Robust design optimization for uncertain complex dynamical systems

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Robust design optimization with respect to model and data uncertainties in computational mechanics

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ABSTRACT: In this paper, a probabilistic approach is proposed to solve the robust design optimization problem of complex dynamical systems not only with respect to data uncertainties but also to model uncertainties. The possible designs are represented by a numerical finite element model whose parameters belong to an admissible set of design parameters. A recent nonparametric probabilistic model of uncertainties is used for taking into account model uncertainties and data uncertainties. The robust design optimization problem is formulated as a multi-objective optimization problem which consist to minimize a cost function including a target with respect to an admissible set of design parameters. The theory is presented followed by a numerical application.

Keywords: robust design, model uncertainties, structural dynamics

1 INTRODUCTION

Design optimization has become a major subject of interest in many industrial areas. The main challenge consists in including the effects of uncertainties in the design optimization problem (Taguchi et al., 1989) called the robust design. In the context of mechanical engineering, the robust design leads to solve a nonlinear constrained optimization problem with numerical models which are little sensitive to uncertainties in the vicinity of the design point (see for instance (Parkinson et al., 1993; Ramakrishnan and Rao, 1996) for the early works and (Lee and Park, 2001; Jung and Lee, 2002; Sandgren and Cameron, 2002; Doltsinis and Kang, 2004; Zang et al., 2005; Papadrakakis et al., 2005) for the most recent advances concerning this research area). The main difficulty of such robust design optimization problems concerns the probabilistic model of uncertainties, whose relevance is a major factor for the robust design optimization problem. The use of an erroneous probabilistic model yields an erroneous optimal design. However there is no reason for that the responses of the real dynamical system which is manufactured from this erroneous optimal design correspond to the performance objective. Until now, most of the published papers concerning robust design have been carried out in the context of static performances using parametric probabilistic models for modeling data uncertainties in the mechanical system. The robust design in the dynamic field is relatively recent (Doltsinis and Kang, 2004; Zang et al., 2005). In any case, all the works published until now concern robust design with respect to data uncertainties and not with respect to model uncertainties.

In the present paper, a robust design with respect to nonhomogeneous model uncertainties and adapted to the context of complex dynamical systems is proposed. Model uncertainties are taken into account by using the nonparametric probabilistic approach (Soize, 2000; Soize, 2001; Soize, 2005b). Several experimental validations (Duchereau and Soize 2005; Chebli and Soize, 2004; Durand et al., 2005, Chen et al., 2006) and numerical validations (Soize, 2005b, Soize, 2005a; Capiez-Lernout et al., 2005; Capiez-Lernout et al., 2006) have proved the capability of the nonparametric probabilistic approach to take into account model uncertainties and data uncertainties. The cost function used to formulate the robust design optimization problem is then defined as a function of the design parameter. Concerning the formulation of the cost function, the performance objective includes not only the target but also the robustness with respect to model uncertainties and data uncertainties.

In section 2, the set of mean reduced matrix models related to the set of all the feasible designs is constructed by using the Benfield and Hruda substructuring technique (Benfield and Hruda, 1971). In section 3, the design optimization problem is formulated assuming no uncertainties in the model of the dynamical system. Section 4 is devoted to the implementation of the nonparametric probabilistic approach for model and data uncertainties. Section 5 concerns the construction of the cost function describing the target and the sensitivity of the dynamical system to uncertainties (performance objectives) in order to formulate the robust design optimization problem with respect 1 to model and data uncertainties. Finally, a numerical application is presented in section 6.

2 MEAN DYNAMICAL SYSTEM

The dynamical system is made up of a given master system (a structure) coupled with a subsystem (a substructure) which has to be designed. The dynamical system is assumed to be linear and slightly damped. The equations are discretized by the finite element method and are written in the frequency domain. The frequency band of analysis is denoted by $\mathbb{B}$. It is assumed that the master system has no rigid body displacements and that the subsystem is free with r rigid body modes. Let $\mathbf{p} = (p_1, \ldots, p_s)$ be the $\mathbb{C}^s$-vector of the design parameters (geometry, elasticity properties, boundary conditions, etc.). The vector of the design parameters belongs to an admissible set $\mathcal{P}$ defined by the set of constraints prescribed by the design. For $\mathbf{p}$ fixed in $\mathcal{P}$ and for ω fixed in $\mathbb{B}$, the equation of the mean dynamical system is written as

$$\left(\left[\underline{A}^1(\omega)\right]+\left[\underline{A}^2(\mathbf{p},\omega)\right]\right)\underline{\mathbf{u}}(\mathbf{p},\omega)=\mathbf{f}(\mathbf{p},\omega), \tag{1}$$

in which $\underline{\mathbf{u}}(\mathbf{p}, \omega)$ is the $\mathbb{C}^n$-vector of the n DOF and $\mathbf{f}(\mathbf{p}, \omega)$ is the $\mathbb{C}^n$-vector induced by the external forces. In Eq. (1), the symmetric $n \times n$ complex matrices $[\underline{A}^1(\omega)]$ and $[\underline{A}^2(\mathbf{p}, \omega)]$ are the dynamical stiffness matrices of the master system and of the subsystem. It is assumed that vector $\mathbf{f}(\mathbf{p}, \omega)$ and matrix are $[\underline{A}^2(\mathbf{p},\omega)]$ written as

$$\mathbf{f}(\mathbf{p},\omega)=\mathbf{f}^0(\omega)+\sum_{i=1}^{s} p_i \mathbf{f}^i(\omega), \tag{2}$$

$$\left[\underline{A}^2(\mathbf{p},\omega)\right]=\left[\underline{A}^{2,0}(\omega)\right]+\sum_{i=1}^{s} p_i\left[\underline{A}^{2,i}(\omega)\right]. \tag{3}$$

