v_{i_{2k-1}})$ . Moreover, since  $k \geq 2$ ,  $G - (X, Y)$  is traceable according to Lemma 12, there exists a realization  $(V_{2k}, \dots, V_{p-1})$  of  $(\tau_{2k}, \dots, \tau_{p-1})$  in  $G - (X, Y)$ . Finally,  $(V_0, \dots, V_{p-1})$  is a correct realization of  $\tau$  in  $G$  under  $P$ .

Let us now suppose that, without loss of generality,  $\sum_{j=0}^{k-1} \tau_j > a_1 + a_3 + k$  and  $\sum_{j=k}^{2k-1} \tau_j < a_2 + a_4 + k$ . If  $\sum_{j=0}^{2k-1} \tau_j \geq a_1 + a_2 + 2k + 1$ , then we can find two subsets of consecutive vertices  $X, Y \subseteq V(G)$  such that, for some  $\gamma \in [i_{k-1} + 1, i_k - 1]$ ,  $v_\gamma \in X$ ,  $v_\gamma^+ \in Y$ ,  $\{v_{i_0}, \dots, v_{i_{k-1}}\} \subseteq X$ ,  $\{v_{i_k}, \dots, v_{i_{2k-1}}\} \subseteq Y$ ,  $|X| = \sum_{j=0}^{k-1} \tau_j$  and  $|Y| = \sum_{j=k}^{2k-1} \tau_j$ . By Theorem 1, we know that there exists a realization  $(V_0, \dots, V_{k-1})$  of  $(\tau_0, \dots, \tau_{k-1})$  in  $G[X]$  under the prescription  $(v_{i_0}, \dots, v_{i_{k-1}})$ , and a realization  $(V_k, \dots, V_{2k-1})$  of  $(\tau_k, \dots, \tau_{2k-1})$  in  $G[Y]$  under  $(v_{i_k}, \dots, v_{i_{2k-1}})$ . Finally,  $G - (X, Y)$  is isomorphic to the  $k^{\text{th}}$  power of a path, and thus there exists a realization  $(V_{2k}, \dots, V_p)$  of the remaining sequence  $(\tau_{2k}, \dots, \tau_{p-1})$  in it. We get that  $(V_0, \dots, V_{p-1})$  is a realization of  $\tau$  in  $G$  under  $P$ .

If it is not possible to choose the subsets  $X$  and  $Y$  in such a way that they have two neighbouring consecutive vertices along the arc of size  $a_1 + a_2 + 1$  of  $G$ , then we can necessarily find two such subsets along the arc of size  $a_3 + a_4 + 1$  of  $G$ . Indeed, in such a situation we have  $\sum_{j=0}^{k-1} \tau_j > a_1 + a_3 + k$  by hypothesis, but  $\sum_{j=0}^{k-1} \tau_j < a_1 + a_2 + k + 1$ . This implies that  $a_2 \geq a_3$ , and, since  $a_1 + a_3 = a_2 + a_4$ , that  $a_1 \geq a_4$ . Hence, we get  $\sum_{j=0}^{k-1} \tau_j > a_4 + a_3 + k$  and it is therefore possible to choose the two subsets  $X$  and  $Y$  along the arc of size  $a_3 + a_4 + 1$  of  $G$  to deduce a realization of  $\tau$  in  $G$  under  $P$ .  $\blacksquare$

### 5.3 Construction 3: $k$ and $n$ are odd

The Harary graph  $G = H_{2k+1, n+1}$ , where  $k \geq 2$  and  $n + 1 \geq 2k + 1$  are odd, is spanned by  $C_n^k$ . Thus, it follows, according to Lemmas 9 and 10, that there always exists a realization of  $\tau$  in  $G$  under a  $2k$ -prescription which admits at most one maximal block in  $G$  with size at least  $k$ . Moreover,  $H_{2k+1, n}$  and  $H_{2k+1, n+1}$  differ by their diagonal edges, but the arguments we used to prove *Case 3* of Theorem 13 also hold when considering  $H_{2k+1, n+1}$  instead of  $H_{2k+1, n}$ .

Thus, we can directly derive the following result.

