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► **To cite this version:**

| Valeriy G. Bardakov, Paolo Bellingeri. Groups of virtual and welded links. 2012. hal-00687526

**HAL Id: hal-00687526**

**<https://hal.science/hal-00687526>**

Preprint submitted on 13 Apr 2012

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# GROUPS OF VIRTUAL AND WELDED LINKS

VALERIY G. BARDAKOV AND PAOLO BELLINGERI

ABSTRACT. We define new notions of groups of virtual and welded knots (or links) and we study their relations with other invariants, in particular the Kauffman group of a virtual knot.

## 1. INTRODUCTION

Virtual knot theory has been introduced by Kauffman [21] as a generalization of classical knot theory. Virtual knots (and links) are represented as generic immersions of circles in the plane (virtual link diagrams) where double points can be classical (with the usual information on overpasses and underpasses) or virtual. Virtual link diagrams are equivalent under ambient isotopy and some types of local moves (generalized Reidemeister moves): classical Reidemeister moves (Figure 1), virtual Reidemeister moves and mixed Reidemeister moves (Figures 2 and 3).

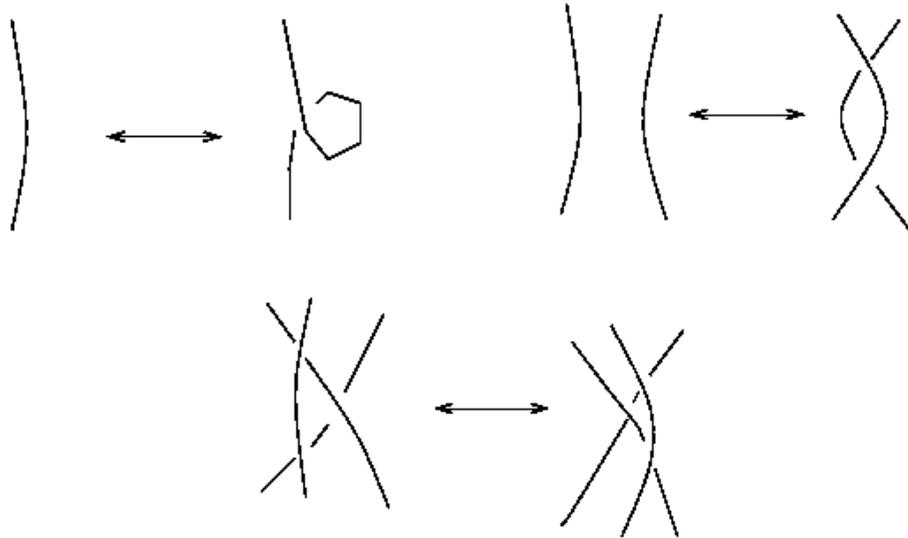


FIGURE 1. Classical Reidemeister moves

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1991 *Mathematics Subject Classification.* Primary 20F36.

*Key words and phrases.* Braid groups, virtual braid, welded braids, virtual knots, welded knots, group of knot.

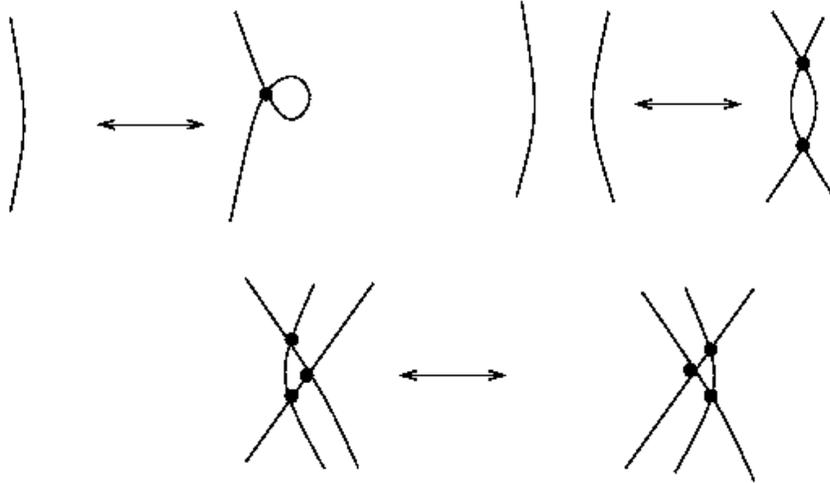


FIGURE 2. Virtual Reidemeister moves

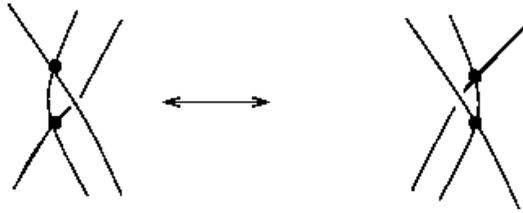


FIGURE 3. Mixed Reidemeister moves

A Theorem of Goussarov, Polyak and Viro [15, Theorem 1B] states that if two classical knot diagrams are equivalent under generalized Reidemeister moves, then they are equivalent under the classical Reidemeister moves. In this sense virtual link theory is a nontrivial extension of classical theory. This Theorem is a straightforward consequence of the fact that the knot group (more precisely the *group system* of a knot, see for instance [11, 18]) is a complete knot invariant which can be naturally extended in the realm of virtual links. Nevertheless this notion of invariant does not appear satisfactory for virtual objects (see Section 4): the main goal of this paper is to explore new invariants for virtual links using braids and their virtual generalizations.

In fact, using virtual generalized Reidemeister moves we can introduce a notion of “virtual” braids (see for instance [21, 30]). Virtual braids on  $n$  strands form a group, usually denoted by  $VB_n$ . The relations between virtual braids and virtual knots (and links) are completely determined by a generalization of Alexander and Markov Theorems [19].

To the generalized Reidemeister moves on virtual diagrams one could add the following local moves, called *forbidden moves* of type  $F1$  and  $F2$  (Figure 4):

We can include one or both of them to obtain a “quotient” theory of the theory of virtual links. If we allow the move  $F1$ , then we obtain the theory of *Welded links* whose interest is

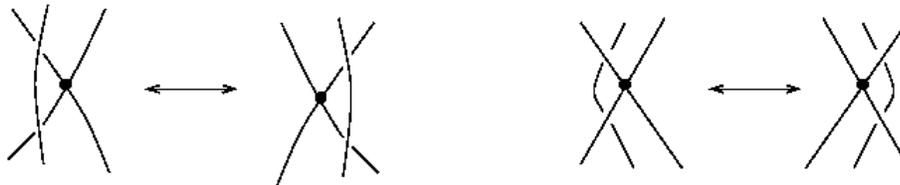


FIGURE 4. Forbidden moves of type  $F1$  (on the left) and type  $F2$  (on the right)

growing up recently, in particular because of the fact that the welded braid counterpart can be defined in several equivalent ways (for instance in terms of configuration spaces, mapping classes and automorphisms of free groups). The theory with both forbidden moves added is called the theory of *Fused links* but this theory is trivial, at least at the level of knots, since any knot is equivalent to the trivial knot [20, 27].

