



**HAL**  
open science

## Torsion of chiral Cosserat elastic rods

D. Iesan

► **To cite this version:**

D. Iesan. Torsion of chiral Cosserat elastic rods. *European Journal of Mechanics - A/Solids*, 2010, 29 (6), pp.990. 10.1016/j.euromechsol.2010.08.001 . hal-00687343

**HAL Id: hal-00687343**

**<https://hal.science/hal-00687343>**

Submitted on 13 Apr 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Accepted Manuscript

Title: Torsion of chiral Cosserat elastic rods

Authors: D. Iesan

PII: S0997-7538(10)00099-9

DOI: [10.1016/j.euromechsol.2010.08.001](https://doi.org/10.1016/j.euromechsol.2010.08.001)

Reference: EJMSOL 2634

To appear in: *European Journal of Mechanics / A Solids*

Received Date: 30 November 2009

Revised Date: 19 July 2010

Accepted Date: 4 August 2010

Please cite this article as: Iesan, D. Torsion of chiral Cosserat elastic rods, *European Journal of Mechanics / A Solids* (2010), doi: [10.1016/j.euromechsol.2010.08.001](https://doi.org/10.1016/j.euromechsol.2010.08.001)

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



# Torsion of chiral Cosserat elastic rods

D. Ieşan

*Department of Mathematics, "Al.I. Cuza" University and "O. Mayer" Institute of Mathematics, Romanian Academy, 700506 Iaşi, Romania*

---

## Abstract

This paper is concerned with the torsion of isotropic chiral Cosserat elastic cylinders. First, the generalized plane strain problem is defined and an existence result is presented. Then, the three-dimensional problem is reduced to the study of some generalized plane strain problems. In general, the torsion of the cylinder is accompanied by bending and extension. The method is applied to study the torsion of a circular cylinder.

*Key words:* Chiral materials; Cosserat cylinder; Torsion; Bending

---

## 1 Introduction

In recent years the behaviour of chiral materials has been the object of intensive research. **A body which is not isotropic with respect to inversion is called chiral. Chirality can be observed in some carbon nanotubes, biological molecules (DNA), bones, honeycomb structures as well as in composites with inclusions.**

It is well-known that chiral effects cannot be described within classical elasticity (see Lakes, 2001). The theory of Cosserat elasticity is adequate to describe the mechanical behaviour of chiral elastic bodies (Lakes, 2001; Donescu et al., 2009). The Cosserat theory studies continua with oriented particles which have the six degrees of freedom of a rigid body. The theory of Cosserat elastic solids was extensively studied in the last decades. For the historical development and the analysis of various results on the subject see Nowacki (1981), Ciarletta and Ieşan (1993), Eringen (1999), Rubin (2000), Dyszlewicz (2004), Jasiuk and Ostoja-Starzewski (2004). A large number of papers have been devoted to the study of chiral Cosserat elastic bodies (Nowacki, 1977; Lakes and

---

*Email address:* [iesan@uaic.ro](mailto:iesan@uaic.ro) (D. Ieşan).

Benedict, 1982; Dyszlewicz and Kolodziej, 1987). This class of materials continues to attract attention both from the theoretical and the technical point of view (Healey, 2002; Dyszlewicz, 2004; Teodorescu et al., 2005; Natroshvili et al., 2006; Khurana and Tomar, 2009). A Cosserat material is called isotropic chiral if its symmetry group equals the proper orthogonal group. In the linear theory an isotropic chiral Cosserat elastic material is characterized by nine constitutive coefficients.

In this paper we study the deformation of homogeneous and isotropic chiral Cosserat elastic cylinders subjected to torsion, extension and bending. This work is motivated by the recent interest in the using of the isotropic chiral Cosserat elastic solid as model for cellular solids and composite materials (Prall and Lakes, 1997; Lakes, 1998, 2001; Donescu et al., 2009; Chandraseker et al., 2009) and for bones (Lakes et al., 1983). We note that Lakes and Benedict (1982) studied the problem of a chiral Cosserat elastic rod of circular cross section, stretched axially and free to twist. In this paper we consider the problem of extension, bending and torsion of a right cylinder with general cross section. First, we introduce the generalized plane strain problem and present an existence result. In contrast with the case of centrosymmetric solids, in the generalized plane strain of a chiral Cosserat elastic cylinder, all components of displacement vector and microrotation vector are different from zero. We introduce four special generalized plane strain problems characterized by external data which depend only on the constitutive coefficients. The solutions of these problems are used to solve the torsion problem. We show that, in general, the torsion of an isotropic chiral Cosserat elastic cylinder is accompanied by extension and bending. The results are applied to study the torsion of a circular cylinder. We determine the solutions of the auxiliary generalized plane strain problems for a circular domain. Then, the displacement and microrotation vector fields in the torsion problem are determined.

## 2 Basic equations

Throughout this paper  $B$  denotes the interior of a right cylinder of length  $h$  with open cross-section  $\Sigma$  and lateral boundary  $\Pi$  (Fig. 1). We call  $\partial B$  the boundary of  $B$ , and designate by  $n_i$  the components of the outward unit normal of  $\partial B$ . Throughout this paper a rectangular cartesian coordinate system  $Ox_k$ , ( $k = 1, 2, 3$ ), is used. The rectangular cartesian coordinate frame is chosen such that the  $x_3$ -axis is parallel to the generators of  $B$  and the  $x_1Ox_2$  plane contains one of terminal cross-sections. We denote by  $\Sigma_1$  and  $\Sigma_2$ , respectively, the cross-section located at  $x_3 = 0$  and  $x_3 = h$ .

Fig. 1. A general prismatic rod

We assume that the generic cross-section  $\Sigma$  is a simply connected regular region. Let  $L$  be the boundary of the region  $\Sigma$ . We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers  $(1, 2)$ , whereas Latin subscripts (unless otherwise specified) to the range  $(1, 2, 3)$ ; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. Letters in boldface stand for tensors of an order  $p \geq 1$ , and if  $\mathbf{v}$  has the order  $p$ , we write  $v_{ij\dots s}$  ( $p$  subscripts) for the components of  $\mathbf{v}$  in the cartesian coordinate frame.

Throughout this paper we consider the linear theory of homogeneous and isotropic chiral Cosserat elastic bodies. Let  $\mathbf{u}$  be the displacement vector field on  $B$ . We denote by  $\boldsymbol{\varphi}$  the microrotation vector field. The strain measures are given by

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad (1)$$

where  $\varepsilon_{ijk}$  is the alternating symbol. Let  $t_{ij}$  be the stress tensor and let  $m_{ij}$  be the couple stress tensor. The equilibrium equations, in the absence of body loads, are

$$t_{ji,j} = 0, \quad m_{ji,j} + \varepsilon_{irs}t_{rs} = 0. \quad (2)$$

The constitutive equations can be presented in the form (Lakes and Benedict, 1982; Dyszlewicz, 2004)

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu + \kappa)e_{ij} + \mu e_{ji} + C_1\kappa_{ss}\delta_{ij} + C_2\kappa_{ji} + C_3\kappa_{ij}, \\ m_{ij} &= \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + C_1e_{rr}\delta_{ij} + C_2e_{ji} + C_3e_{ij}, \end{aligned} \quad (3)$$

where  $\delta_{ij}$  is Kronecker's delta, and  $\gamma, \mu, \kappa, \alpha, \beta, \gamma, C_1, C_2$  and  $C_3$  are constitutive constants. In the case of a centrosymmetric material the coefficients  $C_1, C_2$  and  $C_3$  are equal to zero. The surface force and the surface moment acting at a regular point of  $\partial B$  are given by

$$t_i = t_{ji}n_j, \quad m_i = m_{ji}n_j.$$

We assume that the considered cylinder is free of lateral loads. Thus, we have the following conditions

$$t_{\alpha i}n_\alpha = 0, \quad m_{\alpha i}n_\alpha = 0 \quad \text{on } \Pi. \quad (4)$$

We suppose that the cylinder  $B$  is subjected to extension, bending and torsion. Let  $\mathbf{R} = (0, 0, R_3)$  and  $\mathbf{M} = (M_1, M_2, M_3)$  be prescribed vectors representing the resultant force and the resultant moment about  $O$  of the tractions acting on  $\Sigma_1$ . On  $\Sigma_2$  there are tractions applied so as to satisfy the equilibrium conditions of the body. Consequently, for  $x_3 = 0$  we have the conditions

$$\int_{\Sigma_1} t_{3\alpha} da = 0, \quad (5)$$

$$\int_{\Sigma_1} t_{33} da = -R_3, \quad (6)$$

$$\int_{\Sigma_1} (x_\alpha t_{33} - \varepsilon_{3\alpha\beta} m_{3\beta}) da = \varepsilon_{\alpha\beta 3} M_\beta, \quad (7)$$

$$\int_{\Sigma_1} (\varepsilon_{\alpha\beta 3} x_\alpha t_{3\beta} + m_{33}) da = -M_3. \quad (8)$$

The problem consists in finding of the functions  $u_i$  and  $\varphi_i$  of class  $C^2(B) \cap C^1(\bar{B})$  which satisfy the equations (1)-(3) on  $B$ , the conditions (4) on the lateral surface, and the conditions (5)-(8) on the end  $\Sigma_1$ . We have introduced the mechanical loads  $R_3$  and  $M_\alpha$  in order to emphasize the torsional effects on the deformation of the cylinder.