It should be noted that the theory presented in this paper is also valuable when this linear assumption is removed, but then requires the numerical calculation of the gradient and the Hessian of $\mathbf{f}(\mathbf{p}, \omega)$ and $[\underline{A}^2(\mathbf{p}, \omega)]$ for each $\mathbf{p}$ in $\mathcal{P}$. In this paper, the Benfield and Hruda dynamic substructuring method (Benfield and Hruda, 1971) is used. Let $[H(\mathbf{p})]$ be the projection basis corresponding to the Benfield and Hruda dynamic substructuring method and such that

$$\underline{\mathbf{u}}(\mathbf{p},\omega)=\left[\underline{H}(\mathbf{p})\right]\begin{bmatrix}\underline{\mathbf{q}}^1(\mathbf{p},\omega)\\ \underline{\mathbf{q}}^2(\mathbf{p},\omega)\end{bmatrix}, \tag{4}$$

in which $\underline{\mathbf{q}}^i(\mathbf{p}, \omega)$ is the $\mathbb{C}^{N_i}$-vector of the reduced coordinates, for $i = \{1, 2\}$. In the robust design optimization context, the probabilistic model of uncertainties must be independent of the value of the design parameter $\mathbf{p}$. This implies that the projection basis for the subsystem must be independent of $\mathbf{p}$. Consequently, a numerical value $\mathbf{p}_0$ of $\mathbf{p}$ is chosen as an initial design value. Let V_N be the subspace of $\mathbb{R}^n$ spanned by the $N = N_1 + N_2$ columns of $[\underline{H}(\mathbf{p}_0)]$. The value of N is chosen (in studying the convergence) such that $\underline{\mathbf{u}}(\mathbf{p}, \omega)$ belongs to V_N for all $\mathbf{p}$ in $\mathcal{P}$. Equation (4) is then replaced by

$$\underline{\mathbf{u}}(\mathbf{p},\omega)=\left[\underline{H}(\mathbf{p}_0)\right]\begin{bmatrix}\underline{\mathbf{q}}^1(\mathbf{p},\omega)\\ \underline{\mathbf{q}}^2(\mathbf{p},\omega)\end{bmatrix}. \tag{5}$$

Projecting Eq. (1) in using Eq. (5) yields the mean reduced matrix equation

$$\left(\left[\underline{A}^1_{\mathrm{red}}(\omega)\right]+\left[\underline{A}^2_{\mathrm{red}}(\mathbf{p},\omega)\right]\right)\begin{bmatrix}\underline{\mathbf{q}}^1(\mathbf{p},\omega)\\ \underline{\mathbf{q}}^2(\mathbf{p},\omega)\end{bmatrix}=\mathbf{f}_{\mathrm{red}}(\mathbf{p},\omega), \tag{6}$$

in which $\mathbf{f}_{\mathrm{red}}(\mathbf{p}, \omega) = [\underline{H}(\mathbf{p}_0)]^T \mathbf{f}(\mathbf{p}, \omega)$ and where $[\underline{A}^1_{\mathrm{red}}(\omega)]$ and $[\underline{A}^2_{\mathrm{red}}(\mathbf{p}, \omega)]$ are symmetric $N \times N$ complex matrices. Note that $[\underline{A}^1_{\mathrm{red}}(\omega)]$ is a full matrix and that matrix $[\underline{A}^2_{\mathrm{red}}(\mathbf{p}, \omega)]$ is written as

$$[\underline{A}^2_{\mathrm{red}}(\mathbf{p},\omega)] = \begin{bmatrix} [\mathbb{O}] & [\mathbb{O}] \\ [\mathbb{O}] & [\underline{A}^2_s(\mathbf{p},\omega)] \end{bmatrix}. \tag{7}$$

3 FORMULATION FOR THE DESIGN OPTIMIZATION PROBLEM WITH A NUMERICAL MODEL WITH NO UNCERTAINTIES

In this Section, we remind a formulation to solve the design optimization problem assuming that there is no uncertainties. This formulation will be used to compare the solution of this deterministic design optimization problem with the robust design optimization solution presented in Section 5. Let $\underline{\mathbf{w}}(\mathbf{p}, \omega)$ be the vector in $\mathbb{C}^k$ of the observations of the mean dynamical system, defined as a function of the displacement vector $\underline{\mathbf{u}}(\mathbf{p}, \omega)$ such that

$$\underline{\mathbf{w}}(\mathbf{p},\omega)=b_{\omega}(\underline{u}(p,\omega)), \tag{8}$$

where $\mathbf{b}_{\omega}$ is a given function from $\mathbb{C}^n$ into $\mathbb{C}^k$ depending on the frequency ω. The performance objectives for the observations in the frequency band $\mathbb{B}_1 \subset \mathbb{B}$ will be defined as the "target". This target is then represented by the function $\omega \mapsto \mathbf{g}(\omega)$ from $\mathbb{B}_1$ into $\mathbb{C}^k$.

The cost function $\underline{j}(\mathbf{p})$ is formulated as a distance between the target $\underline{\mathbf{g}}$ and the observation $\underline{\mathbf{w}}(\mathbf{p}, \cdot)$ and is written as

$$\underline{j}(\mathbf{p}) = \frac{\left\| \underline{\mathbf{w}}(\mathbf{p},\cdot) - \mathbf{g} \right\|^2_{\mathbb{B}_1}}{\left\| \mathbf{g} \right\|^2_{\mathbb{B}_1}}, \tag{9}$$

in which $\|\mathbf{g}\|^2_{\mathbb{B}_1} = \int_{\mathbb{B}_1} \|\mathbf{g}(\omega)\|^2 \, d\omega$ with $\|\mathbf{g}(\omega)\|$ the hermitian norm of $\mathbf{g}(\omega)$. The design optimization problem is formulated as the minimization of the cost function $\underline{j}(\mathbf{p})$ with respect to the design parameter $\mathbf{p}$ in the admissible set $\mathcal{P}$: find $\mathbf{p}^{\mathrm{D}}$ in $\mathcal{P}$ such that $\underline{j}(\mathbf{p}^{\mathrm{D}}) \leq \underline{j}(\mathbf{p})$ for all $\mathbf{p}$ in $\mathcal{P}$.

4 STOCHASTIC DYNAMICAL SYSTEM WITH MODEL AND DATA UNCERTAINTIES

As explained in the Introduction, the robust design optimization problem is formulated with respect to the model uncertainties and data uncertainties existing in the mean model of the dynamical system. In this Section, we introduce this nonparametric probabilistic approach of uncertainties (Soize, 2000; Soize, 2001; Soize, 2005b). It is assumed that the mean model of the master system and subsystem contain model uncertainties and data uncertainties. The level of uncertainties of these two systems is a priori different and will be then characterized by different values of the dispersion parameters defined below.