The paper is organized as follows: in Sections 2 and 3 we recall some definitions and classical results and we construct a representation of  $VB_n$  into  $Aut F_{n+1}$ ; using this representation we define (Section 4) a new notion of group of a virtual knot (or link) and we compare our invariant to other known invariants. In the case of welded objects (Section 5) our construction gives an invariant which is a straightforward generalization to welded knots of Kauffman notion of group of virtual knots. We conclude with some observations on the analogous of Wada groups in the realm of welded links.

**Acknowledgements.** The research of the first author was partially supported by RFBR-10-01-00642. The research of the second author was partially supported by French grant ANR-11-JS01-002-01.

This work started during the staying of the second author at the University of Caen, december 2010, in the framework of the French - Russian grant 10-01-91056. The first author would like to thank the members of the Laboratory of Mathematics of the University of Caen for their kind hospitality.

## 2. BRAIDS VS VIRTUAL AND WELDED BRAIDS

During last twenty years several generalizations of braid groups were defined and studied, according to their definition as "diagrams" in the plane: in particular singular braids [1], virtual braids [21, 30] and welded braids [13].

It is worth to mention that for all of these generalizations it exists an Alexander-like theorem, stating that any singular (respectively virtual or welded) link can be represented as the closure of a singular (respectively virtual or welded) braid. Moreover, there are generalizations of classical Markov's theorem for braids giving a characterization for two singular (respectively virtual or welded) braids whose closures represent the same singular (respectively virtual or welded) link [14, 19].

In the following we introduce virtual and welded braid groups as quotients of free product of braid groups and corresponding symmetric groups.

Virtual and welded braid groups have several other definitions, more intrinsic, see for instance [6, 19] for the virtual case and [9, 13, 19] for the welded one.

The braid group  $B_n$ ,  $n \geq 2$ , on  $n$  strings can be defined as the group generated by  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ , with the defining relations:

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| \geq 2.\end{aligned}$$

The virtual braid group  $VB_n$  can be defined as the group generated by the elements  $\sigma_i, \rho_i$ ,  $i = 1, 2, \dots, n-1$  with the defining relations:

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| \geq 2, \\ \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \dots, n-2, \\ \rho_i \rho_j &= \rho_j \rho_i, \quad |i-j| \geq 2, \\ \rho_i^2 &= 1, \quad i = 1, 2, \dots, n-1; \\ \sigma_i \rho_j &= \rho_j \sigma_i, \quad |i-j| \geq 2, \\ \rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \dots, n-2.\end{aligned}$$

It is easy to verify that  $\rho_i$ 's generate the symmetric group  $S_n$  and that the  $\sigma_i$ 's generate the braid group  $B_n$  (see Remark 1).

In [15] it was proved that the relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i$$

corresponding to the forbidden moves F1 and F2 for virtual link diagram, are not fulfilled in  $VB_n$ .

According to [13] the welded braid group  $WB_n$  is generated by  $\sigma_i, \alpha_i$ ,  $i = 1, 2, \dots, n-1$ . Elements  $\sigma_i$  generate the braid group  $B_n$  and elements  $\alpha_i$  generate the symmetric group  $S_n$ , and the following mixed relations hold

$$\begin{aligned}\alpha_i \sigma_j &= \sigma_j \alpha_i, \quad |i-j| \geq 2, \\ \alpha_{i+1} \alpha_i \sigma_{i+1} &= \sigma_i \alpha_{i+1} \alpha_i, \quad i = 1, 2, \dots, n-2, \\ \alpha_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \alpha_{i+1}, \quad i = 1, 2, \dots, n-2.\end{aligned}$$

Comparing the defining relations of  $VB_n$  and  $WB_n$ , we see that the group presentation of  $WB_n$  can be obtained from the group presentation of  $VB_n$  replacing  $\rho_i$  by  $\alpha_i$  and adding relations of type  $\alpha_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \alpha_{i+1}$ ,  $i = 1, 2, \dots, n-2$  which are related to F1 moves.

Notice that if we add to relations of  $VB_n$  the relations related to F2 moves:

$$\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, \quad i = 1, 2, \dots, n-2,$$

we get a group,  $WB'_n$ , which is isomorphic to  $WB_n$ : this isomorphism is given by the map  $\iota_n : WB'_n \rightarrow WB_n$  that sends  $\rho_i$  in  $\alpha_i$  and  $\sigma_i$  in  $\sigma_i^{-1}$ .

## 3. BRAIDS AS AUTOMORPHISMS OF FREE GROUPS AND GENERALIZATIONS

As remarked by Artin, the braid group  $B_n$  may be represented as a subgroup of  $\text{Aut}(F_n)$  by associating to any generator  $\sigma_i$ , for  $i = 1, 2, \dots, n-1$ , of  $B_n$  the following automorphism of  $F_n$ :

$$\sigma_i : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_l \mapsto x_l, \end{cases} \quad l \neq i, i+1.$$

Artin proved a stronger result (see for instance [23, Theorem 5.1]), by giving a characterization of braids as automorphisms of free groups. He proved that any automorphism  $\beta$  of  $\text{Aut}(F_n)$ <sup>1</sup> corresponds to an element of  $B_n$  if and only if  $\beta$  satisfies the following conditions:

- i)  $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$ ,  $1 \leq i \leq n$ ,
- ii)  $\beta(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n$ ,

where  $\pi \in S_n$  and  $a_i \in F_n$ .

The group of conjugating automorphisms  $C_n$  consists of automorphisms satisfying the first condition. In [13] it was proved that  $WB_n$  is isomorphic to  $C_n$  and therefore the group  $WB_n$  can be also considered as a subgroup of  $\text{Aut}(F_n)$ . More precisely the generators  $\sigma_1, \dots, \sigma_{n-1}$  of  $WB_n$  correspond to previous automorphisms of  $F_n$  while any generator  $\alpha_i$ , for  $i = 1, 2, \dots, n-1$  is associated to the following automorphism of  $F_n$ :

$$\alpha_i : \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i, \\ x_l \mapsto x_l, \end{cases} \quad l \neq i, i+1.$$

**Remark 1.** *As noticed by Kamada [21], the above representation of  $WB_n$  as conjugating automorphisms and the fact  $WB_n$  is a quotient of  $VB_n$  imply that the  $\sigma_i$ 's generate the braid group  $B_n$  in  $VB_n$ .*

On the other hand the construction of an embedding of  $VB_n$  into  $\text{Aut}(F_m)$  for some  $m$  remains an open problem.

**Theorem 2.** [2] *There is a representation  $\psi$  of  $VB_n$  in  $\text{Aut}(F_{n+1})$ ,  $F_{n+1} = \langle x_1, x_2, \dots, x_n, y \rangle$  which is defined by the following actions on the generators of  $VB_n$ :*

$$\psi(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_l \mapsto x_l, \quad l \neq i, i+1; \\ y \mapsto y, \end{cases} \quad \psi(\rho_i) : \begin{cases} x_i \mapsto y x_{i+1} y^{-1}, \\ x_{i+1} \mapsto y^{-1} x_i y, \\ x_l \mapsto x_l, \quad l \neq i, i+1, \\ y \mapsto y, \end{cases}$$

for all  $i = 1, 2, \dots, n-1$ .

