



HAL
open science

A new bound for the $2/3$ conjecture

Daniel Král', Chun-Hung Liu, Jean-Sébastien Sereni, Peter Whalen, Zelealem Yilma

► **To cite this version:**

Daniel Král', Chun-Hung Liu, Jean-Sébastien Sereni, Peter Whalen, Zelealem Yilma. A new bound for the $2/3$ conjecture. 2012. hal-00686989v1

HAL Id: hal-00686989

<https://hal.science/hal-00686989v1>

Submitted on 11 Apr 2012 (v1), last revised 3 Jan 2013 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A new bound for the $2/3$ conjecture

Daniel Král’* Chun-Hung Liu† Jean-Sébastien Sereni‡
Peter Whalen§ Zelealem B. Yilma¶

Abstract

We show that any n -vertex complete graph with edges colored with three colors contains a set of at most four vertices such that the number of the neighbors of these vertices in one of the colors is at least $2n/3$. The previous best value, proved by Erdős *et al.* in 1989, is 22. It is conjectured that three vertices suffice.

1 Introduction

Erdős and Hajnal [5] made the observation that for fixed positive integer t , positive real ϵ , and graph G on $n > n_0$ vertices, there is a set of t vertices that have a neighborhood of size at least $(1 - (1 + \epsilon)(2/3)^t)n$ in either G or its complement. They further inquired whether $2/3$ may be replaced by $1/2$. This was answered in the affirmative by Erdős, Faudree, Gyárfás and Schelp [3], who not only proved the

*Institute of Computer Science (IUUK), Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague 1, Czech Republic. E-mail: kral@kam.mff.cuni.cz. The work of this author leading to this invention has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.

†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA. E-mail: cliu87@math.gatech.edu.

‡CNRS (LIAFA, Université Denis Diderot), Paris, France. E-mail: sereni@kam.mff.cuni.cz. This author’s work was partially supported by the French *Agence Nationale de la Recherche* under reference ANR 10 JCJC 0204 01.

§School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA. E-mail: pwhalen3@math.gatech.edu.

¶LIAFA (Université Denis Diderot), Paris, France. E-mail: Zelealem.Yilma@liafa.jussieu.fr. This author’s work was supported by the French *Agence Nationale de la Recherche* under reference ANR 10 JCJC 0204 01.

result but also dispensed with the $(1 + \epsilon)$ factor. They also phrased the question as a problem of vertex domination in a multicolored graph.

Given a color c in an r -coloring of the edges of the complete graph, a subset A of the vertex set c -dominates another subset B if, for every $y \in B \setminus A$, there exists a vertex $x \in A$ such that the edge xy is colored c . The subset A *strongly* c -dominates B if such a vertex $x \in A$ exists for every $y \in B$. (Thus, the two notions coincide when $A \cap B = \emptyset$.) The result of Erdős *et al.* [3] may then be stated as follows.

Theorem 1. *For any fixed positive integer t and any 2-coloring of the edges of the complete graph K_n on n vertices, there exist a color c and a subset X of size at most t such that all but at most $n/2^t$ vertices of K_n are c -dominated by X .*

In a more general form, they asked: *Given positive integers r , t , and n along with an r -coloring of the edges of the complete graph K_n on n vertices, what is the largest subset B of the vertices of K_n necessarily monochromatically dominated by some t -element subset of K_n ?* However, in the same paper [3], the authors presented a 3-coloring of the edges of K_n — attributed to Kierstead — which shows that if $r \geq 3$, then it is not possible to monochromatically dominate all but a small fraction of the vertices with any fixed number t of vertices. This 3-coloring is defined as follows: the vertices of K_n are partitioned into three sets V_1, V_2, V_3 of equal sizes and an edge xy with $x \in V_i$ and $y \in V_j$ is colored i if $1 \leq i \leq j \leq 3$. Observe that, if t is fixed, then at most $2n/3$ vertices may be monochromatically dominated.

In the other direction, it was shown in the follow-up paper of Erdős, Faudree, Gould, Gyárfás, Rousseau and Schelp [4], that if $t \geq 22$, then, indeed, at least $2n/3$ vertices are monochromatically dominated in any 3-coloring of the edges of K_n . The authors then ask if 22 may be replaced by a smaller number (specifically, 3). We prove here that $t \geq 4$ is sufficient.

Theorem 2. *For any 3-coloring of the edges of K_n , where $n \geq 2$, there exist a color c and a subset A of at most four vertices of K_n such that A strongly c -dominates at least $2n/3$ vertices of K_n .*

Our proof suggests that Kierstead's coloring is in some sense extremal, giving more credence to the conjecture that three vertices would suffice to monochromatically dominate a set of size $2n/3$ in any 3-coloring of the edges of K_n .

We note that there exists 3-colorings of the edges of K_n such that no pair of vertices monochromatically dominate $2n/3 + O(1)$ vertices. This can be seen by realizing that in a random 3-coloring, the probability that an arbitrary pair of vertices monochromatically dominate more than $5n/9 + o(n)$ vertices is $o(n^{-2})$ by Chernoff's bound.

Our proof of Theorem 2 utilizes the flag algebra theory introduced by Razborov, which, recently, has led to numerous results in extremal graph and hypergraph theory. In the following section, we present a brief introduction to the flag algebra framework. The proof of Theorem 2 is presented in Section 3.

We end this introduction by pointing out another interesting question: what happens when one increases r , the number of colors? Constructions in the vein of that of Kierstead — for example, partitioning K_n into s parts and using $r = \binom{s}{2}$ colors — show that the size of dominated sets decreases with increasing r . While it may be difficult to determine the minimum value of t dominating a certain proportion of the vertices, it would be interesting to find out whether such constructions do, in fact, give the correct bounds.

2 Flag Algebras

Flag algebras were introduced by Razborov [11] as a tool based on the graph limit theory of Lovász and Szegedy [10] and Borgs *et al.* [2] to approach problems pertaining to extremal graph theory. This tool has been successfully applied to various topics, such as Turán-type problems [13], super-saturation questions [12], jumps in hypergraphs [1], the Caccetta-Häggkvist conjecture [9], the chromatic number of common graphs [7] and the number of pentagons in triangle-free graphs [6, 8].

Let us now introduce the terminology related to flag algebras needed in this paper. Since we deal with 3-colorings of the edges of complete graphs, we restrict our attention only to this particular case. Let us define a *tricolored graph* to be a complete graph whose edges are colored with 3 colors. If G is a tricolored graph, then $V(G)$ is its vertex-set. For a set F , we define $\mathbf{R}F$ to be the set of all formal linear combinations of elements of F with real coefficients. Let \mathbb{F}_ℓ be the set of non-isomorphic tricolored graphs with ℓ vertices (two tricolored graphs are considered to be isomorphic if they differ by a permutation of the vertices and a permutation of the edge colors). The elements of \mathbb{F}_3 are shown in Figure 1. We set $\mathbb{F} := \cup_{\ell \in \mathbf{N}} \mathbb{F}_\ell$. Given a tricolored graph σ , we define \mathbb{F}_ℓ^σ to be the set of tricolored graphs F on ℓ vertices with a fixed embedding of σ , that is, an injective mapping ν from $V(\sigma)$ to $V(F)$ such that $\text{Im}(\nu)$ induces in F a subgraph that differs from σ only by a permutation of the edge colors. We set $\mathbb{F}^\sigma := \cup_{\ell \in \mathbf{N}} \mathbb{F}_\ell^\sigma$.

The central notions are factor algebras of \mathbb{F} and \mathbb{F}^σ equipped with addition and multiplication. Let us start with the simpler case of \mathbb{F} . Let \mathcal{F} be $\mathbf{R}\mathbb{F}$ factorised by

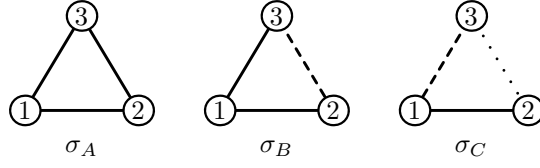


Figure 1: The elements of \mathbb{F}_3 . The edges of color 1, 2 and 3 are represented by solid, dashed and dotted lines, respectively.

the subspace of $\mathbf{R}\mathbb{F}$ generated by all combinations of the form

$$H - \sum_{H' \in \mathbb{F}_{|H|+1}} p(H, H')H',$$

where $p(H, H')$ is the probability that a randomly chosen subset of $|V(H)|$ vertices of H' induces a subgraph isomorphic to H . We set $\mathcal{A} := \mathbf{R}\mathcal{F}$.

