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# GROUP SCHEMES OUT OF BIRATIONAL GROUP LAWS, NÉRON MODELS

by

Bas Edixhoven & Matthieu Romagny

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**Abstract.** — In this note, we present the theorem of extension of birational group laws in both settings of classical varieties (Weil) and schemes (Artin). We improve slightly the original proof and result with a more direct construction of the group extension, a discussion of its separation properties, and the systematic use of algebraic spaces. We also explain the important application to the construction of Néron models of abelian varieties. This note grew out of lectures given by Ariane Mézard and the second author at the Summer School "Schémas en groupes" held in the CIRM (Luminy) from 29 August to 9 September, 2011.

**Résumé.** — Dans cette note, nous présentons le théorème d'extension d'une loi de groupe birationnelle en un groupe algébrique, dans le cadre des variétés algébriques classiques (Weil) et des schémas (Artin). Nous améliorons légèrement le résultat original et sa preuve en donnant une construction plus directe du groupe, en apportant des compléments sur ses propriétés de séparation, et en utilisant systématiquement les espaces algébriques. Nous expliquons aussi l'application importante à la construction des modèles de Néron des variétés abéliennes. Cette note est issue des cours donnés par Ariane Mézard et le second auteur à l'École d'été "Schémas en groupes" qui s'est tenue au CIRM (Luminy) du 29 août au 9 septembre 2011.

## 1. Introduction

This paper is devoted to an exposition of the generalization to group schemes of Weil's theorem in [Wei2] on the construction of a group from a birational group law, as can be found in Artin's Exposé XVIII in SGA3 [Art]. In addition, we show how this theorem is used by Néron in order to produce canonical smooth models (the famous *Néron models*) of abelian varieties.

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**Key words and phrases.** — group scheme, birational group law, Néron model.

**MSC 2010:** 14L15, 11G05, 11G10, 14K99.

The content of Weil’s theorem is to extend a given “birational group law” on a scheme  $X$  to an actual multiplication on a group scheme  $G$  birational to  $X$ . The original motivation of Weil was the algebraic construction of the Jacobian varieties of curves [Wei1]. This construction was extended by Rosenlicht to generalized Jacobians [Ros]. Weil’s ideas were later used by Demazure in his thesis in order to show the existence of split reductive groups over the ring  $\mathbf{Z}$  of integers [Dem2] and then by Néron in order to study minimal models of abelian varieties [N]. To our knowledge, these are the three main applications of the extension theorem.

The construction of split reductive groups by Demazure uses a version of Weil’s theorem written by Artin, valid for flat (maybe non-smooth) finitely presented group schemes. There, the set-theoretic arguments of Weil are replaced by sheaf-theoretic arguments. The main point then is to show that Weil’s procedure gives a sheaf which is representable; since this sheaf is defined as a quotient by an fppf equivalence relation, the natural sense in which it is representable is as an algebraic space (i.e. a quotient of an étale equivalence relation of schemes, see 3.17). However, at the time when Artin figured out his adaptation of Weil’s result, he had not yet discovered algebraic spaces. Thus he had to resort at times to ad hoc statements; for example, his main statement (Theorem 3.7 of [Art]) is a bit unsatisfying. Nowadays it is more natural to use the language of algebraic spaces, and this is what we shall do. As an aside, it is clear that one may as well start from a birational group law on an *algebraic space*, but we do not develop this idea.

Another feature of Artin’s proof is that he constructs  $G$  let us say “in the void”, and that needs a lot of verifications that moreover are not so structured. We give a more structured proof of Theorem 3.7 of [Art]. The idea is to construct the group space  $G$  as a subfunctor of the  $S$ -functor in groups  $\mathcal{R}$  that sends  $T$  to the group of  $T$ -birational maps from  $X_T$  to itself, as in Section 5.2 of Bosch-Lütkebohmert-Raynaud [BLR]. We push the construction of [BLR] a bit further: we show that  $\mathcal{R}$  is a sheaf and we define  $G$  to be the subsheaf of groups generated by the image of  $X$  under a morphism that sends  $a$  in  $X(T)$  to the rational left-translation by  $a$  on  $X_T$ .

One technical detail is that whereas Artin requires  $X$  to be of finite presentation, we allow it to be only locally so (that is, maybe not quasi-compact and quasi-separated). This turns out to need no modification of our proofs, and may be interesting for instance for the treatment of Néron models of semi-abelian varieties, since these fail to be of finite type.

A significant difference between [Art] and Section 5.2 of [BLR] is that [BLR] treats descent only in Chapter 6, after the construction of groups from birational ones. So, Chapter 5 of [BLR] is more geometric and less sheaf-theoretic than [Art]. It is a good thing to compare the two accounts. Here are some considerations.

1. In [Art],  $S$  is arbitrary, and  $X/S$  is faithfully flat and of finite presentation, with separated fibres without embedded components. The conclusion is that  $G/S$  is an algebraic space.
2. In Theorem 5.1/5 of [BLR], the scheme  $S$  is the spectrum of a field or of a discrete valuation ring, and  $X/S$  is separated, smooth and quasi-compact, and surjective.
3. In Theorem 6.1/1 of [BLR],  $S$  is arbitrary,  $X/S$  is smooth, separated, quasi-compact. The conclusion is that  $G/S$  is a scheme. For the proof of this theorem, whose main ideas come from Raynaud [Ra2], Theorem 3.7 of [Art] is admitted, although it is also said that if  $S$  is normal, then it can be obtained as in Chapter 5 of [BLR].
4. In [Art] the birational group law is “strict”. Proposition 5.2/2 of [BLR] and [Wei2] reduce, under certain conditions, the case of a birational group law to a strict one.

Let us now briefly describe what we say on the application to Néron models. While Néron’s original paper was written in the old language of Weil’s Foundations and quite hard to read, the book [BLR] is a modern treatment that provides all details and more on this topic. It is however quite demanding for someone who wishes to have a quick overview of the construction. In this text, we tried to show to the reader that it is in fact quite simple to see not only the skeleton but also almost all the flesh of the complete construction. Thus we bring out the main ideas of Néron to produce a model of the abelian variety one started with, endowed with a strict birational group law. Then Weil’s extension theorem finishes the job. The few things that we do not prove are:

1. the decreasing of Néron’s measure for the defect of smoothness under blow-up of suitable singular strata (Lemma 5.5),
2. the theorem of Weil on the extension of morphisms from smooth schemes to smooth separated group schemes (proof of Proposition 6.4).

In both cases, using these results as black boxes does not interrupt the main line of the proof, and moreover there was nothing we could add to the proofs of these facts in [BLR].

The exposition of Weil’s theorem occupies sections 2 and 3 of the paper, while the application to Néron models occupies sections 4 to 6.

## 2. A case treated by André Weil

Let  $k$  be an algebraically closed field. An *algebraic variety* over  $k$  will mean a  $k$ -scheme that is locally of finite type, separated, and reduced. For such an  $X$ , we denote  $X(k)$  by  $X$  itself, that is, we forget about the non-closed points. A subvariety of  $X$  is said to be *dense* if it is topologically dense.

Let, in this paragraph,  $G$  be an algebraic variety over  $k$  with an algebraic group structure. Then the graph of the multiplication map from  $G \times G$  to  $G$  is a closed subvariety  $\Gamma$  of  $G \times G \times G$ ; it is the set of  $(a, b, c)$  in  $G \times G \times G$  such that  $c = ab$ . For every  $i$  and  $j$  in  $\{1, 2, 3\}$  with  $i < j$  the projection  $\text{pr}_{i,j}: \Gamma \rightarrow G \times G$  is an isomorphism, hence  $\Gamma$  is the graph of a morphism  $f_{i,j} := \text{pr}_k \circ \text{pr}_{i,j}^{-1}$ , where  $\{i, j, k\} = \{1, 2, 3\}$ , from  $G \times G$  to  $G$ . We have  $f_{1,2}(a, b) = ab$ ,  $f_{1,3}(a, c) = a^{-1}c$  and  $f_{2,3}(b, c) = cb^{-1}$ . For  $X$  a dense open subvariety of  $G$  and  $W$  a dense open subvariety of  $\Gamma$  contained in  $X \times X \times X$ , the pair  $(X, W)$  is a strict birational group law as in the following definition. Theorem 2.11 shows in fact that each strict birational group law is in fact obtained in this way.

**Definition 2.1.** — Let  $X$  be an algebraic variety over  $k$ , not empty. A *strict birational group law on  $X$*  is a subvariety (locally closed, by definition)  $W$  of  $X \times X \times X$ , that satisfies the following conditions.

1. For every  $i$  and  $j$  in  $\{1, 2, 3\}$  with  $i < j$  the projection  $\text{pr}_{i,j}: W \rightarrow X \times X$  is an open immersion whose image, denoted  $U_{i,j}$ , is dense in  $X \times X$ . For each such  $(i, j)$ , we let  $f_{i,j}: U_{i,j} \rightarrow X$  be the morphism such that  $W$  is its graph. For every such  $(i, j)$  and for every  $x = (x_1, x_2, x_3)$  in  $X^3$  the condition  $x \in W$  is equivalent to:  $(x_i, x_j) \in U_{i,j}$  and  $x_k = f_{i,j}(x_i, x_j)$ , with  $\{i, j, k\} = \{1, 2, 3\}$ . We denote the morphism  $f_{1,2}: U_{1,2} \rightarrow X$  by  $(a, b) \mapsto ab$ . Hence, for  $(a, b, c)$  in  $X^3$  we have  $(a, b, c) \in W$  if and only if  $(a, b) \in U_{1,2}$  and  $c = ab$ .
2. For every  $a$  in  $X$ , and for every  $i$  and  $j$  in  $\{1, 2, 3\}$  with  $i < j$  the inverse images of  $U_{i,j}$  under the morphisms  $(a, \text{id}_X)$  and  $(\text{id}_X, a): X \rightarrow X \times X$  are dense in  $X$  (in other words,  $U_{i,j} \cap (\{a\} \times X)$  is dense in  $\{a\} \times X$  and  $U_{i,j} \cap (X \times \{a\})$  is dense in  $X \times \{a\}$ ).
3. For all  $(a, b, c) \in X^3$  such that  $(a, b)$ ,  $(b, c)$ ,  $(ab, c)$  and  $(a, bc)$  are in  $U_{1,2}$ , we have  $a(bc) = (ab)c$ .

From now on,  $X$  is an algebraic variety over  $k$  with a strict rational group law  $W$ . The idea in what follows is that we can let  $X$  act on itself by left and right translations, which are rational maps. Left translations commute with right translations, and the group we want to construct can be obtained as the group of birational maps from  $X$  to  $X$  that is generated by the left translations, or, equivalently, the group of birational maps from  $X$  to  $X$  that commute with the right translations.

**Definition 2.2.** — We let  $\mathcal{R}$  be the set of birational maps from  $X$  to itself, that is, the set of equivalence classes of  $(U, f, V)$ , where  $U$  and  $V$  are open and dense in  $X$  and  $f: U \rightarrow V$  is an isomorphism, where  $(U, f, V)$  is equivalent to  $(U', f', V')$  if and only if  $f$  and  $f'$  are equal on  $U \cap U'$  (note that  $X$  is separated,

this is needed for transitivity of the relation). For each element  $g$  of  $\mathcal{R}$  there is a maximal dense open subset  $\text{Dom}(g)$  of  $X$  on which it is a morphism.