<sup>1</sup>In the following we will consider the action of (classical, virtual or welded) braids from left to right and  $\beta_1 \beta_2(x_i)$  will denote  $((x_i) \beta_1) \beta_2$ .

This representation was independently considered in [25].

Remark that the group  $WB_n$  can be considered as a quotient of  $\psi(VB_n)$ : in fact a straightforward verification shows that:

**Proposition 3.** *Let  $q_n : VB_n \rightarrow WB_n$  be the projection defined by  $q_n(\sigma_i) = \sigma_i$  and  $q_n(\rho_i) = \alpha_i$  for  $i = 1, \dots, n$ . Let  $F_{n+1} = \langle x_1, x_2, \dots, x_n, y \rangle$  and  $F_n = \langle x_1, x_2, \dots, x_n \rangle$ . The projection  $p_n : F_{n+1} \rightarrow F_{n+1}/\langle\langle y \rangle\rangle \simeq F_n$  induces a map  $p_n^\# : \psi(VB_n) \rightarrow WB_n$  such that  $p_n^\# \circ \psi = q_n$ .*

The faithfulness of the representation given in Theorem 2 is evident for  $n = 2$  since in this case  $VB_2 \simeq C_2 \simeq \mathbb{Z} * \mathbb{Z}_2$ . In fact, from the defining relations it follows that  $VB_2 \simeq C_2$  and if we consider composition of  $\psi$  with  $p_2 : F_3 = \langle x_1, x_2, y \rangle \rightarrow F_2 = \langle x_1, x_2 \rangle$  we get the representation of  $C_2$  by automorphisms of  $F_2$ .

For  $n > 2$  we do not know if above representation is faithful: since  $\psi(VB_n) \subseteq WB_{n+1}$  the faithfulness of  $\psi$  for any  $n$  would imply that virtual braid groups can be considered as subgroup of welded groups, whose structure and applications in finite type invariants theory is much more advanced (see for instance [6, 8]).

We remark also that a quite tedious computation shows that image by  $\psi$  of the *Kishino braid* :  $Kb = \sigma_2\sigma_1\rho_2\sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1}\rho_2\sigma_1\sigma_2\sigma_1\rho_2\sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1}\rho_2\sigma_1\sigma_2$  is non trivial while its Alexander invariant is trivial [7].

Notice that  $Kb = \sigma_2^{-1}b_1^2\sigma_2$  where  $b_1 = \sigma_2^2\sigma_1\rho_2\sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1}\rho_2\sigma_1$ .

#### 4. GROUPS OF VIRTUAL LINKS

In the classical case the group of a link  $L$  is defined as the fundamental group  $\pi_1(S^3 \setminus N(L))$  where  $N(L)$  is a tubular neighborhood of the link in  $S^3$ . To find a group presentation of this group we can use Wirtinger method as follows.

One can consider the oriented diagram of the link as the union of oriented arcs in the plane. Define a base point for  $\pi_1(S^3 \setminus N(L))$  and associate to any arc a loop starting from the base point, which goes straight to the chosen arc, encircles it with linking number +1 and returns straight to the base point. Let us consider the loops  $a_i, a_j, a_k$  around three arcs in a crossing of the diagram as in Figure 5:

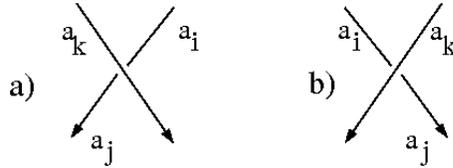


FIGURE 5. Arcs around two types of crossings

One can easily verify that in the first case the loop  $a_j$  is homotopic to  $a_k a_i a_k^{-1}$  and in the second case the loop  $a_j$  is homotopic to  $a_k^{-1} a_i a_k$ .

The group  $\pi_1(S^3 \setminus N(L))$  admits the following presentation: Let  $D$  be an oriented diagram of a link  $L$  and  $A_1, \dots, A_n$  be the arcs determined by  $D$ . The group  $\pi_1(S^3 \setminus N(L))$ , admits the following group presentation:

**Generators:**  $\{a_1, \dots, a_n\}$ , where  $a_j$  is the loop associated to the arc  $A_j$ .

**Relations:** To each crossing corresponds a relation as follows

$$\begin{aligned} a_j a_k &= a_k a_i && \text{if } a_i, a_j, a_k \text{ meet in a crossing like in case a) of figure 5;} \\ a_k a_j &= a_i a_k && \text{if } a_i, a_j, a_k \text{ meet in a crossing like in case b) of figure 5.} \end{aligned}$$

We recall that this presentation is usually called upper Wirtinger presentation while the lower Wirtinger presentation is obtained applying Wirtinger method to the diagram where all crossings are reversed; these presentations are generally different but the corresponding groups are evidently isomorphic because of their geometrical meaning.

Another way to obtain a group presentation for  $\pi_1(S^3 \setminus N(L))$  is to consider a braid  $\beta \in B_n$  such that its Alexander closure is isotopic to  $L$ ; therefore the group  $\pi_1(S^3 \setminus N(L))$  admits the presentation:

$$\pi_1(S^3 \setminus N(L)) = \langle x_1, x_2, \dots, x_n \mid x_i = \beta(x_i), i = 1, \dots, n \rangle,$$

where we consider  $\beta$  as an automorphism of  $F_n$  (this a consequence of van Kampen's Theorem, see for instance [31]).

Given a virtual link  $vL$ , according to [21] the group of the virtual knot  $vL$ , denoted with  $G_{K,v}(vL)$ , is the group obtained extending the Wirtinger method to virtual diagrams, forgetting all virtual crossings. This notion of group of a virtual knot (or link) does not seem satisfactory: for instance if  $vT$  is the virtual trefoil knot with two classical crossings and one virtual crossing and  $U$  is unknot then  $G_{K,v}(vT) \simeq G_{K,v}(U) \simeq \mathbb{Z}$ , although that  $vT$  is not equivalent to  $U$ . In addition, as noted Goussarov-Polyak-Viro [15], the upper Wirtinger group of a virtual knot is not necessary isomorphic to the corresponding lower Wirtinger group.

We introduce another notion of group  $G_v(vL)$  of a virtual link  $vL$ . Let  $vL = \widehat{\beta}_v$  be a closed virtual braid, where  $\beta_v \in VB_n$ <sup>2</sup>. Define

$$G_v(vL) = \langle x_1, x_2, \dots, x_n, y \mid x_i = \psi(\beta_v)(x_i), i = 1, \dots, n \rangle.$$

We will consider the action from left to right and for simplicity of notation we will write  $x_i \beta_v$  instead of  $\psi(\beta_v)(x_i)$ .

**Notation.** In the following we use the notations  $[a, b] = a^{-1}b^{-1}ab$  and  $a^b = b^{-1}ab$ .

The following Theorem was announced in [2].

**Theorem 4.** *The group  $G_v(vL)$  is an invariant of the virtual link  $vL$ .*

*Proof.* According to [19] two virtual braids have equivalent closures as virtual links if and only if they are related by a finite sequence of the following moves:

- 1) a braid move (which is a move corresponding to a defining relation of the virtual braid group),
- 2) a conjugation in the virtual braid group,

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<sup>2</sup>The indices  $v$  and  $w$  are given to precise when we are considering virtual or welded braids

- 3) a right stabilization of positive, negative or virtual type, and its inverse operation,
- 4) a right/left virtual exchange move.