Next, we define the multiplication on \mathcal{A} based on the elements of \mathbb{F} as

$$H_1 \cdot H_2 := \sum_{H \in \mathbb{F}_{|H_1|+|H_2|}} p(H_1, H_2; H),$$

where $p(H_1, H_2; H)$ is the probability that two randomly chosen disjoint subsets of vertices of H with sizes $|V(H_1)|$ and $|V(H_2)|$ induce subgraphs isomorphic to H_1 and H_2 , respectively. The multiplication is linearly extended to $\mathbf{R}\mathbb{F}$. Further, it can be shown that the result of the multiplication falls into the same class of \mathcal{A} independently of the choice of the elements of \mathbb{F} in their classes.

The definition of \mathcal{A}^σ follows the same lines. Let H and H' be two tricolored graphs in \mathbb{F}^σ with embeddings ν and ν' of σ . Let $p(H, H')$ be the probability that $\nu'(V(\sigma))$ together with a randomly chosen subset of $|V(H)| - |V(\sigma)|$ vertices in $V(H') \setminus \nu'(V(\sigma))$ induce a subgraph that is isomorphic to H through an isomorphism f that preserves the embeddings, that is, $\nu' = f \circ \nu$. The set \mathcal{A}^σ is composed of all formal real linear combinations of elements of $\mathbf{R}\mathbb{F}^\sigma$ factorised by the subspace of $\mathbf{R}\mathbb{F}^\sigma$ generated by all combinations of the form

$$H - \sum_{H' \in \mathbb{F}_{|H|+1}^\sigma} p(H, H')H'.$$

Similarly, $p(H_1, H_2; H)$ is the probability that $\nu(V(\sigma))$ together with two randomly chosen disjoint subsets of $|V(H_1)| - |V(\sigma)|$ and $|V(H_2)| - |V(\sigma)|$ vertices in $V(H) \setminus$

$\nu(V(\sigma))$ induce subgraphs isomorphic to H_1 and H_2 , respectively, with the isomorphisms preserving the embeddings of σ . The definition of the product is then analogous to that in \mathcal{A} .

Consider an infinite sequence $(G_i)_{i \in \mathbf{N}}$ of tricolored graphs with an increasing number of vertices. Recall that if $H \in \mathbb{F}$, then $p(H, G_i)$ is the probability that a randomly chosen subset of $|V(H)|$ vertices of G_i induces a subgraph isomorphic to H . The sequence $(G_i)_{i \in \mathbf{N}}$ is *convergent* if $p(H, G_i)$ has a limit for every $H \in \mathbb{F}$. A standard argument (using Tychonoff's theorem [14]) yields that every infinite sequence of tricolored graphs has a convergent (infinite) subsequence.

Fix now a convergent sequence $(G_i)_{i \in \mathbf{N}}$ of tricolored graphs. We set $q(H) := \lim_{i \rightarrow \infty} p(H, G_i)$ for every $H \in \mathbb{F}$, and we linearly extend q to \mathcal{A} . The key property is that q is a homomorphism from \mathcal{A} to \mathbf{R} . Moreover, for $\sigma \in \mathbb{F}$ and an embedding ν of σ in G_i , define $p'_i(H) := p(H, G_i)$. So, for every $i \in \mathbf{N}$, the mappings p'_i form a random distribution of mappings from \mathcal{A} to \mathbf{R} , where randomness comes from the choice of ν . Since $p(H, G_i)$ converges (as i tends to infinity) for every $H \in \mathbb{F}$, the sequence of these distributions also converges. In what follows, q^σ will be a randomly chosen mapping from \mathcal{A}^σ to \mathbf{R} based on the limit distribution. It can be shown that such a mapping is a homomorphism from \mathcal{A}^σ to \mathbf{R} . In fact, q fully determines the random distributions of q^σ for all σ .

Let us now have a closer look at the relation between q and q^σ . The ‘‘averaging’’ operator $\llbracket \cdot \rrbracket_\sigma : \mathcal{A}^\sigma \rightarrow \mathcal{A}$ is a linear operator defined on the elements of \mathbb{F}^σ by $\llbracket H \rrbracket_\sigma := p \cdot H'$, where H' is the (unlabeled) tricolored graph in \mathbb{F} corresponding to H and p is the probability that a random injective mapping from $V(\sigma)$ to $V(H')$ is an embedding of σ in H' yielding H . The key relation between q and q^σ is the following:

$$\forall H \in \mathcal{A}^\sigma, \quad q(\llbracket H \rrbracket_\sigma) = \int q^\sigma(H), \quad (1)$$

where the integration is over the probability space given by the limit random distribution of q^σ . We immediately conclude that if $q^\sigma(H) \geq 0$ almost surely, then $q(\llbracket H \rrbracket_\sigma) \geq 0$. In particular,

$$\forall H \in \mathcal{A}^\sigma, \quad q(\llbracket H^2 \rrbracket_\sigma) \geq 0. \quad (2)$$

2.1 Particular Notation Used in our Proof

Before presenting the proof of Theorem 2, we need to introduce some notation and several lemmas. Recall that σ_A, σ_B and σ_C , the elements of \mathbb{F}_3 , are given in Figure 1. For $i \in \{A, B, C\}$ and a triple $t \in \{1, 2, 3\}^3$, let F_t^i be the element of $\mathbb{F}_4^{\sigma_i}$ in which the unlabeled vertex of F_t^i is joined by an edge of color t_j to the image of the j -th

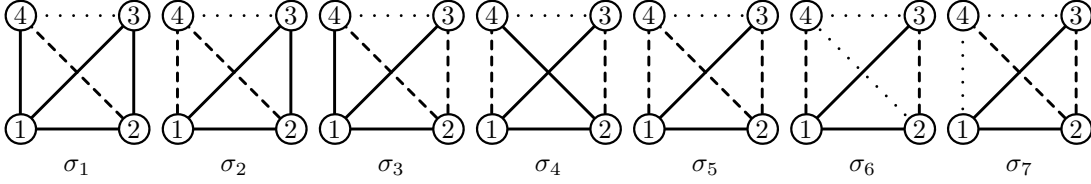


Figure 2: The elements $\sigma_1, \dots, \sigma_7$ of \mathbb{F}_4 . The edges of color 1, 2 and 3 are represented by solid, dashed and dotted lines, respectively.

vertex of σ_i for $j \in \{1, 2, 3\}$. Two elements of \mathcal{A}^{σ_B} and two of \mathcal{A}^{σ_C} will be of interest in our further considerations:

$$\begin{aligned}
w_B &:= 165F_{113}^B + 165F_{333}^B - 279F_{123}^B - 44F_{131}^B + 328F_{133}^B + 10F_{233}^B + 421F_{323}^B, \\
w'_B &:= -580F_{113}^B - 580F_{333}^B + 668F_{123}^B - 264F_{131}^B + 10F_{133}^B + 725F_{233}^B + 632F_{323}^B, \\
w_C &:= 100F_{112}^C + 100F_{312}^C - 100F_{113}^C - 100F_{133}^C + 162F_{122}^C + 163F_{221}^C, \quad \text{and} \\
w'_C &:= -10F_{112}^C - 10F_{312}^C + 10F_{113}^C + 10F_{133}^C - 77F_{122}^C + 89F_{221}^C.
\end{aligned}$$

We make use of seven elements $\sigma_1, \dots, \sigma_7$ out of the 15 elements of \mathbb{F}_4 . They are depicted in Figure 2. For $i \in \{1, \dots, 7\}$ and a quadruple $q \in \{1, 2, 3\}^4$, let F_q^i be the element of $\mathbb{F}_5^{\sigma_i}$ such that the unlabeled vertex of F_q^i is joined by an edge of color q_j to the j -th vertex of σ_i for $j \in \{1, 2, 3, 4\}$. If $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$, then $F_{(c)}^i$ is the element of \mathcal{A}^{σ_i} that is the sum of all the five-vertex σ_i -flags F_q^i such that the unlabeled vertex is joined by an edge of color c to at least one of the vertices of σ_i , i.e., at least one of the entries of q is c .

Finally, we define H_1, \dots, H_{142} to be the elements of \mathbb{F}_5 in the way depicted in Appendix A.

3 Proof of Theorem 2

In this section, we prove Theorem 2 by contradiction: in a series of lemma, we shall prove some properties of a counterexample, which eventually allows us to establish the nonexistence of counterexamples.

Let G be a tricolored complete graph. For a vertex v of G , let A_v be the set of the colors of the edges incident with v . Consider a sequence of graphs $(G_k)_{k \in \mathbb{N}}$, obtained from G by replacing each vertex v of G with a complete graph of order k with edges colored uniformly at random with colors in A_v ; the colors of the edges between the complete graphs corresponding to the vertices v and v' of G being assigned the color

| | i=1 | i=2 | i=3 | i=4 | i=5 | i=6 | i=7 |
|---------|------|-----|------|------|------|------|-----|
| $c = 1$ | -1/3 | 0 | -1/3 | -1/3 | 0 | 0 | 0 |
| $c = 2$ | 1/2 | 0 | 1/6 | -1/3 | -1/3 | -1/3 | 0 |
| $c = 3$ | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 0 | 0 |

Table 1: The values $\varepsilon_c(\sigma_i)$ for $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$.

of the edge vv' . This sequence of graphs converges asymptotically almost surely; let q_G be the corresponding homomorphism from \mathcal{A} to \mathbf{R} .

