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Abstract

Impedance matrices allow for the coupling of domains with very different properties, and possibly modeled with different methods. This paper presents the construction of a probabilistic model for such matrices, using a nonparametric method that allows for the consideration of data errors as well as model errors. To enable the application of this method in the case of a domain for which no Finite Element model is available, the identification of a "hidden state variables model" - from the knowledge of the impedance matrix at a discrete set of frequencies - is also described. Finally the construction of the probabilistic model of the impedance of a pile foundation on a layered unbounded soil illustrates the possibilities of the method.

Key words: Computational stochastic mechanics, Impedance matrix, Unbounded domains, Nonparametric probabilistic method, Modal identification.

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1 Introduction

In many fields of applied mechanics and engineering, the problems considered are composed of several parts with (very) different properties. A classical resolution scheme consists in using domain decomposition techniques, splitting the global problem into several local ones interacting through boundary impedances. When these problems are defined on bounded domains, a finite element (FE) model is classically constructed, yielding, under hypothesis of linearity, the matrices of mass, damping and stiffness for the whole domain.

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Afterwards, the boundary impedance matrix is computed using the appropriate condensation on the boundary of the dynamic stiffness matrix.

When the local problem is defined on a domain which is unbounded or the size of which impedes the computation of the entire FE model, other means have to be used, bypassing the need to mesh the entire domain. Such is the case for example in geodynamics, with applications in seismology and oil prospecting, and in hydrodynamics and aerodynamics for marine and automotive engineering or meteorology. The most classical methods to compute the impedance matrix for that class of problems are the Boundary Element (BE) method [1], and the FE method with absorbing boundaries or infinite elements [2–4]. Other means [5] aim at developing simple approximate mechanical systems, with few degrees of freedom (DOFs), highlighting the physical phenomena taking place inside the unbounded domain.

Both for the bounded and unbounded domains, these deterministic techniques are often not sufficient to characterize in a realistic way the influence of the domain on the boundary. Many sources of uncertainty exist which call for the computation of the set of marginal laws of the impedance matrix, rather than just the one mean value. In geotechnics for example [6], the natural variability on the properties of the soil, the measurement errors, the scarcity of the available data and the simplicity of the models (linearity, among other simplifications) are factors that both make some quantification of uncertainty unavoidable and hamper it. In Fluid Mechanics, the complexity of the physical phenomenon of turbulence [7] also urges on the use of probabilistic methods. Stochastic approaches have therefore been proposed to take into account, in the computation of the impedance matrix, randomness or uncertainty on the properties of the materials, on the boundary conditions, and, more recently, on the models.

Many different computational methods have been considered [8], and amongst them, the Stochastic FE method is the most widely used. It was first introduced by Cornell [9] and was later derived under more efficient forms [10]. These methods are called parametric, as they describe the resolution of a problem with uncertain parameters, modeled as stochastic random variables or fields. They propagate the randomness of the parameters to the solution of the problem. Incidentally, they are well suited to take into account data uncertainties, when these can be acurately modeled, but are not appropriate for the consideration of model errors. Additionally, they require the meshing of the entire domain. In [11], this limitation was addressed through the coupling of a bounded random domain with an unbounded mean domain, but the limit between the deterministic and random domains remains rather artificial. In general, particularly in geotechnics, the available statistics on the data are scarce and polluted, and the model errors are important, therefore other techniques should be considered.

A nonparametric method has been recently introduced, which can, for a given mean model, take into account as a whole both model uncertainties and data uncertainties. It was originally developped in linear structural dynamics [12,13] with applications in vibrations and transient elastodynamics, and was extended to nonlinear dynamical systems [14] and to the medium frequency range [15]. The coupling of structures with different levels of uncertainty has also been considered in [16,17] and a nonparametric-parametric approach has been presented in [18] to model each source of uncertainty with the most appropriate method. The main concept of this method is to identify, for each problem, the unquestionable information, and to use the maximum-entropy principle to derive a probabilistic model using only that available information. This information is scarcer than that used in the parametric methods and includes algebraic properties of the random matrices considered and a mean model of the system.