Let $[\underline{M}^1_{\mathrm{red}}]$, $[\underline{D}^1_{\mathrm{red}}]$, $[\underline{K}^1_{\mathrm{red}}]$ and $[\underline{M}^2_{\mathrm{s}}(\mathbf{p})]$, $[\underline{D}^2_{\mathrm{s}}(\mathbf{p})]$, $[\underline{K}^2_{\mathrm{s}}(\mathbf{p})]$ be the mean reduced mass, damping, stiffness matrices of the mean master system and of the mean subsystem respectively. The dynamic stiffness reduced matrices are such that $[\underline{A}^1_{\mathrm{red}}(\omega)] = -\omega^2 [\underline{M}^1_{\mathrm{red}}] + i\omega[\underline{D}^1_{\mathrm{red}}] + [\underline{K}^1_{\mathrm{red}}]$ and $[\underline{A}^2_{\mathrm{s}}(\mathbf{p}, \omega)] = -\omega^2 [\underline{M}^2_{\mathrm{s}}(\mathbf{p})] + i\omega [\underline{D}^2_{\mathrm{s}}(\mathbf{p})] + [\underline{K}^2_{\mathrm{s}}(\mathbf{p})]$. The matrices $[\underline{M}^1_{\mathrm{red}}]$, $[\underline{D}^1_{\mathrm{red}}]$, $[\underline{K}^1_{\mathrm{red}}]$ are positive symmetric $N \times N$ matrices whose rank is $N_1 = N - n_2$ (which is assumed to be positive) whereas $[\underline{M}^2_{\mathrm{s}}(\mathbf{p})]$ is a positive-definite symmetric $N_2 \times N_2$ matrix and $[\underline{D}^2_{\mathrm{s}}(\mathbf{p})]$, $[\underline{K}^2_{\mathrm{s}}(\mathbf{p})]$ are positive symmetric $N_2 \times N_2$ matrices whose rank is $N_2 - r$. The methodology of the nonparametric probabilistic approach consists in replacing the matrices $[\underline{M}^1_{\mathrm{red}}]$, $[\underline{D}^1_{\mathrm{red}}]$, $[\underline{K}^1_{\mathrm{red}}]$ and $[\underline{M}^2_{\mathrm{s}}(\mathbf{p})]$, $[\underline{D}^2_{\mathrm{s}}(\mathbf{p})]$, $[\underline{K}^2_{\mathrm{s}}(\mathbf{p})]$ by the random matrices $[\mathbf{M}^1_{\mathrm{red}}]$, $[\mathbf{D}^1_{\mathrm{red}}]$, $[\mathbf{K}^1_{\mathrm{red}}]$ and $[\mathbf{M}^2_{\mathrm{s}}(\mathbf{p})]$, $[\mathbf{D}^2_{\mathrm{s}}(\mathbf{p})]$, $[\mathbf{K}^2_{\mathrm{s}}(\mathbf{p})]$ such that $\varepsilon\{[\mathbf{M}^1_{\mathrm{red}}]\} = [\underline{M}^1_{\mathrm{red}}]$, $\varepsilon\{[\underline{\mathbf{D}}^1_{\mathrm{red}})]\} = [\underline{D}^1_{\mathrm{red}})]$, $\varepsilon\{[\mathbf{K}^1_{\mathrm{red}}]\} = [\underline{K}^1_{\mathrm{red}}]$, and $\varepsilon\{[\mathbf{M}^2_{\mathrm{s}}(\mathbf{p})]\} = [\underline{M}^2_{\mathrm{s}}(\mathbf{p})]$, $\varepsilon\ \{[\mathbf{D}^2_{\mathrm{s}}(\mathbf{p})]\} = [\underline{D}^2_{\mathrm{s}}(\mathbf{p})]$, $\{[\mathbf{K}^2_{\mathrm{s}}(\mathbf{p})]\} = [\underline{K}^2_{\mathrm{s}}(\mathbf{p})]$ in which ε is the mathematical expectation.

The probability model for each one of these random matrices is briefly recalled below. Let $[\underline{E}^i(\mathbf{p})]$, $i = \{1, 2\}$ be the positive symmetric $n \times n$ real matrix of rank m representing one of the matrices of the set $\{[\underline{M}^1_{\mathrm{red}}]$, $[\underline{D}^1_{\mathrm{red}}]$, $[\underline{K}^1_{\mathrm{red}}]\}$ when $i = 1$ or of the set $\{[\underline{M}^2_{\mathrm{s}}(\mathbf{p})], [\underline{D}^2_{\mathrm{s}}(\mathbf{p})], [\underline{K}^2_{\mathrm{s}}(\mathbf{p})]\}$ when $i = 2$. Using the nonparametric probabilistic approach, the matrix $[\underline{E}^i(\mathbf{p})]$ is replaced by the random matrix $[\mathbf{E}^i(\mathbf{p})]$ such that

$$\left[\mathbf{E}^i(\mathbf{p})\right] = \left[\underline{L}^i_E(\mathbf{p})\right]^T \left[\mathbf{G}^i_E\right]\left[\underline{L}^i_E(\mathbf{p})\right], \tag{10}$$

in which $[\underline{L}^i_E(\mathbf{p})]$ is a $m \times n$ rectangular real matrix such that $[\underline{E}^i(\mathbf{p})] = [\underline{L}^i_E(\mathbf{p})]^{\mathrm{T}} [\underline{L}^i_E(\mathbf{p})]$ and where $[\mathbf{G}^i_E]$ is a random matrix with value in the set of all the positive-definite symmetric $m \times m$ matrices. All the details concerning the construction of the probability model of random matrix $[\mathbf{G}^i_E]$ can be found in (Soize, 2000; Soize, 2001; Soize, 2005b). The dispersion of the random matrix $[\mathbf{G}^i_E]$ is controlled by one real positive parameter δ^i_E called the dispersion parameter. This means that the dispersion parameters related to random matrices $[\mathbf{G}^1_M]$, $[\mathbf{G}^1_D]$, $[\mathbf{G}^1_K]$ and $[\mathbf{G}^2_M]$, $[\mathbf{G}^2_D]$, $[\mathbf{G}^2_K]$ are δ^1_M, δ^1_D, δ^1_K and δ^2_M, δ^2_D, δ^2_K. It should be noted that as a result of this theory, these six random matrices are independent random matrices. In addition, there exists an algebraic representation of these random matrices useful to the Monte Carlo numerical simulation.