Here a *right stabilization of positive, negative or virtual type* is a replacement of  $b \in VB_n$  by  $b\sigma_n$ ,  $b\sigma_n^{-1}$  or  $b\rho_n \in VB_{n+1}$ , respectively, a *right virtual exchange move* is a replacement

$$b_1\sigma_n^{-1}b_2\sigma_n \longleftrightarrow b_1\rho_nb_2\rho_n \in VB_{n+1}$$

and a *left virtual exchange move* is a replacement

$$s(b_1)\sigma_1^{-1}s(b_2)\sigma_1 \longleftrightarrow s(b_1)\rho_1s(b_2)\rho_1 \in VB_{n+1},$$

where  $b_1, b_2 \in VB_n$  and  $s : VB_n \longrightarrow VB_{n+1}$  is the shift map i.e.  $s(\sigma_i) = \sigma_{i+1}$ .

We have to check that under all moves 1) - 4), the group  $G_v(vL)$  does not change. Let  $vL = \widehat{\beta}_v$  where  $\beta_v \in VB_n$ .

1) If  $\beta'_v \in VB_n$  is another braid such that  $\beta_v = \beta'_v$  in  $VB_n$ , then  $\psi(\beta_v) = \psi(\beta'_v)$  since  $\psi$  is a homomorphism. Hence  $G(\widehat{\beta}_v) = G(\widehat{\beta}'_v)$  and the first move does not change the group of virtual link.

2) Evidently it is enough to consider only conjugations by the generators of  $VB_n$ . Let

$$G_1 = G(\widehat{\beta}_v) = \langle x_1, x_2, \dots, x_n, y \mid x_i = x_i\beta_v, i = 1, 2, \dots, n \rangle$$

and

$$G_2 = G(\widehat{\sigma_k\beta_v\sigma_k^{-1}}) = \langle x_1, x_2, \dots, x_n, y \mid x_i = x_i(\sigma_k\beta_v\sigma_k^{-1}), i = 1, 2, \dots, n \rangle,$$

where  $k \in \{1, 2, \dots, n-1\}$ . To prove that  $G_2 \simeq G_1$  we rewrite the defining relations of  $G_2$  in the form

$$x_i\sigma_k = x_i(\sigma_k\beta_v), i = 1, 2, \dots, n.$$

If  $i \neq k, k-1$  then this relation is equivalent to

$$x_i = x_i\beta_v$$

since  $x_i\sigma_k = x_i$ . But it is a relation in  $G_1$ . Hence, we have to consider only two relations:

$$x_k\sigma_k = x_k(\sigma_k\beta_v), x_{k+1}\sigma_k = x_{k+1}(\sigma_k\beta_v).$$

By the definition of  $\psi$  these relations are equivalent to

$$x_kx_{k+1}x_k^{-1} = (x_kx_{k+1}x_k^{-1})\beta_v, x_k = x_k\beta_v.$$

The second relation is a relation from  $G_1$ . Rewrite the first relation:

$$x_kx_{k+1}x_k^{-1} = (x_k\beta_v)(x_{k+1}\beta_v)(x_k^{-1}\beta_v)$$

and using the second relation we get

$$x_{k+1} = x_{k+1}\beta_v,$$

which is a relation from  $G_1$ . Hence we proved that any relation from  $G_1$  is true in  $G_2$  and analogously one can prove that any relation from  $G_2$  is true in  $G_1$ .

Consider the conjugation by element  $\rho_k$ . In this case we have

$$G_2 = G(\widehat{\rho_k\beta_v\rho_k}) = \langle x_1, x_2, \dots, x_n, y \mid x_i = x_i(\rho_k\beta_v\rho_k), i = 1, 2, \dots, n \rangle.$$

Rewrite the relations of  $G_2$  in the form

$$x_i \rho_k = x_i(\rho_k \beta_v), \quad i = 1, 2, \dots, n.$$

If  $i \neq k, k - 1$  then we have

$$x_i = x_i \beta_v,$$

since  $x_i \rho_k = x_i$ . But it is a relation in  $G_1$ . Hence, we have to consider only two relations:

$$x_k \rho_k = x_k(\rho_k \beta_v), \quad x_{k+1} \rho_k = x_{k+1}(\rho_k \beta_v).$$

By the definition of  $\psi$  these relations are equivalent to

$$y x_{k+1} y^{-1} = (y x_{k+1} y^{-1}) \beta_v, \quad y^{-1} x_k y = (y^{-1} x_k y) \beta_v$$

or

$$y x_{k+1} y^{-1} = y(x_{k+1} \beta_v) y^{-1}, \quad y^{-1} x_k y = y^{-1}(x_k \beta_v) y.$$

These relations are equivalent to

$$x_{k+1} = x_{k+1} \beta_v, \quad x_k = x_k \beta_v$$

which are relations from  $G_1$ . Hence we proved that the set of relations from  $G_2$  is equivalent to the set of relations from  $G_1$ .

3) Consider the move from a braid  $\beta_v = \beta_v(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho_1, \rho_2, \dots, \rho_{n-1}) \in VB_n$  to the braid  $\beta_v \sigma_n^{-1} \in VB_{n+1}$ . We have two groups:

$$G_1 = G(\widehat{\beta_v}) = \langle x_1, x_2, \dots, x_n, y \mid x_i = x_i \beta_v, \quad i = 1, 2, \dots, n \rangle$$

and

$$G_2 = G(\widehat{\beta_v \sigma_n^{-1}}) = \langle x_1, x_2, \dots, x_n, x_{n+1}, y \mid x_i = x_i(\beta_v \sigma_n^{-1}), \quad i = 1, 2, \dots, n+1 \rangle$$

and we need to prove that they are isomorphic.

Rewrite the relations of  $G_2$  in the form

$$x_i \sigma_n = x_i \beta_v, \quad i = 1, 2, \dots, n+1.$$

If  $i = 1, 2, \dots, n-1$  then we have

$$x_i = x_i \beta_v,$$

which is a relation in  $G_1$ . Hence, we have to consider only two relations:

$$x_n \sigma_n = x_n \beta_v, \quad x_{n+1} \sigma_n = x_{n+1} \beta_v,$$

these relations are equivalent to

$$x_n x_{n+1} x_n^{-1} = x_n \beta_v, \quad x_n = x_{n+1}.$$

Using the second relation rewrite the first in the form

$$x_n = x_n \beta_v,$$

which is a relation from  $G_1$ . Also we can remove  $x_{n+1}$  from the set of generators of  $G_2$ . Hence we proved that the set of relations from  $G_2$  is equivalent to the set of relations from  $G_1$ .

The move from a braid  $\beta_v \in VB_n$  to the braid  $\beta_v \sigma_n$  is similar.