The objective of this paper is to present the construction of a probabilistic model for the impedance matrix following the same pattern. The main difference between the probabilistic modeling of the impedance matrix and the cases that have been treated in the references cited in the previous paragraph is that its dependency on frequency is *a priori* unknown. The definition of the impedance matrix and its algebraic properties are therefore first recalled in Sec. 3, with a special highlight on the property of causality. Different possible models of the impedance enforcing that property are then studied, and the choice is finally set upon one based on underlying matrices of mass, damping and stiffness. Once the algebraic properties of the impedance matrix, based on that of the matrices of mass, damping and stiffness, have been identified, and given a mean value and a reduced set of dispersion parameters, the construction of the probabilistic model is then described in Sec. 4. Very often, for engineering applications, the only information that is really available consists in the algebraic properties of the impedance, its mean value at a discrete set of frequencies, and a general idea of the uncertainty of the problem. The issue of the identification of the mean values of the mass, damping and stiffness matrices of the hidden variables model using only such a mean impedance matrix is therefore discussed in Sec. 5. The identification from experimental results is left to be discussed in a future paper, and the choice of the dispersion parameters is not presented here as it has been described in other publications [19,20]. Finally, an example illustrating the construction of the probabilistic model of the impedance matrix of a circular superficial foundation resting on piles in a horizontally layered soil is presented in Sec. 6.

It should be noted that, to increase the readability, some results have been taken out of the main body of this paper and written in separate appendices. This should not be understood as a sign that they are well-known results, unworthy of the reader's attention. Particularly, App. B and C describe two problems that are, to the knowledge of the authors, original, and are central to the understanding of this paper.

2 Notations

In this short section, the main notations that will be used in this paper for vectors and matrices are recalled. The implicit Einstein summation over repeated indices is used throughout. Let $\mathbf{x} = (x_1, ..., x_n)$ be a vector of the euclidian space $\mathbb{R}^n$, equipped with the usual scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ and the associated euclidian norm $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$. The hermitian space $\mathbb{C}^n$ is equipped with the hermitian scalar product $(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \overline{\mathbf{y}} \rangle$ and its associated norm $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = \langle \mathbf{x}, \overline{\mathbf{x}} \rangle^{1/2}$, where $\overline{\mathbf{x}}$ is the complex conjugate of $\mathbf{x}$. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ when equivalent relations exist in both cases. $\mathbb{M}_{mn}(\mathbb{K})$ is the space of $n \times m$ matrices $[A]$ whose elements A_{ij} are in $\mathbb{M}_n(\mathbb{K})$. The vectors of $\mathbb{K}^n$ are identified to the column matrices in $\mathbb{M}_{n1}(\mathbb{K})$. $[I_{nm}]$ is the identity matrix in $\mathbb{M}_{nm}(\mathbb{K})$, whose elements are such that $I_{ii} = 1$ and $I_{ij,j\neq i} = 0$, and $[0_{nm}]$ denotes the null matrix in $\mathbb{M}_{nm}(\mathbb{K})$, whose elements are all equal to 0. When $n = m$, $\mathbb{M}_n(\mathbb{K}) = \mathbb{M}_{nn}(\mathbb{K})$, $[I_n] = [I_{nn}]$ and $[0_n] = [0_{nn}]$. The determinant, the trace, the transpose and the adjoint of a matrix $[A]$ in $\mathbb{M}_n(\mathbb{K})$ are denoted $\det[A]$, $\mathrm{tr}[A] = \sum_{j=1}^{n} A_{jj}$, $[A]^T$ and $[A]^* = [\overline{A}]^T$. The subset of $\mathbb{M}_n(\mathbb{K})$ of symmetric matrices (verifying $[A] = [A]^T$) is denoted $\mathbb{M}_n^S(\mathbb{K})$. The subset of $\mathbb{M}_n^S(\mathbb{K})$ of positive definite matrices (respectively semi-positive definite), such that $\langle [A]\mathbf{x}, \mathbf{x} \rangle > 0$ (respectively $\langle [A]\mathbf{x}, \mathbf{x} \rangle \geq 0$) $\forall \mathbf{x} \in \mathbb{K}^n \backslash \{[0_{n1}]\}$. With the Frobenius (or Hilbert-Schmidt) norm, defined for a matrix $[A]$ in $\mathbb{M}_n(\mathbb{K})$ by $\|[A]\|_F = (\mathrm{tr}\{[A][A]^*\})^{1/2}$, the set $\mathbb{M}_n(\mathbb{K})$ is a Hilbert space. The subset of $\mathbb{M}_n(\mathbb{K})$ of the non-singular (invertible) matrices is denoted $\mathbb{M}_n^*(\mathbb{K})$.