**Remark 2.3.** — The elements of  $\mathcal{R}$  can be composed, they have inverses, and so  $\mathcal{R}$  is a group. For  $(U, f, V)$  as above, let  $g$  be  $f^{-1}: V \rightarrow U$ , then  $f$  and  $g$  induce inverse morphisms between  $f^{-1}\text{Dom}(g)$  and  $g^{-1}\text{Dom}(f)$ , and therefore  $(f^{-1}\text{Dom}(g), f, g^{-1}\text{Dom}(f))$  is a maximal representative of the equivalence class of  $(U, f, V)$  (see the proof of Lemma 3.6 for details).

**Lemma 2.4.** — For  $a$  in  $X$ , let  $U_a := (a, \text{id}_X)^{-1}U_{1,2}$  and  $V_a := (a, \text{id}_X)^{-1}U_{1,3}$ . Then  $U_a$  and  $V_a$  are open and dense in  $X$ , and  $f_{1,2} \circ (a, \text{id}_X): U_a \rightarrow X$ ,  $x \mapsto ax$ , and  $f_{1,3} \circ (a, \text{id}_X): V_a \rightarrow X$  induce inverse morphisms between  $U_a$  and  $V_a$ .

*Proof.* — Let  $a \in X$ . For  $b$  and  $c$  in  $X$ , the condition  $(a, b, c) \in W$  is equivalent to  $((a, b) \in U_{1,2}$  and  $c = f_{1,2}(a, b))$ , and to  $((a, c) \in U_{1,3}$  and  $b = f_{1,3}(a, c))$ . But  $(a, b) \in U_{1,2}$  means, by definition, that  $b \in U_a$ . And  $(a, c) \in U_{1,3}$  means that  $c \in V_a$ .  $\square$

**Definition 2.5.** — For  $a$  in  $X$ , we let  $\phi(a)$  denote the element of  $\mathcal{R}$  given by  $(U_a, f_{1,2} \circ (a, \text{id}_X), V_a)$ . Hence:  $\phi(a): U_a \rightarrow V_a$  is the isomorphism  $x \mapsto ax$ . We have  $\phi: X \rightarrow \mathcal{R}$ , a map of sets, from  $X$  to the group  $\mathcal{R}$ . We let  $G$  be the subgroup of  $\mathcal{R}$  generated by  $\phi(X)$ . For  $a$  in  $X$ , let  $\psi(a)$  be the element of  $\mathcal{R}$  given by  $(U'_a, f_{1,2} \circ (\text{id}_X, a), V'_a)$ , where  $U'_a = (\text{id}_X, a)^{-1}U_{1,2}$  and  $V'_a = (a, \text{id}_X)^{-1}U_{2,3}$ . Then  $\psi(a)$  is the isomorphism  $x \mapsto xa$  from  $U'_a$  to  $V'_a$ .

**Lemma 2.6.** — For all  $(a, b) \in U_{1,2}$  we have  $\phi(a) \circ \phi(b) = \phi(ab)$ . For all  $a$  and  $b$  in  $X$ , we have  $\phi(a) \circ \psi(b) = \psi(b) \circ \phi(a)$ . Every  $g$  in  $G$  commutes with every  $\psi(b)$  ( $b \in X$ ).

*Proof.* — The first two statements follow from the associativity of the birational group law. The last statement follows from the definition of  $G$ : it is generated by  $\{\phi(a) : a \in X\}$ .  $\square$

**Lemma 2.7.** — Let  $g$  be in  $G$  and  $x \in \text{Dom}(g)$  such that  $g(x) = x$ . Then  $g = \text{id}_X$ .

*Proof.* — For every  $b \in X$  such that  $b \in U_x$  and  $xb \in \text{Dom}(g)$ , we have

$$g(xb) = (g \circ \psi(b))x = (\psi(b) \circ g)x = xb.$$

Hence  $g(y) = y$  for all  $y$  in a dense open subset of  $X$ .  $\square$

**Lemma 2.8.** — The map  $\phi: X \rightarrow G$  is injective.

*Proof.* — Let  $a$  and  $b$  be in  $X$ , such that  $\phi(a) = \phi(b)$ . Then, for all  $x \in U_a \cap U_b$ , we have  $(a, x, ax) \in W$  and  $(b, x, bx) \in W$ . But these two points of  $W$  have the same image under  $\text{pr}_{2,3}$ , as  $ax = bx$ . As  $\text{pr}_{2,3}$  is injective,  $a = b$ .  $\square$

In order to make  $G$  into a group variety, the idea is now simply to use translates of  $\phi: X \rightarrow G$  as charts.

**Definition 2.9.** — For  $g$  in  $G$ , let  $\phi_g: X \rightarrow G$  be given by  $a \mapsto g \circ \phi(a)$ . Note that  $\phi_g$  is  $\phi: X \rightarrow G$  followed by left-multiplication by  $g$  on  $G$ .

The  $\phi_g$  cover  $G$  because  $X$  is not empty. The next lemma shows that these charts are compatible: for  $g_1$  and  $g_2$  in  $G$ ,  $\phi_{g_2}^{-1}(\phi_{g_1}(X))$  is open in  $X$ , and the map  $\phi_{g_1}^{-1} \circ \phi_{g_2}: \phi_{g_2}^{-1}(\phi_{g_1}(X)) \rightarrow X$  sending  $x$  to  $\phi_{g_1}^{-1}(\phi_{g_2}(x))$ , is a morphism.

**Lemma 2.10.** — For  $g_1$  and  $g_2$  in  $G$ , the set of  $(a, b)$  in  $X^2$  such that  $\phi_{g_1}(a) = \phi_{g_2}(b)$  is the transpose of the graph of  $g_1^{-1}g_2: \text{Dom}(g_1^{-1}g_2) \rightarrow X$ ,  $b \mapsto (g_1^{-1}g_2)(b)$ , that is, the set  $\{(a, b) \in X^2 : b \in \text{Dom}(g_1^{-1}g_2) \text{ and } a = (g_1^{-1}g_2)b\}$ .

*Proof.* — Let  $g := g_1^{-1}g_2$ . We want to know for which  $(a, b) \in X^2$  the condition  $\phi(a) = g \circ \phi(b)$  holds. Let  $(a, b)$  be in  $X^2$ .

Assume that  $\phi(a) = g \circ \phi(b)$ . Then there is an  $x$  in  $X$  such that  $(a, x) \in U_{1,2}$ ,  $(b, x) \in U_{1,2}$ ,  $(b, x) \in f_{1,2}^{-1} \text{Dom}(g)$ , and  $ax = g(bx)$ . Let  $x$  be such.

Then  $a \in \text{Dom}(\psi(x))$  and  $(\psi(x))(a) = ax$ ,  $b \in \text{Dom}(\psi(x))$  and  $(\psi(x))(b) = bx$ , and  $(\psi(x))(b) = bx \in \text{Dom}(g)$ , and  $(\psi(x))(a) = g((\psi(x))(b))$ , and  $(\psi(x))(a) = ax \in \text{Dom}(\psi(x)^{-1})$ .

Hence  $b \in \text{Dom}(\psi(x))$ ,  $(\psi(x))(b) \in \text{Dom}(g)$ ,  $g((\psi(x))(b)) \in \text{Dom}(\psi(x)^{-1})$ , and  $a = (\psi(x)^{-1} \circ g \circ \psi(x))(b)$ . Now note that  $\psi(x)^{-1} \circ g \circ \psi(x) = g$  in  $\mathcal{R}$ , hence  $b \in \text{Dom}(g)$  and  $a = g(b)$ .

Now assume that  $b \in \text{Dom}(g)$  and  $a = g(b)$ . We must prove that  $\phi(a) = g \circ \phi(b)$  in  $G$ .

Let  $x$  be in  $X$  such that  $(a, x) \in U_{1,2}$ ,  $(b, x) \in U_{1,2}$ , and  $bx \in \text{Dom}(g)$ . Then  $(\phi(a))(x) = ax$  because  $(a, x) \in U_{1,2}$ . We have  $ax = g(b)x$ , because  $a = g(b)$ . We have  $(g(b)x) = \psi(x)(g(b))$  because  $(g(b), x) \in U_{1,2}$ . We have  $\psi(x)(g(b)) = (\psi(x) \circ g)(b)$  because  $b \in \text{Dom}(g)$ ,  $g(b) = a$  and  $(a, x) \in U_{1,2}$ . We have  $\psi(x) \circ g = g \circ \psi(x)$  in  $\mathcal{R}$ , hence  $(\psi(x) \circ g)(b) = (g \circ \psi(x))(b)$ . We have  $(g \circ \psi(x))(b) = g((\psi(x))(b))$  because  $b \in \text{Dom}(\psi(x))$  and  $(\psi(x))(b) \in \text{Dom}(g)$ . We have  $g((\psi(x))(b)) = g(bx) = g((\phi(b))(x)) = (g \circ \phi(b))(x)$ . We conclude, using Lemma 2.7, that  $\phi(a) = g \circ \phi(b)$  in  $G$ .  $\square$

**Theorem 2.11.** — Let  $k$  be an algebraically closed field. Let  $X$  be an algebraic variety over  $k$ , that is, the variety of  $k$ -points of a  $k$ -scheme that is locally of finite type, separated and reduced. Let  $W$  be a strict birational group law on  $X$ . Let  $G$  be the group constructed as above.

1. The charts  $\phi_g: X \rightarrow G$  are compatible, and each of them is an open immersion with dense image. They give  $G$  the structure of a  $k$ -scheme that is reduced, and locally of finite type.

2. The group law on  $G$  extends the birational group law on  $X$  that is given by  $W$ . As a  $k$ -group scheme, locally of finite type,  $G$  is separated.
3. The map  $X \times X \rightarrow G$ ,  $(a, b) \mapsto \phi(a)\phi(b)^{-1}$  is surjective, and the fibre over  $g$  in  $G$  is the set  $\{(a, b) \in X^2 : b \in \text{Dom}(g) \text{ and } a = g(b)\}$ .
4. Every  $g$  in  $G$  induces an isomorphism between  $\text{Dom}(g)$  and  $\text{Dom}(g^{-1})$ .
5. If  $X$  is of finite type, then  $G$  is of finite type.

*Proof.* — For 1, apply Lemma 2.10.

For 2, use Lemma 2.6, and that the diagonal in  $G \times G$  is the fibre over the unit element of the morphism  $G \times G \rightarrow G$ ,  $(x, y) \mapsto x^{-1}y$ .

For 3, use Lemma 2.10.

For 4, use 3.

For 5, note that  $G$  is locally of finite type, and, as the image of  $X^2$ , quasi-compact. □

**Remark 2.12.** — Let  $A, B$  be two commutative connected algebraic groups. Let  $\text{Ext}(A, B)$  be the set of classes of classes of extensions of  $A$  by  $B$ . A rational map  $f: A \times A \rightarrow B$  satisfying the identity

$$f(y, z) - f(x + y, z) + f(x, y + z) - f(x, y) = 0$$

for all  $x, y, z \in A$  is called a *rational factor system*. It is called *symmetric* if  $f(x, y) = f(y, x)$ ,  $x, y \in A$ . Such a system will be called *trivial* if there exists a rational map  $g: A \rightarrow B$  such that  $f = \delta g$ , where

$$\delta g(x, y) = g(x, y) = g(x + y) - g(x) - g(y).$$

The classes of symmetric factor systems modulo the trivial factor systems form a group denoted  $H_{\text{rat}}^2(A, B)_s$ . As an application of Weil's theorem, one can show that  $H_{\text{rat}}^2(A, B)_s$  is isomorphic to the subgroup of  $\text{Ext}(A, B)$  given by the extensions which admit a rational section (see [Ser], Chap. VII, § 1, no 4., Prop. 4).