Consider the move from a braid  $\beta_v \in VB_n$  to the braid  $\beta_v \rho_n \in VB_{n+1}$ . We have two groups:

$$G_1 = G(\widehat{\beta_v}) = \langle x_1, x_2, \dots, x_n, y \mid x_i = x_i \beta_v, i = 1, 2, \dots, n \rangle$$

and

$$G_2 = G(\widehat{\beta_v \rho_n}) = \langle x_1, x_2, \dots, x_n, x_{n+1}, y \mid x_i = x_i (\beta_v \rho_n), i = 1, 2, \dots, n+1 \rangle.$$

For  $i = n$  and  $i = n+1$  we have the following relations in  $G_2$ :

$$x_n \rho_n = x_n \beta_v, \quad x_{n+1} \rho_n = x_{n+1} \beta_v,$$

which are equivalent to

$$y x_{n+1} y^{-1} = x_n \beta_v, \quad y^{-1} x_n y = x_{n+1}.$$

Rewrite the second relation in the form  $x_n = y x_{n+1} y^{-1}$  and substituting in the first relation we have

$$x_n = x_n \beta_v,$$

which is a relation from  $G_1$ . Also we can remove  $x_{n+1}$  from the set of generators of  $G_2$ . Hence we proved that the set of relations from  $G_2$  is equivalent to the set of relations from  $G_1$ .

4) Finally, consider the exchange move <sup>3</sup>

$$b_1 \sigma_n^{-1} b_2^{-1} \sigma_n \longleftrightarrow b_1 \rho_n b_2^{-1} \rho_n, \quad b_1, b_2 \in VB_n.$$

We have two groups:

$$G_1 = G(\widehat{b_1 \sigma_n^{-1} b_2^{-1} \sigma_n}) = \langle x_1, x_2, \dots, x_{n+1}, y \mid x_i = x_i (b_1 \sigma_n^{-1} b_2^{-1} \sigma_n), i = 1, 2, \dots, n+1 \rangle$$

and

$$G_2 = G(\widehat{b_1 \rho_n b_2^{-1} \rho_n}) = \langle x_1, x_2, \dots, x_{n+1}, y \mid x_i = x_i (b_1 \rho_n b_2^{-1} \rho_n), i = 1, 2, \dots, n+1 \rangle$$

and we need to prove that they are isomorphic.

Rewrite the defining relations from  $G_1$  in the form

$$x_i (\sigma_n^{-1} b_2) = x_i (b_1 \sigma_n^{-1}), \quad i = 1, 2, \dots, n+1,$$

and defining relations from  $G_2$  in the form

$$x_i (\rho_n b_2) = x_i (b_1 \rho_n), \quad i = 1, 2, \dots, n+1.$$

---

<sup>3</sup>In this formula we take  $b_2^{-1}$  instead  $b_2$  for convenience.

Let  $b_1$  and  $b_2$  are the following automorphisms

$$b_1 : \left\{ \begin{array}{l} x_1 \mapsto x_{\pi(1)}^{a_1}, \\ x_2 \mapsto x_{\pi(2)}^{a_2}, \\ \dots\dots\dots \\ x_n \mapsto x_{\pi(n)}^{a_n}, \\ x_{n+1} \mapsto x_{n+1}, \\ y \mapsto y, \end{array} \right. \quad b_2 : \left\{ \begin{array}{l} x_1 \mapsto x_{\tau(1)}^{c_1}, \\ x_2 \mapsto x_{\tau(2)}^{c_2}, \\ \dots\dots\dots \\ x_n \mapsto x_{\tau(n)}^{c_n}, \\ x_{n+1} \mapsto x_{n+1}, \\ y \mapsto y, \end{array} \right. ,$$

where  $\pi, \tau \in S_n$  and  $a_i, c_i \in \langle x_1, x_2, \dots, x_n, y \rangle$ . Then the automorphism  $\sigma_n^{-1}b_2$  has the form

$$\sigma_n^{-1}b_2 : \left\{ \begin{array}{l} x_1 \mapsto x_{\tau(1)}^{c_1}, \\ x_2 \mapsto x_{\tau(2)}^{c_2}, \\ \dots\dots\dots \\ x_{n-1} \mapsto x_{\tau(n-1)}^{c_{n-1}}, \\ x_n \mapsto x_{n+1}, \\ x_{n+1} \mapsto x_{n+1}^{-1}x_{\tau(n)}^{c_n}x_{n+1}, \\ y \mapsto y, \end{array} \right.$$





$$G_2 = \langle x_1, x_2, \dots, x_n, z_{n+1}, y \mid x_{\tau(1)}^{c_1} = (z_{n+1})^{a'_1}, x_{\tau(2)}^{c_2} = (x_{\pi(2)})^{a'_2}, \dots, \\ x_{\tau(n-1)}^{c_{n-1}} = (x_{\pi(n-1)})^{a'_{n-1}}, z_{n+1} = (x_{\pi(n)})^{a'_n}, x_{\tau(n)}^{c_n} = x_n \rangle.$$

and therefore they are isomorphic. □

**Example 0.** For the unknot  $U$  we have

$$G_v(U) = \langle x, y \rangle \simeq F_2.$$

**Example 1.** For the virtual trefoil  $vT$  we have that  $vT = \widehat{\sigma_1^2 \rho_1}$  and then

$$G_v(vT) = \langle x, y \mid x(yxy^{-2}xy) = (yxy^{-2}xy)x \rangle \not\simeq F_2.$$

Hence, we obtain a new proof of the fact that  $vT$  a non-trivial virtual knot.

**Example 2.** The group  $G_v(K)$  is not a complete invariant for virtual knots. Let  $c = \rho_1 \sigma_1 \sigma_2 \sigma_1 \rho_1 \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \in VB_3$ ; the closure  $\widehat{c}$  is equivalent to the Kishino knot (see Figure 6). The Kishino knot is a non trivial virtual knot [10] with trivial Jones polynomial and trivial fundamental group ( $G_{K,v}(\widehat{c}) = \mathbb{Z}$ ). For this knot we have that  $G_v(\widehat{c}) = F_2$ .

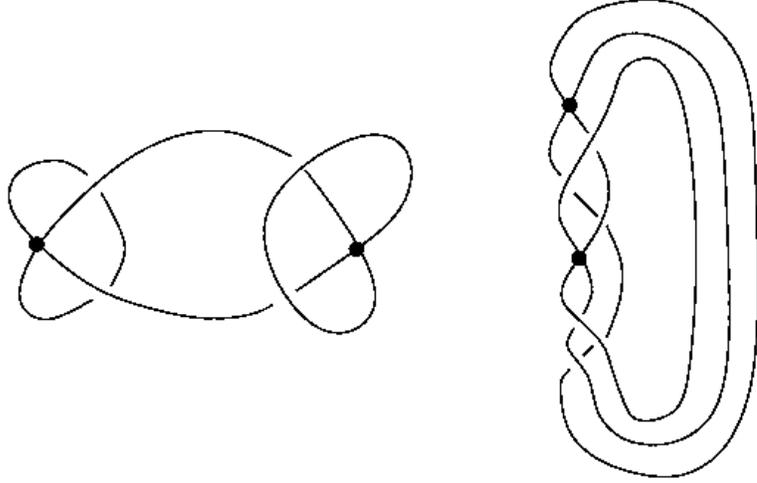


FIGURE 6. The Kishino knot: usual diagram and as the closure of a virtual braid.