3 Definition and properties of the impedance

In this section, the definition of the impedance matrix of a bounded or unbounded domain with respect to a bounded boundary is recalled. The property of causality is highlighted, giving rise to a model in terms of mass, damping and stiffness matrices. The equations are presented for an elastodynamic problem but could have been equally derived in acoustics or fluid dynamics. In these cases, it is customary to define the impedance matrix with respect to a velocity field rather than a displacement field, as will be done here.

3.1 Setting of the boundary value problem

Let Ω be an open, bounded or unbounded, subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, and Ω_s an open bounded domain of $\mathbb{R}^3$, with a smooth boundary $\partial\Omega_s$, and such that $\Omega \cap \Omega_s = \emptyset$. Let Σ be the coupling boundary defined by $\Sigma = \partial\Omega \cap \partial\Omega_s$.

[Fig. 1 about here.]

Let $\boldsymbol{u} = [u_i]_{1\leq i\leq 3}$ be a displacement field defined on Ω, and $\boldsymbol{\sigma} = [\sigma_{ij}]_{1\leq i,j\leq 3}$ and $\boldsymbol{\epsilon} = [\epsilon_{ij}]_{1\leq i,j\leq 3}$ the corresponding linear stress and strain tensors. Let $\boldsymbol{C}^e = [C^e_{ijk\ell}]_{1\leq i,j,k,\ell\leq 3}$ and $\boldsymbol{C}^d = [C^d_{ijk\ell}]_{1\leq i,j,k,\ell\leq 3}$ be respectively the fourth order elastic and damping tensors of the materials in Ω, having the usual properties of symmetry ($C^e_{ijk\ell} = C^e_{jik\ell} = C^e_{k\ell ij}$ and $C^d_{ijk\ell} = C^d_{jik\ell} = C^d_{k\ell ij}$) and positive-definiteness ($C^e_{ijk\ell}e_{ij}e_{k\ell} \geq \alpha e_{ij}e_{ij}$ and $C^d_{ijk\ell}e_{ij}e_{k\ell} \geq \beta e_{ij}e_{ij}$, with $\alpha, \beta > 0$ for any second order real symmetric tensor e).

The local harmonic boundary value problem (BVP) in Ω consists in finding, for each ω in $\mathbb{R}$, a displacement field $\boldsymbol{u}$ such that

$$\begin{cases} \sigma_{ij,j}(\boldsymbol{u}) + \rho\omega^2 u_i = 0 & \text{in } \Omega \\ \sigma_{ij}(\boldsymbol{u})\, n_j = 0 & \text{on } \partial\Omega\backslash\Sigma \,, \\ u_i = \phi_i & \text{on } \Sigma \end{cases} \tag{1}$$

where $\boldsymbol{\phi} = [\phi_i]_{1\leq i\leq 3}$ is a given displacement field imposed on the boundary Σ. It should be noted that, in the linear case, the study of a problem with incident waves radiating from infinity - as in seismology - or with sources in Ω, can be brought back to the study of a BVP, via superposition.