We assume that  $\Sigma$  is  $C^\infty$ -smooth. We have chosen this hypothesis in order to best describe the method for the solution of the problem. We consider only a "  $C^\infty$ -theory" but it is possible to get a classical solution for more general assumptions of regularity. In what follows we suppose that the elastic potential is a positive definite quadratic form. The restrictions imposed by this hypothesis on the constitutive coefficients have been presented in various works (Lakes, 1982; Dyszlewicz, 2004). We note only the following inequalities

$$\begin{aligned} \kappa > 0, 2\mu + \kappa > 0, \gamma + \beta > 0, \gamma - \beta > 0, \lambda + 2\mu + \kappa > 0, \\ (\lambda + 2\mu + \kappa)(\alpha + \beta + \gamma) - (C_1 + C_2 + C_3)^2 > 0. \end{aligned} \quad (9)$$

### 3 Generalized plane strain

Let us assume now that a body force  $\mathbf{f}$  and a body couple  $\mathbf{g}$  are prescribed on  $B$ . Moreover, we consider that on the lateral boundary of the cylinder there are prescribed the surface force  $\tilde{\mathbf{t}}$  and the surface moment  $\tilde{\mathbf{m}}$ . We suppose that  $\mathbf{f}, \mathbf{g}, \tilde{\mathbf{t}}$  and  $\tilde{\mathbf{m}}$  are all independent of the axial coordinate. We define the state of generalized plane strain of the cylinder to be that state in which the displacement vector and microrotation vector are independent of the axial coordinate,

$$u_i = u_i(x_1, x_2), \quad \varphi_i = \varphi_i(x_1, x_2), \quad (x_1, x_2) \in \Sigma_1. \quad (10)$$

The above restrictions, in conjunction with the geometrical equations (1) and the constitutive equations (2), imply that  $e_{ij}, \kappa_{ij}, t_{ij}$  and  $m_{ij}$  are all independent of the axial coordinate. The equations (1) reduce to

$$e_{\alpha i} = u_{i,\alpha} + \varepsilon_{i\alpha j} \varphi_j, \quad e_{3i} = \varepsilon_{i3\beta} \varphi_\beta, \quad \kappa_{\alpha i} = \varphi_{i,\alpha}, \quad \kappa_{3i} = 0. \quad (11)$$

The constitutive equations for a generalized plane strain are given by

$$\begin{aligned}
t_{\alpha\beta} &= \lambda e_{\rho\rho} \delta_{\alpha\beta} + (\mu + \kappa) e_{\alpha\beta} + \mu e_{\beta\alpha} + C_1 \kappa_{\rho\rho} \delta_{\alpha\beta} + C_2 \kappa_{\beta\alpha} + C_3 \kappa_{\alpha\beta}, \\
t_{\alpha 3} &= (\mu + \kappa) e_{\alpha 3} + \mu e_{3\alpha} + C_3 \kappa_{\alpha 3}, \\
t_{3\alpha} &= (\mu + \kappa) e_{3\alpha} + \mu e_{\alpha 3} + C_2 \kappa_{\alpha 3}, \quad t_{33} = \lambda e_{\rho\rho} + C_1 \kappa_{\rho\rho}, \\
m_{\nu\eta} &= \alpha \kappa_{\rho\rho} \delta_{\nu\eta} + \beta \kappa_{\eta\nu} + \gamma \kappa_{\nu\eta} + C_1 e_{\rho\rho} \delta_{\nu\eta} + C_2 e_{\eta\nu} + C_3 e_{\nu\eta}, \\
m_{\alpha 3} &= \gamma \kappa_{\alpha 3} + C_2 e_{3\alpha} + C_3 e_{\alpha 3}, \\
m_{3\alpha} &= \beta \kappa_{\alpha 3} + C_2 e_{\alpha 3} + C_3 e_{3\alpha}, \quad m_{33} = \alpha \kappa_{\rho\rho} + C_1 e_{\rho\rho}.
\end{aligned} \tag{12}$$

The equations of equilibrium can be written as

$$t_{\beta i, \beta} + f_i = 0, \quad m_{\beta i, \beta} + \varepsilon_{irst} t_{rs} + g_i = 0, \tag{13}$$

on  $\Sigma_1$ . The conditions on the lateral surface become

$$t_{\beta i} n_\beta = \tilde{t}_i, \quad m_{\beta i} n_\beta = \tilde{m}_i \text{ on } L. \tag{14}$$

We assume that  $f_i, g_i, \tilde{t}_i$  and  $\tilde{m}_i$  are functions of class  $C^\infty$ .

The generalized plane strain problem consists in finding of the displacement  $u_i$  and microrotations  $\varphi_i$  which satisfy the equations (11)-(13) on  $\Sigma_1$ , and the boundary conditions (14) on  $L$ . In view of (11) and (12), the equations of equilibrium can be expressed in terms of the functions  $u_i$  and  $\varphi_i$  in the form

$$\begin{aligned}
(\mu + \kappa) \Delta u_\alpha + (\lambda + \mu) u_{\beta, \beta\alpha} + \kappa \varepsilon_{\alpha\beta 3} \varphi_{3, \beta} + C_3 \Delta \varphi_\alpha + (C_1 + C_2) \varphi_{\beta, \beta\alpha} + f_\alpha &= 0, \\
(\mu + \kappa) \Delta u_3 + \kappa \varepsilon_{3\beta\alpha} \varphi_{\alpha, \beta} + C_3 \Delta \varphi_3 + f_3 &= 0, \\
C_3 \Delta u_\nu + (C_1 + C_2) u_{\rho, \rho\nu} + \kappa \varepsilon_{\nu\beta 3} u_{3, \beta} + \gamma \Delta \varphi_\nu + \\
+ (\alpha + \beta) \varphi_{\rho, \rho\nu} + 2(C_3 - C_2) \varepsilon_{\nu\eta 3} \varphi_{3, \eta} - 2\kappa \varphi_\nu + g_\nu &= 0, \\
C_3 \Delta u_3 + \kappa \varepsilon_{3\nu\eta} u_{\eta, \nu} + \gamma \Delta \varphi_3 + 2(C_3 - C_2) \varepsilon_{3\nu\eta} \varphi_{\eta, \nu} - 2\kappa \varphi_3 + g_3 &= 0,
\end{aligned} \tag{15}$$

on  $\Sigma_1$ , where  $\Delta$  is the two-dimensional Laplacian.

In the case of the generalized plane strain the elastic potential  $W$  is given by

$$\begin{aligned}
2W &= \lambda e_{\rho\rho} e_{\nu\nu} + (\mu + \kappa) e_{\alpha\beta} e_{\alpha\beta} + \mu e_{\beta\alpha} e_{\alpha\beta} + (\mu + \kappa) e_{\alpha 3} e_{\alpha 3} + \\
&+ 2\mu e_{\alpha 3} e_{3\alpha} + (\mu + \kappa) e_{3\alpha} e_{3\alpha} + \alpha \kappa_{\rho\rho} \kappa_{\nu\nu} + \beta \kappa_{\eta\nu} \kappa_{\nu\eta} + \\
&+ \gamma \kappa_{\nu\eta} \kappa_{\nu\eta} + \gamma \kappa_{\alpha 3} \kappa_{\alpha 3} + 2C_1 \kappa_{\rho\rho} e_{\nu\nu} + 2C_2 e_{\alpha\beta} \kappa_{\beta\alpha} + \\
&+ 2C_3 e_{\alpha\beta} \kappa_{\alpha\beta} + 2C_2 e_{3\alpha} \kappa_{\alpha 3} + 2C_3 e_{\alpha 3} \kappa_{\alpha 3}.
\end{aligned} \tag{16}$$

Let

$$u_1^* = c_1 - c_4 x_2, \quad u_2^* = c_2 + c_4 x_1, \quad u_3^* = c_3, \quad \varphi_1^* = 0, \quad \varphi_2^* = 0, \quad \varphi_3^* = c_4, \tag{17}$$

where  $c_s$ , ( $s = 1, 2, 3, 4$ ), are arbitrary constants. Then  $U^* = (u_1^*, u_2^*, u_3^*, \varphi_1^*, \varphi_2^*, \varphi_3^*)$  is called a plane rigid vector field.