**Theorem 14** *For every odd  $k \geq 2$  and odd  $n \geq 2k + 1$ , the Harary graph  $H_{2k+1, n}$  is AP+2k.*

## 6 On the existence of optimal AP+2-graphs

The proof of Theorems 13 and 14 relies on the fact that some particular subgraphs of a Harary graph  $H_{k, n}$  necessarily satisfy the conditions of Lemma 12 when  $k \neq 3$ . Although, one can easily check that this lemma cannot be used when  $H_{k, n}$  is not connected enough, in particular when  $k = 3$ . Hence, our proof cannot be used directly to prove that a Harary graph  $H_{3, n}$  is AP+2 for some  $n$ .

Besides, these graphs are not all AP+2 anyway:

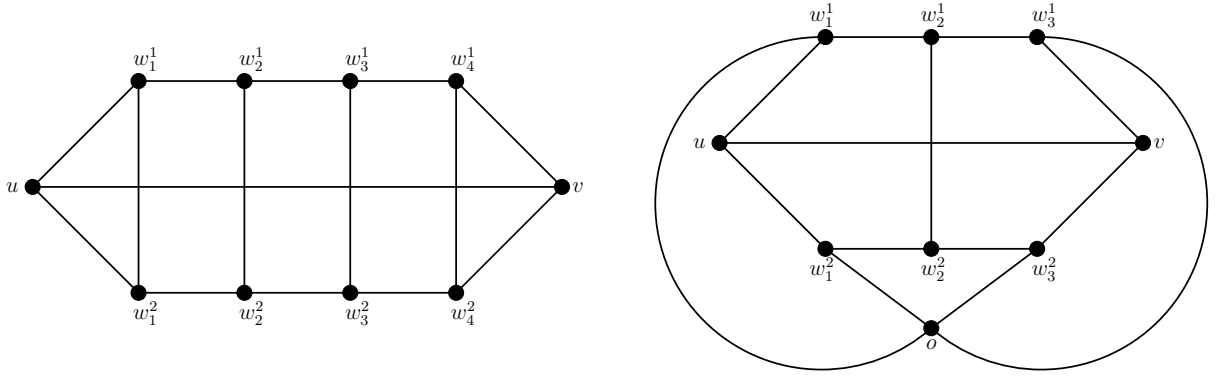


Figure 2: The graphs  $Pr_{10}$  and  $Pr_9$

**Observation 15** For every  $n \equiv 2 \pmod{4}$ , the Harary graph  $H_{3,n}$  is not AP+2.

Indeed, let  $n \equiv 2 \pmod{4}$  and  $G = H_{3,n}$ . One can check that the subgraph  $G - \{v_0, v_2\}$  does not admit a perfect matching. Therefore, the sequence  $(1, 1, 2, \dots, 2)$  is not realizable in  $G$  under  $(v_0, v_2)$ .

In order to prove that there actually exist optimal AP+2-graphs, we introduce a new class of 3-connected graphs.

**Definition 16** Let  $n \geq 4$ . The graph  $Pr_n$  is constructed as follows:

- If  $n$  is even,  $Pr_n$  is obtained from the cycle  $C_n$ , whose vertices are successively denoted by  $u, w_1^1, \dots, w_{\frac{n-2}{2}}^1, v, w_{\frac{n-2}{2}}^2, \dots, w_1^2$ , by adding it the edge  $uv$  and all edges  $w_i^1 w_i^2$ , for every  $i \in [1, \frac{n-2}{2}]$ .
- If  $n$  is odd,  $Pr_n$  is obtained by first removing the edges  $w_1^1 w_1^2$  and  $w_{\frac{n-3}{2}}^1 w_{\frac{n-3}{2}}^2$  from  $Pr_{n-1}$ , and then adding it a new vertex  $o$  linked to  $w_1^1, w_1^2, w_{\frac{n-3}{2}}^1$ , and  $w_{\frac{n-3}{2}}^2$ .

The two graphs  $Pr_9$  and  $Pr_{10}$  are drawn in Figure 2.

For every  $n \geq 4$ , the graph  $Pr_n$  is 3-connected and has the least possible number of edges, that is  $\lceil \frac{3n}{2} \rceil$ . To now prove that such a graph is AP+2, we consider the following sufficient condition for a graph to be AP+2 which is easier to determine for  $Pr$  graphs. Recall that a graph  $G$  is *Hamiltonian-connected* if, for every two distinct vertices  $u$  and  $v$  of  $G$ , there exists a Hamiltonian path in  $G$  with endvertices  $u$  and  $v$ .

**Lemma 17** If a graph  $G$  is Hamiltonian-connected, then  $G$  is AP+2.

This lemma obviously holds since, by Lemma 6, we can partition  $P_n$  under every 2-prescription  $(u, v)$  as long as  $u$  and  $v$  are the endvertices of  $P_n$ .