3.2 Variational formulation of the BVP

The classical Sobolev space on Ω, of the square integrable functions defined on Ω, with square integrable first derivative, is denoted $H^1(\Omega)$.

$$H^1(\Omega) = \left\{ u \in L^2(\Omega)\ ,\ D^1 u \in L^2(\Omega) \right\}, \tag{2}$$

where D^1 denotes the first order partial derivation operator. $H^1(\Omega)$ is a Hilbert space when associated with the norm, $\|\cdot\|_{H^1(\Omega)}$, defined for u in $H^1(\Omega)$ by

$$\|u\|_{H^1(\Omega)} = \left[\int_\Omega \|u\|^2 + \sum_i \|\partial_i u\|^2 dx \right]^{1/2} . \tag{3}$$

The space of the admissible solutions for the variational formulation of the BVP is $V_\Omega = \{\boldsymbol{u} \in [H^1(\Omega)]^3\}$. The continuous hermitian positive definite sesquilinear forms of mass, damping and stiffness are defined in $V_\Omega \times V_\Omega$, respectively, by

$$m(\boldsymbol{u}, \delta\boldsymbol{u}) = \int_\Omega \rho u_i \overline{\delta u_i} d\boldsymbol{x}, \tag{4}$$

$$k(\boldsymbol{u}, \delta\boldsymbol{u}) = \int_\Omega C^e_{ijk\ell} \epsilon_{ij}(\boldsymbol{u}) \epsilon_{k\ell}(\overline{\delta\boldsymbol{u}}) d\boldsymbol{x}, \tag{5}$$

$$d(\boldsymbol{u}, \delta\boldsymbol{u}) = \int_\Omega C^d_{ijk\ell} \epsilon_{ij}(\boldsymbol{u}) \epsilon_{k\ell}(\overline{\delta\boldsymbol{u}}) d\boldsymbol{x}. \tag{6}$$

It should be noted that in the case of a bounded domain Ω, the sesquilinear forms of stiffness and damping are not necessarily definite as the imposed displacement field $\boldsymbol{\phi}$ might allow rigid body modes in Ω. This particular case is treated extensively in [21] and will not be considered further in this paper. For a given frequency ω in $\mathbb{R}$, the continuous hermitian sesquilinear form of dynamic stiffness is defined in $V_\Omega \times V_\Omega$ by

$$s(\boldsymbol{u}, \delta\boldsymbol{u}; \omega) = -\omega^2 m(\boldsymbol{u}, \delta\boldsymbol{u}) + i\omega d(\boldsymbol{u}, \delta\boldsymbol{u}) + k(\boldsymbol{u}, \delta\boldsymbol{u}). \tag{7}$$

Considering an element $\boldsymbol{u}$ of V_Ω, its restriction to boundary Σ, defined as a limit for positions in Ω tending towards positions in Σ, is called the trace of $\boldsymbol{u}$ on Σ, and denoted $\boldsymbol{u}_{|\Sigma}$. The space of the traces of the elements of V_Ω on Σ is denoted V_Σ. The functional spaces V_Ω^ϕ and V_Ω^0 are then defined by

$$V_\Omega^\phi = \{\boldsymbol{u} \in V_\Omega, \boldsymbol{u}_{|\Sigma} = \boldsymbol{\phi}\}, \text{ and } V_\Omega^0 = \{\boldsymbol{u} \in V_\Omega, \boldsymbol{u}_{|\Sigma} = \boldsymbol{0}\}. \tag{8}$$

The variational formulation of the BVP consists in finding, for ω in $\mathbb{R}$, $\boldsymbol{u} \in V_\Omega^\phi$ such that

$$s(\boldsymbol{u}, \delta\boldsymbol{u}; \omega) = 0 \ , \ \forall \delta\boldsymbol{u} \in V_\Omega^0. \tag{9}$$