In fact, it is not difficult to prove that  $c$  defines the following automorphism of  $F_4$

$$\psi(c) : \begin{cases} x_1 \mapsto y^2 x_3^{-1} x_2 x_3 y^{-2} x_3 y^2 x_3^{-1} x_2^{-1} x_3 y^{-2}, \\ x_2 \mapsto x_3^{-1} x_2 x_3 y^{-2} x_3 y x_3^{-1} x_2^{-1} x_1 x_2 x_3 y^{-1} x_3^{-1} y^2 x_3^{-1} x_2^{-1} x_3, \\ x_3 \mapsto y x_3^{-1} x_2 x_3 y^{-1}, \\ y \mapsto y, \end{cases}$$

and therefore the image of  $c$  as automorphism of  $F_4$  is non trivial: nevertheless we have that

$$G_v(\widehat{c}) = \langle x_1, x_2, x_3, y \mid x_1 = y^2 x_3^{-1} x_2 x_3 y^{-2} x_3 y^2 x_3^{-1} x_2^{-1} x_3 y^{-2}, \\ x_2 = x_3^{-1} x_2 x_3 y^{-2} x_3 y x_3^{-1} x_2^{-1} x_1 x_2 x_3 y^{-1} x_3^{-1} y^2 x_3^{-1} x_2^{-1} x_3, x_3 = y x_3^{-1} x_2 x_3 y^{-1} \rangle.$$

Using the first relation we can remove  $x_1$  and using the third relation we can remove  $x_2$ . We get

$$G_v(\widehat{c}) = \langle x, y \mid xy^{-1}xyx^{-1} = xy^{-1}xyx^{-1} \rangle \simeq F_2,$$

where  $x = x_3$

**Example 3.** Let  $b_1 = b_2^{-1} = \sigma_1 \rho_1 \sigma_1$  and  $b = b_1 \rho_2 b_2 \rho_2$ . It is easy to check that for this virtual braid its group

$$G_v(\widehat{b}) = \langle x_1, x_2, y \mid yx_1y^{-1} = x_2 \rangle$$

is an HNN-extension of the free group  $\langle x_1, x_2 \rangle$  with cyclic associated subgroups. Therefore the closure of  $b$  is a non trivial virtual link.

The following Propositions establish the relations between the different notions of groups of (virtual) knots.

**Proposition 5.** *Let  $K$  be a classical knot then*

$$G_v(K) = \mathbb{Z} * \pi_1(S^3 \setminus N(K)), \quad \mathbb{Z} = \langle y \rangle.$$

*Proof.* As previously recalled, if  $\beta \in B_n$  is a braid such that its Alexander closure is isotopic to  $K$ , the group  $\pi_1(S^3 \setminus N(K))$  admits the presentation  $\langle x_1, x_2, \dots, x_n \mid x_i = \beta(x_i), i = 1, \dots, n \rangle$ .

The claim follows therefore from the remark that the representation  $\psi$  of  $VB_n$  in  $\text{Aut}(F_{n+1})$  restraint to  $B_n$  coincides with the usual Artin representation composed with the natural inclusion  $\iota : \text{Aut}(F_n) \rightarrow \text{Aut}(F_{n+1})$ .  $\square$

The Proposition below will be proved at the end of Section 5.

**Proposition 6.** *Let  $vK$  be a virtual knot.*

- (1) *The group  $G_v(vK)/\langle\langle y \rangle\rangle$  is isomorphic to  $G_{K,v}(vK)$ ;*
- (2) *The abelianization of  $G_v(vK)$  is isomorphic to  $\mathbb{Z}^2$ .*

We recall that a group  $G$  is residually finite if for any nontrivial element  $g \in G$  there exists a finite-index subgroup of  $G$  which does not contain  $g$ .

According to [17] every knot group of a classical knot is residually finite, while the virtual knot groups defined by Kauffman does not need to have this property (see [29]).

**Proposition 7.** *Let  $K$  be a classical knot then  $G_v(K)$  is residually finite.*

*Proof.* The free product of two residually finite groups is residually finite [16].  $\square$

We do not know whether  $G_v(vK)$  is residually finite for any virtual knot  $vK$ .

**Remark 8.** We can modify Wirtinger method to virtual links using previous representation of  $VB_n$ . Let us now consider each virtual crossing as the common endpoint of four different arcs and mark the obtained arcs with labels  $x_1, \dots, x_m$  and add an element  $y$  of this set. Now consider the group  $G_y(vK)$  generated by elements  $x_1, \dots, x_m, y$  under the usual Wirtinger relations for classical crossings plus the following relations for virtual crossing: the labeling of arcs  $x_k, x_l, x_i, x_j$  meeting in a virtual crossing as in Figure 7 respect the relations  $x_i = y^{-1}x_l y$  and  $x_j = yx_k y^{-1}$ . One can easily check that  $G_y(vK)$  is actually invariant under virtual and mixed Reidemeister moves and hence is an invariant for virtual knots. Arguments in Proposition 11 can be therefore easily adapted to this case to prove that  $G_y(vK)$  is isomorphic  $G_v(vK)$ .

**Remark 9.** Using the "Wirtinger like" labeling proposed in previous Remark it is also possible to extend the notion of group system to  $G_v(vK)$ . We recall that the group system of a classical knot  $K$  is given by the knot group, a meridian and its corresponding longitude: in the case of  $G_v(vK)$  we can call meridian the generator corresponding to any arc. The longitude corresponding to this meridian is defined as follows: we go along the diagram starting from this arc (say  $a_i$ ) and we write  $a_k^{-1}$  when passing under  $a_k$  as in Figure 5 a) and  $a_k$  when passing under  $a_k$  as in Figure 5 b). On the other hand if we encounter a virtual crossing according Figure 7 we write  $y$  when we pass from  $x_l$  to  $x_i$  and we write  $y^{-1}$  when we pass from  $x_k$  to  $x_j$ . Finally we write  $a_i^{-m}$  where  $m$  is the length (the sum of exponents) of the word that we wrote following the diagram. It is easy to verify that such an element belongs to the commutator subgroup of  $G_v(vK)$  and that meridian and longitude are well defined under generalized Reidemeister moves.

## 5. GROUPS OF WELDED LINKS

As in the case of virtual links, Wirtinger method can be naturally adapted also to welded links: it suffices to check that also the forbidden relation  $F1$  is preserved by Wirtinger labeling. Given a welded link  $wL$  we can therefore define the group of the welded link  $wL$ ,  $G_{K,w}(wL)$ , as the group obtained extending the Wirtinger method to welded diagrams, forgetting all welded crossings. This fact has been already remarked: see for instance [32], where the notion of *group system* is extended to welded knot diagrams.

Notice that the forbidden move  $F2$  is not preserved by above method and then that the lower Wirtinger presentation does not extend to welded link diagrams.