Before showing that  $G = Pr_n$  is Hamiltonian-connected for every  $n \geq 4$ , we first introduce some notations to deal with the vertices of such a graph. Let  $q = \frac{n-2}{2}$  if  $n$  is even (resp.  $q = \frac{n-3}{2}$  if  $n$  is odd); given two distinct integers  $x$  and  $y$  in  $[1, q]$  (resp. in  $[2, q-1]$ ) such that  $x < y$ , we denote by  $P_{x,y}^{\nearrow}(G)$  and  $P_{x,y}^{\searrow}(G)$  the paths of  $G$  defined as follows.

$$P_{x,y}^{\nearrow}(G) = \begin{cases} w_x^2 w_x^1 & \text{if } x = y, \\ w_x^2 w_x^1 P_{x+1,y}^{\searrow}(G) & \text{otherwise.} \end{cases}$$

$$P_{x,y}^{\searrow}(G) = \begin{cases} w_x^1 w_x^2 & \text{if } x = y, \\ w_x^1 w_x^2 P_{x+1,y}^{\nearrow}(G) & \text{otherwise.} \end{cases}$$

$s$	$t$	$P$
$u$	$v$	$uP_{1,q}^{\nearrow}(G)v$
$u$	$w_i^1$	$uP_{1,i-1}^{\nearrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ if $i-1$ is even $uP_{1,i-1}^{\searrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ otherwise
$w_i^1$	$w_j^1$	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\searrow}(G)uvP_{q,j}^{\nwarrow}(G)$ if $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\swarrow}(G)$ otherwise
$w_i^1$	$w_j^2$	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\searrow}(G)uvP_{q,j}^{\nwarrow}(G)$ if $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\nwarrow}(G)$ otherwise

Table 1: Proof that  $Pr_n$  is Hamiltonian-connected for every  $n \geq 4$  even

The paths  $P_{x,y}^{\nwarrow}(G)$  and  $P_{x,y}^{\swarrow}(G)$  of  $G$  are defined analogously from right to left when  $x > y$ . For every  $\alpha \in \{1, 2\}$ , we additionally define  $P_{x,y}^{\alpha,\rightarrow}(G)$  for  $x < y$  (resp.  $P_{x,y}^{\alpha,\leftarrow}(G)$  for  $x > y$ ) to be the path  $w_x^\alpha w_{x+1}^\alpha \dots w_y^\alpha$  of  $G$  (resp. the path  $w_x^\alpha w_{x-1}^\alpha \dots w_y^\alpha$  of  $G$ ). For convenience, let us assume that  $P_{x,y}^{\nearrow}(G) = \emptyset$ ,  $P_{x,y}^{\searrow}(G) = \emptyset$  and  $P_{x,y}^{\alpha,\rightarrow}(G) = \emptyset$  (resp.  $P_{x,y}^{\nwarrow}(G) = \emptyset$ ,  $P_{x,y}^{\swarrow}(G) = \emptyset$  and  $P_{x,y}^{\alpha,\leftarrow}(G) = \emptyset$ ) whenever  $x$  or  $y$  do not belong to the interval above or when  $x > y$  (resp. when  $x < y$ ). Using these notations, we get, for instance, that  $uP_{1,4}^{1,\rightarrow}(Pr_{10})vP_{4,1}^{2,\leftarrow}(Pr_{10})$  and  $uP_{1,4}^{\nearrow}(Pr_{10})v$  are Hamiltonian paths of  $Pr_{10}$ .

We are now ready to prove that every  $Pr_n$  graph is Hamiltonian-connected, and thus AP+2 according to Lemma 17.

**Proposition 18** *For every  $n \geq 4$ , the graph  $Pr_n$  is Hamiltonian-connected.*

**Proof.** Let  $G = Pr_n$ , and  $q = \frac{n-2}{2}$  if  $n$  is even,  $q = \frac{n-3}{2}$  otherwise. Table 1 (resp. Table 2) exhibits, given two distinct vertices  $s$  and  $t$  of  $G$ , a Hamiltonian path  $P$  of  $G$  whose endvertices are  $s$  and  $t$  when  $n$  is even (resp.  $n$  is odd). In Table 1 (resp. Table 2), it is assumed that  $1 \leq i \leq q$  when  $j$  is not defined (resp.  $1 < i < q$ ), and  $0 \leq i < j \leq q$  otherwise (resp.  $1 < i < j < q$ ). All the Hamiltonian paths which do not appear in these two tables can be deduced from other Hamiltonian paths using the symmetries of  $G$ . ■

**Corollary 19** *For every  $n \geq 4$ , the graph  $Pr_n$  is AP+2.*

## 7 Conclusions

We summarize Corollaries 11 and 19 and Theorems 13 and 14 in this concluding theorem.