### 3. The case treated by Michael Artin in SGA3

In order to state and prove a relative version of the extension theorem for birational group laws, a short reminder on  $S$ -rational maps is useful. Definitions 3.1 and Proposition 3.2 below can be found, some parts in a broader generality, in [Art], Section 1, except for the statements where other references are given.

**Definition 3.1.** — Let  $S$  be a scheme. Let  $X, Y$  be  $S$ -schemes.

1. An open subscheme  $U \subset X$  is *S-dense* if  $U \times_S S'$  is schematically dense in  $X \times_S S'$  for all morphisms  $S' \rightarrow S$ .

2. An *S-rational map*  $f: X \dashrightarrow Y$  is an equivalence class of morphisms  $U \rightarrow Y$  with  $U \subset X$  open and  $S$ -dense, where  $U \rightarrow Y$  and  $V \rightarrow Y$  are equivalent if they agree on an  $S$ -dense open subscheme  $W \subset U \cap V$ .
3. An *S-birational map* is an  $S$ -rational map that can be represented by a morphism  $U \rightarrow Y$  inducing an isomorphism with an  $S$ -dense open subscheme of  $Y$ .

**Proposition 3.2.** — *Let  $S$  be a scheme. Let  $X$  and  $Y$  be  $S$ -schemes that are flat and locally of finite presentation.*

1. By [EGA] IV.11.10.10, an open subscheme  $U$  of  $X$  is  $S$ -dense if and only if for all  $s \in S$ ,  $U_s$  is schematically dense in  $X_s$  (that is,  $U_s$  contains the associated points of  $X_s$ ).
2. Unions of non-empty families and finite intersections of  $S$ -dense opens are  $S$ -dense open.
3. If  $U \subset X$  and  $V \subset Y$  are  $S$ -dense open subschemes, then  $U \times_S V$  is an  $S$ -dense open subscheme of  $X \times_S Y$ .
4. Let  $f$  and  $g$  be  $S$ -morphisms from  $X$  to  $Y$ . Assume that the fibres of  $Y \rightarrow S$  are separated, and that  $f$  and  $g$  are equal on an  $S$ -dense open subscheme  $U$  of  $X$ . Then  $f = g$ .
5. Let  $f: X \dashrightarrow Y$  be an  $S$ -rational map. Assume that the fibres of  $Y \rightarrow S$  are separated. Then there is a maximal  $S$ -dense open subscheme  $U \subset X$  with a morphism  $U \rightarrow Y$  representing  $f$ , called the domain of definition of  $f$  and denoted  $\text{Dom}_S(f)$ . Its reduced complement is called the exceptional locus of  $f$ . For  $S' \rightarrow S$  flat and locally of finite presentation,  $\text{Dom}_{S'}(f \times_S S') = \text{Dom}_S(f) \times_S S'$ .

From now on and until the rest of this section, we put ourselves in the situation of Theorem 3.7 of [Art], namely we make the following:

**Assumptions 3.3.** — *Let  $S$  be a scheme, and  $f: X \rightarrow S$  be an  $S$ -scheme that is faithfully flat and locally of finite presentation, whose fibres are separated and have no embedded components (condition  $(\diamond)$  in [Art, 3.0]). Note that it is equivalent to require these conditions for the geometric fibres, see [EGA] IV.4.2.7.*

It is clear that for any  $T \rightarrow S$ , an open subset  $U$  of  $X_T$  is  $T$ -dense if and only if for all  $t$  in  $T$  its fibre  $U_t$  is topologically dense in  $X_t$  (the “no embedded components” condition is used here).

We can now generalise Definition 2.1 to the present situation. Unlike Artin, we do not assume the immersion  $W \rightarrow X \times_S X \times_S X$  below to be of finite presentation.

**Definition 3.4.** — *A strict  $S$ -birational group law on  $X/S$  is a (locally closed) subscheme  $W$  of  $X \times_S X \times_S X$  that satisfies the following conditions.*

1. For every  $i$  and  $j$  in  $\{1, 2, 3\}$  with  $i < j$  the projection  $\text{pr}_{i,j}: W \rightarrow X \times_S X$  is an open immersion whose image, denoted  $U_{i,j}$ , is  $S$ -dense in  $X \times_S X$  (condition  $(*)$  in [Art, 3.0]). For each such  $(i, j)$ , we let  $f_{i,j}: U_{i,j} \rightarrow X$  be the  $S$ -morphism such that  $W$  is its graph. For every such  $(i, j)$ , for every  $T \rightarrow S$  and for every  $x = (x_1, x_2, x_3)$  in  $X(T)^3$  the condition  $x \in W(T)$  is equivalent to:  $(x_i, x_j) \in U_{i,j}(T)$  and  $x_k = f_{i,j}(x_i, x_j)$  in  $X(T)$ , with  $\{i, j, k\} = \{1, 2, 3\}$ . We denote the  $S$ -morphism  $f_{1,2}: U_{1,2} \rightarrow X$  by  $(a, b) \mapsto ab$ . Hence, for  $(a, b, c)$  in  $X(T)^3$  we have  $(a, b, c) \in W(T)$  if and only if  $(a, b) \in U_{1,2}(T)$  and  $c = ab$  in  $X(T)$ .
2. For every  $T \rightarrow S$ , for every  $a$  in  $X(T)$ , and for every  $i$  and  $j$  in  $\{1, 2, 3\}$  with  $i < j$  the inverse images of  $(U_{i,j})_T$  under the morphisms  $(a, \text{id}_{X_T})$  and  $(\text{id}_{X_T}, a): X_T \rightarrow (X \times_S X)_T$  are  $T$ -dense in  $X_T$ .
3. For every  $T \rightarrow S$  and all  $(a, b, c) \in X(T)^3$  such that  $(a, b)$ ,  $(b, c)$ ,  $(ab, c)$  and  $(a, bc)$  are in  $U_{1,2}(T)$ , we have  $a(bc) = (ab)c$  in  $X(T)$ .

Let  $W$  be a strict birational group law on  $X/S$ . Then  $(X, W)$  is a *group germ over  $S$*  as in Definition 3.1 of [Art].

**Definition 3.5.** — For  $T \rightarrow S$ , let  $\mathcal{R}(T)$  be the set of  $T$ -rational maps from  $X_T$  to itself that have a representative  $(U, f)$  with  $U \subset X_T$  open and  $T$ -dense, and  $f$  an isomorphism from  $U$  to an open  $T$ -dense open subset  $V$  of  $X_T$ . As in Section 2.5 of [BLR], we call elements of  $\mathcal{R}(T)$   *$T$ -birational maps* from  $X_T$  to itself.

The next lemma says that every  $f$  in  $\mathcal{R}(T)$  has a unique maximal representative.

**Lemma 3.6.** — *Let  $T \rightarrow S$ ,  $U$  and  $V$  be open and  $T$ -dense in  $X_T$ , and  $f: U \rightarrow V$  a  $T$ -isomorphism, and let  $g: V \rightarrow U$  be its inverse. Let  $\text{Dom}(f)$  and  $\text{Dom}(g)$  be their domains of definition. Then we have  $f: \text{Dom}(f) \rightarrow X_T$  and  $g: \text{Dom}(g) \rightarrow X_T$ , and  $f$  and  $g$  induce inverse morphisms between  $f^{-1}\text{Dom}(g)$  and  $g^{-1}\text{Dom}(f)$ , and  $U \subset f^{-1}\text{Dom}(g)$  and  $V \subset g^{-1}\text{Dom}(f)$ .*

*Proof.* — By Proposition 3.2,  $f$  and  $g$  extend uniquely to  $\text{Dom}(f)$  and  $\text{Dom}(g)$ , respectively. Note that  $U \subset \text{Dom}(f)$  and  $V \subset \text{Dom}(g)$ , hence  $U \subset f^{-1}\text{Dom}(g)$  and  $V \subset g^{-1}\text{Dom}(f)$ . By definition, we have  $g \circ f: f^{-1}\text{Dom}(g) \rightarrow X_T$ , and on  $U$  this is equal to the inclusion morphism, hence  $g \circ f$  is the inclusion morphism of  $f^{-1}\text{Dom}(g)$  in  $X_T$ . But then  $g \circ f$  factors through  $\text{Dom}(f)$ , hence  $f: f^{-1}\text{Dom}(g) \rightarrow \text{Dom}(g)$  factors through  $g^{-1}\text{Dom}(f)$ . So we have morphisms  $f: f^{-1}\text{Dom}(g) \rightarrow g^{-1}\text{Dom}(f)$  and  $g: g^{-1}\text{Dom}(f) \rightarrow f^{-1}\text{Dom}(g)$ . Then  $g \circ f$  is equal to the identity on  $U$ , and therefore is the identity, and similarly for  $f \circ g$ .  $\square$

**Lemma 3.7.** — *For all  $T \rightarrow S$ , the elements of  $\mathcal{R}(T)$  can be composed, have a two-sided inverse, and this makes  $T \mapsto \mathcal{R}(T)$  into a presheaf of groups on  $S$ . It is a sheaf for the fppf topology.*

*Proof.* — For composition of  $f$  and  $g$  in  $\mathcal{R}(T)$  (which are equivalence classes), choose representatives  $(U, f, U')$  and  $(V, g, V')$ . Then  $U' \cap V$  is  $S$ -dense in  $X_T$ . This gives  $(f^{-1}(U' \cap V), g \circ f, g(U' \cap V))$ . Its equivalence class does not depend on the choices of  $(U, f, U')$  and  $(V, g, V')$  because of Proposition 3.2. For the inverse of  $f$ : take a representative  $(U, f, U')$ , then  $f^{-1}$  is the equivalence class of  $(U', f^{-1}, U)$  (independent of choice). As composition is associative, and as we have the identity map,  $\mathcal{R}(T)$  with this composition is a group.

For  $f$  in  $\mathcal{R}(T)$  and  $T_1 \rightarrow T$ , we get  $f_1 = f_{T_1}$  in  $\mathcal{R}(T_1)$  by base change as follows. Let us denote by a subscript 1 the pullbacks along  $T_1 \rightarrow T$ . Let  $f$  be represented by  $(U, f, U')$ , then  $U_1$  and  $U'_1$  are  $T_1$ -dense in  $X_1$  (this is by definition). We let  $f_1$  be the equivalence class of  $(U_1, f_1, U'_1)$ ; this is independent of the choice of  $(U, f, U')$ . Pullback is functorial, hence  $\mathcal{R}$  is a presheaf of groups.