3.3 The impedance operator

In the static case (resp., in the dynamic case), the ellipticity of the elastic tensor C^e (resp., of the damping tensor C^d) ensures that the dynamic stiffness form $s(\cdot, \cdot)$ (resp., $is(\cdot, \cdot)$) is coercive. The Lax-Milgram theorem [22], then ensures that the variational formulation of the BVP has a unique solution, therefore defining for each ω in $\mathbb{R}$, an unique operator $\mathcal{T}$ from V_Σ to V_Ω^ϕ such that

$$\boldsymbol{u} = \mathcal{T}(\omega)\boldsymbol{\phi}. \tag{10}$$

The impedance operator is then defined, for each ω in $\mathbb{R}$, from V_Σ into its dual V_Σ', for each ω in $\mathbb{R}$, by

$$\langle \mathcal{Z}(\omega)\boldsymbol{\phi}, \delta\boldsymbol{\phi} \rangle_\Sigma = s(\mathcal{T}(\omega)\boldsymbol{\phi}, \mathcal{T}(\omega)\delta\boldsymbol{\phi}; \omega), \tag{11}$$

where $\langle ., . \rangle_\Sigma$ is the antiduality product between V_Σ' and V_Σ.

$\boldsymbol{u}$ and $\delta\boldsymbol{u}$ can be approximated, with any desired level of accuracy, by their expansion on a finite Hilbert basis of functions defined on Ω. The coordinates of these expansions are denoted $\mathbf{u}$ and $\delta\mathbf{u}$. In that basis, the sesquilinear forms of mass, damping and stiffness are approximated respectively by the symmetric positive definite real matrices $[M]$, $[D]$ and $[K]$ (rigid body modes are not physically acceptable on Ω as they would correspond to displacements fields out of V_Ω, of infinite energy), and the sesquilinear form of dynamic stiffness can be approximated by a second order polynomial with real matrix coefficients $[S(\omega)] = -\omega^2[M] + i\omega[D] + [K]$. For a given $\omega \in \mathbb{R}$, $\omega \neq 0$, $[S(\omega)]$ is symmetric and its imaginary part is positive-definite, and for $\omega \neq 0$ sufficiently small, its real part is also positive-definite. $\boldsymbol{\phi}$ and $\delta\boldsymbol{\phi}$ can also be expanded on a Hilbert basis of functions defined on Σ, and compatible with the Hilbert basis defined on Ω, and their projections are denoted $\mathbf{\Phi}$ and $\delta\mathbf{\Phi}$. The previous matrices can then be block-decomposed, separating the DOFs defined on Σ from the DOFs defined in the interior of Ω.

$$\begin{bmatrix} [S_\Sigma(\omega)] & [S_c(\omega)] \\ [S_c(\omega)]^T & [S_h(\omega)] \end{bmatrix} = -\omega^2 \begin{bmatrix} [M_\Sigma] & [M_c] \\ [M_c]^T & [M_h] \end{bmatrix} + i\omega \begin{bmatrix} [D_\Sigma] & [D_c] \\ [D_c]^T & [D_h] \end{bmatrix} + \begin{bmatrix} [K_\Sigma] & [K_c] \\ [K_c]^T & [K_h] \end{bmatrix}. \tag{12}$$

For every frequency $\omega \in \mathbb{R}$, the impedance matrix $[Z(\omega)]$, approximation of the impedance operator $\mathcal{Z}(\omega)$ in the Hilbert basis defined on Σ, is then the Schur complement of $[S_h(\omega)]$ in $[S(\omega)]$.

$$[Z(\omega)] = [S_\Sigma(\omega)] - [S_c(\omega)][S_h(\omega)]^{-1}[S_c(\omega)]^T. \tag{13}$$

In Eq. (12), $[M_\Sigma]$, $[D_\Sigma]$ and $[K_\Sigma]$ are in $\mathbb{M}^+_{n_\Sigma}(\mathbb{R})$, the set of real $n_\Sigma \times n_\Sigma$ positive-definite matrices, $[M_c]$, $[D_c]$ and $[K_c]$ are in $\mathbb{M}_{n_\Sigma n_h}(\mathbb{R})$, the set of $n_\Sigma \times n_H$ real matrices, and $[M_h]$, $[D_h]$ and $[K_h]$ are in $\mathbb{M}^+_{n_h}(\mathbb{R})$, the set of real $n_H \times n_H$ positive-definite matrices. $[Z(\omega)]$ is a symmetric matrix whose imaginary part can be demonstrated to be positive [23]. For ω sufficiently small its real part is also a positive matrix.