Let $n \geq 2$. We define a *counterexample* to be a tricolored graph with n vertices such that for every color $c \in \{1, 2, 3\}$, each set W of at most four vertices strongly c -dominates less than $2n/3$ vertices of G . A counterexample readily satisfies the following.

Lemma 3. *If G is a counterexample, then every vertex is incident with edges of at least two different colors.*

In the next lemma, we establish an inequality that q_G satisfies if G is a counterexample. To do so, define the quantity $\varepsilon_c(\sigma_i)$ for $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$ to be $1/2$ if σ_i contains a single edge with color c , $-1/3$ if each vertex of σ_i is incident with an edge colored c , $1/6$ if σ_i contains at least two edges with color c and a vertex incident with edges of a single color different from c , and 0 , otherwise. These values are gathered in Table 1.

Lemma 4. *Let G be a counterexample with n vertices. For every $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$, a homomorphism $q_G^{\sigma_i}$ from \mathcal{A}^{σ_i} to \mathbf{R} almost surely satisfies the inequality*

$$q_G^{\sigma_i}(F_{(c)}^i) \leq \frac{2}{3} + \frac{\varepsilon_c(\sigma_i)}{n}.$$

Proof. Fix $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$. Consider the graph G_k for sufficiently large k . Let (w_1, w_2, w_3, w_4) be a quadruple of vertices of G_k inducing a subgraph isomorphic to σ_i . Further, let W be the set of vertices strongly c -dominated by $\{w_1, \dots, w_4\}$. We show that $|W| \leq \frac{2nk}{3} + \varepsilon_c(\sigma_i)k + o(k)$ with probability tending to one as k tends to infinity. This will establish the inequality stated in the lemma.

For $i \in \{1, 2, 3, 4\}$, let v_i be the vertex of G corresponding to the clique W_i of G_k containing w_i . Let V be the set of vertices of G that are strongly c -dominated by $\{v_1, \dots, v_4\}$. Since G is a counterexample, $|V| < 2n/3$, and hence, $|V| \leq 2n/3 - 1/3$.

If w_j and $w_{j'}$ are joined by an edge of color c and, furthermore, $v_j = v_{j'}$, then v_j is added to V as well. Since V is still strongly c -dominated by a quadruple of vertices in G (replace $v_{j'}$ by any of its c -neighbors), it follows that $|V| \leq 2n/3 - 1/3$.

The set W can contain the $|V|k$ vertices of the cliques corresponding to the vertices of V , and, potentially, it also contains some additional vertices if w_i has no c -neighbors among w_1, \dots, w_4 . In this case, the additional vertices in W are the c -neighbors of w_i in W_i . There are at most $k/3 + o(k)$ such vertices if v_i is incident with edges of all three colors in G , and at most $k/2 + o(k)$ if v_i is incident with edges of only two colors in G .

If $\varepsilon_c(\sigma_i) = -1/3$, then all the vertices w_1, \dots, w_4 have a c -neighbor among w_1, \dots, w_4 and thus W contains only vertices of the cliques corresponding to the vertices V . We conclude that $|W| \leq \frac{(2n-1)k}{3} + o(k)$, as required.

If $\varepsilon_c(\sigma_i) = 0$, then all but one of the vertices w_1, \dots, w_4 have a c -neighbor among w_1, \dots, w_4 and the vertex w_j that has none is incident in σ_i with edges of the two colors different from c . In particular, either w_j has no c -neighbors inside W_j or v_j is incident with edges of three distinct colors in G . This implies that $|W| \leq \frac{(2n-1)k}{3} + o(k)$ in the former case and $|W| \leq \frac{2nk}{3} + o(k)$ in the latter case. So, the bound holds.

If $\varepsilon_c(\sigma_i) = 1/6$, then all but one of the vertices among w_1, \dots, w_4 have a c -neighbor among w_1, \dots, w_4 . Let w_j be the exceptional vertex. Since w_j has at most $k/2 + o(k)$ c -neighbors in W_j , it follows that $|W| \leq \frac{2nk}{3} + \frac{k}{6} + o(k)$.

Finally, if $\varepsilon_c(\sigma_i) = 1/2$, then two vertices w_j and $w_{j'}$ among w_1, \dots, w_4 have no c -neighbors in $\{w_1, \dots, w_4\}$. The vertices w_j and $w_{j'}$ have at most $k/2 + o(k)$ c -neighbors each in W_j and $W_{j'}$, respectively. Moreover, since σ_i contains edges of all three colors, one of w_j and $w_{j'}$ is incident in σ_i with edges of the two colors different from c . Hence, this vertex has at most $k/3 + o(k)$ c -neighbors in W_j . We conclude that the set W contains at most $|V|k + 5k/6 + o(k) \leq \frac{2nk}{3} + \frac{k}{2} + o(k)$ vertices. \square

As a consequence of (1), we have the following corollary of Lemma 4.

Lemma 5. *Let G be a counterexample with n vertices. For every $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$ such that $\varepsilon_c(\sigma_i) \leq 0$, it holds that*

$$q_G(\llbracket 2\sigma_i/3 - F_{(c)}^i \rrbracket_{\sigma_i}) \geq 0.$$

We now prove that in a counterexample, at most two colors are used to color the edges incident with any given vertex. As we shall see, this structural property of counterexamples directly implies their nonexistence, thereby proving Theorem 2.

Lemma 6. *No counterexample contains a vertex incident with edges of all three colors.*

Proof. Let G be a counterexample and $w_3 \in \mathcal{A}$ be the sum of all elements of \mathbb{F}_5 that contain a vertex incident with at least three colors. By the definition of q_G , the graph G has a vertex incident with edges of all three colors if and only if $q_G(w_3) > 0$. Lemma 5 implies that $q_G(H)$ is non-negative for each element H of \mathcal{A} corresponding to any column of Table 2 (in Appendix B). In addition, (2) ensures that $q_G(H)$ is also non-negative for each element H of \mathcal{A} corresponding to any of the first four columns of Table 3 (in Appendix B). Summing these columns with coefficients

$$\begin{array}{r}
\underline{23457815885978657985} \\
1029505785512512 \\
15852088219609163945 \\
\underline{514752892756256} \\
3956624143678293415 \\
\underline{772129339134384} \\
74313622711306287405 \\
\underline{2059011571025024} \\
15977347300925119 \\
\underline{32944185136400384}
\end{array}, \quad
\begin{array}{r}
\underline{134730108347752975} \\
4596007971038 \\
\underline{196791037567187109905} \\
12354069426150144 \\
\underline{30762195734543710715} \\
772129339134384 \\
\underline{48968798259015} \\
514752892756256 \\
\underline{8880723226482731} \\
24708138852300288
\end{array}, \quad
\begin{array}{r}
\underline{134730108347752975} \\
4596007971038 \\
\underline{33245823856447882025} \\
24708138852300288 \\
\underline{20816545085118359705} \\
4118023142050048 \\
\underline{39315342699665} \\
6177034713075072
\end{array},$$

respectively, yields an element w_0 of \mathcal{A} given in the very last column of Table 3. As q_G is a homomorphism from \mathcal{A} to \mathbf{R} , it follows that $q_G(w_0) \geq 0$. Since $w_0 \leq -w_3$, we deduce that $q_G(w_3) \leq 0$, which therefore implies that $q_G(w_3) = 0$, as desired. \square

We are now in a position to prove Theorem 2, whose statement is recalled below.