Let us prove that  $\mathcal{R}$  is a sheaf for the fppf topology on  $\text{Sch}/S$ . By Proposition 6.3.1 of [Dem1], it suffices to consider a cover  $q: S' \rightarrow S$  with  $q$  faithfully flat and locally of finite presentation. Let  $S'' := S' \times_S S'$ , with the two projection morphisms  $\text{pr}_1$  and  $\text{pr}_2: S'' \rightarrow S'$ . Let  $f'$  be in  $\mathcal{R}(S')$ , such that  $\text{pr}_1^* f' = \text{pr}_2^* f'$  in  $\mathcal{R}(S'')$ . Then we have  $\text{Dom}(\text{pr}_1^* f') = \text{pr}_1^{-1} \text{Dom}(f')$  by Proposition 3.2, and the same for  $\text{pr}_2$ . Therefore,  $\text{pr}_1^* \text{Dom}(f') = \text{pr}_2^* \text{Dom}(f')$ . Therefore, with  $U := q(\text{Dom}(f'))$ , which is open in  $X$  because  $q: X_{S'} \rightarrow X$  is faithfully flat and locally of finite presentation hence open, we have  $\text{Dom}(f') = q^{-1}U$ , and  $U \subset X$  is  $S$ -dense because for every  $s$  in  $S$  the fibre  $U_s$  is dense in  $X_s$ . Hence (fully faithfulness of the pullback functor  $q^*$  from  $\text{Sch}/S$  to the category of  $S'$ -schemes with descent datum to  $S$ , see Proposition 6.3.1(iii)+(iv) of [Dem1]) there is a unique  $S$ -morphism  $f: U \rightarrow X$  such that  $q^* f = f': \text{Dom}(f') \rightarrow X_{S'}$ . Let  $g'$  in  $\mathcal{R}(S')$  be the inverse of  $f'$ . Then we also have  $V := q(\text{Dom}(g'))$  open and  $S$ -dense in  $X$  and a unique  $S$ -morphism  $g: V \rightarrow X$  such that  $g' = q^* g$ . Now the formation of  $f^{-1} \text{Dom}(g)$  and  $g^{-1} \text{Dom}(f)$  commutes with the base change  $S' \rightarrow S$  by Proposition 3.2. Over  $S'$  we have that  $f'$  and  $g'$  are inverses on these two  $S'$ -dense open subsets of  $X_{S'}$ , and therefore (using again that  $q^*$  is fully faithful and the fibre-wise criterion for  $S$ -denseness)  $f$  and  $g$  are inverses on  $f^{-1} \text{Dom}(g)$  and  $g^{-1} \text{Dom}(f)$  and these two opens are  $S$ -dense in  $X$ . This proves that  $f$  is in  $\mathcal{R}(S)$ .  $\square$

**Lemma 3.8.** — *For  $T \rightarrow S$  and  $a \in X(T)$ , let  $U_a$  be the inverse image of  $(U_{1,2})_T$  under  $(a, \text{id}_{X_T}): X_T \rightarrow (X \times_S X)_T$ , and let  $V_a := (a, \text{id}_{X_T})^{-1}(U_{1,3})_T$ . Then  $U_a$  and  $V_a$  are open and  $T$ -dense in  $X_T$ , and  $(f_{1,2})_T \circ (a, \text{id}_{X_T}): U_a \rightarrow X_T$  and  $(f_{1,3})_T \circ (a, \text{id}_{X_T}): V_a \rightarrow X_T$  induce inverse morphisms between  $U_a$  and  $V_a$ .*

*Proof.* — The  $T$ -density of the opens  $U_a$  and  $V_a$  comes from Definition 3.4. For  $T' \rightarrow T$ ,  $b$  and  $c$  in  $X(T')$ , the condition  $(a, b, c) \in W(T')$  is equivalent to  $((a, b) \in U_{1,2}(T')$  and  $c = f_{1,2}(a, b)$  in  $X(T')$ ), and to  $((a, c) \in U_{1,3}(T')$  and  $b = f_{1,3}(a, c)$  in  $X(T')$ ). But  $(a, b) \in U_{1,2}(T')$  means, by definition, that  $b \in U_a(T')$ . And  $(a, c) \in U_{1,3}(T')$  means that  $c \in V_a(T')$ .  $\square$

**Definition 3.9.** — For  $T \rightarrow S$  and  $a$  in  $X(T)$ , we let  $\phi(a)$  denote the element of  $\mathcal{R}(T)$  given by  $(U_a, f_{1,2} \circ (a, \text{id}_X), V_a)$ . Hence, for every  $T' \rightarrow T$ ,  $\phi(a): U_a(T') \rightarrow V_a(T')$  is the bijection  $x \mapsto ax$ . We have  $\phi: X \rightarrow \mathcal{R}$ , a morphism of sheaves on  $S_{\text{fppf}}$ , from the sheaf of sets  $X$  to the sheaf of groups  $\mathcal{R}$ . We let  $G$  be the subsheaf of groups of  $\mathcal{R}$  generated by  $\phi(X)$ .

**Lemma 3.10.** — For every  $T \rightarrow S$ , for all  $(a, b) \in U_{1,2}(T)$  we have  $\phi(a) \circ \phi(b) = \phi(ab)$ . For  $T \rightarrow S$  and  $a$  in  $X(T)$  we have  $\psi(a)$  in  $\mathcal{R}(T)$  given by  $x \mapsto xa$  on some appropriate  $T$ -dense open subschemes of  $X_T$ . For every  $T \rightarrow S$  and all  $a$  and  $b$  in  $X(T)$ , we have  $\phi(a) \circ \psi(b) = \psi(b) \circ \phi(a)$  in  $\mathcal{R}(T)$ . For every  $T \rightarrow S$ , every  $g$  in  $G(T)$  commutes with every  $\psi(b)$  ( $b \in X(T)$ ).

*Proof.* — The first statement is the associativity of the birational group law. The statement concerning  $\psi$  is proved just as for  $\phi$ . The third statement is again the associativity of the birational group law. The last statement follows from the definition of  $G$ : fppf locally on  $T$ ,  $g$  is in the subgroup of  $\mathcal{R}(T)$  generated by  $\{\phi(a) : a \in X(T)\}$ .  $\square$

**Lemma 3.11.** — Let  $T \rightarrow S$ ,  $g$  be in  $G(T)$  and  $x \in \text{Dom}(g)(T)$  such that  $gx = x$  in  $X(T)$ . Then  $g = \text{id}_{X_T}$ .

*Proof.* — For every  $T' \rightarrow T$  and  $b \in X(T')$  such that  $b \in U_x(T')$  and  $xb \in \text{Dom}(g)(T')$ , we have

$$g(xb) = (g \circ \psi(b))x = (\psi(b) \circ g)x = xb \quad \text{in } X(T').$$

It follows that  $g$  is the identity on the  $T$ -dense open subset of  $X_T$  that is the image of  $(\text{id}_{X_T}, b)^{-1}(U_{1,2})_T$  under the open immersion  $\psi(b): (\text{id}_{X_T}, b)^{-1}(U_{1,2})_T \rightarrow X_T$ . Hence  $g = \text{id}_{X_T}$ .  $\square$

**Lemma 3.12.** — The morphism of sheaves  $\phi: X \rightarrow G$  is injective.

*Proof.* — Let  $T \rightarrow S$ , and  $a$  and  $b$  be in  $X(T)$ , such that  $\phi(a) = \phi(b)$  in  $G(T)$ . Then, for all  $T' \rightarrow T$  and all  $x \in (U_a \cap U_b)(T')$ , we have  $(a, x, ax) \in W(T')$  and  $(b, x, bx) \in W(T')$ . But these two points of  $W(T')$  have the same image in  $(X \times_S X)(T')$  under  $\text{pr}_{2,3}$ , as  $ax = (\phi a)x = (\phi b)x = bx$ . As  $\text{pr}_{2,3}: W \rightarrow X^2$  is an open immersion,  $a = b$  in  $T'$ . Now take  $T' := U_a \cap U_b$ , which is  $T$ -dense in  $X_T$ , hence faithfully flat over  $T$ , and take for  $x$  the identity. Then we get  $a = b$  in  $X(T')$ , hence  $a = b$  in  $X(T)$ .  $\square$

**Lemma 3.13.** — *The morphism of sheaves  $\phi: X \rightarrow G$  is representable, and an open immersion.*

*Proof.* — Let  $T \rightarrow S$  and  $g \in G(T)$ . What we must show is that there is an open subscheme  $V \subset T$  and an  $S$ -morphism  $a_V: V \rightarrow X$  such that  $\phi(a_V) = g$  in  $G(V)$  and such that for all  $T' \rightarrow T$ ,  $a \in X(T')$  with  $\phi(a) = g$  in  $G(T')$ ,  $T' \rightarrow T$  factors through  $V$  and  $a = a_V$  in  $X(T')$ .

Let us first produce  $V$  and  $a_V$ . We have  $\text{Dom}(g) \subset X_T$ , open and  $T$ -dense, and we have  $(\text{id}_{X_T}, g): \text{Dom}(g) \rightarrow (X \times_S X)_T$ . This gives us the open subset  $V' := (\text{id}_{X_T}, g)^{-1}(\text{U}_{2,3})_T$  of  $\text{Dom}(g)$ . As  $f_T: X_T \rightarrow T$  is open (being flat and locally of finite presentation), we get an open subset  $V$  of  $T$  as  $V := f_T V'$ . On  $V'$  we have

$$a_{V'}: V' \xrightarrow{(\text{id}_{X_T}, g)} (\text{U}_{2,3})_V \xrightarrow{(f_{2,3})_V} X_V, \quad x \longmapsto f_{2,3}(x, gx).$$

Then  $(a_{V'}, \text{id}_{X_T}, g)$  in  $X(V')^3$  is in  $W(V')$ , hence  $g = \phi(a_{V'})$  in  $G(V')$ . We claim that there is a unique  $a_V$  in  $G(V)$  that is mapped to  $a_{V'}$  in  $G(V')$  under  $f_V: V' \rightarrow V$ , that is,  $a_{V'} = f_V^* a_V$ . Now note that  $f_V: V' \rightarrow V$  is faithfully flat and locally of finite presentation, being the composition of the  $V$ -dense open immersion of  $V'$  into  $X_V$ , and the morphism  $X_V \rightarrow V$ . Therefore (Proposition 6.3.1(iii)+(iv) of [Dem1])  $f_V: V' \rightarrow V$  is a morphism of descent, that is, the pullback functor  $f_V^*$  from the category of schemes over  $V$  to the category of schemes over  $V'$  with descent datum for  $V' \rightarrow V$  is fully faithful. Let  $V'' := V' \times_V V'$ , and let  $\text{pr}_1$  and  $\text{pr}_2$  be the two projections from  $V''$  to  $V'$ . Then we have  $\text{pr}_1^* a_{V'}$  and  $\text{pr}_2^* a_{V'}$  in  $X(V'')$ , and we have, in  $G(V'')$ ,

$$\phi(\text{pr}_1^* a_{V'}) = \text{pr}_1^* \phi(a_{V'}) = \text{pr}_1^* g = \text{pr}_2^* g = \text{pr}_2^* \phi(a_{V'}) = \phi(\text{pr}_2^* a_{V'}).$$

Lemma 3.12 then gives  $\text{pr}_1^* a_{V'} = \text{pr}_2^* a_{V'}$  in  $X(V'')$ . But then, by the fully faithfulness of  $f_V^*$ , there is a unique  $a_V$  in  $X(V)$  that gives  $a_{V'}$  by pullback. Then we have, in  $G(V)$ ,  $g = \phi(a_V)$ , because  $f_V^* g = \phi(a_{V'}) = f_V^* \phi(a_V)$ .