**Theorem 1.** *If the elastic potential  $W$  is a positive definite quadratic form, then any two solutions of the generalized plane strain problem (11) – (14) are equal modulo a plane rigid vector field.*

**Proof.** In view of (11)-(13) and (16), we get

$$\begin{aligned} 2W &= t_{\beta i} e_{\beta i} + t_{3i} e_{3i} + m_{\beta i} \kappa_{\beta i} = \\ &= (t_{\beta i} u_i + m_{\beta i} \varphi_i)_{,\beta} + f_i u_i + g_i \varphi_i. \end{aligned}$$

With the help of the divergence theorem we obtain

$$2 \int_{\Sigma_1} W da = \int_L (t_{\beta i} u_i + m_{\beta i} \varphi_i) n_{\beta} ds + \int_{\Sigma_1} (f_i u_i + g_i \varphi_i) da. \quad (18)$$

Let  $(u'_i, \varphi'_i)$  and  $(u''_i, \varphi''_i)$  be two solutions of the problem (11)-(14). We denote  $u_i^0 = u'_i - u''_i$  and  $\varphi_i^0 = \varphi'_i - \varphi''_i$ . The functions  $u_i^0$  and  $\varphi_i^0$  satisfy the generalized plane strain problem corresponding to null data. Thus, from (18) we find that

$$\int_{\Sigma_1} W^0 da = 0, \quad (19)$$

where  $W^0$  is the elastic potential corresponding to  $u_i^0$  and  $\varphi_i^0$ . Since  $W^0$  is positive definite, we find  $e_{ji}^0 = 0$ , and  $\kappa_{\alpha i}^0 = 0$  on  $\Sigma_1$ . These equations imply that  $(u_1^0, u_2^0, u_3^0, \varphi_1^0, \varphi_2^0, \varphi_3^0)$  is a plane rigid vector field.  $\square$

The six-dimensional vector  $U^* = (u_1^*, u_2^*, u_3^*, \varphi_1^*, \varphi_2^*, \varphi_3^*)$  defined by (17) can be expressed in the form

$$U^* = \sum_{k=1}^4 c_k U^{(k)},$$

where

$$\begin{aligned} U^{(1)} &= (1, 0, 0, 0, 0, 0), U^{(2)} = (0, 1, 0, 0, 0, 0), \\ U^{(3)} &= (0, 0, 1, 0, 0, 0), U^{(4)} = (-x_2, x_1, 0, 0, 0, 1). \end{aligned} \quad (20)$$

Let us consider the equations (11)-(13) on  $\Sigma_1$  with the boundary conditions

$$t_{\beta i} n_{\beta} = 0, m_{\beta i} n_{\beta} = 0 \text{ on } L. \quad (21)$$

Following Fichera (1972), a  $C^\infty$  solution of the boundary valued problem (11)-(13), (21) exists if and only if

$$\int_{\Sigma_1} \sum_{j=1}^4 H_j U_j^{(k)} da = 0, \quad (k = 1, 2, 3, 4), \quad (22)$$

where  $H = (H_1, H_2, \dots, H_6)$  with  $H_i = f_i$ ,  $H_{3+i} = g_i$ , and  $U^{(k)} = (U_1^{(k)}, U_2^{(k)}, \dots, U_6^{(k)})$ ,  $U_1^{(1)} = 1$ ,  $U_{1+s}^{(1)} = 0$ , ( $s = 1, 2, \dots, 5$ ),  $U_2^{(2)} = 1$ ,  $U_1^{(2)} = U_{2+m}^{(2)} = 0$ ,



( $m = 1, 2, 3, 4$ ),  $U_3^{(3)} = 1$ ,  $U_\alpha^{(3)} = U_{3+p}^{(3)} = 0$ ,  $U_1^{(4)} = -x_2$ ,  $U_2^{(4)} = x_1$ ,  $U_6^{(4)} = 1$ ,  $U_{2+k}^{(4)} = 0$ . The conditions (22) reduce to

$$\int_{\Sigma_1} f_i a = 0, \quad \int_{\Sigma_1} (x_1 f_2 - x_2 f_1 + g_3) da = 0.$$

It is a simple matter to see that in the case of the boundary value problem (11)-(14), the above conditions are replaced by

$$\int_{\Sigma_1} f_i da + \int_L \tilde{t}_i ds = 0, \quad \int_{\Sigma} (\varepsilon_{\alpha\beta\gamma} x_\alpha f_\beta + g_3) da + \int_L (\varepsilon_{\alpha\beta\gamma} x_\alpha \tilde{t}_\beta + \tilde{m}_3) ds = 0. \quad (23)$$

Thus, we have the following existence result.

**Theorem 2.** *The boundary value problem (11) – (14) has solutions if and only if the relations (23) hold.*

In the next section we will have occasion to use four generalized plane strain problems  $P^{(k)}$ , ( $k = 1, 2, 3, 4$ ). The problem  $P^{(1)}$  corresponds to the body loads

$$f_i^{(1)} = \lambda \delta_{1i}, \quad g_i^{(1)} = (C_1 + C_3 - C_2) \delta_{1i}, \quad (24)$$

and to the boundary data

$$\tilde{t}_\alpha^{(1)} = -\lambda x_1 n_\alpha, \quad \tilde{t}_3^{(1)} = C_2 n_2, \quad \tilde{m}_\rho^{(1)} = -C_1 x_1 n_\rho, \quad \tilde{m}_3^{(1)} = \beta n_2. \quad (25)$$

In the problem  $P^{(2)}$  the body loads and the boundary data are

$$\begin{aligned} f_i^{(2)} &= \lambda \delta_{2i}, \quad g_i^{(2)} = (C_1 + C_3 - C_2) \delta_{2i}, \\ \tilde{t}_\alpha^{(2)} &= -\lambda x_2 n_\alpha, \quad \tilde{t}_3^{(2)} = -C_2 n_1, \quad \tilde{m}_\alpha^{(2)} = -C_1 x_2 n_\alpha, \quad \tilde{m}_3^{(2)} = -\beta n_1. \end{aligned} \quad (26)$$

The problem  $P^{(3)}$  corresponds to the following loading

$$f_i^{(3)} = 0, \quad g_i^{(3)} = 0, \quad \tilde{t}_\alpha^{(3)} = -\lambda n_\alpha, \quad \tilde{t}_3^{(3)} = 0, \quad \tilde{m}_\beta^{(3)} = -C_1 n_\beta, \quad \tilde{m}_3^{(3)} = 0. \quad (27)$$

We denote by  $P^{(4)}$  the problem characterized by the following body loads and boundary tractions

$$\begin{aligned} f_i^{(4)} &= 0, \quad g_\alpha^{(4)} = -\kappa x_\alpha, \quad g_3^{(4)} = 0, \\ \tilde{t}_\alpha^{(4)} &= -C_1 n_\alpha, \quad \tilde{t}_3^{(4)} = -\mu \varepsilon_{3\rho\alpha} x_\rho n_\alpha, \quad \tilde{m}_\rho^{(4)} = -\alpha n_\rho, \quad \tilde{m}_3^{(4)} = C_2 \varepsilon_{3\rho\nu} x_\nu n_\rho. \end{aligned} \quad (28)$$

We note that the necessary and sufficient conditions (23) for the existence of the solution are satisfied for each boundary value problem  $P^{(k)}$ , ( $k = 1, 2, 3, 4$ ). We denote by  $u_i^{(k)}$ ,  $\varphi_i^{(k)}$ ,  $e_{ij}^{(k)}$ ,  $\kappa_{ij}^{(k)}$ ,  $t_{ij}^{(k)}$  and  $m_{ij}^{(k)}$  the displacement, microrotation, strain measures, stress tensor and couple stress tensor in the problem