Theorem 7. *Let $n \geq 2$. Every tricolored graph with n vertices contains a subset of at most four vertices that strongly c -dominates at least $2n/3$ vertices for some color c .*

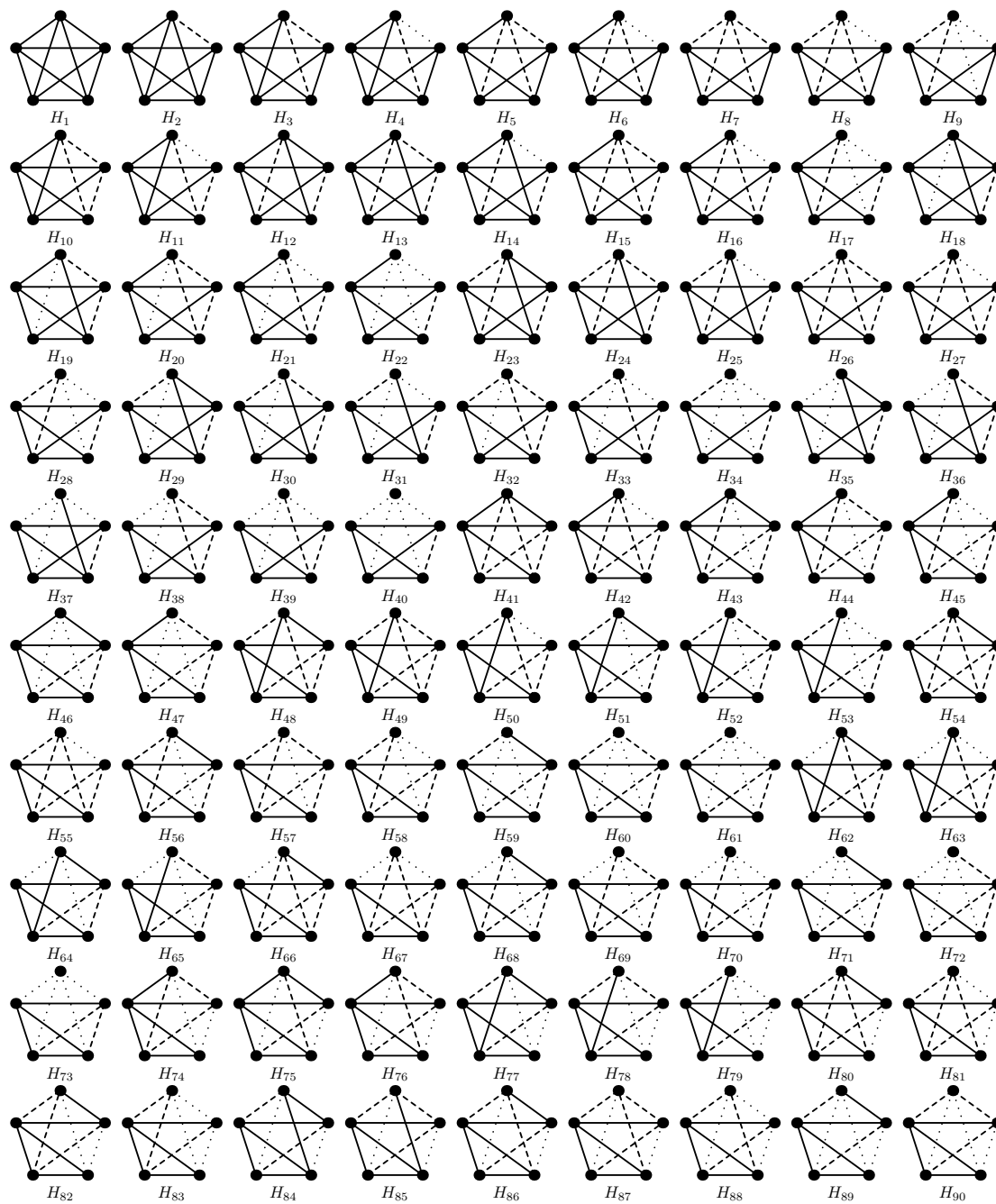
Proof. Suppose, on the contrary, that there exists a counterexample G . Recall that A_v is the set of colors that appear on the edges incident to the vertex v . Now, by Lemmas 3 and 6, it holds that $|A_v| = 2$ for every vertex v of G . Hence, $V(G)$ can be partitioned into three sets V_1, V_2 and V_3 , where $v \in V_i$ if and only if $i \notin A_v$. Without loss of generality, assume that $|V_1| \geq |V_2| \geq |V_3|$. Pick $u \in V_1$ and $v \in V_2$. As $A_u \cap A_w = \{3\}$ for all $w \in V_2$, we observe that V_2 is 3-dominated by $\{u\}$. Similarly, V_1 is 3-dominated by $\{v\}$. Therefore, the set $\{u, v\}$ strongly 3-dominates $V_1 \cup V_2$, which has size at least $2n/3$. \square

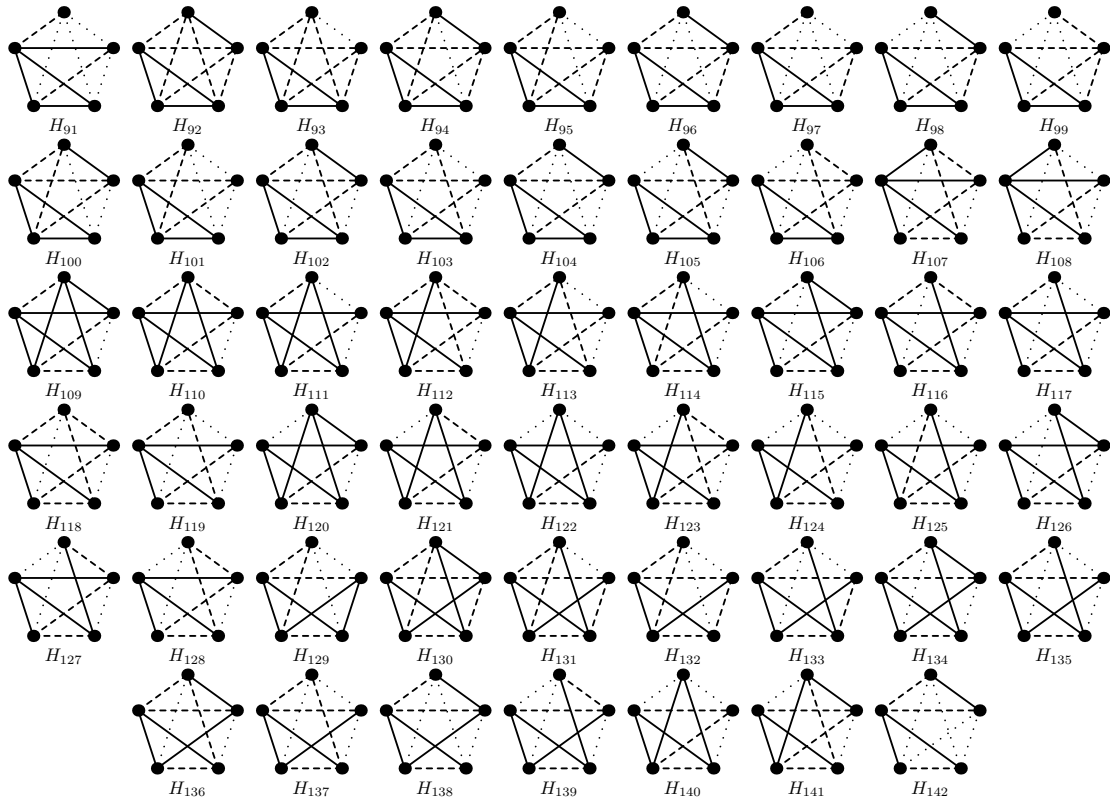
References

- [1] R. BABER AND J. TALBOT, *Hypergraphs do jump*, *Combin. Probab. Comput.*, 20 (2011), pp. 161–171.
- [2] C. BORGS, J. T. CHAYES, L. LOVÁSZ, V. T. SÓS, AND K. VESZTERGOMBI, *Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing*, *Adv. Math.*, 219 (2008), pp. 1801–1851.

- [3] P. ERDŐS, R. FAUDREE, A. GYÁRFÁS, AND R. H. SCHELP, *Domination in colored complete graphs*, J. Graph Theory, 13 (1989), pp. 713–718.
- [4] P. ERDŐS, R. J. FAUDREE, R. J. GOULD, A. GYÁRFÁS, C. ROUSSEAU, AND R. H. SCHELP, *Monochromatic coverings in colored complete graphs*, in Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989), vol. 71 of Congr. Numer., 1990, pp. 29–38.
- [5] P. ERDŐS AND A. HAJNAL, *Ramsey-type theorems*, Discrete Appl. Math., 25 (1989), pp. 37–52. Combinatorics and complexity (Chicago, IL, 1987).
- [6] A. GRZESIK, *On the maximum number of C_5 's in a triangle-free graph*, J. Comb. Theory Ser. B, forthcoming.
- [7] H. HATAMI, J. HLADKÝ, D. KRÁL', S. NORINE, AND A. RAZBOROV, *Non-three-colorable common graphs exist*, Combin. Probab. Comput., forthcoming.
- [8] —, *On the number of pentagons in triangle-free graphs*. Submitted for publication.
- [9] J. HLADKÝ, D. KRÁL', AND S. NORINE, *Counting flags in triangle-free digraphs*. Submitted for publication.
- [10] L. LOVÁSZ AND B. SZEGEDY, *Limits of dense graph sequences*, J. Combin. Theory Ser. B, 96 (2006), pp. 933–957.
- [11] A. RAZBOROV, *Flag algebras*, J. Symbolic Logic, 72 (2007), pp. 1239–1282.
- [12] —, *On the minimal density of triangles in graphs*, Combin. Probab. Comput., 17 (2008), pp. 603–618.
- [13] —, *On 3-hypergraphs with forbidden 4-vertex configurations*, SIAM J. Discrete Math., 24 (2010), pp. 946–963.
- [14] A. TYCHONOFF, *Über die topologische erweiterung von räumen*, Mathematische Annalen, 102 (1930), pp. 544–561.