Let us now show that  $V \subset T$  and  $a_V \in X_V(V) = X_V$  have the universal property mentioned above. So let  $T' \rightarrow T$ , and  $a \in X(T')$  such that  $\phi(a) = g$  in  $G(T')$ . What is to be shown is that  $T' \rightarrow T$  factors through  $V \subset T$  and that  $a = a_V$  in  $X(T')$ . For this, we may work locally on  $T'$  in the fppf topology, hence we may assume that we have  $x$  in  $X(T')$  such that  $x \in (\text{Dom } g)(T')$  and  $x \in (a, \text{id}_X)^{-1} \text{U}_{1,2}$ . Then we have  $(a, x, gx) \in W(T')$ , hence  $(x, gx) \in \text{U}_{2,3}(T')$  and  $a = f_{2,3}(x, gx)$  in  $X(T')$ . But this means that  $T' \rightarrow T$  factors through  $V' = (\text{id}_{X_T}, g)^{-1}(\text{U}_{2,3})_T$ , and hence through  $V$  because  $V$  is the image of  $V'$  in  $T$ .  $\square$

**Definition 3.14.** — For  $T \rightarrow S$  and  $g$  in  $G(T)$  let  $\phi_g := (g \cdot) \circ \phi_T: X_T \rightarrow G_T$ , that is,  $\phi_g$  is  $\phi_T: X_T \rightarrow G_T$ , followed by left multiplication by  $g$  on  $G_T$ . For  $T' \rightarrow T$  and  $x$  in  $X(T')$ , it sends  $x$  to  $g \circ (\phi x)$  in  $G(T')$ .

By Lemma 3.13, the  $\phi_g$  are open immersions. For  $T \rightarrow S$  and  $g \in G(T)$  the fibred product of  $\phi_T: X_T \rightarrow G_T$  and  $\phi_{g^{-1}}: X_T \rightarrow G_T$  is their intersection  $X_T \cap (g^{-1}\cdot)X_T$ . Composing each of them with  $(g\cdot): G_T \rightarrow G_T$  gives an isomorphism  $(g\cdot)$  from  $X_T \cap (g^{-1}\cdot)X_T$  to  $(g\cdot)X_T \cap X_T$ .

**Lemma 3.15.** — *The formation of  $(g\cdot)X_T \cap X_T$  from  $g$  commutes with every base change  $T' \rightarrow T$ . For  $T \rightarrow S$ , and  $g \in G(T)$ , we have  $X_T \cap (g^{-1}\cdot)X_T = \text{Dom}(g)$ ,  $(g\cdot)X_T \cap X_T = \text{Dom}(g^{-1})$ , and  $g$  and  $g^{-1}$  are inverse morphisms between them.*

*Proof.* — The first statement follows directly from the definitions. Let us prove the second statement. Let  $t \rightarrow T$  be a geometric point. Then  $X_{t,\text{réd}}$  is a dense open subvariety of  $G_{t,\text{réd}}$  (Theorem 2.11)  $X_{t,\text{réd}} \cap (g^{-1}\cdot)X_{t,\text{réd}}$  is precisely the open subset of  $X_{t,\text{réd}}$  that is mapped into  $X_{t,\text{réd}}$  under left-multiplication by  $g_t$ . Therefore,  $X_{t,\text{réd}} \cap (g^{-1}\cdot)X_{t,\text{réd}}$  equals  $\text{Dom}(g_t)$ : it is contained in  $\text{Dom}(g_t)$  because  $g_t$  is a morphism on it, and the points in its complement are mapped, by  $g_t$ , outside  $X_{t,\text{réd}}$ . We conclude that the open subschemes  $X_T \cap (g^{-1}\cdot)X_T$  and  $\text{Dom}(g)$  have the same geometric fibres over  $T$ , and hence are equal.  $\square$

**Lemma 3.16.** — *Let  $\delta: X \times_S X \rightarrow G$  be the morphism of sheaves given by  $(a, b) \mapsto \phi(a) \circ \phi(b)^{-1}$ . For  $T \rightarrow S$  and  $g$  in  $G(T)$ , the fibre of  $\delta$  over  $g$  is the transpose of the graph of  $(g\cdot): X_T \cap (g^{-1}\cdot)X_T \rightarrow (g\cdot)X_T \cap X_T$ . The morphism  $\delta$  is representable, faithfully flat and locally of finite presentation.*

*Proof.* — The first statement is true by definition on the sets of  $T'$ -points for all  $T' \rightarrow T$ , and therefore it is true. The second statement follows from the first, plus the facts that  $X_T \cap (g^{-1}\cdot)X_T = \text{Dom}(g_T)$  is open and  $T$ -dense in  $X_T$  and that  $X \rightarrow S$  is faithfully flat and locally of finite presentation.  $\square$

**Convention 3.17.** — *By an algebraic space we mean the quotient of an étale equivalence relation of schemes, as in [RG] I.5.7 or in the Stacks Project [SP]. Thus in contrast with Knutson [Kn], our algebraic spaces are not necessarily quasi-separated; see also Appendix A of [CLO] for further comments on this point.*

We can now state a variant of Theorem 3.7 of [Art].

**Theorem 3.18.** — *Let  $S$  be a scheme and  $(X, W)$  a strict  $S$ -birational group law, with  $X \rightarrow S$  faithfully flat locally of finite presentation and whose geometric fibres are separated and have no embedded components. With the notation as above, we have the following.*

1. *Existence: (i)  $G \rightarrow S$  is a group algebraic space and its formation from  $(X, W)/S$  commutes with base change on  $S$ .*

- (ii)  $\phi: X \rightarrow G$  is representable by  $S$ -dense open immersions, and is compatible with (rational) group laws on  $X$  and  $G$ .
- (iii) The morphism  $\delta: X \times_S X \rightarrow G$ ,  $(a, b) \mapsto \phi(a)\phi(b)^{-1}$  is representable, faithfully flat and locally of finite presentation.
2. Uniqueness: the properties in 1. determine  $G$  and  $\phi: X \rightarrow G$  up to unique isomorphism.
  3. Properties:  $G \rightarrow S$  is faithfully flat and locally of finite presentation. If  $X \rightarrow S$  is smooth (resp. quasi-compact, resp. with geometrically irreducible fibres), then  $G \rightarrow S$  also.
  4. Separation:  $G \rightarrow S$  is locally separated, that is, its diagonal is an immersion. If  $X \rightarrow S$  is quasi-separated, or if  $S$  is locally noetherian, then  $G \rightarrow S$  is quasi-separated. If  $X \rightarrow S$  is separated, then  $G \rightarrow S$  also.
  5. Representability: if  $X \rightarrow S$  is of finite presentation, then locally on  $S$  for the fppf topology, the algebraic space  $G \rightarrow S$  is a scheme, faithfully flat and of finite presentation.

*Proof.* — Let us write  $X^2$  for  $X \times_S X$  and  $G^2$  for  $G \times_S G$ .

1. In Definition 3.9 we defined  $G$  to be a sheaf. Lemma 3.13 says that  $\phi$  is an open immersion. By Lemma 3.15 it is  $S$ -dense (in the obvious sense). Lemma 3.10 gives the compatibility with group laws. This proves (ii). Assertion (iii) is part of Lemma 3.16. Then it follows from Artin's theorem on flat equivalence relations, proved without quasi-separatedness condition in [SP], Theorem 04S6, that  $G$  is an algebraic space. The fact that its formation commutes with base change is true by definition.

2. Suppose that  $G'$  and  $\phi': X \rightarrow G'$  satisfy the properties in 1. Properties (i) and (ii) give an action by  $G'$  on  $X$  by  $S$ -birational maps. This action is faithful, hence gives an injective morphism of sheaves from  $G'$  to  $\mathcal{R}$ . As morphisms to  $\mathcal{R}$ ,  $\phi$  and  $\phi'$  are equal. Therefore  $G$  and  $G'$  are equal, being generated by the images of  $\phi$  and  $\phi'$  by property (iii). Hence we have an isomorphism between  $G$  and  $G'$  compatible with  $\phi$  and  $\phi'$ . There is at most one such an isomorphism because of property (iii).

3. Since  $X \rightarrow S$  is faithfully flat locally of finite presentation, then  $X^2 \rightarrow S$  also is. If moreover  $X \rightarrow S$  is smooth or quasi-compact or with geometrically irreducible fibres, then  $X^2$  also. In any case, using the presentation  $\delta: X^2 \rightarrow G$ , we see that  $G$  inherits the properties.

4. Let  $\delta^*\Delta_G: X^2 \times_G X^2 \rightarrow X^2 \times_S X^2$  be the monomorphism of schemes obtained by pulling back the diagonal  $\Delta_G: G \rightarrow G^2$  along  $\delta \times \delta$ , and  $\pi = \text{pr}_{234} \circ \delta^*\Delta_G: X^2 \times_G X^2 \xrightarrow{X} \times_S X^2$ . By Lemma 3.15 and Lemma 3.16 applied with  $T = X^2$  and  $g = \delta$ , the map  $\pi$  is an open immersion. Let  $\mathcal{U} \subset X \times_S X^2$  be the image of  $\pi$  and  $\mathcal{V} = \text{pr}_{234}^{-1}(\mathcal{U}) \subset X^2 \times_S X^2$ . What we just said means that  $X^2 \times_G X^2$  is a section of  $\mathcal{V} \rightarrow \mathcal{U}$ . Since sections of morphisms of schemes are immersions, this proves that  $\delta^*\Delta_G$  is an immersion,

hence  $\Delta_G$  is an immersion (note that the property of being an immersion is fppf local on the target by [SP], Lemma 02YM). If  $X \rightarrow S$  is quasi-separated, then  $X^2 \rightarrow S$  also and we see using the presentation  $\delta: X^2 \rightarrow G$  that  $G \rightarrow S$  is quasi-separated. Now let us assume  $S$  locally noetherian. Let  $V$  be an open affine in  $S$  and  $U_1, U_2, U_3, U_4$  open affines in  $X_V$ . The inverse image of  $U_1 \times_S U_2 \times_S U_3 \times_S U_4$  in  $(X \times_S X) \times_G (X \times_S X)$  is isomorphic to an open subset of the affine noetherian scheme  $U_2 \times_V U_3 \times_V U_4$ , and is therefore quasi-compact, i.e.  $G \rightarrow S$  is quasi-separated.

Finally we prove that  $X \rightarrow S$  separated implies  $G \rightarrow S$  separated. For this, we use some notions on algebraic spaces that either have a treatment in the existing literature (e.g. in [Kn], [SP]) or translate immediately from the analogous notions for schemes (e.g., the schematic image of a morphism can be defined by descent from an fppf presentation of the target). In order to show that  $\Delta_G: G \rightarrow G^2$  is a closed immersion, we may pass to a covering of  $S$  in the fppf topology; we use  $T := X \rightarrow S$ , and we denote  $g \in G_T(T)$  the tautological point given by  $\phi: T \rightarrow G$ . Now let  $D$  be the image of  $\Delta_G$ , let  $\overline{D}$  be its schematic closure inside  $G \times_S G$ , and let  $\partial D = \overline{D} \setminus D$  be the boundary. The formation of  $\partial D$  commutes with flat base change on  $G^2$ , because it is the case for images of immersions and schematic images of arbitrary morphisms. This has two consequences. The first is that  $(\partial D)_T$  is the boundary of the image of  $\Delta_{G_T}$ . Hence  $(\partial D)_T$ , inside  $G_T \times_T G_T$ , is invariant by the action of  $g$  by simultaneous left translation  $((x, y) \mapsto (gx, gy))$ , because  $g$  gives an automorphism of  $\Delta_{G_T}: G_T \rightarrow G_T^2$  by simultaneous left-translation. The second consequence is that  $(\partial D) \cap X^2 = \emptyset$ , because that intersection is the boundary of  $D \cap X^2$  inside  $X^2$ , hence is empty because  $X \rightarrow S$  is separated. Now assume  $\partial D \neq \emptyset$ . Then there is an algebraically closed field  $k$ , an  $s$  in  $S(k)$  and a  $(x, y) \in (\partial D)(k)$  over  $s$ . Let  $T_s$  denote the fiber of  $T \rightarrow S$  over  $s$ , it is a  $k$ -scheme, locally of finite type (recall that it is  $X_s$ ). Then the sets  $\{t \in T_s(k) : g(t)x \in X_s(k)\}$  and  $\{t \in T_s(k) : g(t)y \in X_s(k)\}$  are both open and dense in  $T_s(k)$ , hence their intersection is non-empty. This contradicts that  $(\partial D)_T \cap X_T^2 = \emptyset$ . Thus  $\partial D = \emptyset$  and we are done.