In coherence with the notation of Section 2, let $\mathbf{U}(\mathbf{p}, \omega)$ be the $\mathbb{C}^n$-valued random vector of the displacement. The equations of the stochastic reduced system corresponding to the nonparametric probabilistic model of uncertainties are given by

$$\mathbf{U}(\mathbf{p},\omega) = \left[\underline{H}(\mathbf{p}_0)\right]\begin{bmatrix}\mathbf{Q}^1(\mathbf{p},\omega)\\ \mathbf{Q}^2(\mathbf{p},\omega)\end{bmatrix}, \tag{11}$$

where $\mathbf{Q}^i(\mathbf{p}, \omega)$ is the $\mathbb{C}^{N_i}$-valued random vector of the reduced coordinates, for $i \in \{1, 2\}$, solution of the random matrix equation

$$\left(\left[\mathbf{A}^1_{\mathrm{red}}(\omega)\right] + \left[\mathbf{A}^2_{\mathrm{red}}(\mathbf{p},\omega)\right]\right)\begin{bmatrix}\mathbf{Q}^1(\mathbf{p},\omega)\\ \mathbf{Q}^2(\mathbf{p},\omega)\end{bmatrix} = \mathbf{f}_{\mathrm{red}}(\mathbf{p},\omega), \tag{12}$$

in which the matrix $[\mathbf{A}^1_{\mathrm{red}}(\omega)]$ is such that $[\mathbf{A}^1_{\mathrm{red}}(\omega)] = -\omega^2 [\mathbf{M}^1_{\mathrm{red}}] + i\omega [\mathbf{D}^1_{\mathrm{red}}] + [\mathbf{K}^1_{\mathrm{red}}]$ and where the matrix $[\mathbf{A}^2_{\mathrm{red}}(\mathbf{p}, \omega)]$ is such that

$$\left[\mathbf{A}^2_{\mathrm{red}}(\mathbf{p},\omega)\right] = \begin{bmatrix} [\mathbb{0}] & [\mathbb{0}] \\ [\mathbb{0}] & [\mathbf{A}^2_s(\mathbf{p},\omega)] \end{bmatrix}, \tag{13}$$

with

$$\left[\mathbf{A}^2_s(\mathbf{p},\omega)\right] = -\omega^2\left[\mathbf{M}^2_s(\mathbf{p})\right] + i\omega\left[\mathbf{D}^2_s(\mathbf{p})\right] + \left[\mathbf{K}^2_s(\mathbf{p})\right]. \tag{14}$$

5 FORMULATION OF THE ROBUST DESIGN OPTIMIZATION PROBLEM WITH RESPECT TO MODEL UNCERTAINTIES AND DATA UNCERTAINTIES

In this Section, the robust design optimization problem is formulated with respect to model uncertainties and data uncertainties using the nonparametric probabilistic approach described in Section 4. The robust design optimization problem deals with the minimization of a cost function with respect to the design parameter. The cost function is constructed with an uncertain numerical model. Contrary to the design optimization problem described in Section 3, the cost function is not defined for the performance of the mean dynamical system but is defined with respect to the performance of the stochastic dynamical system representing the real manufactured system. For the robust problem, the performance objectives are double: (1) minimizing the distance between the mean value of the stochastic observation and the target and (2) minimizing the sensitivity of the stochastic observation with respect to model uncertainties and data uncertainties. The solution of this robust design optimization problem yields an optimal value of the design parameter which corresponds to an optimal dynamical system from which the real manufactured system fulfills the performance objectives.

In coherence with the notation introduced in Section 3, let $\mathbf{W}(\mathbf{p}, \omega)$ be the $\mathbb{C}^k$-valued random variable modeling the random observation of the stochastic dynamical system. It is defined as a function of the random displacement vector $\mathbf{U}(\mathbf{p}, \omega)$ such that

$$\mathbf{W}(\mathbf{p},\omega) = \mathbf{b}_{\omega}(\mathbf{U}(\mathbf{p},\omega)). \tag{15}$$

The cost function is then defined by

$$j(\mathbf{p},\alpha) = \alpha j_1(\mathbf{p}) + (1-2\alpha)\sigma^2(\mathbf{p}), \tag{16}$$

where α belonging to $[0, 1/2]$ is the weighting factor. In Eq. (16), the functions $j_1(\mathbf{p})$ and $\sigma^2(\mathbf{p})$ are given by

$$j_1(\mathbf{p}) = \frac{1}{\left\|\mathbf{g}\right\|^2_{\mathbb{B}_1}} \varepsilon\left\{\left\|\mathbf{W}(\mathbf{p},\cdot) - \mathbf{g}\right\|^2_{\mathbb{B}_1}\right\} \tag{17}$$

$$\sigma^2(\mathbf{p}) = \frac{1}{\left\|\mathbf{g}\right\|^2_{\mathbb{B}1}} \varepsilon\left\{\left\|\mathbf{W}(\mathbf{p},\cdot) - \boldsymbol{\mu}(\mathbf{p},\cdot)\right\|^2_{\mathbb{B}_1}\right\} \tag{18}$$

in which $\boldsymbol{\mu}(\mathbf{p}, \omega) = \varepsilon\{\mathbf{W}(\mathbf{p}, \omega)\}$. Note that $j_1(\mathbf{p})$ is related to a distance between the stochastic observation and the target. It allows the two performance objectives (1) and (2) to be simultaneously achieved. It can be shown that Eq. (16) can be rewritten as

$$j(\mathbf{p},\alpha) = \frac{1}{\|\mathbf{g}\|^2_{\mathbb{B}_1}}\Big(\alpha\,\|\boldsymbol{\mu}(\mathbf{p},\cdot) - \mathbf{g}\|^2_{\mathbb{B}_1} + (1-\alpha)\sigma^2(\mathbf{p})\Big). \tag{19}$$

The robust design optimization problem is formulated as a multi-objective optimization problem which consists to minimize the cost function $\mathbf{p} \mapsto j(\mathbf{p}, \alpha)$ with respect to the admissible set $\mathcal{P}$ of the design parameter $\mathbf{p}$. For given dispersion parameters δ^1_M, δ^1_D, δ^1_K and δ^2_M, δ^2_D, δ^2_K and for a given value of $\alpha \in [0, 1/2]$, the robust design optimization problem is written as: find $\mathbf{p}_{\text{RD}}$ in $\mathcal{P}$ such that

$$j(\mathbf{p}^{\text{RD}},\alpha) \leq j(\mathbf{p},\alpha), \text{ for all } \mathbf{p} \text{ in } \mathcal{P} \tag{20}$$