$P^{(k)}$ , respectively. Clearly, the problems  $P^{(k)}$  are characterized by the geometrical equations

$$e_{\alpha i}^{(k)} = u_{i,\alpha}^{(k)} + \varepsilon_{i\alpha j} \varphi_j^{(k)}, e_{3i}^{(k)} = \varepsilon_{i3\beta} \varphi_\beta^{(k)}, \kappa_{\alpha i}^{(k)} = \varphi_{i,\alpha}^{(k)}, \quad (29)$$

the constitutive equations

$$\begin{aligned} t_{\alpha\beta}^{(k)} &= \lambda e_{\rho\rho}^{(k)} \delta_{\alpha\beta} + (\mu + \kappa) e_{\alpha\beta}^{(k)} + \mu e_{\beta\alpha}^{(k)} + C_1 \kappa_{\rho\rho}^{(k)} \delta_{\alpha\beta} + C_2 \kappa_{\beta\alpha}^{(k)} + C_3 \kappa_{\alpha\beta}^{(k)}, \\ t_{\alpha 3}^{(k)} &= (\mu + \kappa) e_{\alpha 3}^{(k)} + \mu e_{3\alpha}^{(k)} + C_3 \kappa_{\alpha 3}^{(k)}, \\ t_{3\alpha}^{(k)} &= (\mu + \kappa) e_{3\alpha}^{(k)} + \mu e_{\alpha 3}^{(k)} + C_2 \kappa_{\alpha 3}^{(k)}, t_{33}^{(k)} = \lambda e_{\rho\rho}^{(k)} + C_1 \kappa_{\rho\rho}^{(k)}, \\ m_{\nu\eta}^{(k)} &= \alpha \kappa_{\rho\rho}^{(k)} \delta_{\nu\eta} + \beta \kappa_{\eta\nu}^{(k)} + \gamma \kappa_{\nu\eta}^{(k)} + C_1 e_{\rho\rho}^{(k)} \delta_{\nu\eta} + C_2 e_{\eta\nu}^{(k)} + C_3 e_{\nu\eta}^{(k)}, \\ m_{\alpha 3}^{(k)} &= \gamma \kappa_{\alpha 3}^{(k)} + C_2 e_{3\alpha}^{(k)} + C_3 e_{\alpha 3}^{(k)}, m_{3\alpha}^{(k)} = \beta \kappa_{\alpha 3}^{(k)} + C_2 e_{\alpha 3}^{(k)} + C_3 e_{3\alpha}^{(k)}, \\ m_{33}^{(k)} &= \alpha \kappa_{\rho\rho}^{(k)} + C_1 e_{\rho\rho}^{(k)}, \end{aligned} \quad (30)$$

and the equations of equilibrium

$$t_{\beta i, \beta}^{(k)} + f_i^{(k)} = 0, m_{\beta i, \beta}^{(k)} + \varepsilon_{irs} t_{rs}^{(k)} + g_i^{(k)} = 0, \quad (31)$$

on  $\Sigma_1$ , and the following boundary conditions

$$t_{\beta i}^{(k)} n_\beta = \tilde{t}_i^{(k)}, m_{\beta i}^{(k)} n_\beta = \tilde{m}_i^{(k)} \text{ on } L, \quad (32)$$

( $k = 1, 2, 3, 4$ ). Here,  $f_i^{(k)}, g_i^{(k)}, \tilde{t}_i^{(k)}$  and  $\tilde{m}_i^{(k)}$  are defined by (24)-(28).

#### 4 Solution of the problem

In this section we reduce the three-dimensional problem to the study of four generalized plane strain problems. Following Ieşan (1986) we seek the solution of the problem formulated in Section 2 in the form

$$\begin{aligned} u_\alpha &= -\frac{1}{2} a_\alpha x_3^2 + \varepsilon_{3\beta\alpha} a_4 x_\beta x_3 + \sum_{k=1}^4 a_k u_\alpha^{(k)}, \\ u_3 &= (a_1 x_1 + a_2 x_2 + a_3) x_3 + \sum_{k=1}^4 a_k u_3^{(k)}, \\ \varphi_\alpha &= \varepsilon_{3\alpha\beta} a_\beta x_3 + \sum_{k=1}^4 a_k \varphi_\alpha^{(k)}, \\ \varphi_3 &= a_4 x_3 + \sum_{k=1}^4 a_k \varphi_3^{(k)}, \quad (x_1, x_2, x_3) \in B, \end{aligned} \quad (33)$$

where  $u_i^{(k)}$  and  $\varphi_i^{(k)}$  are the displacements and microrotations from the problems  $P^{(k)}$  introduced in Section 3, and  $a_k$ , ( $k = 1, 2, 3, 4$ ), are unknown constants. It follows from (1) and (30) that

$$\begin{aligned} e_{\alpha i} &= \sum_{k=1}^4 a_k e_{\alpha i}^{(k)}, \quad e_{3\alpha} = \varepsilon_{3\beta\alpha} x_\beta a_4 + \sum_{k=1}^4 a_k e_{3\alpha}^{(k)}, \\ e_{33} &= a_1 x_1 + a_2 x_2 + a_3, \quad \kappa_{\alpha i} = \sum_{k=1}^4 a_k \kappa_{\alpha i}^{(k)}, \quad \kappa_{3\alpha} = \varepsilon_{3\alpha\beta} a_\beta, \quad \kappa_{33} = a_4, \end{aligned} \quad (34)$$

where  $e_{ij}^{(k)}$  and  $\kappa_{ij}^{(k)}$  are given by (29). By (34) and the constitutive equations (3) we obtain

$$\begin{aligned} t_{\alpha\beta} &= \lambda(a_1 x_1 + a_2 x_2 + a_3) \delta_{\alpha\beta} + C_1 \delta_{\alpha\beta} a_4 + \sum_{k=1}^4 a_k t_{\alpha\beta}^{(k)}, \\ t_{\alpha 3} &= \mu \varepsilon_{3\beta\alpha} x_\beta a_4 + C_2 \varepsilon_{3\alpha\beta} a_\beta + \sum_{k=1}^4 a_k t_{\alpha 3}^{(k)}, \\ t_{3\alpha} &= (\mu + \kappa) \varepsilon_{3\beta\alpha} x_\beta a_4 + C_3 \varepsilon_{3\alpha\beta} a_\beta + \sum_{k=1}^4 a_k t_{3\alpha}^{(k)}, \\ t_{33} &= (\lambda + \mu + \kappa)(a_1 x_1 + a_2 x_2 + a_3) + (C_1 + C_2 + C_3) a_4 + \sum_{k=1}^4 a_k t_{33}^{(k)}, \quad (35) \\ m_{\nu\eta} &= [\alpha a_4 + C_1(a_1 x_1 + a_2 x_2 + a_3)] \delta_{\nu\eta} + \sum_{k=1}^4 a_k m_{\nu\eta}^{(k)}, \\ m_{\nu 3} &= \beta \varepsilon_{3\nu\rho} a_\rho + C_2 \varepsilon_{3\rho\nu} x_\rho a_4 + \sum_{k=1}^4 a_k m_{\nu 3}^{(k)}, \\ m_{3\nu} &= \gamma \varepsilon_{3\nu\rho} a_\rho + C_3 \varepsilon_{3\rho\nu} x_\rho a_4 + \sum_{k=1}^4 a_k m_{3\nu}^{(k)}, \\ m_{33} &= (\alpha + \beta + \gamma) a_4 + (C_1 + C_2 + C_3)(a_1 x_1 + a_2 x_2 + a_3) + \sum_{k=1}^4 a_k m_{33}^{(k)}, \end{aligned}$$

where  $t_{ij}^{(k)}$  and  $m_{ij}^{(k)}$  are the stress tensor and the couple stress tensor from the problem  $P^{(k)}$ , ( $k = 1, 2, 3, 4$ ).

It is a simple matter to see that the equations of equilibrium (2) and the boundary conditions (4) are satisfied on the basis of equations (31) and the boundary conditions (32). Let us study the end conditions (5). With the help of equilibrium equations (2) and boundary conditions (4) we can write

$$\begin{aligned} \int_{\Sigma_1} t_{31} da &= \int_{\Sigma_1} (t_{13} - m_{\rho 2, \rho}) da = \int_{\Sigma_1} [(x_1 t_{\rho 3})_{,\rho} - m_{\alpha 2, \alpha}] da = \\ &= \int_L (x_1 t_{\alpha 3} n_\alpha - m_{\alpha 2} n_\alpha) ds = 0. \end{aligned}$$

Similarly we can prove that the second condition from (5) is satisfied.