5. We refer to Theorem 3.7(iii) of [Art]. To prove that, Artin shows that for each  $s$  in  $S$  there is an open neighborhood  $V$  of  $s$  and a  $T \rightarrow V$  that is faithfully flat and of finite presentation such that  $G_T$  is covered by the open immersions  $\phi_g: X_T \rightarrow G_T$ , where  $g$  varies in a finite subset of  $G(T)$ . Then  $G_T$  is a scheme, faithfully flat and of finite presentation over  $T$ . The fact that Artin assumes  $W \rightarrow X^3$  to be of finite presentation is harmless; he used this assumption only in the proof of Proposition 3.5 of [Art], which says that the projections from  $X^2 \times_G X^2$  to  $X^2$  are of finite presentation, and we have proved that result in 3. and 4. above.  $\square$

As a complement, we indicate with reference to the literature the state of the art concerning the question of representability by a scheme of the group algebraic space  $G$  of Theorem 3.18.

**Theorem 3.19.** — *In Theorem 3.18, assume moreover that  $S$  is locally noetherian. Then the group algebraic space  $G$  over  $S$  is representable by a scheme in the following cases:*

1.  $S$  has dimension 0, e.g. the spectrum of a field or of an artinian ring,
2.  $S$  has dimension 1 and  $X \rightarrow S$  is separated,
3.  $X \rightarrow S$  is smooth.

*Proof.* — 1. Assume that  $S$  is the spectrum of a field. Then  $G$  is quasi-separated by 3.18.4, and the set  $U$  consisting of all points of  $G$  admitting a scheme-like neighborhood is topologically dense by [RG] 5.7.7. Using finite Galois descent, one shows that  $U$  is invariant under  $G$ , since any finite set of points of  $U$  is contained in an affine open subscheme of  $U$ . In our case,  $G = GU$  so that  $U = G$  is a scheme. If  $S$  is local artinian, the result follows from the previous argument since any finite subset of a group over a local artinian ring is contained in an affine open (Lemma 5.6.1 in [Ber]).

2. This is a result of Anantharaman, see [An] IV.4.B.

3. As discussed in [BLR] Chapter 6, this is an application of the theorem of the square, the quasi-projectivity of torsors in the case of a normal base as proved in [Ra2], and a suitable criterion for effectivity of descent proved in [Ra1], Theorem 4.2. More precisely, according to Theorem 3.18, the diagonal of  $G$  is a quasi-compact immersion hence  $G$  is a smooth algebraic space in the sense used in [BLR]. It follows that the connected component along the unit section  $G^0 \rightarrow S$  is a well-defined open subspace: indeed, for group schemes this result is [Ber], th. 3.10, (iii)  $\Rightarrow$  (iv) and the arguments of the proof in *loc. cit.* work verbatim, replacing [EGA].IV.15.6.5 by [Rom] 2.2.1 for the finitely presented  $W \rightarrow T$  that appears in [Ber]. Applying part (b) of [BLR] 6.6/2 with  $G := G^0$  acting by translations on the space  $X := G$  with open  $S$ -dense subspace  $Y := X$  shows that  $G$  is a scheme.  $\square$

We end this section with some remarks and examples.

**Remark 3.20.** — 1. Let  $G$  be the quotient  $\mathbf{G}_m/\mathbf{Z}$  of the multiplicative group over a field by the subsheaf generated by a section  $x \in \mathbf{G}_m(k)$  that is of infinite order. This is a group algebraic space whose diagonal is not an immersion (see 3.18.4) and hence is not representable by a group scheme, even after base change (see 3.18.5).

2. By 3.19.3, if  $S$  is locally noetherian the following conditions on a smooth group algebraic space  $G \rightarrow S$  are equivalent:  $G$  comes from a strict birational group law on a scheme  $X \rightarrow S$ ;  $G$  is representable by a scheme;  $G$  contains an

S-dense open subspace which is a scheme. In [Ra2] X.14, one finds an example showing that these conditions are not always satisfied. In this example, the base  $S$  is the affine plane over a field of characteristic 2 and the group algebraic space  $G$  is a quotient of the square of the additive group  $(\mathbf{G}_{a,S})^2$  by an étale closed subgroup.

3. We do not know an example of a strict  $S$ -birational group law  $(X, W)$  as in Theorem 3.18 such that  $G \rightarrow S$  is not representable by a scheme.

#### 4. Application to Néron models

Let  $S$  be a Dedekind scheme (a noetherian, integral, normal scheme of dimension 1) with field of rational functions  $K$ , and let  $A_K$  be a  $K$ -abelian variety.

A *model* of  $A_K$  over  $S$  is a pair composed of an  $S$ -scheme  $A$  and a  $K$ -isomorphism  $A \times_S \text{Spec}(K) \simeq A_K$ . Usually, one refers to such a model by the letter  $A$  alone. If  $A$  is an  $S$ -model of  $A_K$ , we often say that its generic fibre "is"  $A_K$ . The nicest possible model one can have is a proper smooth  $S$ -model, but unfortunately this does not exist in general. In the search for good models for abelian varieties, Néron's tremendous idea is to abandon the requirement of properness, insisting on smoothness and existence of a group structure. He was led to the following notion.

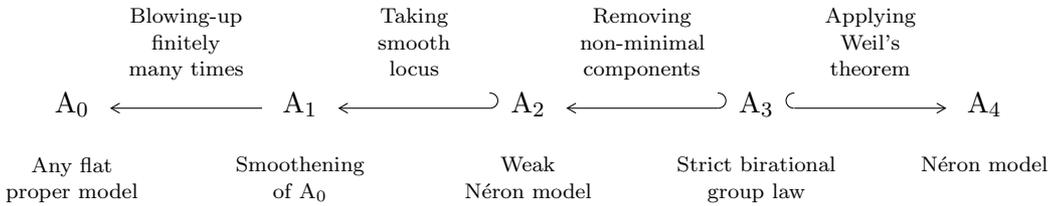
**Definition 4.1.** — A *Néron model* of  $A_K$  over  $S$  is a smooth, separated model of finite type  $A$  that satisfies the *Néron mapping property*: each  $K$ -morphism  $u_K: Z_K \rightarrow A_K$  from the generic fibre of a smooth  $S$ -scheme  $Z$  extends uniquely to an  $S$ -morphism  $u: Z \rightarrow A$ .

Our aim is to prove that a Néron model exists. Note that once existence is established, the universal property implies that the Néron model  $A$  is unique up to canonical isomorphism; it implies also that the law of multiplication extends, so that  $A$  is an  $R$ -group scheme. Therefore it could seem that for the construction of the Néron model, we may forget the group structure and recover it as a bonus. The truth is that things go the other way round: the Néron model is constructed first and foremost as a group scheme, and then one proves that it satisfies the Néron mapping property.

An important initial observation is that  $A_K$  extends to an abelian scheme over the complement in  $S$  of a finite number of closed points  $s$ , so one can reduce the construction of the Néron model in general to the construction in the local case by glueing this abelian scheme together with the finitely many local Néron models (i.e. over the spectra of the local rings  $\mathcal{O}_{S,s}$ ). Therefore it will be enough for us to consider the case where  $S$  is the spectrum of a discrete valuation ring  $R$  with field of fractions  $K$  and residue field  $k$ . We fix

a separable closure  $k \rightarrow k^s$  and a strict henselisation  $R \rightarrow R^{\text{sh}}$ ; we have an extension of fractions fields  $K \rightarrow K^{\text{sh}}$ .

If  $A$  is a smooth, finite type and separated model satisfying the extension property of the above definition only for  $Z$  étale, we say that it is a *weak Néron model*. Alternatively, it is equivalent to require that  $A$  satisfies the extension property for  $Z = \text{Spec}(R^{\text{sh}})$ , as one can see using the fact that  $R^{\text{sh}}$  is the inductive limit of “all” the discrete valuation rings  $R'$  that are étale over  $R$ . In contrast with Néron models, weak Néron models are not unique since their special fibre contains in general plenty of extraneous components, as we shall see. The Néron model will be obtained as the rightmost scheme in the following chain (hooked arrows denote open immersions):



## 5. Néron's smoothening process

One way to start the construction of the Néron model is to choose an embedding of  $A_K$  into some projective space  $\mathbf{P}_K^N$  (this is possible by a classical consequence of the theorem of the square). Then the schematic closure of  $A_K$  inside  $\mathbf{P}_R^N$  is a proper flat  $R$ -model  $A_0$ . Then, the valuative criterion of properness implies that the canonical map  $A_0(R^{\text{sh}}) \rightarrow A_K(K^{\text{sh}})$  is surjective. Thus if  $A_0$  happened to be smooth, it would be a weak Néron model of  $A_K$ . It is known that the special fibre  $A_0 \otimes k$  is proper and geometrically connected (see [EGA] IV.15.5.9), unfortunately it may be singular and even nonreduced. In order to recover smoothness at least at integral points, in Theorem 5.7 below we will produce a *smoothening* of  $A_0$  as defined in the following.

**Definition 5.1.** — Let  $A$  be a flat  $R$ -scheme of finite type with smooth generic fibre. A *smoothening* of  $A$  is a proper morphism  $A' \rightarrow A$  which is an isomorphism on the generic fibres and such that the canonical map  $A'_{\text{sm}}(R^{\text{sh}}) \rightarrow A(R^{\text{sh}})$  is bijective, where  $A'_{\text{sm}}$  is the smooth locus of  $A'$ .

In order to construct a smoothening, we will repeatedly blow up  $A$  along geometrically reduced closed subschemes of the special fibre containing the specializations of the points of  $A(R^{\text{sh}})$  that are “maximally singular”, in a sense that we shall define soon. This leads to consider that the natural object

to start with is a pair  $(A, E)$  where  $E$  is a given subset of  $A(\mathbb{R}^{\text{sh}})$ . Note that for any proper morphism  $A' \rightarrow A$  which is an isomorphism on the generic fibres, the set  $E$  lifts uniquely to  $A'(\mathbb{R}^{\text{sh}})$  and we will identify it with its image. The sense in which the singularity is maximal is measured by two invariants  $\delta(A, E)$  and  $t(A, E)$  which we now introduce.

**Definition 5.2.** — Let  $A$  be a flat  $\mathbb{R}$ -scheme of finite type with smooth generic fibre and let  $E$  be a subset of  $A(\mathbb{R}^{\text{sh}})$ . For each  $a: \text{Spec}(\mathbb{R}^{\text{sh}}) \rightarrow A$  in  $E$ , we set

$$\delta(a) = \text{the length of the torsion submodule of } a^*\Omega_{A/\mathbb{R}}^1.$$

The integer  $\delta(A, E) = \max\{\delta(a), a \in E\} \geq 0$  is called *Néron's measure for the defect of smoothness*.