A The Elements of \mathbb{F}_5





B Vectors Used in the Proof of Lemma 6

Table 2: The first ten vectors

| | $\llbracket 2\sigma_1/3 - F_{(1)}^1 \rrbracket_{\sigma_1}$ | $\llbracket 2\sigma_2/3 - F_{(1)}^2 \rrbracket_{\sigma_2}$ | $\llbracket 2\sigma_2/3 - F_{(2)}^2 \rrbracket_{\sigma_2}$ | $\llbracket 2\sigma_3/3 - F_{(1)}^3 \rrbracket_{\sigma_3}$ | $\llbracket 2\sigma_4/3 - F_{(1)}^4 \rrbracket_{\sigma_4}$ | $\llbracket 2\sigma_4/3 - F_{(2)}^4 \rrbracket_{\sigma_4}$ | $\llbracket 2\sigma_5/3 - F_{(1)}^5 \rrbracket_{\sigma_5}$ | $\llbracket 2\sigma_5/3 - F_{(2)}^5 \rrbracket_{\sigma_5}$ | $\llbracket 2\sigma_6/3 - F_{(1)}^6 \rrbracket_{\sigma_6}$ | $\llbracket 2\sigma_7/3 - F_{(2)}^7 \rrbracket_{\sigma_7}$ |
|----------|--|--|--|--|--|--|--|--|--|--|
| H_1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_4 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_6 | -1/90 | -1/180 | 1/90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_8 | 0 | -1/60 | -1/60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_9 | 0 | -1/45 | 2/45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{10} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{11} | 0 | 0 | 0 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{12} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{13} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{14} | -1/90 | 0 | 0 | 0 | -1/180 | 1/90 | 0 | 0 | 0 | 0 |
| H_{15} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{16} | -1/180 | 0 | 0 | -1/180 | 0 | 0 | -1/360 | 1/180 | 0 | 0 |
| H_{17} | -1/90 | 0 | 0 | -1/180 | 0 | 0 | 0 | 0 | 1/180 | 0 |
| H_{18} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{19} | -1/180 | 0 | 0 | 0 | 0 | 0 | 1/180 | -1/360 | 0 | 0 |
| H_{20} | -1/90 | 1/90 | -1/180 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{21} | -1/180 | 0 | 0 | -1/180 | 0 | 0 | 0 | 0 | 0 | 1/120 |
| H_{22} | 0 | 0 | 0 | 1/180 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{23} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{24} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{25} | 0 | -1/180 | -1/180 | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 |
| H_{26} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{27} | 0 | -1/180 | -1/180 | 0 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{28} | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 | 0 | 1/90 | 0 |

Continued on next page

Table 2 – *Continued from previous page*

| | $\left[\left[\frac{2\sigma_1}{3} - F_{(1)}^1 \right] \right]_{\sigma_1}$ | $\left[\left[\frac{2\sigma_2}{3} - F_{(1)}^2 \right] \right]_{\sigma_2}$ | $\left[\left[\frac{2\sigma_2}{3} - F_{(2)}^2 \right] \right]_{\sigma_2}$ | $\left[\left[\frac{2\sigma_3}{3} - F_{(1)}^3 \right] \right]_{\sigma_3}$ | $\left[\left[\frac{2\sigma_4}{3} - F_{(1)}^4 \right] \right]_{\sigma_4}$ | $\left[\left[\frac{2\sigma_4}{3} - F_{(2)}^4 \right] \right]_{\sigma_4}$ | $\left[\left[\frac{2\sigma_5}{3} - F_{(1)}^5 \right] \right]_{\sigma_5}$ | $\left[\left[\frac{2\sigma_5}{3} - F_{(2)}^5 \right] \right]_{\sigma_5}$ | $\left[\left[\frac{2\sigma_6}{3} - F_{(1)}^6 \right] \right]_{\sigma_6}$ | $\left[\left[\frac{2\sigma_7}{3} - F_{(2)}^7 \right] \right]_{\sigma_7}$ |
|----------|--|--|--|--|--|--|--|--|--|--|
| H_{29} | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{30} | -1/180 | -1/180 | -1/180 | 0 | 0 | 0 | -1/360 | -1/360 | 0 | 0 |
| H_{31} | -1/180 | -1/180 | 1/90 | 0 | -1/180 | 1/90 | 1/180 | -1/360 | 0 | 0 |
| H_{32} | 0 | -1/60 | -1/60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{33} | 0 | -1/90 | 1/180 | 0 | 0 | 0 | -1/360 | 1/180 | 0 | 0 |
| H_{34} | 0 | -1/90 | -1/90 | 1/90 | 0 | 0 | 0 | 0 | -1/90 | 0 |
| H_{35} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{36} | 0 | -1/180 | 1/90 | 0 | 0 | 0 | 1/90 | -1/180 | 0 | 0 |
| H_{37} | 0 | 0 | 0 | 0 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{38} | 0 | 1/90 | 1/90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{39} | 0 | -1/180 | -1/180 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{40} | 0 | 0 | 0 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{41} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{42} | 0 | 0 | 0 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{43} | -1/90 | 0 | 0 | 0 | 1/90 | -1/180 | 0 | 0 | 0 | 0 |
| H_{44} | 1/180 | 0 | 0 | -1/180 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{45} | -1/180 | 0 | 0 | -1/90 | 0 | 0 | 1/180 | 1/180 | 0 | 0 |
| H_{46} | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1/45 | 0 |
| H_{47} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{48} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{49} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{50} | 0 | 0 | 0 | 0 | -1/180 | -1/180 | -1/180 | -1/180 | 0 | 0 |
| H_{51} | -1/180 | 0 | 0 | 0 | -1/180 | -1/180 | -1/360 | -1/360 | 0 | 0 |
| H_{52} | -1/180 | 0 | 0 | -1/180 | 0 | 0 | -1/360 | -1/360 | 0 | 0 |
| H_{53} | -1/180 | 0 | 0 | 0 | -1/180 | -1/180 | 0 | 0 | 1/180 | 0 |
| H_{54} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{55} | 0 | 0 | 0 | 0 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{56} | 0 | -1/180 | -1/180 | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 |
| H_{57} | -1/180 | -1/180 | -1/180 | 0 | 0 | 0 | -1/360 | -1/360 | 0 | 0 |
| H_{58} | 0 | -1/180 | -1/180 | 0 | 0 | 0 | 1/360 | -1/180 | 1/180 | 0 |
| H_{59} | 0 | -1/180 | 1/90 | 0 | -1/90 | -1/90 | 0 | 0 | 1/180 | 0 |