On the other hand, considering welded links as closure of welded braids we can deduce as in the virtual case another possible definition of group of a welded link. Let  $wL = \widehat{\beta_w}$  be the closure of the welded braid  $\beta_w \in WB_n$ . Define

$$G_w(wL) = \langle x_1, x_2, \dots, x_n \mid x_i = \beta_w(x_i), i = 1, \dots, n \rangle.$$

**Theorem 10.** *The group  $G_w(wL)$  is an invariant of the virtual link  $wL$ .*

*Proof.* We recall that two welded braids have equivalent closures as welded links if and only if they are related by a finite sequence of the following moves [19]:

- 1) a braid move (which is a move corresponding to a defining relation of the welded braid group),
- 2) a conjugation in the welded braid group,
- 3) a right stabilization of positive, negative or welded type, and its inverse operation,

We have to check that under all moves 1) - 3), the group  $G_w(wL)$  does not change: the move 1) is evident and for the moves 2) and 3) we can repeat verbatim the proof of Theorem 4.  $\square$

**Proposition 11.** *Let  $wK$  be a welded knot. The groups  $G_w(wK)$  and  $G_{K,w}(wK)$  are isomorphic.*

*Proof.* Any generator of  $WB_n$  acts trivially on generators of  $F_n$  except a pair of generators: more precisely we have respectively that

$$\begin{aligned} x_i \cdot \sigma_i &= x_i x_{i+1} x_i^{-1} &:= u^+(x_i, x_{i+1}), \\ x_{i+1} \cdot \sigma_i &= x_i &:= v^+(x_i, x_{i+1}), \\ x_j \cdot \sigma_i &= x_j & \quad j \neq i, i+1, \end{aligned}$$

$$\begin{aligned} x_i \cdot \sigma_i^{-1} &= x_{i+1} &:= u^-(x_i, x_{i+1}), \\ x_{i+1} \cdot \sigma_i^{-1} &= x_i^{-1} x_i x_{i+1} &:= v^-(x_i, x_{i+1}), \\ x_j \cdot \sigma_i^{-1} &= x_j & \quad j \neq i, i+1, \end{aligned}$$

$$\begin{aligned} x_i \cdot \alpha_i &= x_{i+1} &:= u^\circ(x_i, x_{i+1}), \\ x_{i+1} \cdot \alpha_i &= x_i &:= v^\circ(x_i, x_{i+1}), \\ x_j \cdot \alpha_i &= x_j & \quad j \neq i, i+1. \end{aligned}$$

Let  $\beta_w$  a welded braid with closure equivalent to  $wK$ . Let us regard to the diagram representing the closure of  $\beta_w$ ,  $\widehat{\beta_w}$ , as a directed graph and denote the edges of  $\widehat{\beta_w}$ , the generators of  $G_{K,w}(wK)$  by labels  $x_1, \dots, x_m$ : according to Wirtinger method recalled in Section 4, for each crossing of  $\widehat{\beta_w}$  (see Figure 7) we have the following relations:

- (1) If the crossing is positive  $x_i = x_l$  and  $x_j = x_l^{-1} x_k x_l$ ;
- (2) If the crossing is negative  $x_i = x_k x_l x_k^{-1}$  and  $x_j = x_k$  where the word  $u^+(x_k, x_l)$  and  $v^+(x_k, x_l)$  are the words defined above;
- (3) If the crossing is welded  $x_i = x_l$  and  $x_j = x_k$  are the words defined above.

Now let us remark that we have that  $x_l = u^-(x_k, x_l)$ ,  $x_l^{-1} x_k x_l = v^-(x_k, x_l)$ ,  $x_k x_l x_k^{-1} = u^+(x_k, x_l)$ ,  $x_k = v^+(x_k, x_l)$ ,  $x_l = u^\circ(x_k, x_l)$  and  $x_k = v^\circ(x_k, x_l)$  and let us recall that the action of welded braids is from left to right ( $\beta_1 \beta_2(x_i)$  denotes  $((x_i) \beta_1) \beta_2$ ). Therefore if we label  $x_i$  (for  $i = 1, \dots, n$ ) the arcs on the top of the braid  $\beta_w$ , the arcs on the bottom will be labelled by  $\beta_w^{-1}(x_i)$  (for  $i = 1, \dots, n$ ). Since we are considering the closure of  $\beta_w$ , we identify

labels on the bottom with corresponding labels on the top and we deduce that a possible presentation of  $G_{K,w}(wK)$  is :

$$G_{K,w}(wK) = \langle x_1, x_2, \dots, x_n \mid x_i = \beta_w^{-1}(x_i), i = 1, \dots, n \rangle.$$

and therefore  $G_{K,w}(wK)$  is clearly isomorphic to  $G_w(wK)$ .  $\square$

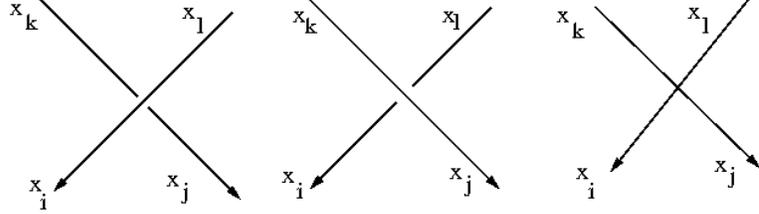


FIGURE 7. Wirtinger-like labelling

Let  $\mathcal{D}$  be a diagram representing the immersion of a circle in the plane, where double points can be presented with two different labelings:

- with the usual overpasses/underpasses information;
- as "singular" points.

Clearly  $\mathcal{D}$  will represent a virtual knot diagram if we allow virtual local moves or respectively a welded knot diagram if we allow welded local moves.

In the following we will write  $G_v(\mathcal{D})$  and  $G_{K,v}(\mathcal{D})$  when we consider  $\mathcal{D}$  as a virtual knot diagram, while we will set  $G_w(\mathcal{D})$  and  $G_{K,w}(\mathcal{D})$  when we see  $\mathcal{D}$  as a welded knot diagram. The following Proposition is a consequence of the fact that Wirtinger labeling is preserved by  $F1$  moves.

**Proposition 12.** *Let  $\mathcal{D}$  be a diagram as above. Then  $G_{K,v}(\mathcal{D}) = G_{K,w}(\mathcal{D})$ .*

Remark also that if  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are equivalent as welded diagrams then  $G_{K,v}(\mathcal{D}_1) = G_{K,v}(\mathcal{D}_2)$  even if  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are not equivalent as virtual diagrams.

*Proof of Proposition 6.* Let  $\beta_v$  be a virtual braid with closure equivalent to  $vK$ . Let  $F_{n+1} = \langle x_1, x_2, \dots, x_n, y \rangle$  and  $F_n = \langle x_1, x_2, \dots, x_n \rangle$ . As in recalled in Proposition 3, the projection  $p_n : F_{n+1} \rightarrow F_{n+1}/\langle\langle y \rangle\rangle \simeq F_n$  induces a surjective map  $p_n^\# : \psi(VB_n) \rightarrow WB_n$  and therefore that  $G_v(vK)/\langle\langle y \rangle\rangle$  is isomorphic to  $G_w(wK) = \langle x_1, x_2, \dots, x_n \mid x_i = \beta_w(x_i), i = 1, \dots, n \rangle$ , where  $\beta_w = p_n^\#(\psi(\beta_v))$ .

The first claim is therefore a straightforward consequence of Proposition 12.