The value of the weighting factor characterizes the importance of each performance objective with respect to the other one and is adjusted in order to obtain the better solution. When $\alpha = 1/2$, the weight of the performance objectives (1) and (2) are the same. For small values of α, the performance objective related to the robustness with respect to model and data uncertainties becomes more important with respect to the performance objective related to the target. Since the normalisation does not change the optimization problem, it should be noted that the formulation used is coherent with the usual formulation of the robust design optimization problem (Doltsinis and Kang, 2004; Zang et al., 2005; Papadrakakis et al., 2005) when the target is not taken into account ($\mathbf{g} = 0$) and for the mono-dimensional case ($k = 1$). Note that the definition of the robust design optimization problem is coherent with respect to the deterministic design optimization problem given in Section 3, i.e., $\lim_{\|\delta\|\mapsto \mathbf{0}} \mathbf{p}^{\text{RD}} = \mathbf{p}^{\text{D}}$ in which $\boldsymbol{\delta} = \left(\delta^1_M, \delta^1_D, \delta^1_K, \delta^2_M, \delta^2_D, \delta^2_K\right)$. Finally, the robust design optimization problem is solved by using the sequential quadratic optimization algorithm (Fletcher, 1980; Powell, 1983) coupled with the Monte Carlo numerical simulation.

6 NUMERICAL APPLICATION

6.1 *Mean finite element model of the dynamical system*

The mean master system is an heterogeneous system made up of a plate with two attached lumped masses, one attached spring and 51 attached single DOF linear oscillators.

The plate is a thin plate in bending mode and is located in the plane (Ox, Oy) of a cartesian coordinate system ($Oxyz$). The out-plane displacements are only considered. The plate is made of a homogeneous, isotropic elastic material with mass density 7800 $\text{Kg} \times \text{m}^{-3}$, Poisson ratio 0.29, Young modulus 2×10^{11} $\text{N} \times \text{m}^{-2}$. The plate has constant thickness

0.4×10^{-3} m, length 0.5 m and width 0.4 m. The plate is simply supported on three edges and is free on the fourth edge corresponding to $y = 0$. The mean finite element model of the plate is constituted of 2000 bending plate elements (with 4 nodes) and is shown in Fig. 1. A damping model is added to the plate corresponding to a hysteretic model with a mean loss factor 0.02. The two lumped masses have mass 4 Kg and 1 Kg, located at points (0.15, 0.15, 0) and (0.2, 0, 0) respectively (see Fig. 1). The attached spring has stiffness 2.388×10^{11} N $\times$ m^{-1} and is located at point (0.06, 0.23, 0) (see Fig. 1). The attached oscillators are grouped by sets of 3 oscillators (see Fig. 1). The eigenfrequencies of these oscillators are in the band [350, 750] Hz. The mean subsystem to be optimized is constituted of 9 identical vibration absorbers, each one being made up of 5 single DOF linear oscillators in parallel (see Fig. 1). The eigenfrequencies of the 5 oscillators are 560, 565, 570, 575, 580 Hz. The critical damping rate is the same for the 5 oscillators and is 0.01. The five oscillators of a vibration absorber have the same mass m which has to be optimized. The total mass of the vibration absorbers is then defined by $\mathbb{m} = 45\,m$ m and is the design parameter of the mechanical system. The finite element model of the mean dynamical system (master system coupled with the five vibration absorbers) is thus constituted of $n = 6106$ DOF with $n_1 = 6052$ internal DOF of the mean master system, $n_2 = 45$ internal DOF for the mean subsystem and $n_\Sigma = 9$ coupling interface DOF. The mean dynamical system is submitted to a given deterministic unit transverse load constant in frequency band [5, 1200] Hz with amplitude 1 (see Fig. 1). The observation chosen for the dynamic analysis is the signal energy

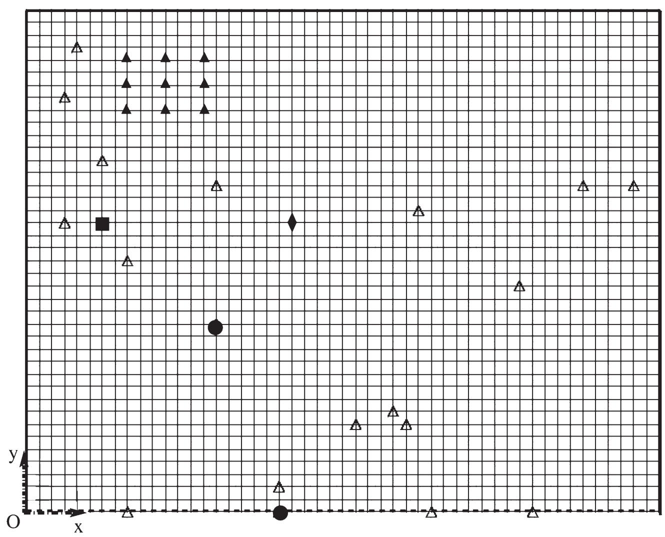

Figure 1. Finite element mesh of the dynamical system: attached spring (■), attached lumped mass (●), attached set of 3 single DOF linear oscillators (△), vibration absorbers (▲), excitation node (◆), simply supported boundary (thick line), free boundary (thick dashed line).

related to the out-plane accelerations of the plate such that $\underline{w}(\mathbf{p}, \omega) = b_\omega(\underline{\mathbf{u}}(\mathbf{p}, \omega)) = \omega^2 \|\underline{\mathbf{u}}^{\text{plate}}(\mathbf{p}, \omega)\|$ in which $\underline{\mathbf{u}}^{\text{plate}}(\mathbf{p}, \omega)$ is the complex vector constituted of the 1960 out-plane displacements of the plate.