Continued on next page

Table 2 – Continued from previous page

| | $\left[\left[\frac{2\sigma_1}{3} - F_{(1)}^1 \right] \right]_{\sigma_1}$ | $\left[\left[\frac{2\sigma_2}{3} - F_{(1)}^2 \right] \right]_{\sigma_2}$ | $\left[\left[\frac{2\sigma_2}{3} - F_{(2)}^2 \right] \right]_{\sigma_2}$ | $\left[\left[\frac{2\sigma_3}{3} - F_{(1)}^3 \right] \right]_{\sigma_3}$ | $\left[\left[\frac{2\sigma_4}{3} - F_{(1)}^4 \right] \right]_{\sigma_4}$ | $\left[\left[\frac{2\sigma_4}{3} - F_{(2)}^4 \right] \right]_{\sigma_4}$ | $\left[\left[\frac{2\sigma_5}{3} - F_{(1)}^5 \right] \right]_{\sigma_5}$ | $\left[\left[\frac{2\sigma_5}{3} - F_{(2)}^5 \right] \right]_{\sigma_5}$ | $\left[\left[\frac{2\sigma_6}{3} - F_{(1)}^6 \right] \right]_{\sigma_6}$ | $\left[\left[\frac{2\sigma_7}{3} - F_{(2)}^7 \right] \right]_{\sigma_7}$ |
|----------|--|--|--|--|--|--|--|--|--|--|
| H_{60} | 0 | -1/180 | 1/90 | -1/180 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{61} | 0 | -1/180 | -1/180 | 1/90 | 0 | 0 | 0 | 0 | -1/45 | 0 |
| H_{62} | -1/180 | 0 | 0 | 0 | 0 | 0 | -1/360 | -1/360 | 0 | 0 |
| H_{63} | -1/180 | 0 | 0 | 0 | 0 | 0 | 1/360 | 1/360 | 0 | 0 |
| H_{64} | 0 | 0 | 0 | 0 | 0 | 0 | 1/90 | -1/180 | 0 | 0 |
| H_{65} | 0 | 0 | 0 | 1/90 | 0 | 0 | -1/360 | -1/360 | 0 | 0 |
| H_{66} | 0 | -1/180 | -1/180 | 0 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{67} | 0 | -1/180 | -1/180 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{68} | 0 | -1/180 | 1/90 | 0 | 1/90 | -1/180 | 1/360 | -1/180 | 0 | 0 |
| H_{69} | 1/90 | -1/90 | 1/180 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{70} | 0 | -1/180 | -1/180 | 1/90 | 0 | 0 | -1/360 | 1/180 | 0 | -1/120 |
| H_{71} | 0 | 0 | 0 | 0 | 0 | 0 | -1/180 | -1/180 | 1/180 | 0 |
| H_{72} | 0 | 0 | 0 | 1/90 | 0 | 0 | 0 | 0 | 0 | -1/60 |
| H_{73} | 0 | 0 | 0 | -1/60 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{74} | -1/90 | 1/90 | 1/90 | -1/180 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{75} | -1/45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{76} | -1/90 | 0 | 0 | -1/90 | 1/90 | 1/90 | 0 | 0 | 0 | 0 |
| H_{77} | -1/90 | 0 | 0 | 0 | 0 | 0 | 1/90 | -1/180 | 0 | 0 |
| H_{78} | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{79} | -1/90 | 1/45 | -1/90 | 0 | -1/180 | 1/90 | 0 | 0 | 0 | 0 |
| H_{80} | -1/180 | 0 | 0 | 0 | -1/180 | -1/180 | -1/360 | -1/360 | 0 | 0 |
| H_{81} | 1/90 | 0 | 0 | 0 | 0 | 0 | 1/360 | -1/180 | 0 | 0 |
| H_{82} | -1/180 | -1/180 | -1/180 | 0 | 0 | 0 | 1/180 | -1/360 | 1/180 | 0 |
| H_{83} | 1/90 | 1/60 | -1/60 | 0 | 0 | 0 | -1/360 | 1/180 | 0 | 0 |
| H_{84} | -1/90 | 0 | 0 | 0 | -1/90 | 1/45 | 0 | 0 | 0 | 0 |
| H_{85} | -1/90 | 0 | 0 | 0 | 0 | 0 | -1/180 | 1/360 | -1/90 | 0 |
| H_{86} | -1/180 | -1/180 | -1/180 | 0 | -1/180 | -1/180 | -1/360 | -1/360 | 0 | 0 |
| H_{87} | 1/90 | -1/180 | -1/180 | -1/180 | -1/180 | 1/90 | -1/360 | -1/360 | 0 | 0 |
| H_{88} | 1/90 | -1/180 | -1/180 | 0 | 1/90 | -1/180 | 0 | 0 | -1/90 | -1/120 |
| H_{89} | -1/180 | -1/180 | -1/180 | 0 | 1/180 | 1/180 | -1/360 | -1/360 | 0 | 0 |
| H_{90} | 1/90 | -1/180 | -1/180 | 0 | 0 | 0 | -1/180 | 1/90 | 0 | -1/120 |

Continued on next page

Table 2 – *Continued from previous page*

| | $\left[\left[\frac{2\sigma_1}{3} - F_{(1)}^1 \right] \right]_{\sigma_1}$ | $\left[\left[\frac{2\sigma_2}{3} - F_{(1)}^2 \right] \right]_{\sigma_2}$ | $\left[\left[\frac{2\sigma_2}{3} - F_{(2)}^2 \right] \right]_{\sigma_2}$ | $\left[\left[\frac{2\sigma_3}{3} - F_{(1)}^3 \right] \right]_{\sigma_3}$ | $\left[\left[\frac{2\sigma_4}{3} - F_{(1)}^4 \right] \right]_{\sigma_4}$ | $\left[\left[\frac{2\sigma_4}{3} - F_{(2)}^4 \right] \right]_{\sigma_4}$ | $\left[\left[\frac{2\sigma_5}{3} - F_{(1)}^5 \right] \right]_{\sigma_5}$ | $\left[\left[\frac{2\sigma_5}{3} - F_{(2)}^5 \right] \right]_{\sigma_5}$ | $\left[\left[\frac{2\sigma_6}{3} - F_{(1)}^6 \right] \right]_{\sigma_6}$ | $\left[\left[\frac{2\sigma_7}{3} - F_{(2)}^7 \right] \right]_{\sigma_7}$ |
|-----------|--|--|--|--|--|--|--|--|--|--|
| H_{91} | 1/45 | -1/90 | -1/90 | 0 | 0 | 0 | 0 | 0 | -1/90 | 0 |
| H_{92} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{93} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{94} | 0 | -1/180 | -1/180 | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 |
| H_{95} | 0 | 1/90 | -1/180 | 0 | 0 | 0 | 1/90 | -1/180 | 0 | 0 |
| H_{96} | 0 | -1/180 | 1/90 | 0 | -1/90 | -1/90 | 0 | 0 | 1/180 | 0 |
| H_{97} | 0 | 1/90 | -1/180 | 1/90 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{98} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1/30 | 0 |
| H_{99} | 0 | 0 | 0 | -1/60 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{100} | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 | 0 | 1/90 | 0 |
| H_{101} | 0 | 2/45 | -1/45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{102} | 0 | -1/90 | -1/90 | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 |
| H_{103} | 0 | 1/45 | -1/90 | 0 | 1/90 | -1/180 | -1/180 | -1/180 | 0 | 0 |
| H_{104} | 0 | -1/90 | -1/90 | 0 | 1/45 | -1/90 | 0 | 0 | -1/90 | 0 |
| H_{105} | 0 | -1/90 | -1/90 | 0 | 0 | 0 | 0 | 0 | -1/90 | 0 |
| H_{106} | 0 | 1/45 | -1/90 | -1/180 | -1/90 | -1/90 | 0 | 0 | 0 | 0 |
| H_{107} | 0 | 0 | 0 | -1/45 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{108} | 0 | 0 | 0 | -1/45 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{109} | 0 | 0 | 0 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{110} | 0 | 0 | 0 | -1/180 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{111} | 0 | 0 | 0 | -1/180 | -1/180 | -1/180 | 1/180 | -1/360 | 0 | 0 |
| H_{112} | 0 | 0 | 0 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{113} | 0 | 0 | 0 | -1/180 | -1/180 | -1/180 | 1/180 | -1/360 | 0 | 0 |
| H_{114} | 0 | 0 | 0 | -1/90 | 0 | 0 | 1/360 | -1/180 | 0 | 0 |
| H_{115} | 0 | 0 | 0 | -1/180 | -1/90 | -1/90 | 0 | 0 | -1/90 | 0 |
| H_{116} | 0 | 0 | 0 | -1/180 | -1/90 | 1/180 | -1/180 | -1/180 | 0 | 0 |
| H_{117} | 0 | 0 | 0 | -1/180 | -1/180 | -1/180 | -1/180 | 1/360 | -1/90 | 0 |
| H_{118} | 0 | 0 | 0 | 1/180 | 0 | 0 | -1/180 | -1/180 | 0 | 0 |
| H_{119} | 0 | 0 | 0 | 1/90 | 0 | 0 | -1/360 | -1/360 | -1/90 | -1/120 |
| H_{120} | 0 | 0 | 0 | -1/90 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{121} | 0 | 0 | 0 | -1/180 | -1/180 | 1/90 | -1/180 | 1/360 | 0 | 0 |