3.5 The causality of the impedance

A very important property of the impedance is that it corresponds in the time domain to a causal function. This represents the natural condition that an effect should never take place before the cause that creates it, that a displacement of the boundary does not happen before a load is applied to it. Mathematically, the condition is written in the time domain (with $\mathcal{F}\{\cdot\}$ the Fourier transform and $\mathcal{F}^{-1}\{\cdot\}$ the corresponding inverse operation):

$\forall \mathbf{\Phi}$ *such that* $\hat{\mathbf{\Phi}} = \mathcal{F}\{\mathbf{\Phi}\}$ *and* $\mathcal{F}^{-1}\{[Z]\mathbf{\Phi}\}$ *exist in the sense of the distributions,*

$$\mathbf{\Phi}(t) = \mathbf{0}\,, \forall t < 0 \Rightarrow \mathcal{F}^{-1}\{[Z]\hat{\mathbf{\Phi}}\}(t) = \mathbf{0}\,, \forall t < 0. \tag{14}$$

As we wish to work in the frequency domain, this definition is not appropriate. In App. A, the impedance matrix, as defined in the frequency-domain in the previous section, is shown to be causal.

3.6 Construction of a model of the impedance matrix

From what was seen in the previous section, any reasonable model of the impedance matrix must ensure causality. This can be done in different ways, which are examined in this section. First, the expansion of the impedance matrix on a basis of Hardy functions, which are the Laplace transforms of a subset of the causal functions, is studied; then a model based on the Kramers-Kronig relations, linking the real and imaginary parts of the Fourier transform of a causal function, is presented; finally, as none of these two models seems constructive, a model based on the underlying model of the dynamic stiffness matrix is chosen. The construction of a model of the impedance matrix then consists in the choice of the parameters of that underlying model.

3.6.1 Expansion on a basis of Hardy functions

A first approach consists in trying to model the impedance as an expansion on a basis of some subspace of all Laplace transforms of causal functions, the space of Hardy functions. A function f defined on $\mathbb{C}^+ = \{p = \xi + i\eta, \xi > 0, \eta \in \mathbb{R}\}$ is said to be a Hardy function if it is holomorphic on $\mathbb{C}^+$ and $\sup_{a>0} \int_{\mathbb{R}} |f(\xi + i\eta)|^2 d\eta < +\infty$. It can be shown [24] that f is a Hardy function if and only if it is the Laplace transform of some square integrable causal function. A nice feature of that functional space is that it is equipped with an explicit orthonormal basis, the functions $\{p \mapsto e_\ell(p)\}_{\ell \geq 0}$, with

$$e_\ell(p) = \frac{(-1)^{\ell+1}}{\sqrt{\pi}} \left(\frac{1}{1-p}\right) \left(\frac{1+p}{1-p}\right)^\ell . \tag{15}$$

This space only contains square integrable functions, whereas the pseudodifferential part of the impedance matrix is not. The expansion of the impedance matrix in Eq. (13) would therefore be in the form $[Z(p)] = p^2\,[R_2] + p\,[R_1] + [R_0] + \sum_{\ell\geq 0} [Z_\ell] e_\ell(p)$. Unfortunately, there is no algebraic information on the matrices $\{[Z_\ell]\}_{\ell\geq 0}$. Also, since this construction is purely mathematical, the coefficients of this expansion are not easily related to physical parameters of the system. More important the rate of convergence of the infinite sum is not known so a very large number of terms might be required to yield a good approximation.