The functions  $t_{ij}$  and  $m_{ij}$  from (35) can be expressed in the form

$$t_{ij} = \sum_{k=1}^4 a_k s_{ij}^{(k)}, \quad m_{ij} = \sum_{k=1}^4 a_k \pi_{ij}^{(k)}, \quad (36)$$

where

$$\begin{aligned} s_{\alpha\beta}^{(\nu)} &= \lambda x_\nu \delta_{\alpha\beta} + t_{\alpha\beta}^{(\nu)}, \quad s_{\alpha\beta}^{(3)} = \lambda \delta_{\alpha\beta} + t_{\alpha\beta}^{(3)}, \quad s_{\alpha\beta}^{(4)} = C_1 \delta_{\alpha\beta} + t_{\alpha\beta}^{(4)}, \\ s_{\alpha 3}^{(\nu)} &= C_2 \varepsilon_{3\alpha\nu} + t_{\alpha 3}^{(\nu)}, \quad s_{\alpha 3}^{(3)} = t_{\alpha 3}^{(3)}, \\ s_{\alpha 3}^{(4)} &= \mu \varepsilon_{3\rho\alpha} x_\rho + t_{\alpha 3}^{(4)}, \quad s_{3\alpha}^{(\nu)} = C_3 \varepsilon_{3\alpha\nu} + t_{3\alpha}^{(\nu)}, \quad s_{3\alpha}^{(3)} = t_{3\alpha}^{(3)}, \\ s_{3\alpha}^{(4)} &= (\mu + \kappa) \varepsilon_{3\rho\alpha} x_\rho + t_{3\alpha}^{(4)}, \\ s_{33}^{(\nu)} &= (\lambda + 2\mu + \kappa) x_\nu + t_{33}^{(\nu)}, \\ s_{33}^{(3)} &= \lambda + 2\mu + \kappa + t_{33}^{(3)}, \quad s_{33}^{(4)} = C_1 + C_2 + C_3 + t_{33}^{(4)}, \\ \pi_{\alpha\beta}^{(\nu)} &= C_1 x_\nu \delta_{\alpha\beta} + m_{\alpha\beta}^{(\nu)}, \quad \pi_{\alpha\beta}^{(3)} = C_1 \delta_{\alpha\beta} + m_{\alpha\beta}^{(3)}, \quad \pi_{\nu\eta}^{(4)} = \alpha \delta_{\nu\eta} + m_{\nu\eta}^{(4)}, \\ \pi_{3\nu}^{(\rho)} &= \gamma \varepsilon_{3\nu\rho} + m_{3\nu}^{(\rho)}, \quad \pi_{3\nu}^{(3)} = m_{3\nu}^{(3)}, \quad \pi_{3\nu}^{(4)} = C_3 \varepsilon_{3\rho\nu} x_\rho + m_{3\nu}^{(4)}, \\ \pi_{33}^{(\rho)} &= (C_1 + C_2 + C_3) x_\rho + m_{33}^{(\rho)}, \quad \pi_{33}^{(3)} = C_1 + C_2 + C_3 + m_{33}^{(3)}, \\ \pi_{33}^{(4)} &= \alpha + \beta + \gamma + m_{33}^{(4)}. \end{aligned} \quad (37)$$

In view of (36), the conditions (6)-(8) reduce to the following system for the constants  $a_1, a_2, a_3$  and  $a_4$ ,

$$\sum_{k=1}^4 D_{\alpha k} a_k = \varepsilon_{\alpha\beta 3} M_\beta, \quad \sum_{k=1}^4 D_{3k} a_k = -R_3, \quad \sum_{k=1}^4 D_{4k} a_k = -M_3, \quad (38)$$

where we have introduced the notations

$$\begin{aligned} D_{\alpha k} &= \int_{\Sigma_1} (x_\alpha s_{33}^{(k)} - \varepsilon_{3\alpha\beta} \pi_{3\beta}^{(k)}) da, \quad D_{3k} = \int_{\Sigma_1} s_{33}^{(k)} da, \\ D_{4k} &= \int_{\Sigma_1} [\varepsilon_{3\alpha\beta} x_\alpha s_{3\beta}^{(k)} + \pi_{33}^{(k)}] da. \end{aligned} \quad (39)$$

We note that the constants  $D_{rs}$ , ( $r, s = 1, 2, 3, 4$ ), depend only on the cross section and the constitutive coefficients. As in classical theory (Ieşan, 2009) we can prove that

$$\det(D_{rs}) > 0, \quad D_{rs} = D_{sr}. \quad (40)$$

The inequality from (40) follows from (36) and the positive definiteness of the elastic potential. The symmetry of  $D_{rs}$  is a direct consequence of the reciprocal theorem. The system (38) uniquely determines the constants  $a_k$ , ( $k = 1, 2, 3, 4$ ).

We conclude that the solution of the problem has the form (33), where the

constants  $a_k$  are given by (38) and  $(u_j^{(k)}, \varphi_j^{(k)})$  are the solutions of the two-dimensional problems  $P^{(k)}$ , ( $k = 1, 2, 3, 4$ ). Generally, the torsion of an isotropic chiral Cosserat elastic cylinder is accompanied by bending and extension.

## 5 Torsion of a circular cylinder

In this section we use the solution given in Section 4 in order to study the torsion of a circular cylinder made of an isotropic chiral Cosserat elastic material. We assume that the cylinder  $B$  is defined by  $B = \{x : x_1^2 + x_2^2 < a^2, 0 < x_3 < h\}$ , ( $a > 0$ ), and suppose that  $M_1 = M_2 = 0$ . The solution has the form (33) where the functions  $u_i^{(k)}$  and  $\varphi_i^{(k)}$  are the displacements and microrotations in the problem  $P^{(k)}$ , ( $k = 1, 2, 3, 4$ ) defined by (29)-(32), and the constants  $a_k$  are given by the system (38).

First, we study the generalized plane strain problem  $P^{(3)}$  characterized by the external loading (27). We seek the solution of this problem in the form

$$u_\alpha^{(3)} = x_\alpha G(r), u_3^{(3)} = 0, \varphi_\alpha^{(3)} = x_\alpha \Phi(r), \varphi_3^{(3)} = 0, \quad (41)$$

where  $r = (x_1^2 + x_2^2)^{1/2}$ , and  $G$  and  $\Phi$  are unknown functions. Clearly,

$$\begin{aligned} u_{\alpha,\beta}^{(3)} &= \delta_{\alpha\beta} G + x_\alpha x_\beta r^{-1} G', u_{\rho,\rho}^{(3)} = 2G + rG', \\ u_{\rho,\rho\alpha}^{(3)} &= \Delta u_\alpha^{(3)} = x_\alpha (G'' + r^{-1} G'), \\ \varphi_{\alpha,\beta}^{(3)} &= \delta_{\alpha\beta} \Phi + x_\alpha x_\beta r^{-1} \Phi', \Delta \varphi_\alpha^{(3)} = x_\alpha (\Phi'' + 3r^{-1} \Phi'), \end{aligned} \quad (42)$$

where  $f' = df/dr$ . It follows from (29) and (42) that

$$\begin{aligned} e_{\alpha\beta}^{(3)} &= e_{\beta\alpha}^{(3)} = u_{\beta,\alpha}^{(3)}, e_{\rho\beta}^{(3)} = \varepsilon_{3\rho\beta} x_\beta \Phi = -e_{3\rho}^{(3)}, e_{33}^{(3)} = 0, \\ \kappa_{\rho\rho}^{(3)} &= 2\Phi + r\Phi', \kappa_{\alpha\beta}^{(3)} = \kappa_{\beta\alpha}^{(3)} = \varphi_{\alpha,\beta}^{(3)}, \kappa_{\alpha 3}^{(3)} = 0, \kappa_{3i}^{(3)} = 0. \end{aligned} \quad (43)$$

In view of (42), the equations of equilibrium (15) for the problem  $P^{(3)}$ , reduce to

$$\begin{aligned} (\lambda + 2\mu + \kappa)(G'' + 3r^{-1}G') + (C_1 + C_2 + C_3)(\Phi'' + 3r^{-1}\Phi') &= 0, \\ (C_1 + C_2 + C_3)(G'' + 3r^{-1}G') + (\alpha + \beta + \gamma)(\Phi'' + 3r^{-1}\Phi' - s^2\Phi) &= 0, \end{aligned} \quad (44)$$

where

$$s^2 = \frac{2\kappa}{\alpha + \beta + \gamma}. \quad (45)$$

If we eliminate the function  $G$ , from (44) we find that  $\Phi$  satisfies the equation

$$\Phi'' + 3r^{-1}\Phi' - q^2\Phi = 0, \quad (46)$$

where

$$q^2 = 2\kappa(\lambda + 2\mu + \kappa)Q^{-1}, Q = (\lambda + 2\mu + \kappa)(\alpha + \beta + \gamma) - (C_1 + C_2 + C_3)^2. \quad (47)$$

Clearly, from (9) we find that  $s^2 > 0$ ,  $Q > 0$  and  $q^2 > 0$ . We introduce the function  $\Omega$  by

$$\Omega = r\Phi. \quad (48)$$

Then, the equation (46) becomes

$$\Omega'' + \frac{1}{r}\Omega' - \left(\frac{1}{r^2} + q^2\right)\Omega = 0. \quad (49)$$