Continued on next page

Table 3: The last six vectors

| | $[[w_B \cdot w_B]]_{\sigma_B}$ | $[[w'_B \cdot w'_B]]_{\sigma_B}$ | $[[w_C \cdot w_C]]_{\sigma_C}$ | $[[w'_C \cdot w'_C]]_{\sigma_C}$ | w_3 | w_0 |
|----------|--------------------------------|----------------------------------|--------------------------------|----------------------------------|-------|---|
| H_1 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_2 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_3 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_4 | 0 | 0 | 0 | 0 | 1 | $\frac{-1563854392398577199}{6177034713075072}$ |
| H_5 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_6 | 29161/60 | 101524/15 | 0 | 0 | 1 | -1 |
| H_7 | 0 | 0 | 0 | 0 | 0 | 0 |
| H_8 | 0 | 0 | 2000 | 20 | 0 | 0 |
| H_9 | 0 | 0 | -4000/3 | -40/3 | 0 | 0 |
| H_{10} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{11} | 1815/2 | 33640/3 | 0 | 0 | 1 | $\frac{-10173977739002723}{55152095652456}$ |
| H_{12} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{13} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{14} | -242 | 5104 | 0 | 0 | 1 | $\frac{-734882450141728337}{2316388017403152}$ |
| H_{15} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{16} | -9922/15 | -422/3 | 0 | 0 | 1 | $\frac{-5722046702587908817}{37062208278450432}$ |
| H_{17} | 57013/60 | -65634/5 | 0 | 0 | 1 | $\frac{-57717650068438077139}{148248833113801728}$ |
| H_{18} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{19} | 0 | 0 | 0 | 0 | 1 | $\frac{-703462682135213465}{3369291661677312}$ |
| H_{20} | 29161/60 | 101524/15 | 0 | 0 | 1 | -1 |
| H_{21} | 781/2 | -37700/3 | 0 | 0 | 1 | $\frac{-5891700664190917297}{148248833113801728}$ |
| H_{22} | -1804 | -580/3 | 0 | 0 | 1 | $\frac{-32614888977443071}{18531104139225216}$ |
| H_{23} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{24} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{25} | 0 | 0 | 19723/20 | 2047/20 | 1 | $\frac{-15461491234942018543}{5929953324552069120}$ |
| H_{26} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{27} | 0 | 0 | 540 | 77/3 | 1 | $\frac{-88140807390257339}{289548502175394}$ |
| H_{28} | 10/3 | 105125/6 | 0 | 0 | 1 | $\frac{-35834405989042100849}{74124416556900864}$ |
| H_{29} | 0 | 0 | 0 | 0 | 1 | $\frac{-1563854392398577199}{6177034713075072}$ |
| H_{30} | 0 | 0 | 270 | 77/6 | 1 | $\frac{-5456161254717178191}{12354069426150144}$ |
| H_{31} | 0 | 0 | 0 | 0 | 1 | $\frac{-4427934211353668633}{37062208278450432}$ |
| H_{32} | 0 | 0 | 2000 | 20 | 0 | 0 |
| H_{33} | 0 | 0 | -1810/3 | -97/6 | 1 | $\frac{-1570031427111652271}{6177034713075072}$ |
| H_{34} | -328/3 | 725/3 | 2000/3 | 20/3 | 1 | $\frac{-327323049775204219}{6738583323354624}$ |
| H_{35} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{36} | 0 | 0 | 0 | 0 | 1 | $\frac{-2040849950139277}{1323650295658944}$ |

Continued on next page

Table 3 – Continued from previous page

| | $[[w_B \cdot w_B]]_{\sigma_B}$ | $[[w'_B \cdot w'_B]]_{\sigma_B}$ | $[[w_C \cdot w_C]]_{\sigma_C}$ | $[[w'_C \cdot w'_C]]_{\sigma_C}$ | w_3 | w_0 |
|----------|--------------------------------|----------------------------------|--------------------------------|----------------------------------|-------|---|
| H_{37} | 0 | 0 | 2187/5 | 5929/60 | 1 | $\frac{-324486989357699}{150414806324880}$ |
| H_{38} | 0 | 0 | -4000/3 | -40/3 | 0 | 0 |
| H_{39} | 177241/60 | 99856/15 | 0 | 0 | 1 | $\frac{-14389006173379021}{6177034713075072}$ |
| H_{40} | 53792/15 | 10/3 | 0 | 0 | 1 | -1 |
| H_{41} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{42} | 53792/15 | 10/3 | 0 | 0 | 1 | -1 |
| H_{43} | -242 | 5104 | 0 | 0 | 1 | $\frac{-2444189262506217731}{32944185136400384}$ |
| H_{44} | -19723/60 | 8018 | 0 | 0 | 1 | $\frac{-130980504818216225}{5294601182635776}$ |
| H_{45} | 19251/20 | -46426/5 | 0 | 0 | 1 | $\frac{-767941410949255531}{4118023142050048}$ |
| H_{46} | 0 | 0 | 0 | 0 | 1 | $\frac{-5219817337367791253}{37062208278450432}$ |
| H_{47} | -4743/20 | -27388/5 | 0 | 0 | 1 | $\frac{-354709127257189891}{6177034713075072}$ |
| H_{48} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{49} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{50} | 0 | 0 | 0 | 0 | 1 | $\frac{-102522009006261748933}{296497666227603456}$ |
| H_{51} | 0 | 0 | 4401/10 | -6853/60 | 1 | $\frac{-103816701767414115797}{592995332455206912}$ |
| H_{52} | 1331/4 | 24476/3 | 0 | 0 | 1 | $\frac{-1794830760611264087}{5294601182635776}$ |
| H_{53} | -7157/30 | 72838/15 | 0 | 0 | 1 | $\frac{-55226415700070668835}{296497666227603456}$ |
| H_{54} | 0 | 0 | 0 | 0 | 0 | 0 |
| H_{55} | 0 | 0 | 2187/5 | 5929/60 | 1 | $\frac{-324486989357699}{150414806324880}$ |
| H_{56} | 0 | 0 | 1630/3 | -89/3 | 1 | $\frac{-148706888944854103}{561548610279552}$ |
| H_{57} | 0 | 0 | 270 | 77/6 | 1 | $\frac{-5456161234717178191}{12354069426150144}$ |
| H_{58} | 0 | 0 | 0 | 0 | 1 | $\frac{-74826771055029195907}{148248833113801728}$ |
| H_{59} | 0 | 0 | 0 | 0 | 1 | -1 |
| H_{60} | 0 | 0 | -540 | -77/3 | 1 | $\frac{-127346913837154513}{240663690119808}$ |
| H_{61} | -93 | 48430/3 | 0 | 0 | 1 | $\frac{-10466732649688555}{5294601182635776}$ |
| H_{62} | 0 | 0 | 0 | 0 | 1 | $\frac{-9320739160958665481}{37062208278450432}$ |
| H_{63} | 0 | 0 | 0 | 0 | 1 | $\frac{-62387193432797713}{-62387193432797713}$ |
| H_{64} | 0 | 0 | 0 | 0 | 1 | $\frac{37062208278450432}{-217609023306544037}$ |
| H_{65} | -34522/15 | 316/3 | 0 | 0 | 1 | $\frac{1323650295658944}{-1}$ |
| H_{66} | 0 | 0 | 540 | 77/3 | 1 | $\frac{-88140807390257339}{289548502175394}$ |
| H_{67} | 177241/60 | 99856/15 | 0 | 0 | 1 | $\frac{-14389006173379021}{6177034713075072}$ |
| H_{68} | 0 | 0 | -815/3 | 89/6 | 1 | $\frac{-42621413028711205}{37062208278450432}$ |
| H_{69} | 4631/4 | -18328/3 | -1000/3 | -10/3 | 1 | $\frac{-1330374174734201}{1029505785512512}$ |
| H_{70} | -39153/20 | 105544/15 | -270 | -77/6 | 1 | $\frac{-5834645898385742195}{16472092568200192}$ |
| H_{71} | 0 | 0 | 0 | 0 | 1 | $\frac{-3652233205897755459}{16472092568200192}$ |
| H_{72} | -39153/10 | 211088/15 | 0 | 0 | 1 | $\frac{-40194399986166687563}{74124416556900864}$ |
| H_{73} | 17391/4 | 114896/15 | 0 | 0 | 1 | $\frac{-67376462435613401}{1323650295658944}$ |