3.6.2 Kramers-Kronig relations

The Kramers-Kronig relations were originally developped for electromagnetic problems to link the real and imaginary parts of the polarization coefficients [25] and of the index of refraction [26], and were proved without any explicit reference to causality. Later, they were shown (see [27] for example for a demonstration) to be verified by any complex causal function, and state that the real and imaginary parts of its Fourier transform $\omega \mapsto f(\omega)$ verify, equivalently,

$$\Re\{f(\omega)\} = \frac{1}{\pi} \oint_{\mathbb{R}} \frac{\Im\{f(\omega')\}}{\omega - \omega'} d\omega', \text{ or } \Im\{f(\omega)\} = -\frac{1}{\pi} \oint_{\mathbb{R}} \frac{\Re\{f(\omega')\}}{\omega - \omega'} d\omega'. \quad (16)$$

They are widely used in experimental physics because, in many applications, the imaginary part of a quantity of interest can be measured experimentally, and then the whole function constructed with Eq. (16). Unfortunately, numerically, the real and imaginary parts of the impedance matrix cannot be computed separately. Therefore these relations can be used to prolongate an impedance matrix that was computed on a bounded frequency interval to the entire frequency domain [28], but are not constructive for the modeling of an impedance matrix from scratch. Moreover these formulas apply only for functions f in L^1, which is not the case for $[Z]$, and the accurate evaluation of the singular integral might require a large band of frequency, highly refined, to be studied.

3.6.3 Hidden state variables model

Ultimately, a given algebraic structure can be proposed to construct an approximation of the boundary impedance matrix, satisfying *a priori* the desired causality condition. Following [29], the structure of the boundary impedance matrix of a mechanical system whose vibrations in the time domain are governed by a second-order differential equation with constant coefficients is chosen. This algebraic structure may not span the entire space of possible boundary impedance matrices arising from real physical systems and it is only a sufficient condition for causality to be satisfied. However, the similarities between this structure and the underlying system considered in Eq. (12) is appealing in the sense that the approach may give some interesting insights on the mechanical system hidden behind the boundary impedance matrix.

Let us therefore consider such a mechanical system and its discretization in n DOFs. We will denote $n = n_\Sigma + n_H$, where n_Σ is the number of DOFs of the part of the boundary with respect to which the boundary impedance matrix is constructed. In the frequency domain, the hypothesis in the time domain means that the dynamic stiffness $\omega \mapsto [S(\omega)]$ is a second-order polynomial in $(i\omega)$ with real matrix coefficients. Denoting $[M]$, $[D]$ and $[K]$ in $\mathbb{M}_n^+(\mathbb{R})$ those coefficients, we have

$$[S(\omega)] = -\omega^2[M] + i\omega[D] + [K], \tag{17}$$

and $[Z(\omega)]$ is the condensation on the first n_Σ DOFs of $[S(\omega)]$. These matrices $[M]$, $[D]$ and $[K]$ correspond to generalized mass, damping and stiffness matrices, but it should be clear that they are not necessarily the classical mass, damping and stiffness matrices, e.g. constructed using a FE method. As the only quantity we are eventually interested in is the impedance, and since there are infinitely many mechanical systems yielding, by condensation on Σ, the same impedance, we are free to choose the set of generalized matrices $[M]$, $[D]$ and $[K]$ as we wish in the set of matrices yielding the appropriate impedance. This will be discussed in more details in Sec. 5.2.1. For the same reasons, the DOFs on which these matrices are defined are not necessarily physical degrees of freedom, but rather state variables that are hidden in the background of the physical model; hence the name "hidden state variables model".

4 Construction of the nonparametric model of random uncertainties for the impedance matrix

In the previous section, we constructed a model of the impedance matrix which ensures that the causality condition is satisfied. It was implicitely assumed that the problem (mechanical model, constitutive equation, mechanical parameters, boundary conditions) was perfectly deterministic. The real-life experiments are very far from that situation. The models are approximate, as much from our lack of understanding of the underlying physics as from economic or time limitations, and the scarcity and pollution of the data hinders our ability to feed these models. That uncertainty in the model of the impedance matrix that was constructed in Sec. 3 should therefore be considered, and a probabilistic model prefered to a deterministic one. Using one of the classical parametric methods would require the identification of the uncertain parameters and their statistical caracterization, which is often out of reach. Also, a complete parametric probabilistic model of the impedance matrix would require the determination of all the marginal laws. Obviously, as the number of variables to be considered generally increases with the size of the domain, this approach is not feasible for the construction of the probabilistic model of the impedance matrix of a domain possibly unbounded.