In what follows we denote by  $I_n$  and  $K_n$  the modified Bessel functions of order  $n$ . The general solution of the equation (49) is

$$\Omega = B_1I_1(qr) + B_2K_1(qr),$$

where  $B_1$  and  $B_2$  are arbitrary constants. Since the functions  $\varphi_\alpha^{(3)}$  must be bounded for  $r = 0$ , we have  $B_2 = 0$ . Thus, we find that

$$\Phi = r^{-1}B_1I_1(qr). \quad (50)$$

It follows from (44) that the function  $G$  satisfies the equation

$$\left(\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - q^2\right)G = 0. \quad (51)$$

We can write  $G = G_1 + G_2$  where the functions  $G_1$  and  $G_2$  satisfy the equations

$$G_1'' + 3r^{-1}G_1' = 0, G_2'' + 3r^{-1}G_2' - q^2G_2 = 0.$$

Clearly,

$$G_1 = A_1 + A_2r^{-2}, G_2 = r^{-1}[D_1I_1(qr) + D_2K_1(qr)],$$

where  $A_\alpha$  and  $D_\alpha$  are arbitrary constants. Since the functions  $u_\alpha^{(3)}$  are bounded for  $r = 0$  we must take  $A_2 = 0$  and  $D_2 = 0$ . Thus the function  $G$  has the form

$$G = A_1 + D_1r^{-1}I_1(qr). \quad (52)$$

We note that the functions  $G$  and  $\Phi$ , given by (50) and (52), must satisfy the system of second-order (44). Substituting  $G$  and  $\Phi$  into the equations (44) we obtain the relations

$$\begin{aligned} (\lambda + 2\mu + \kappa)D_1 + (C_1 + C_2 + C_3)B_1 &= 0, \\ (C_1 + C_2 + C_3)q^2D_1 + (\alpha + \beta + \gamma)q^2B_1 - 2\kappa B_1 &= 0. \end{aligned} \quad (53)$$

The first equation from (53) implies that

$$D_1 = -\sigma B_1, \quad (54)$$

where

$$\sigma = (C_1 + C_2 + C_3)(\lambda + 2\mu + \kappa)^{-1}. \quad (55)$$

It is a simple matter to see that the second equation from (53) is identically satisfied on the basis of the relations (47) and (55). We conclude that

$$G = A_1 - \sigma r^{-1} B_1 I_1(qr), \quad \Phi = r^{-1} B_1 I_1(qr). \quad (56)$$

Let us impose the boundary conditions (32) for  $k = 3$ . It follows from (43) and the constitutive equations (30) that

$$\begin{aligned} t_{\alpha\beta}^{(3)} &= \lambda(2G + rG')\delta_{\alpha\beta} + (2\mu + \kappa)(G\delta_{\alpha\beta} + x_\alpha x_\beta r^{-1}G') + \\ &\quad + C_1(2\Phi + r\Phi')\delta_{\alpha\beta} + (C_2 + C_3)(\Phi\delta_{\alpha\beta} + x_\alpha x_\beta r^{-1}\Phi'), \\ t_{\alpha 3}^{(3)} &= \kappa \varepsilon_{3\alpha\beta} x_\beta \Phi = -t_{3\alpha}^{(3)}, \quad t_{33}^{(3)} = \lambda(2G + rG') + C_1(2\Phi + r\Phi'), \\ m_{\nu\eta}^{(3)} &= \alpha(2\Phi + r\Phi')\delta_{\nu\eta} + (\beta + \gamma)(\Phi\delta_{\nu\eta} + x_\nu x_\eta r^{-1}\Phi') + \\ &\quad + C_1(2G + rG') + (C_2 + C_3)(G\delta_{\nu\eta} + x_\nu x_\eta r^{-1}G'), \\ m_{\nu 3}^{(3)} &= -m_{3\nu}^{(3)} = (C_3 - C_2)\varepsilon_{3\nu\beta} x_\beta \Phi, \quad m_{33}^{(3)} = \alpha(2\Phi + r\Phi') + C_1(2G + rG'). \end{aligned} \quad (57)$$

On the boundary  $L$  we have  $r = a$  and  $n_\alpha = x_\alpha/a$ . From (57) we obtain

$$\begin{aligned} t_{\alpha\beta}^{(3)} n_\alpha &= a^{-1} x_\beta [(2\lambda + 2\mu + \kappa)G(a) + (\lambda + 2\mu + \kappa)aG'(a) + \\ &\quad + (2C_1 + C_2 + C_3)\Phi(a) + (C_1 + C_2 + C_3)a\Phi'(a)], \\ t_{\alpha 3}^{(3)} n_\alpha &= \kappa a^{-1} \varepsilon_{3\alpha\beta} x_\alpha x_\beta \Phi'(a) = 0, \\ m_{\alpha\beta}^{(3)} n_\alpha &= a^{-1} x_\beta [(2\alpha + \beta + \gamma)\Phi(a) + (\alpha + \beta + \gamma)a\Phi'(a) + \\ &\quad + (2C_1 + C_2 + C_3)G(a) + (C_1 + C_2 + C_3)aG'(a)], \\ m_{\alpha 3}^{(3)} n_\alpha &= (C_3 - C_2)\varepsilon_{3\alpha\beta} x_\alpha x_\beta a^{-1} \Phi(a) = 0 \text{ on } L. \end{aligned} \quad (58)$$

In view of (58), the boundary conditions (32), for  $k = 3$ , reduce to

$$\begin{aligned} (2\lambda + 2\mu + \kappa)G(a) + (\lambda + 2\mu + \kappa)aG'(a) + \\ + (2C_1 + C_2 + C_3)\Phi(a) + (C_1 + C_2 + C_3)a\Phi'(a) &= -\lambda, \\ (2\alpha + \beta + \gamma)\Phi(a) + (\alpha + \beta + \gamma)a\Phi'(a) + (2C_1 + C_2 + \\ + C_3)G(a) + (C_1 + C_2 + C_3)aG'(a) &= -C_1. \end{aligned} \quad (59)$$

By using the relations (56) and (59) we find that the constants  $A_1$  and  $B_1$  must satisfy the equations

$$\begin{aligned} (2\lambda + 2\mu + \kappa)aA_1 + l_1 B_1 &= -\lambda a, \\ (2C_1 + C_2 + C_3)aA_1 + l_2 B_1 &= -C_1 a, \end{aligned} \quad (60)$$

where

$$\begin{aligned} l_1 &= (C_1 - \sigma\lambda)I_1(qa) + a[C_1 + C_2 + C_3 - \sigma(\lambda + 2\mu + \kappa)]I_1'(qa), \\ l_2 &= (\alpha - \sigma C_1)I_1(qa) + a[\alpha + \beta + \gamma - \sigma(C_1 + C_2 + C_3)]I_1'(qa). \end{aligned} \quad (61)$$

We introduce the notation

$$\Lambda = [(2\lambda + 2\mu + \kappa)l_2 - (2C_1 + C_2 + C_3)l_1]a. \quad (62)$$

From (60) we get

$$A_1 = (C_1 l_1 - \lambda l_2) a \Lambda^{-1}, \quad B_1 = [(C_2 + C_3)\lambda - (2\mu + \kappa)C_1] a^2 \Lambda^{-1}. \quad (63)$$

Let us study now the problem  $P^{(4)}$ . We seek the solution of this problem in the form

$$u_\alpha^{(4)} = x_\alpha F(r), \quad u_3^{(4)} = 0, \quad \varphi_\alpha^{(4)} = x_\alpha \left[ \Psi(r) - \frac{1}{2} \right], \quad \varphi_3^{(4)} = 0, \quad (64)$$

where  $F$  and  $\Psi$  are unknown functions. We note that in this case we have

$$\begin{aligned} e_{\alpha\beta}^{(4)} &= e_{\beta\alpha}^{(4)} = u_{\alpha,\beta}^{(4)} = F\delta_{\alpha\beta} + x_\alpha x_\beta r^{-1} F', \quad e_{33}^{(4)} = 0, \\ e_{\alpha 3}^{(4)} &= -e_{3\alpha}^{(4)} = \varepsilon_{3\alpha\beta} x_\beta \left( \Psi - \frac{1}{2} \right), \quad \kappa_{\alpha 3}^{(4)} = \kappa_{3\alpha}^{(4)} = 0, \\ \kappa_{\alpha\beta}^{(4)} &= \kappa_{\beta\alpha}^{(4)} = \varphi_{\alpha,\beta}^{(4)} = \left( \Psi - \frac{1}{2} \right) \delta_{\alpha\beta} + x_\alpha x_\beta r^{-1} \Psi'. \end{aligned} \quad (65)$$