6.2 *Reference solution for the master system*

The design parameter is the total mass $\mathbb{m}$. Since the eigenfrequency and the critical damping of the oscillators of the vibration absorbers are fixed, the mass, damping and stiffness matrices of the subsystem are linear functions of the design parameter $\mathbb{m}$. In the present case, the excitation does not depend on $\mathbb{m}$. The reference observation $\underline{w}^{\text{master}}(\omega)$ is defined as the response of the mean master system. Figure. 2 displays the graph of $\omega \mapsto 20 \log_{10}(\underline{w}^{\text{master}}(\omega))$. In Fig. 2, it can be seen that the level of the reference solution for the mean master system is lower than 77.5 dB in the frequency band [400,700]Hz except for one single peak whose resonance occurs at 571 Hz with level 80.5 dB, i.e., 3 dB more.

6.3 *Estimation of the numerical parameters for the robust design optimization problem*

Let $\{W^{\text{master}}(\theta_j, \omega), j = 1, \ldots, n_s\}$ be the n_s independent realisations of random variable $W^{\text{master}}(\omega)$. The robust optimization problem needs to solve the stochastic reduced equation Eq. (12). The numerical parameters are then the dimension N of the reduced dynamical system and the number n_s of realizations used in the Monte Carlo numerical simulation. Therefore a convergence analysis has to be performed with respect to N and n_s for the stochastic reduced system. Nevertheless, we have verified that the numerical parameters identified for the stochastic master system give convergent results for the stochastic reduced system. The computation is performed for the dispersion parameters of the master system such that $\delta_M^1 = \delta_D^1 = \delta_K^1 = 0.05$. A stochastic convergence

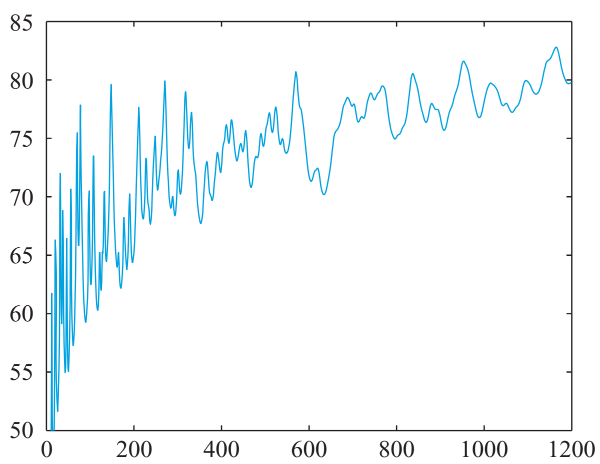

Figure 2. Reference observation of the mean master system. Graph of function $\nu \mapsto 20 \log_{10}(\underline{w}^{\text{master}}(2\,\pi\,\nu))$. Horizontal axis is the frequency ν in Hz.

analysis with $N_2 = n_2 + n_\Sigma = 54$ is carried out in order to define the number N_1 of modes to be kept in the modal reduction and the number n_s of realizations. The mean-square convergence is analyzed by studying the function $(n_s, N_1) \mapsto \text{Conv}\,(n_s, N_1)$ defined by

$$\text{Conv}^2(n_s, N_1) = \frac{1}{n_s} \sum_{j=1}^{n_s} \int_{\mathbb{B}} \| W^{\text{master}}(\theta_j, \omega)) \|^2 \, d\omega, \quad (21)$$

in which $W^{\text{master}}(\theta_j, \omega)$ is calculated with a reduced model of dimension $N = N_1 + N_2$. Figure 3 displays the graph of $n_s \mapsto 20 \log_{10}(\text{Conv}\,(n_s, N_1))$ for $N_1 = 300$. It can be seen that a reasonable convergence is reached for $n_s = 300$. Figure 4 displays the graph $N_1 \mapsto 20 \log_{10}(\text{Conv}\,(n_s, N_1))$ for $n_s = 300$. Convergence is reached for $N_1 = 225$.

6.4 *Target and its comparison with the reference observation*

The robust design optimization is carried out over the frequency band $\mathbb{B}_1 = [500, 600]$Hz in which

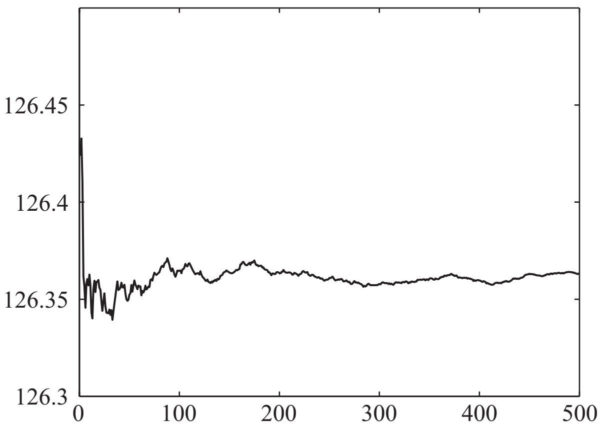

Figure 3. Convergence analysis : Graph of function $n_s \mapsto 20 \log_{10}$ $(\text{Conv}(n_s, N_1))$ for the stochastic master system with $N_1 = 300$. Horizontal axis is n_s.

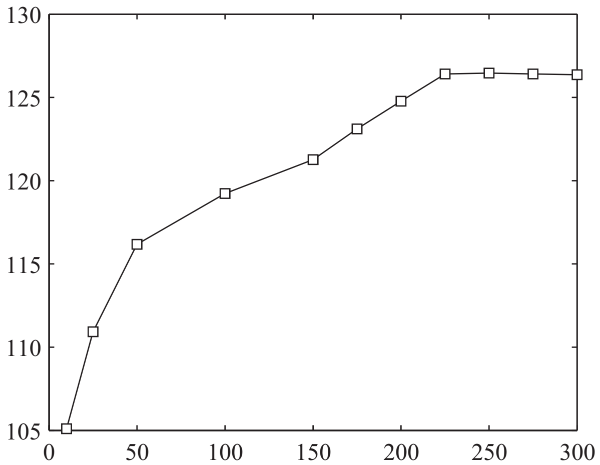

Figure 4. Convergence analysis : Graph of function $N_1 \mapsto 20$ $\log_{10}(\text{Conv}(n_s, N_1))$ for the stochastic master system with $n_s = 300$. Horizontal axis is N_1.

the response of the mean master system presents a resonance the level of which being 80.5 dB (see Fig. 5). The target is defined in order to limit the vibration level in band $\mathbb{B}_1$. Figure 5 shows the target $\omega \mapsto 20 \log_{10}(g(\omega))$ related to the reference observation $\omega \mapsto 20 \log_{10}(\underline{\omega}^{\text{master}}(\omega))$ defined in Section 6.2.