It is easy to see that  $\delta(a)$  remains bounded for  $a \in E$ , so that  $\delta(A, E)$  is finite (see [BLR] 3.3/3). Moreover, this invariant does indeed measure the failure of smoothness:

**Lemma 5.3.** — *We have  $\delta(A, E) = 0$  if and only if  $E \subset A_{\text{sm}}(\mathbb{R}^{\text{sh}})$ .*

*Proof.* — Let  $a \in E$  and let  $d_K = \dim_{a_K}(A_K)$  and  $d_k = \dim_{a_k}(A_k)$  be the local dimensions of the fibres of  $A$ . By the Chevalley semi-continuity theorem, we have  $d_K \leq d_k$ . If  $\delta(a) = 0$  then  $a^*\Omega_{A/\mathbb{R}}^1$  is free generated by  $d_K$  elements. Then, at the point  $a_k$ ,  $\Omega_{A_k/k}^1$  can be generated by  $d_K$  elements, hence also by  $d_k$  elements, so that  $A_k$  is smooth according to [EGA] IV<sub>4</sub>.17.15.5. Being  $\mathbb{R}$ -flat, the scheme  $A$  is smooth at  $a_k$  and  $a \in A_{\text{sm}}(\mathbb{R}^{\text{sh}})$ . Conversely, if  $a \in A_{\text{sm}}(\mathbb{R}^{\text{sh}})$  then  $\Omega_{A/\mathbb{R}}^1$  is locally free in a neighbourhood of  $a_k$  and hence  $\delta(a) = 0$ .  $\square$

Starting from a pair  $(A, E)$  as above, we define geometrically reduced  $k$ -subschemes  $Y^1, U^1, \dots, Y^t, U^t$  of  $A_k$  and the *canonical partition*

$$E = E^1 \sqcup E^2 \sqcup \dots \sqcup E^t$$

as follows:

1.  $Y^1$  is the Zariski closure in  $A_k$  of the specializations of the points of  $E$ ,
2.  $U^1$  is the largest  $k$ -smooth open subscheme of  $Y^1$  where  $\Omega_{A/\mathbb{R}}^1|_{Y^1}$  is locally free,
3.  $E^1$  is the set of points  $a \in E$  whose specialization is in  $U^1$ .

Note that  $Y^1$  is geometrically reduced because it contains a schematically dense subset of  $k^{\text{s}}$ -points (see [EGA] IV<sub>3</sub>.11.10.7) and  $U^1$  is dense by generic smoothness. For  $i \geq 1$ , we remove  $E^1 \sqcup \dots \sqcup E^i$  from  $E$  and we iterate this construction. In this way we define  $Y^{i+1}$  as the Zariski closure in  $A_k$  of the specialization of the points of  $E \setminus (E^1 \sqcup \dots \sqcup E^i)$ ,  $U^{i+1}$  as the largest smooth open subscheme of  $Y^{i+1}$  where  $\Omega_{A/\mathbb{R}}^1$  is locally free, and  $E^{i+1}$  as the set of points  $a \in E$  with specialization in  $U^{i+1}$ . Since  $A_k$  is noetherian, there is an

integer  $t \geq 0$  such that  $Y^{t+1} = U^{t+1} = \emptyset$  and we end up with the canonical partition  $E = E^1 \sqcup E^2 \sqcup \cdots \sqcup E^t$ .

**Definition 5.4.** — We write  $t = t(A, E) \geq 1$  for the *length of the canonical partition*.

The crucial ingredient of the smoothening process is given by the following lemma, due to Néron and Raynaud.

**Lemma 5.5.** — *Let  $a \in E$  be such that  $a_k$  is a singular point of  $A_k$ . Assume that  $a \in E_i$ , let  $A' \rightarrow A$  be the blow-up of  $Y_i$ , and let  $a'$  be the unique lifting of  $a$  to  $A'$ . Then  $\delta(a') < \delta(a)$ .*

*Proof.* — This is an ingenious computation of commutative algebra, which we omit. We refer to [BLR] 3.3/5.  $\square$

For  $E \subset A(\mathbb{R}^{\text{sh}})$ , we denote by  $E_k$  the set of specializations of the points of  $E$  in the underlying topological space of  $A_k$ . We now make a definition that is tailor-made for an inductive proof of the theorem below.

**Definition 5.6.** — Let  $A$  be a flat  $\mathbb{R}$ -scheme of finite type with smooth generic fibre and let  $E$  be a subset of  $A(\mathbb{R}^{\text{sh}})$ . We say that a closed subscheme  $Y \subset A_k$  is *E-permissible* if it is geometrically reduced and the set  $F = Y \cap E_k$  satisfies:

1.  $F$  lies in the smooth locus of  $Y$ ,
2.  $F$  lies in the largest open subscheme of  $Y$  where  $\Omega_{A/\mathbb{R}}^1|_Y$  is locally free,
3.  $F$  is dense in  $Y$ .

We say that the blow-up  $A' \rightarrow A$  with center  $Y$  is *E-permissible* if  $Y$  is *E-permissible*.

Recall that for any proper morphism  $A' \rightarrow A$  which is an isomorphism on the generic fibres, the set  $E$  lifts uniquely to  $A'(\mathbb{R}^{\text{sh}})$  and we identify it with its image.

**Theorem 5.7.** — *Let  $A$  be a flat  $\mathbb{R}$ -scheme of finite type with smooth generic fibre and let  $E$  be a subset of  $A(\mathbb{R}^{\text{sh}})$ . Then there exists a morphism  $A' \rightarrow A$ , a finite sequence of *E-permissible* blow-ups, such that each point  $a \in E$  lifts uniquely to a smooth point of  $A'$ .*

*Proof.* — We proceed by induction on the integer  $\delta(A, E) + t(A, E) \geq 1$ . If  $\delta(A, E) = 0$ , then  $E$  lies in the smooth locus of  $A$  and no blow-up is needed at all; this covers the initial case of the induction. If  $\delta(A, E) \geq 1$ , we consider the canonical partition  $E = E^1 \sqcup \cdots \sqcup E^t$ . The closed subscheme  $Y^t \subset A_k$  is  $E^t$ -permissible, since  $E^{t+1} = \emptyset$  means exactly that the specializations of points of  $E^t$  lie in the open subset of the smooth locus of  $Y^t$  where  $\Omega_{A/\mathbb{R}}^1|_{Y^t}$

is locally free, and are dense in  $Y^t$ . Let  $A' \rightarrow A$  be the blow-up of  $Y^t$ . By Lemma 5.5, we have  $\delta(A', E^t) < \delta(A, E^t)$ . By the inductive assumption, there exists a morphism  $A'' \rightarrow A'$  which is a finite sequence of  $E^t$ -permissible blow-ups such that each point of  $E^t$  lifts uniquely to a point in the smooth locus of  $A''$ . If  $t = 1$ , we are done. Otherwise let  $E'' \subset A''(\mathbb{R}^{\text{sh}})$  be obtained by looking at  $E$  as a subset of  $A''(\mathbb{R}^{\text{sh}})$  and removing  $E^t$ , and for  $1 \leq i \leq t - 1$  let  $(E'')^i$  be the set  $E^i$  viewed in  $E''$ . Since  $A'' \rightarrow A$  is a sequence of  $E^t$ -permissible blow-ups, it does not affect  $E^1 \sqcup \dots \sqcup E^{t-1}$ . In this way one sees that  $E'' = (E'')^1 \sqcup \dots \sqcup (E'')^{t-1}$  is the canonical partition of  $E''$ , therefore  $t(A'', E'') < t(A, E)$ . Applying the inductive assumption once again, we obtain a morphism  $A''' \rightarrow A''$  which is a finite sequence of  $E''$ -permissible blow-ups such that points of  $E''$  lift to smooth points of  $A'''$ . Then  $A''' \rightarrow A$  is the morphism we are looking for.  $\square$

### 6. From weak Néron models to Néron models

Now let us continue the construction of Néron models. We started Section 5 with the schematic closure  $A_0$  of our abelian variety  $A_K$  inside some proper  $\mathbb{R}$ -scheme  $B$ . According to Theorem 5.7 applied with  $E = A_0(\mathbb{R}^{\text{sh}})$ , there exists a proper morphism  $A_1 \rightarrow A_0$  which is an isomorphism on the generic fibre, such that the smooth locus  $A_2 = (A_1)_{\text{sm}}$  is a weak Néron model. We now prove that weak Néron models satisfy a significant positive-dimensional reinforcement of their defining property.

**Proposition 6.1.** — *Let  $A$  be a weak Néron model of  $A_K$ . Then  $A$  satisfies the weak Néron mapping property: each  $K$ -rational map  $u_K: Z_K \dashrightarrow A_K$  from the generic fibre of a smooth  $\mathbb{R}$ -scheme  $Z$  extends uniquely to an  $\mathbb{R}$ -rational map  $u: Z \dashrightarrow A$ .*

Note that conversely, if the extension property of the proposition is satisfied for a smooth and separated model  $A$  of finite type, then one sees that  $A$  is a weak Néron model by taking  $Z = \text{Spec}(\mathbb{R}')$  for varying étale extensions  $\mathbb{R}'/\mathbb{R}$ .

*Proof.* — Since  $A$  is separated, we can first work on open subschemes of  $Z$  with irreducible special fibre and then glue. In this way, we reduce to the case where  $Z$  has irreducible special fibre. Then removing from  $Z$  the scheme-theoretic closure of the exceptional locus of  $u_K$ , we may assume that  $u_K$  is defined everywhere. Let  $\Gamma_K \subset Z_K \times A_K$  be the graph of  $u_K$ , let  $\Gamma \subset Z \times A$  be its scheme-theoretic closure, and let  $p: \Gamma \rightarrow Z$  be the first projection. On the special fibre, the image of  $p_k$  contains all  $k^{\text{s}}$ -points  $z_k \in Z_k$ : indeed, since  $Z$  is smooth each such point lifts to an  $\mathbb{R}^{\text{sh}}$ -point  $z \in Z(\mathbb{R}^{\text{sh}})$  with generic fibre  $z_K$ , and since  $A$  is a weak Néron model the image  $x_K = u_K(z_K)$  extends to a point  $x \in A(\mathbb{R}^{\text{sh}})$ , giving rise to a point  $\gamma = (z, x) \in \Gamma$  such that  $z_k = p_k(\gamma_k)$ . Since

the image of  $p_k$  is constructible, containing the dense set  $Z_k(k^s)$ , it contains an open set of  $Z_k$ .

In particular, the generic point  $\eta$  of  $Z_k$  is the image of a point  $\xi \in \Gamma_k$ . Since the local rings  $\mathcal{O}_{Z,\eta}$  (a discrete valuation ring with the same uniformizer as  $R$ ) and  $\mathcal{O}_{\Gamma,\xi}$  are  $R$ -flat and  $\mathcal{O}_{Z,\eta} \rightarrow \mathcal{O}_{\Gamma,\xi}$  is an isomorphism on the generic fibre, one sees that  $\mathcal{O}_{\Gamma,\xi}$  is included in the fraction field of  $\mathcal{O}_{Z,\eta}$ . Given that  $\mathcal{O}_{\Gamma,\xi}$  dominates  $\mathcal{O}_{Z,\eta}$ , it follows that  $\mathcal{O}_{Z,\eta} \rightarrow \mathcal{O}_{\Gamma,\xi}$  is an isomorphism. The schemes  $Z$  and  $\Gamma$  being of finite presentation over  $R$ , the local isomorphism around  $\xi$  and  $\eta$  extends to an isomorphism  $U \rightarrow V$  between open neighbourhoods  $U \subset \Gamma$  and  $V \subset Z$ . By inverting this isomorphism and composing with the projection  $\Gamma \rightarrow A$ , one obtains an extension of  $u_K$  to  $V$ .  $\square$

In the final step of the construction of the Néron model, we make crucial use of the group structure of  $A_K$  and in particular of the existence of invariant volume forms.