The equations of equilibrium (15) with the body loads  $f_i = 0$ ,  $g_\alpha = -\kappa x_\alpha$ ,  $g_3 = 0$ , reduce to

$$\begin{aligned} (\lambda + 2\mu + \kappa)(F'' + 3r^{-1}F) + (C_1 + C_2 + C_3)(\Psi'' + 3r^{-1}\Psi') &= 0, \\ (C_1 + C_2 + C_3)(F'' + 3r^{-1}F) + (\alpha + \beta + \gamma)(\Psi'' + 3r^{-1}\Psi' - s^2\Psi) &= 0, \end{aligned} \quad (66)$$

where  $s^2$  is given by (45). From these equations we find that  $F$  and  $\Psi$  satisfy the equations

$$\left( \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} \right) \left( \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} - q^2 \right) F = 0, \quad \Psi'' + 3r^{-1}\Psi' - q^2\Psi = 0,$$

where  $q^2$  is defined in (47). The functions  $F$  and  $\Psi$  which are bounded at the origin are given by

$$F = E_1 + N_1 r^{-1} I_1(qr), \quad \Psi = H_1 r^{-1} I_1(qr),$$

where  $E_1, N_1$  and  $H_1$  are arbitrary constants. It is easy to see that  $F$  and  $\Psi$  satisfy the equations (66) only if  $N_1 = -\sigma H_1$ , where  $\sigma$  is given by (55). Thus,



the displacements  $u_\alpha^{(4)}$  and the microrotations  $\varphi_\alpha^{(4)}$  have the form

$$u_\alpha^{(4)} = x_\alpha[E_1 - \sigma H_1 r^{-1} I_1(qr)], \quad \varphi_\alpha^{(4)} = x_\alpha[H_1 r^{-1} I_1(qr) - \frac{1}{2}]. \quad (67)$$

It follows from (65), (29) and (30) that

$$\begin{aligned} t_{\alpha\beta}^{(4)} &= \lambda(2F + rF')\delta_{\alpha\beta} + (2\mu + \kappa)(F\delta_{\alpha\beta} + x_\alpha x_\beta r^{-1} F') + \\ &\quad + C_1(2\Psi + r\Psi' - 1)\delta_{\alpha\beta} + (C_2 + C_3)(\Psi\delta_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta} + x_\alpha x_\beta r^{-1} \Psi'), \\ t_{\alpha 3}^{(4)} &= -t_{3\alpha}^{(4)} = \kappa \varepsilon_{3\alpha\beta} x_\beta (\Psi - \frac{1}{2}), \quad t_{33}^{(4)} = \lambda(2F + rF') + C_1(2\Psi + r\Psi' - 1), \\ m_{\nu\eta}^{(4)} &= \alpha(2\Psi + r\Psi' - 1)\delta_{\nu\eta} + (\beta + \gamma)(\Psi\delta_{\nu\eta} - \frac{1}{2}\delta_{\nu\eta} + x_\nu x_\eta r^{-1} \Psi') + \\ &\quad + C_1(2F + rF')\delta_{\nu\eta} + (C_3 + C_4)(F\delta_{\alpha\beta} + x_\alpha x_\beta r^{-1} F'), \\ m_{\nu 3}^{(4)} &= -m_{3\nu}^{(4)} = (C_3 - C_2)\varepsilon_{3\nu\beta} x_\beta (\Psi - \frac{1}{2}), \\ m_{33}^{(4)} &= \alpha(2\Psi + r\Psi' - 1) + C_1(2F + rF'). \end{aligned} \quad (68)$$

Taking into account (68) and the form of  $F$  and  $\Psi$ , we find that on the boundary  $L$  the following relations hold

$$\begin{aligned} t_{\alpha\beta}^{(4)} n_\alpha &= a^{-2} x_\beta [(2\lambda + 2\mu + \kappa)aE_1 + lH_1 - \frac{1}{2}a(2C_1 + C_2 + C_3)], \\ t_{\alpha 3}^{(4)} n_\alpha &= 0, \quad m_{\alpha 3}^{(4)} n_\alpha = 0, \\ m_{\alpha\rho}^{(4)} n_\alpha &= a^{-2} x_\rho [(2C_1 + C_2 + C_3)aE_1 + l_2 H_1 - \frac{1}{2}a(2\alpha + \beta + \gamma)], \end{aligned} \quad (69)$$

where  $l_1$  and  $l_2$  are defined in (61). Thus, the boundary conditions (32), for  $k = 4$ , reduce to the following system for the constants  $E_1$  and  $H_1$ ,

$$\begin{aligned} (2\lambda + 2\mu + \kappa)aE_1 + l_1 H_1 &= \frac{1}{2}(C_2 + C_3)a, \\ (2C_1 + C_2 + C_3)aE_1 + l_2 H_1 &= \frac{1}{2}(\beta + \gamma)a. \end{aligned} \quad (70)$$

We obtain

$$\begin{aligned} E_1 &= \frac{1}{2}a\Lambda^{-1}[(C_2 + C_3)l_2 - (\beta + \gamma)l_1], \\ H_1 &= -\frac{1}{2}\Lambda^{-1}a^2[(2C_1 + C_2 + C_3)(C_2 + C_3) - (2\lambda + 2\mu + \kappa)(\beta + \gamma)], \end{aligned} \quad (71)$$

where the constant  $\Lambda$  is given by (62).

As the functions  $u_i^{(3)}$ ,  $\varphi_i^{(3)}$ ,  $u_i^{(4)}$  and  $\varphi_i^{(4)}$  are known, from (39) we can determine the constants  $D_{k3}$  and  $D_{k4}$ , ( $k = 1, 2, 3, 4$ ). It follows from (37), (39) and (57)

that

$$\begin{aligned}
D_{\alpha 3} &= \int_{\Sigma_1} x_\alpha [\lambda + 2\mu + \kappa + t_{33}^{(3)} + (C_2 - C_3)\Phi] da, \\
D_{33} &= \int_{\Sigma_1} [\lambda + 2\mu + \kappa + t_{33}^{(3)}] da, \\
D_{\alpha 4} &= \int_{\Sigma_1} [x_\alpha (C_1 + C_2 + C_3 + t_{33}^{(4)}) - \varepsilon_{3\alpha\beta} (C_3 \varepsilon_{3\rho\beta} x_\rho + m_{3\beta}^{(4)})] da, \\
D_{34} &= \int_{\Sigma_1} (C_1 + C_2 + C_3 + t_{33}^{(4)}) da, \\
D_{44} &= \int_{\Sigma_1} [(\mu + \kappa)x_\alpha x_\alpha + \varepsilon_{3\alpha\beta} x_\alpha t_{3\beta}^{(4)} + \alpha + \beta + \gamma + m_{33}^{(4)}] da.
\end{aligned} \tag{72}$$

With the help of (56), (57), (67) and (68), the relations (72) reduce to

$$\begin{aligned}
D_{\alpha 3} &= 0, \quad D_{\alpha 4} = 0, \\
D_{33} &= \pi(\lambda + 2\mu + \kappa)a^2 + 2\pi \int_0^a \{2\lambda A_1 r + \\
&\quad + (C_1 - \lambda\sigma)[I_1(qr) + rI_1'(qr)]B_1\} dr, \\
D_{34} &= \pi(C_2 + C_3)a^2 + 2\pi\lambda a^2 E_1 + 2\pi(C_1 - \lambda\sigma)H_1 \int_0^a [I_1(qr) + rI_1'(qr)] dr, \\
D_{44} &= \pi(\beta + \gamma)a^2 + \frac{1}{4}(2\mu + \kappa)\pi a^4 + 2\pi \int_0^a r[\kappa r^2 \Psi + \\
&\quad + \alpha(2\Psi + r\Psi') + C_1(2F + rF')] dr.
\end{aligned} \tag{73}$$

We note that

$$\begin{aligned}
\int_0^a r^3 \Psi dr &= H_1 \int_0^a r^2 I_1(qr) dr, \\
\alpha(2\Psi + r\Psi') + C_1(2F + rF') &= 2C_1 E_1 + \\
&+ (\alpha - C_1\sigma)H_1[r^{-1}I_1(qr) + I_1'(qr)], \\
[x^2 I_2(x)]' &= x^2 I_1(x), \quad I_1(0) = 0.
\end{aligned} \tag{74}$$

By using the relations (73) and (74) we find

$$\begin{aligned}
D_{33} &= \pi(\lambda + 2\mu + \kappa)a^2 + 2\pi\lambda a^2 A_1 + 2\pi(C_1 - \lambda\sigma)aB_1 I_1(qa), \\
D_{34} &= \pi(C_2 + C_3)a^2 + 2\pi\lambda a^2 E_1 + 2\pi(C_1 - \lambda\sigma)aH_1 I_1(qa), \\
D_{44} &= \pi(\beta + \gamma)a^2 + \frac{1}{4}(2\mu + \kappa)\pi a^4 + 2\pi a^2 C_1 E_1 + \\
&\quad + 2\pi\kappa q^{-1} a^2 H_1 I_2(qa) + 2\pi(\alpha - C_1\sigma)aH_1 I_1(qa).
\end{aligned} \tag{75}$$