6.5 *Robust design optimization*

Below, the robust design optimization is carried out with $\delta^1_M = \delta^1_D = \delta^1_K = 0.05$ for the master system and with $\delta^2_M = \delta^2_D = \delta^2_K = 0$ for the subsystem (no uncertainties in the subsystem) with $N_1 = 225$ and $n_s = 300$. The admissible set for design parameter $\mathbb{m}$ is defined such that $\mathbb{m} \in [4.5 \times 10^7, 1.8 \times 10^{-3}]$Kg. We are interested in comparing the design optimization (no uncertainties) with the robust design optimization (with uncertainties) for a weighting factor α which is chosen as 0.5. The design optimization yields optimal design parameters $\mathbb{m}^{\text{D}} = 2.6 \times 10^{-4}$Kg and $\mathbb{m}^{\text{RD}} = 8.12 \times 10^{-4}$Kg. A stochastic dynamical analysis of each one of the two optimal designs is then carried out in order to analyze the sensitivity of these two optimal designs with respect to model and data uncertainties. Let $\mu^{\text{D}}(\omega)$, $\mu^{\text{RD}}(\omega)$ *and* $\sigma^{\text{D}}(\omega)$, $\sigma^{\text{RD}}(\omega)$ be the mean values and the standard deviations of the random observations $W^{\text{D}}(\omega)$ and $W^{\text{RD}}(\omega)$ defined by $W^{\text{D}}(\omega) = \text{W}(\mathbb{m}^{\text{D}}, \omega)$, $W^{\text{RD}}(\omega) = W(\mathbb{m}^{\text{RD}}, \omega)$. The results are $\|\sigma^{\text{D}}\|_{\mathbb{B}1} = 6.0408 \times 10^{-2} \|\text{g}\|_{\mathbb{B}1}$ and $\|\sigma^{\text{D}}\|_{\mathbb{B}1} = 5.8877 \times 10^{-2} \|\text{g}\|_{\mathbb{B}1}$ which means that the robust design optimization yields an optimal design slightly more robust than the design optimzation. However, we have $\|\mu^{\text{D}}\|_{\mathbb{B}1} = 9.4082 \times 10^{-1} \|\text{g}\|_{\mathbb{B}1}$ and $\|\mu^{\text{RD}}\|_{\mathbb{B}1} = 9.2634 \times 10^{-1} \|\text{g}\|_{\mathbb{B}1}$, which means that the mean value given by the robust design is slightly farther from the target than the mean value given by the design optimization.

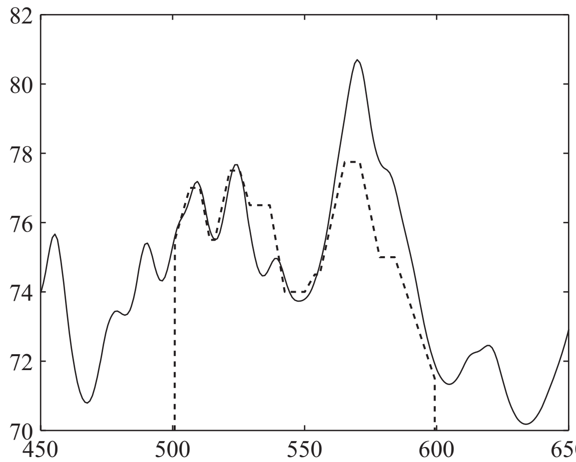

Figure 5. Definition of the target $\nu \mapsto 20 \log_{10}(g(2\pi\nu))$ (thick dashed line). Comparison with the reference observation $\nu \mapsto$ $20 \log_{10}(\underline{w}^{\text{master}}(2\pi\nu))$ (thin solid line) in the frequency band $\mathbb{B}_1 = [500, 600]$ Hz (horizontal axis).

Similarly to the stochastic case, let $\underline{w}^{\mathrm{D}}(\omega) = \underline{w}(\mathbb{m}^{\mathrm{D}}, \omega)$ and $\underline{w}^{\mathrm{RD}}(\omega) = \underline{w}(\mathbb{m}^{\mathrm{RD}}, \omega)$. Fig. 6 displays the comparison of the target with the response of the mean models corresponding to the design optimization and corresponding to the robust design optimization.

For all ω fixed in $\mathbb{B}$, the confidence interval of the random variable $W^{\mathrm{RD}}(\omega)$ is constructed for a probability level $P_c = 0.95$ using the sample quantiles (Serfling, 1980).

Figure 7 compares the reference solution (response of the mean master system) $\omega \mapsto 20 \log_{10} (\underline{w}^{\mathrm{master}}(\omega))$ with the confidence region of the robust design optimization. In particular, the resonance of the reference solution occuring at frequency 571 Hz has been reduced (about of 3 dB or 4 dB) by the robust design optimization process. It can be seen that the response of the mean master system belongs to the confidence region of the response of the stochastic system corresponding to the robust design optimization except in the frequency band $\mathbb{B}_1$ for which the target is active.

Figures 8 and 9 compare the reference solution $\omega \mapsto 20 \log^{10} (\underline{\omega}^{\mathrm{master}}(\omega))$ with the confidence regions of the random responses $\omega \mapsto 20 \log_{10} (W^{\mathrm{D}}(\omega))$ corresponding to the design optimization and $\omega \mapsto 20 \log_{10} (W^{\mathrm{RD}}(\omega))$ corresponding to the robust design optimization for a probability level $P_c = 0.95$ in the frequency band $\mathbb{B}_1$.

In Fig. 9, there are five resonances which occur at frequencies 508 Hz, 524 Hz, 539 Hz, 571 Hz and 583 Hz. In Figure. 9, for peaks number 1 and 4, it can be seen that the robust design optimization yields similar results to the design optimization. For peaks number 2 and 5, the robust design optimization yields lower responses levels. Moreover, the confidence region is particularly narrow in the frequency band [550, 600] Hz which means that the optimum design is more robust with respect to model and data uncertainties than in the frequency band [500, 550] Hz.