**Theorem 20** *For every  $k \geq 1$  and  $n \geq k$ , there exist optimal AP+k-graphs on  $n$  vertices.*

According to Observation 3, we know that the number of optimal AP+k-graphs is upper bounded by the number of edge-minimal  $(k+1)$ -connected graphs. However, it seems difficult to make an estimation on the number of optimal AP+k-graphs in general.

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$s$	$t$	$P$
$o$	$u$	$oP_{1,q}^{1,\rightarrow}(G)vP_{q,1}^{2,\leftarrow}(G)u$
$o$	$w_1^1$	$ow_q^1vw_q^2P_{q-1,2}^{\leftarrow}w_1^2uw_1^1$ if $q$ is even $ow_q^2vw_q^1P_{q-1,2}^{\leftarrow}w_1^2uw_1^1$ otherwise
$o$	$w_i^1$	$ow_1^1uw_1^2P_{2,i-1}^{\leftarrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ if $i$ is even $ow_1^2uw_1^1P_{2,i-1}^{\leftarrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ otherwise
$u$	$v$	$uP_{1,q}^{2,\rightarrow}(G)oP_{1,q}^{1,\rightarrow}(G)v$
$u$	$w_1^1$	$uvP_{q,1}^{2,\leftarrow}(G)oP_{q,1}^{1,\leftarrow}(G)$
$u$	$w_q^1$	$uvP_{q,1}^{2,\leftarrow}(G)oP_{1,q}^{1,\rightarrow}(G)$
$u$	$w_i^1$	$uP_{1,i-1}^{1,\rightarrow}(G)P_{i-1,1}^{2,\leftarrow}(G)ow_q^2vw_q^1P_{q-1,i}^{\leftarrow}(G)$ if $q-i$ is even $uP_{1,i-1}^{1,\rightarrow}(G)P_{i-1,1}^{2,\leftarrow}(G)ow_q^1vw_q^2P_{q-1,i}^{\leftarrow}(G)$ otherwise
$w_1^1$	$w_1^2$	$w_1^1uvP_{q,2}^{1,\leftarrow}(G)P_{2,q}^{2,\rightarrow}(G)ow_1^2$
$w_1^1$	$w_q^1$	$P_{1,q-1}^{1,\rightarrow}(G)P_{q-1,1}^{1,\leftarrow}(G)uvw_q^2ow_q^1$
$w_1^1$	$w_q^2$	$w_1^1ow_1^2uvP_{q,2}^{1,\leftarrow}(G)P_{2,q}^{2,\rightarrow}(G)$
$w_i^1$	$w_j^1$	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^2uw_1^1ow_q^2vw_q^1P_{q-1,j}^{\leftarrow}(G)$ if $i$ and $q-j$ are even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^1uw_1^2ow_q^2vw_q^1P_{q-1,j}^{\leftarrow}(G)$ if $i$ is odd and $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^2uw_1^1ow_q^1vw_q^2P_{q-1,j}^{\leftarrow}(G)$ if $i$ is even and $q-j$ is odd $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^1uw_1^2ow_q^1vw_q^2P_{q-1,j}^{\leftarrow}(G)$ if $i$ and $q-j$ are odd
$w_i^1$	$w_j^2$	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^2uw_1^1ow_q^1vw_q^2P_{q-1,j}^{\leftarrow}(G)$ if $i$ and $q-j$ are even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^1uw_1^2ow_q^2vw_q^1P_{q-1,j}^{\leftarrow}(G)$ if $i$ is odd and $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^2uw_1^1ow_q^2vw_q^1P_{q-1,j}^{\leftarrow}(G)$ if $i$ is even and $q-j$ is odd $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\leftarrow}(G)w_1^1uw_1^2ow_q^1vw_q^2P_{q-1,j}^{\leftarrow}(G)$ if $i$ and $n-j$ are odd

Table 2: Proof that  $Pr_n$  is Hamiltonian-connected for every  $n \geq 5$  odd

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