Quite generally, if  $S$  is a scheme and  $G$  is a smooth  $S$ -group scheme of relative dimension  $d$ , it is known that the sheaf of differential forms of maximal degree  $\Omega_{G/S}^d = \wedge^d \Omega_{G/S}^1$  is an invertible sheaf that may be generated locally by a left-invariant differential form (see [BLR] 4.2). If  $G$  is commutative, left-invariant differential forms are also right-invariant and we call them simply *invariant forms*. Thus on the Néron model of  $A_K$ , provided it exists, there should be an invariant global non-vanishing  $d$ -form, also called an *invariant volume form*, with  $d = \dim(A_K)$ . It is the search for such a form that motivates the following constructions.

We start by choosing an invariant volume form  $\omega$  for  $A_K$ , uniquely determined up to a constant in  $K^*$ . If  $A$  is a model of  $A_K$  which is smooth, separated and of finite type, then all its fibres have pure dimension  $d$  and the sheaf of differential  $d$ -forms  $\Omega_{A/R}^d = \wedge^d \Omega_{A/R}^1$  is invertible. Moreover, if  $\eta$  is a generic point of the special fibre  $A_k$ , its local ring  $\mathcal{O}_{A,\eta}$  is a discrete valuation ring with maximal ideal generated by a uniformizer  $\pi$  for  $R$ . Then the stalk of  $\Omega_{A/R}^d$  at  $\eta$  is a free  $\mathcal{O}_{A,\eta}$ -module of rank one which may be generated by  $\pi^{-r}\omega$  for a unique integer  $r \in \mathbb{Z}$  called the *order of  $\omega$  at  $\eta$*  and denoted  $\text{ord}_\eta(\omega)$ . If  $W$  is the irreducible component with generic point  $\eta$ , this is also called the *order of  $\omega$  along  $W$*  and denoted  $\text{ord}_W(\omega)$ . Moreover, if  $\rho$  denotes the minimum of the orders  $\text{ord}_W(\omega)$  along the various components of  $A_k$ , then by changing  $\omega$  into  $\pi^{-\rho}\omega$  we may and will assume that  $\rho = 0$ . A component  $W$  with  $\text{ord}_W(\omega) = 0$  will be called *minimal*.

In the previous sections, we saw that blowing up in a clever way finitely many times in the special fibre of a model of  $A_K$ , and removing the non-smooth locus, we obtained a weak Néron model  $A_2$ . Now, we consider the open subscheme  $A_3 \subset A_2$  obtained by removing all the non-minimal irreducible components of the special fibre.

**Lemma 6.2.** — *The section  $\omega$  extends to a global section of  $\Omega_{A_2/\mathbb{R}}^d$  and its restriction to  $A_3$  is a global generator of  $\Omega_{A_3/\mathbb{R}}^d$ .*

*Proof.* — Since  $A_2$  is normal and  $\omega$  is defined in codimension  $\leq 1$ , it extends to a global section of  $\Omega_{A_2/\mathbb{R}}^d$ . Now, recall that the zero locus of a nonzero section of a line bundle on an integral scheme has pure codimension 1. Thus since the restriction of  $\omega$  to  $A_3$  does not vanish in codimension  $\leq 1$ , it does not vanish at all and hence extends to a global generator of  $\Omega_{A_3/\mathbb{R}}^d$ .  $\square$

Now we denote by  $m_K: A_K \times A_K \rightarrow A_K$  the multiplication of the abelian variety  $A_K$ .

**Theorem 6.3.** — *The morphism  $m_K: A_K \times A_K \rightarrow A_K$  extends to an  $\mathbb{R}$ -rational map  $m: A_3 \times A_3 \dashrightarrow A_3$  and the  $\mathbb{R}$ -rational maps  $\Phi, \Psi: A_3 \times A_3 \dashrightarrow A_3 \times A_3$  defined by*

$$\begin{aligned}\Phi(x, y) &= (x, xy) \\ \Psi(x, y) &= (xy, y)\end{aligned}$$

*are  $\mathbb{R}$ -birational. In other words,  $m$  is an  $\mathbb{R}$ -birational group law on  $A_3$ .*

*Proof.* — Applying the weak Néron mapping property (Proposition 6.1), we can extend  $m_K$  to an  $\mathbb{R}$ -rational map  $m: A_3 \times A_2 \dashrightarrow A_2$ . We wish to prove that  $m$  induces an  $\mathbb{R}$ -rational map  $A_3 \times A_3 \dashrightarrow A_3$ . Let  $D \subset A_3 \times A_2$  be the domain of definition of  $m$ . We define a morphism  $\varphi: D \rightarrow A_3 \times A_2$  by the formula  $\varphi(x, y) = (x, xy)$  and we view it as a morphism of  $A_3$ -schemes in the obvious way. Denote by the same symbol  $\omega'$  the pullback of  $\omega$  via the projection  $\text{pr}_2: A_3 \times A_2 \rightarrow A_2$  and its restriction to  $D$ . We claim that  $\varphi^*\omega' = \omega'$ : indeed, this holds on the generic fibre because  $\varphi$  is an  $A_3$ -morphism of left translation, so this holds everywhere by density. Now let  $\xi = (\alpha, \beta)$  be a generic point of the special fibre of  $A_3 \times A_3$  and  $\eta = (\alpha, \gamma)$  its image under  $\varphi$ . Let  $r = \text{ord}_\gamma(\omega) = \text{ord}_\eta(\omega') \geq 0$ . Then  $\omega'$  is a generator of  $\Omega_{D/A_3}^d$  at  $\xi$  and  $\pi^{-r}\omega'$  is a generator of  $\Omega_{A_3 \times A_2/A_3}^d$  at  $\eta$ . It follows that  $\varphi^*(\pi^{-r}\omega') = b\omega'$  for some germ of function  $b$  around  $\xi$ . Since  $\varphi^*(\pi^{-r}\omega') = \pi^{-r}\omega'$ , this implies that  $r = 0$  hence  $\eta \in A_3 \times A_3$ . This shows that the set of irreducible components of the special fibre of  $A_3 \times A_3$  is mapped into itself by  $\varphi$ . Setting  $U = D \cap (A_3 \times A_3)$  we obtain morphisms  $\varphi: U \rightarrow A_3 \times A_3$  and  $m = \text{pr}_2 \circ \varphi: U \rightarrow A_3$  that define the sought-for rational maps. Proceeding in the same way with the morphism  $\psi: D \rightarrow A_3 \times A_2$  defined by  $\psi(x, y) = (xy, y)$ , we see that it also induces a morphism  $\psi: U \rightarrow A_3 \times A_3$ . In this way we obtain the  $\mathbb{R}$ -rational maps  $m, \Phi, \Psi$  of the theorem.

In order to prove that  $\Phi$  induces an isomorphism of  $U$  onto an  $\mathbb{R}$ -dense open subscheme, we show that  $\varphi: U \rightarrow A_3 \times A_3$  is an open immersion. We

saw above that the map

$$\varphi^* \Omega_{A_3 \times A_3 / A_3}^d \xrightarrow{\Omega} \frac{d}{U/A_3}$$

takes the generator  $\omega'$  to itself, so it is an isomorphism. This map is nothing else than the determinant of the morphism

$$\varphi^* \Omega_{A_3 \times A_3 / A_3}^1 \xrightarrow{\Omega} \frac{1}{U/A_3}$$

on the level of 1-forms which thus is also an isomorphism. It follows that  $\varphi$  is étale, and in particular quasi-finite. Since it is an isomorphism on the generic fibre, it is an open immersion by Zariski's Main Theorem ([EGA] IV<sub>3</sub>.8.12.10). One proves the required property for  $\Psi$  in a similar way.  $\square$

Let  $U$  be the domain of definition of the  $\mathbb{R}$ -rational map  $m: A_3 \times A_3 \dashrightarrow A_3$ . Then the graph  $W$  of  $m|_U: U \rightarrow A_3$  is a strict  $\mathbb{R}$ -rational group law on  $A_3/\mathbb{R}$  in the sense of Definition 3.4. It follows from Theorems 3.18 and 3.19 that there exists an open  $\mathbb{R}$ -dense immersion of  $A_3$  into a smooth separated  $\mathbb{R}$ -group scheme of finite type  $A_4$ . The last thing we wish to do is to check that  $A_4$  is the Néron model of  $A_K$ :

**Proposition 6.4.** — *The group scheme  $A_4$  is the Néron model of  $A_K$ , that is, each  $K$ -morphism  $u_K: Z_K \rightarrow A_K$  from the generic fibre of a smooth  $\mathbb{R}$ -scheme  $Z$  extends uniquely to an  $\mathbb{R}$ -morphism  $u: Z \rightarrow A_4$ .*

*Proof.* — Let us consider the  $K$ -morphism  $\tau_K: Z_K \times A_K \rightarrow A_K$  defined by  $\tau_K(z, x) = u_K(z)x$ . Applying the weak Néron mapping property, this extends to an  $\mathbb{R}$ -rational map  $\tau_2: Z \times A_2 \dashrightarrow A_2$ . In a similar way as in the proof of 6.3, one proves that the induced  $\mathbb{R}$ -rational map  $Z \times A_2 \dashrightarrow Z \times A_2$  defined by  $(z, x) \mapsto (z, \tau_2(z, x))$  restricts to an  $\mathbb{R}$ -rational map  $Z \times A_3 \dashrightarrow Z \times A_3$ . Since  $A_3$  is  $\mathbb{R}$ -birational to  $A_4$ , the latter may be seen as an  $\mathbb{R}$ -rational map  $Z \times A_4 \dashrightarrow Z \times A_4$ . Composing with the second projection, we obtain an  $\mathbb{R}$ -rational map  $\tau_4: Z \times A_4 \dashrightarrow A_4$  extending the map  $\tau_K$ . By Weil's theorem on the extension of rational maps from smooth  $\mathbb{R}$ -schemes to smooth and separated  $\mathbb{R}$ -group schemes ([BLR] 4.4/1), the latter is defined everywhere and extends to a morphism. Restricting  $\tau_4$  to the product of  $Z$  with the unit section of  $A_4$ , we obtain the sought-for extension of  $u_K$ . The fact that this extension is unique follows immediately from the separation of  $A_4$ .  $\square$

**Remark 6.5.** — Raynaud proved that the Néron model  $A_4$  is quasi-projective over  $\mathbb{R}$ . In fact, one knows that there exists an ample invertible sheaf  $\mathcal{L}_K$  on  $A_K$ . Raynaud proved that there exists an integer  $n$  such that the sheaf  $(\mathcal{L}_K)^{\otimes n}$  extends to an  $\mathbb{R}$ -ample invertible sheaf on  $A_4$ , see [Ra2], theorem VIII.2.

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