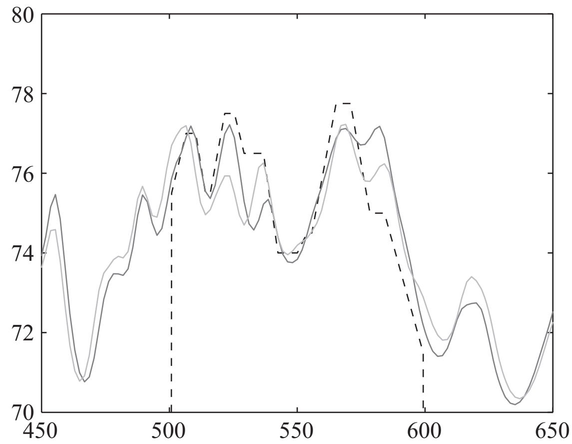

Figure 6. Comparison of the target $\nu \mapsto 20 \log_{10} (g(2\pi \nu))$ (thick dashed line) with the response of the mean model corresponding to the design optimization $\nu \mapsto 20 \log_{10} (\underline{w}^{\mathrm{D}}(2\pi \nu))$ (thin dark gray line) and corresponding to the robust design optimization $\nu \mapsto 20 \log_{10} (\underline{w}^{\mathrm{RD}}(2\pi \nu))$ (thin light gray line) for $\alpha = 1/2$. Horizontal axis is the frequency ν in Hz.

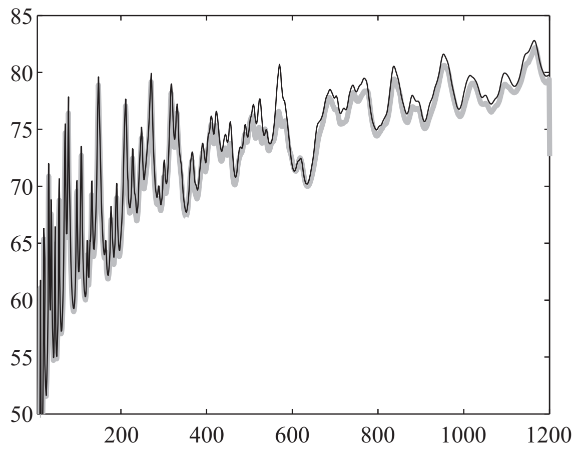

Figure 7. Comparison of the reference observation $\nu \mapsto 20 \log_{10} (\underline{w}^{\mathrm{master}}(2\pi \nu))$ (thin solid line) with the confidence region (light gray region) of random response for the robust design, over the band $\mathbb{B} = [5, 1200]$Hz (horizontal axis) and for $\alpha = 1/2$ and a probability level $P_c = 0.95$. Horizontal axis is the frequency ν in Hz.

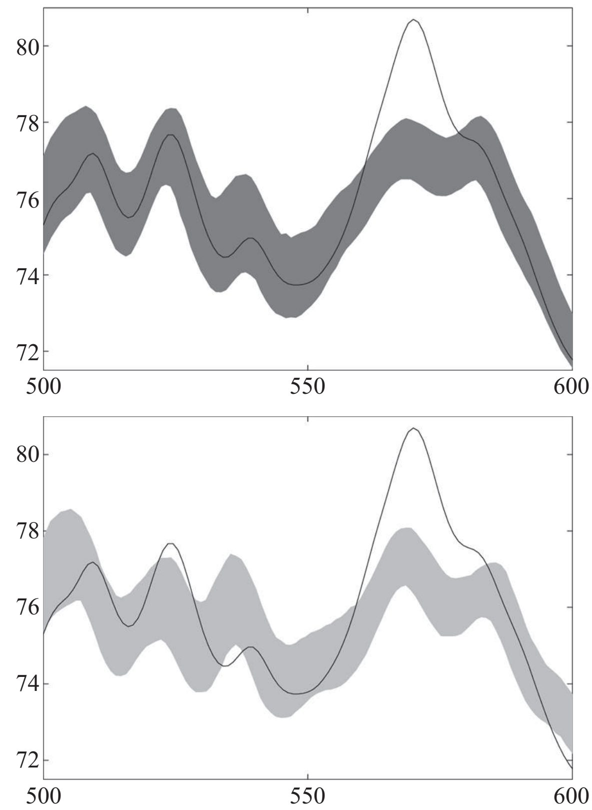

Figure 8. Comparison of the reference solution $\nu \mapsto 20 \log_{10} (\underline{w}^{\mathrm{master}}(2\pi\nu))$ (thin solid line) with the confidence region (dark gray region) of the random response $\nu \mapsto 20 \log_{10} (W^{\mathrm{D}}(2\pi\nu))$ corresponding to the design optimization and with the confidence region (light gray region) of the random response $\nu \mapsto 20 \log_{10} (W^{\mathrm{RD}}(2\pi\nu))$ corresponding to the robust design optimization. Horizontal axis is the frequency ν in Hz.

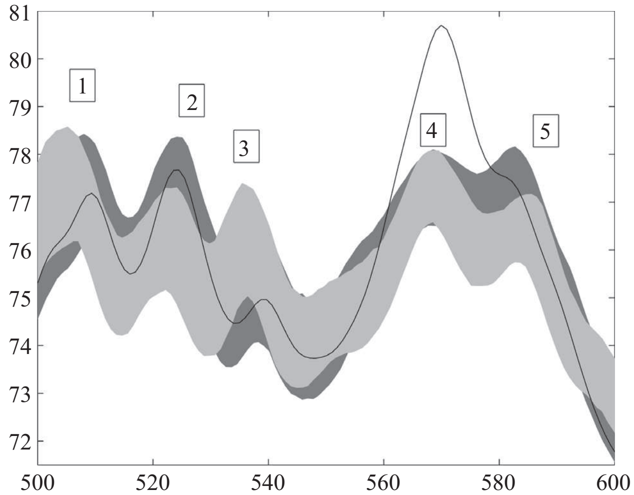

Figure 9. Comparison of the reference solution $\nu \mapsto 20\ \log_{10}(\underline{w}^{\mathrm{master}}(2\pi\nu))$ (thin solid line) with the two confidence regions defined in Fig. 8.

7 CONCLUSION

In this paper, we have presented an approach which allows the robust design optimization problem to be formulated and solved in presence of model uncertainties. Model uncertainties are taken into account with a nonparametric probabilistic approach. Thanks to an adapted algebraic development, the numerical optimization problem is solved with accuracy and with a low numerical cost. The approach proposed is general and can be used for analyzing complex dynamical systems in computational mechanics.

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