Since  $D_{rs} = D_{sr}$ , from (73) we obtain  $D_{43} = D_{34}$ , and

$$D_{3\alpha} = D_{\alpha 3} = 0, \quad D_{4\alpha} = D_{\alpha 4} = 0. \tag{76}$$

The system (38) becomes

$$D_{\alpha\beta} a_\beta = 0, \quad D_{33} a_3 + D_{34} a_4 = -R_3, \quad D_{34} a_3 + D_{44} a_4 = -M_3. \tag{77}$$

Thus, for a right circular cylinder the system (38) reduces to two uncoupled systems: one for the unknowns  $a_1$  and  $a_2$  and the other for the constants  $a_3$  and  $a_4$ . It follows from (76) that in this case the problem of bending can be treated independently. From (77) we obtain

$$a_1 = a_2 = 0, \quad a_3 = d^{-1}(D_{34}M_3 - D_{44}R_3), \quad a_4 = d^{-1}(D_{34}R_3 - D_{33}M_3), \quad (78)$$

where  $d = D_{33}D_{44} - D_{34}^2$ . It follows from (33), (41), (56), (64) and (78) that the solution of the problem of torsion and extension is

$$\begin{aligned} u_\alpha &= \varepsilon_{3\beta\alpha} a_4 x_\beta x_3 + x_\alpha [a_3 A_1 + a_4 E_1 - \sigma r^{-1} (a_3 B_1 + a_4 H_1) I_1(qr)], \\ u_3 &= a_3 x_3, \quad \varphi_3 = a_4 x_3, \\ \varphi_\alpha &= x_\alpha [r^{-1} (a_3 B_1 + a_4 H_1) I_1(qr) - \frac{1}{2} a_4], \end{aligned} \quad (79)$$

where  $A_1, E_1, B_1$  and  $H_1$  are defined in (63), (71), and the constants  $a_3$  and  $a_4$  are given by (78). We conclude that the torsion of a right circular cylinder made of an isotropic chiral Cosserat elastic material is accompanied by extension. Clearly, if  $M_3 = 0$  then we obtain the solution of the extension problem.

The torsion problem for an isotropic centrosymmetric Cosserat elastic cylinder has been studied by Ieşan (1971). In the case of centrosymmetric materials we have  $C_j = 0$ , and the relations (53), (61) and (70) imply that

$$\sigma = 0, \quad l_1 = 0, \quad E_1 = 0. \quad (80)$$

It follows from (75) and (80) that for an isotropic and centrosymmetric circular cylinder we have

$$D_{34} = 0,$$

so that in this case the torsion is not accompanied by extension. Moreover, it is a simple matter to see from (75) and (80) that the constant  $D_{44}$  reduces to the torsional rigidity calculated by Usidus and Sokolovski (1973). The extension, bending and flexure of a circular cylinder made of an isotropic centrosymmetric Cosserat elastic material have been investigated by Reddy and Venkatasubramanian (1976).

## 6 Conclusions

The original results established in this paper can be summarized as follows:

- (a) We present a solution of the torsion, bending and extension problem for a right cylinder made of an isotropic chiral Cosserat elastic material. The three-dimensional problem is reduced to the study of two-dimensional problems.

(b) We study the generalized plane strain problem in the case of chiral materials and present an existence result. We introduce four generalized plane strain problems characterized by external data which depend only on the constitutive coefficients. The solution of the torsion problem is expressed in terms of solutions of these problems.

(c) We prove that, in general, the torsion of an isotropic chiral Cosserat elastic cylinder is accompanied by bending and extension.

(d) We apply the method to study the torsion of a circular cylinder. We establish the solutions of the auxiliary generalized plane strain problems for a circular domain. The displacement and microrotation vector fields in the torsion problem are determined. We show that in this case the torsion is accompanied by extension and a plane deformation. The solution established in the case of a circular cylinder could be of interest for experimental investigations.

**Acknowledgements.** This work was supported by CNCSIS-UEFISCU, project number PN II-IDEI 89/2008. I express my gratitude to the referees for their helpful suggestions.

## References

- Chandraseker, K., Mukherjee, S., Paci, J.T., Schatz, G.C., 2009. An atomistic-continuum Cosserat rod model of carbon nontubes. *J. Mech. Phys. Solids* 57, 932-958.
- Ciarletta, M., Ieşan, D., 1993. *Non-Classical Elastic Solids*, Longman Scientific & Technical, Harlow.
- Donescu, S., Chiroiu, V., Munteanu, L., 2009. On the Young's modulus of an auxetic composite structure. *Mech. Res. Comm.* 36, 294-301.
- Dyzlewicz, J., Kolodziej, C., 1987. Selected one-dimensional problems of elastostatics and thermoelastostatics of the hemitropic micropolar medium. *Bull. Acad. Polon. Sci., Sér. Sci. Techn.* 35, 15-25.
- Dyzlewicz, J., 2004. *Micropolar Theory of Elasticity*, Springer, Berlin, Heidelberg, New York.
- Eringen, A.C., 1999. *Microcontinuum Field Theories, I: Foundations and Solids*, Springer, New York, Berlin, Heidelberg.
- Fichera, G., 1972. Existence Theorems in Elasticity. In: Truesdell, C. (Ed.), *Handbuch der Physik.*, vol. VIa/2, Springer, Berlin, New York.
- Healey, T.J., 2002. Material symmetry and chirality in nonlinearly elastic rods. *Math. Mech. Solids* 7, 405-420.
- Ieşan, D., 1971. Torsion of micropolar elastic beams. *Int. J. Engng. Sci.* 9, 1047-1060.
- Ieşan, D., 1986. On Saint-Venant's problem. *Arch. Rational Mech. Anal.* 91, 363-373.

- Ieşan, D., 2009. *Classical and Generalized Models of Elastic Rods*, Chapman & Hall/CRC Press, London, New York, Boca Raton.
- Jasiuk, I., Ostoja-Starzewski, M., 2004. From lattices and composites to micropolar continua. Analysis of materials with complex microstructure, in: Harik, V.M., Luo, L.S. (Eds.), *Micromechanics and Nanoscale Effects, MEMS, Multi-Scale Materials and Micro-Flows*, Kluwer Acad. Publ., Dordrecht.
- Khurana, A., Tomar, S.K., 2009. Longitudinal wave response of a chiral slab interposed between micropolar solid half-spaces. *Int. J. Solids Struct.* 46, 135-150.
- Krishna Reddy, G.V., Venkatasubramanian, N.K., 1976. Saint-Venant's problems for a micropolar elastic circular cylinder. *Int. J. Engng. Sci.* 14, 1047-1055.
- Lakes, R.S., Benedict, R.L., 1982. Noncentrosymmetry in micropolar elasticity. *Int. J. Engng. Sci.* 29, 1161-1167.
- Lakes, R.S., Yoon, H.S., Katz, J.L., 1983. Slow compressional wave propagation in wet human and bovine cortical bone, *Science* 220, 513-515.
- Lakes, R.S., 1988. Elastic freedom in cellular solids and composite materials, in: Golden, K., Grimmert, G., James, R., Milton, G., Sen, P. (Eds.), *Mathematics of Multiscale Materials*, IMA, vol. 99, Springer, New York, Berlin, pp. 129-153.
- Lakes, R., 2001. Elastic and viscoelastic behaviour of chiral materials. *Int. J. Mech. Sci.* 43, 1579-1589.
- Natroshvili, D., Giorgashvili, L., Stratis, I.G., 2006. Representation formulae of general solutions in the theory of hemitropic elasticity. *Q.J. Mech. Appl. Math.* 59, 451-474.
- Nowacki, J.P., 1977. Green functions for a hemitropic micropolar continuum. *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* 25, 235-241.
- Nowacki, W., 1981. *Theory of Asymmetric Elasticity*, PWN, Warszawa.
- Prall, D., Lakes, R.S., 1997. Properties of a chiral honeycomb with a Poisson's ratio-1. *Int. J. Mech. Sci.* 39, 305-314.
- Rubin, M.B., 2000. *Cosserat Theories: Shells, Rods and Points*, Kluwer Acad. Publ., Dordrecht.
- Teodorescu, P.P., Munteanu, L., Chiroiu, V., 2005. On the wave propagation in chiral media, in: Mihailescu-Suliciu, M., (Ed.), *New Trends in Continuum Mechanics*, Theta Foundation, Bucharest, pp. 303-310.
- Usidus, C., Sokolovski, M., 1973. Torsion of cylindrical shafts made of micropolar materials. *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* 21, 19-23.