Continued on next page

Table 3 – Continued from previous page

| | $[[w_B \cdot w_B]]_{\sigma_B}$ | $[[w'_B \cdot w'_B]]_{\sigma_B}$ | $[[w_C \cdot w_C]]_{\sigma_C}$ | $[[w'_C \cdot w'_C]]_{\sigma_C}$ | w_3 | w_0 |
|-----------|--------------------------------|----------------------------------|--------------------------------|----------------------------------|-------|-------------------------------|
| H_{74} | -3069/2 | -38744/3 | 0 | 0 | 1 | -1 |
| H_{75} | -968 | 20416 | 0 | 0 | 1 | <u>-206704879201250857</u> |
| H_{76} | -13706/15 | -92396/15 | 0 | 0 | 1 | <u>441216765219648</u> |
| H_{77} | 0 | 0 | 0 | 0 | 1 | <u>-8722501888932923387</u> |
| H_{78} | 4631/2 | -36656/3 | 0 | 0 | 1 | <u>16472092568200192</u> |
| H_{79} | 0 | 0 | 0 | 0 | 1 | <u>-703462682135213465</u> |
| H_{80} | 0 | 0 | 0 | 0 | 1 | <u>1684645830838656</u> |
| H_{81} | -4631/15 | -13904/5 | 0 | 0 | 1 | <u>-2050765293919679467</u> |
| H_{82} | 0 | 0 | 0 | 0 | 1 | <u>18531104139225216</u> |
| H_{83} | 0 | 0 | 0 | 0 | 1 | -1 |
| H_{84} | 0 | 0 | 0 | 0 | 1 | <u>-34340368851241376879</u> |
| H_{85} | 121/6 | -30595/3 | 0 | 0 | 1 | <u>98832555409201152</u> |
| H_{86} | 0 | 0 | 0 | 0 | 1 | -1 |
| H_{87} | 1331/4 | 24476/3 | 0 | 0 | 1 | <u>-10725188546965769537</u> |
| H_{88} | -8657/30 | -194687/15 | -815/3 | 89/6 | 1 | <u>21178404730543104</u> |
| H_{89} | 0 | 0 | 270 | 77/6 | 1 | -1 |
| H_{90} | 4631/4 | -18328/3 | -270 | -77/6 | 1 | <u>-7417316739041385395</u> |
| H_{91} | 121/3 | -61190/3 | 2000/3 | 20/3 | 1 | <u>18531104139225216</u> |
| H_{92} | 0 | 0 | 0 | 0 | 0 | <u>-21505715322664188433</u> |
| H_{93} | 0 | 0 | 0 | 0 | 0 | <u>74124416556900864</u> |
| H_{94} | 0 | 0 | 19723/20 | 2047/20 | 1 | <u>-10051575074463385</u> |
| H_{95} | 0 | 0 | 0 | 0 | 1 | <u>18384031884152</u> |
| H_{96} | 0 | 0 | -1630/3 | 89/3 | 1 | <u>-2065655974432544177</u> |
| H_{97} | 0 | 0 | -540 | -77/3 | 1 | <u>9265552069612608</u> |
| H_{98} | 0 | 0 | 0 | 0 | 1 | <u>-10513889487465286471</u> |
| H_{99} | 77841/20 | 111556/5 | 0 | 0 | 1 | <u>21178404730543104</u> |
| H_{100} | 10/3 | 105125/6 | 0 | 0 | 1 | <u>-102492676367157469795</u> |
| H_{101} | 0 | 0 | -4000/3 | -40/3 | 0 | <u>296497666227603456</u> |
| H_{102} | 0 | 0 | 0 | 0 | 0 | <u>-3470421686575164043</u> |
| H_{103} | 0 | 0 | 0 | 0 | 0 | <u>148248833113801728</u> |
| H_{104} | 0 | 0 | 0 | 0 | 0 | <u>-147229716847699567</u> |
| H_{105} | 0 | 0 | 0 | 0 | 1 | <u>9265552069612608</u> |
| H_{106} | 0 | 0 | -1630/3 | 89/3 | 1 | <u>-4163309017023671941</u> |
| H_{107} | 107584/15 | 20/3 | 0 | 0 | 1 | <u>24708138852300288</u> |
| H_{108} | 30504/5 | 1336/3 | 0 | 0 | 1 | -1 |
| H_{109} | 1815/2 | 33640/3 | 0 | 0 | 1 | <u>-35834405989042100849</u> |
| H_{110} | 0 | 0 | 0 | 0 | 1 | <u>74124416556900864</u> |

Continued on next page

Table 3 – Continued from previous page

| | $[[w_B \cdot w_B]]_{\sigma_B}$ | $[[w'_B \cdot w'_B]]_{\sigma_B}$ | $[[w_C \cdot w_C]]_{\sigma_C}$ | $[[w'_C \cdot w'_C]]_{\sigma_C}$ | w_3 | w_0 |
|-----------|--------------------------------|----------------------------------|--------------------------------|----------------------------------|-------|-------------------------------|
| H_{111} | 4631/4 | -18328/3 | 0 | 0 | 1 | <u>-7493427555720786047</u> |
| H_{112} | -1804 | -580/3 | 0 | 0 | 1 | <u>26954333293418496</u> |
| H_{113} | -34522/15 | 316/3 | 0 | 0 | 1 | <u>-4771933910371470719</u> |
| H_{114} | -3069/4 | -19372/3 | 0 | 0 | 1 | <u>9265552069612608</u> |
| H_{115} | 55 | -42050/3 | 26569/60 | 7921/60 | 1 | <u>-56090180193586615063</u> |
| H_{116} | 0 | 0 | 0 | 0 | 1 | <u>98832555409201152</u> |
| H_{117} | -93/2 | 24215/3 | 0 | 0 | 1 | <u>-6146435219180552237</u> |
| H_{118} | -1804 | -580/3 | 2187/5 | 5929/60 | 1 | <u>9265552069612608</u> |
| H_{119} | -70439/30 | 8177 | 0 | 0 | 1 | <u>-476307820942045182061</u> |
| H_{120} | 1815/2 | 33640/3 | 0 | 0 | 1 | <u>1976651108184023040</u> |
| H_{121} | 0 | 0 | 0 | 0 | 1 | <u>-19450524422641811549</u> |
| H_{122} | 4631/4 | -18328/3 | 0 | 0 | 1 | <u>32944185136400384</u> |
| H_{123} | -328/3 | 725/3 | 0 | 0 | 1 | <u>-8220306420511019599</u> |
| H_{124} | 0 | 0 | 0 | 0 | 1 | <u>42356809461086208</u> |
| H_{125} | -27249/10 | 8684/15 | 0 | 0 | 1 | <u>-362958430331557939</u> |
| H_{126} | 55 | -42050/3 | 0 | 0 | 1 | <u>92655520696126080</u> |
| H_{127} | 0 | 0 | 0 | 0 | 1 | <u>-15367554150816847711</u> |
| H_{128} | 170681/60 | 103481/15 | 0 | 0 | 1 | <u>49416277704600576</u> |
| H_{129} | 0 | 0 | 0 | 0 | 1 | <u>-10173977739002723</u> |
| H_{130} | 0 | 0 | 0 | 0 | 1 | <u>55152095652456</u> |
| H_{131} | 0 | 0 | 4401/5 | -6853/30 | 1 | <u>-1003343753899617143</u> |
| H_{132} | 0 | 0 | 0 | 0 | 1 | <u>6177034713075072</u> |
| H_{133} | 421/6 | 22910/3 | 0 | 0 | 1 | <u>-2082303347082636833</u> |
| H_{134} | 0 | 0 | 4401/5 | -6853/30 | 1 | <u>9265552069612608</u> |
| H_{135} | 0 | 0 | 0 | 0 | 1 | <u>-12006211264574547431</u> |
| H_{136} | 0 | 0 | 0 | 0 | 1 | <u>24708138852300288</u> |
| H_{137} | 0 | 0 | 0 | 0 | 1 | <u>-13765701958912919</u> |
| H_{138} | 0 | 0 | 0 | 0 | 1 | <u>2059011571025024</u> |
| H_{139} | 421/6 | 22910/3 | 0 | 0 | 1 | <u>-75624575885139732659</u> |
| H_{140} | 0 | 0 | 0 | 0 | 1 | <u>148248833113801728</u> |
| H_{141} | 0 | 0 | 0 | 0 | 1 | <u>-70683455524198969843</u> |
| H_{142} | 20/3 | 105125/3 | 0 | 0 | 1 | <u>148248833113801728</u> |
| | | | | | 1 | <u>-11080743495118222157</u> |
| | | | | | 1 | <u>21178404730543104</u> |
| | | | | | 1 | <u>-1</u> |
| | | | | | 1 | <u>-1157293995940733471</u> |
| | | | | | 0 | <u>4632776034806304</u> |
| | | | | | 0 | <u>0</u> |
| | | | | | 1 | <u>-1</u> |
| | | | | | 1 | <u>-426427906114141689</u> |
| | | | | | 1 | <u>1029505785512512</u> |
| | | | | | 1 | <u>-1235497822172284283</u> |
| | | | | | 1 | <u>8984777764472832</u> |
| | | | | | 1 | <u>-5617524380783071181</u> |
| | | | | | 1 | <u>32944185136400384</u> |
| | | | | | 1 | <u>-26428837039774952311</u> |
| | | | | | 1 | <u>42356809461086208</u> |
| | | | | | 1 | <u>-73815219170205621743</u> |
| | | | | | 1 | <u>148248833113801728</u> |
| | | | | | 1 | <u>-51576322752518046641</u> |
| | | | | | 1 | <u>296497666227603456</u> |
| | | | | | 1 | <u>-37389454791911250173</u> |
| | | | | | 1 | <u>296497666227603456</u> |
| | | | | | 1 | <u>-17214181054154993319</u> |
| | | | | | 1 | <u>32944185136400384</u> |
| | | | | | 1 | <u>-1542818265173952315</u> |
| | | | | | 1 | <u>8236046284100096</u> |
| | | | | | 1 | <u>-1157293995940733471</u> |
| | | | | | 1 | <u>2316388017403152</u> |
| | | | | | 1 | <u>-1</u> |