The second claim of the Proposition follows easily from the fact that any generator of type  $x_i$  is a conjugated of  $x_{\pi(i)}$  where  $\pi \in S_n$  is of order  $n$ . Therefore in the abelianization all generators of type  $x_i$  have the same image.  $\square$

From Propositions 6 and 11 therefore it follows that:

**Proposition 13.** *Let  $\mathcal{D}$  be a diagram as above. Then  $G_v(\mathcal{D})/\langle\langle y \rangle\rangle = G_w(\mathcal{D})$ .*

## 6. WADA GROUPS FOR VIRTUAL AND WELDED LINKS

In [31] Wada found several representations of  $B_n$  in  $\text{Aut}(F_n)$  which, by the usual braid closure, provide group invariants of links. These representation are of the following special form: any generator (and therefore its inverse) of  $B_n$  acts trivially on generators of  $F_n$  except a pair of generators:

$$\begin{aligned} x_i \cdot \sigma_i &= u^+(x_i, x_{i+1}), \\ x_{i+1} \cdot \sigma_i &= v^+(x_i, x_{i+1}), \\ x_j \cdot \sigma_i &= x_j \quad j \neq i, i+1. \end{aligned}$$

$$\begin{aligned} x_i \cdot \sigma_i^{-1} &= u^-(x_i, x_{i+1}), \\ x_{i+1} \cdot \sigma_i^{-1} &= v^-(x_i, x_{i+1}), \\ x_j \cdot \sigma_i^{-1} &= x_j \quad j \neq i, i+1. \end{aligned}$$

where  $u$  and  $v$  are now words in the generators  $a, b$ , with  $\langle a, b \rangle \simeq F_2$ . In [31] Wada found four families of representations providing group invariants of links (they are types 4 – 7 in Wada's paper):

- Type 1:  $u^+(x_i, x_{i+1}) = x_i^h x_{i+1} x_i^{-h}$  and  $v^+(x_i, x_{i+1}) = x_i$ ;
- Type 2:  $u^+(x_i, x_{i+1}) = x_i x_{i+1}^{-1} x_i$  and  $v^+(x_i, x_{i+1}) = x_i$ ;
- Type 3:  $u^+(x_i, x_{i+1}) = x_i x_{i+1} x_i$  and  $v^+(x_i, x_{i+1}) = x_i^{-1}$ ;
- Type 4:  $u^+(x_i, x_{i+1}) = x_i^2 x_{i+1}$  and  $v^+(x_i, x_{i+1}) = x_{i+1}^{-1} x_i^{-1} x_{i+1}$ .

As in the case of Artin representation we can ask if these representations extend to welded braids providing group invariants for welded links. More precisely, let  $\chi_k : G_{WB_n} \rightarrow \text{Aut}(F_n)$  (for  $k = 1, \dots, 4$ ) be the set map from the set of generators  $G_{WB_n} := \{\sigma_1, \dots, \sigma_{n-1}, \alpha_1, \dots, \alpha_{n-1}\}$  of  $WB_n$  to  $\text{Aut}(F_n)$  which associates to any generators  $\sigma_i$  the Wada representation of type  $k$  and to any generators  $\alpha_i$  the usual automorphism

$$\begin{aligned} x_i \cdot \alpha_i &= x_{i+1} & := u^\circ(x_i, x_{i+1}) \\ x_{i+1} \cdot \alpha_i &= x_i & := v^\circ(x_i, x_{i+1}) \\ x_j \cdot \alpha_i &= x_j & \quad j \neq i, i+1. \end{aligned}$$

**Proposition 14.** *The set map  $\chi_k : G_{WB_n} \rightarrow \text{Aut}(F_n)$  (for  $k = 1, \dots, 4$ ) induces a homomorphism  $\chi_k : WB_n \rightarrow \text{Aut}(F_n)$  if and only if  $k = 1, 2$ .*

*Proof.* For  $k = 1, 2$  the proof is a straightforward verification that relations of  $WB_n$  hold in  $\text{Aut}(F_n)$ : for  $k = 3, 4$  it suffices to remark that relation of type  $\alpha_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \alpha_{i+1}$  is not preserved.  $\square$

When  $k = 1, 2$  we can therefore define for any  $\beta_w \in WB_n$  the group

$$\mathcal{W}_k(\beta_w) = \langle x_1, x_2, \dots, x_n \mid x_i = (\chi_k(\beta_w))(x_i), i = 1, \dots, n \rangle$$

that we will call Wada group of type  $k$  for  $\beta_w \in WB_n$ .

**Theorem 15.** *The Wada group  $\mathcal{W}_k$  is a link invariant for  $k = 1, 2$ .*

*Proof.* One can repeat almost verbatim the arguments from [31] for classical braids. Notice that  $\mathcal{W}_k(\beta_w)$  is the group of co-invariants of  $\beta_w$ , i.e. the maximal quotient of  $F_n$  on which  $\chi_k(\beta_w)$  acts trivially. Since the group of co-invariants is invariant up to conjugation by an automorphism, we deduce that conjugated welded braids have isomorphic Wada groups. To prove the statement it is therefore sufficient to verify that  $\mathcal{W}_k(\beta_w) = \langle x_1, x_2, \dots, x_n \mid x_i = (\chi_k(\beta_w))(x_i), i = 1, \dots, n \rangle$  and  $\mathcal{W}_k^{stab}(\beta_w) = \langle x_1, x_2, \dots, x_n x_{n+1} \mid x_i = (\chi_k(\beta_w \sigma_n))(x_i), i = 1, \dots, n+1 \rangle$  are isomorphic: this a straightforward computation similar to the case 3) in Theorem 4, the key point being that  $(\chi_k(\beta_w \sigma_n))(x_{n+1}) = x_n$  for  $k = 1, 2$  (see also Section 2 of [31]).  $\square$

In [10] Wada representations of type 2, 3, 4 have been extended to group invariants for virtual links using a Wirtinger like presentation of virtual link diagrams: contrarily to the classical case these groups are not necessarily isomorphic.

Analogously it would be interesting to understand the geometrical meaning of Wada groups of welded links. In this perspective, Proposition 16 shows that Wada representations of type 1 and 2 are not equivalent.

We will say that two representations  $\omega_1 : WB_n \rightarrow \text{Aut}(F_n)$  and  $\omega_2 : WB_n \rightarrow \text{Aut}(F_n)$  are equivalent, if there exist automorphisms  $\phi \in \text{Aut}(F_n)$  and  $\mu : WB_n \rightarrow WB_n$  such that

$$\phi^{-1} \omega_1(\beta_w) \phi = \omega_2(\mu(\beta_w)),$$

for any  $\beta_w \in G_{WB_n}$ .

**Proposition 16.** *Wada representations of type 1 and 2 are not equivalent.*

*Proof.* The proof is the same as in Proposition A.1 of [12]: the claim follows from considering the induced action on  $H_1(F_n)$ . Under Wada representations of type 1 a welded braid has evidently finite order as automorphism of  $H_1(F_n)$  while we have that  $\chi_2(\sigma_i^t)[x_i] = (t+1)[x_i] - t[x_2]$  for all  $t \in \mathbb{N}$ , where  $[u]$  denote the equivalence class in  $H_1(F_n)$  of an element  $u \in F_n$ .  $\square$

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