We will use here a nonparametric method in which the uncertainty is not considered on the parameters and propagated to the matrices of the model but where it is rather taken into account directly in these matrices, which are to be chosen in appropriate ensembles of random matrices. Such ensembles, e.g. the Gaussian Orthogonal Ensemble [30], have been used for a long time in theoretical physics, but are not appropriate to model the positive matrices that are encountered in structural mechanics. We will work here with two ensembles that have been more recently introduced and are more adapted to the stiffness, mass and damping matrices of a dynamical system: the normalized positive-definite ensemble SG^+ and the positive-definite ensemble SE^+ [31,12,13]. The probability density functions of the matrices of these ensembles are constructed using the entropy maximization principle [32,33] constrained by knowledge on their algebraic properties and mean value.

The idea here is to replace the matrices $[M]$, $[D]$, $[K]$, $[S(\omega)]$ and $[Z(\omega)]$ of Eq. (12)-(13) by random matrices $[\mathbf{M}]$, $[\mathbf{D}]$, $[\mathbf{K}]$, $[\mathbf{S}(\omega)]$ and $[\mathbf{Z}(\omega)]$, the probability density functions of which have to be derived. The probabilistic model of each of the matrices $[\mathbf{M}]$, $[\mathbf{D}]$ and $[\mathbf{K}]$ is constructed in SE^+, and those of $[\mathbf{S}(\omega)]$ and $[\mathbf{Z}(\omega)]$ are simply derived using Eq. (12)-(13). To improve the readability, we first recall the construction of these two ensembles SG^+ and SE^+, and their important properties. Then, the construction of an approximation of the probability density function of the random impedance matrix $[\mathbf{Z}(\omega)]$ by Monte Carlo techniques is described.

4.1 The normalized positive definite ensemble of random matrices

The normalized positive-definite ensemble of random matrices , denoted SG^+, is defined as the set of the random matrices $[\mathbf{G_n}]$ defined on a probability space $(\mathcal{A}, \mathcal{T}, \mathcal{P})$, with values in $\mathbb{M}_n^+(\mathbb{R})$ verifying:

(1) $[\mathbf{G_n}] \in \mathbb{M}_n^+(\mathbb{R})$, almost surely;
(2) Matrix $[\mathbf{G_n}]$ is a second-order random variable: $E\{\|[\mathbf{G_n}]\|_F^2\} < +\infty$;
(3) The mean value $[\underline{G}_n]$ of $[\mathbf{G_n}]$ is the identity matrix $[I_n]$ in $\mathbb{M}_n^+(\mathbb{R})$: $E\{[\mathbf{G_n}]\} = [\underline{G}_n] = [I_n]$;
(4) $[\mathbf{G_n}]$ is such that $E\{\ln(\det[\mathbf{G_n}])\} = \nu_{[\mathbf{G_n}]}$, $|\nu_{[\mathbf{G_n}]}| < +\infty$.

The last constraint yields the fundamental property that the inverse of $[\mathbf{G_n}]$ is also a second-order random variable: $E\{\|[\mathbf{G_n}]^{-1}\|_F^2\} < +\infty$. For the computation of the probability density function of $[\mathbf{G_n}]$, it was also shown that rather than considering $\nu_{[\mathbf{G_n}]}$, whose exact value does not bear a simple physical meaning, it was interesting to replace it by a parameter δ, which measures the dispersion of the probability model of the random matrix $[\mathbf{G_n}]$ around the mean value $[\underline{G}_n]$.

4.2 The dispersion parameter

The dispersion parameter δ is a real parameter defined by

$$\delta = \left\{ \frac{E\{\|[\mathbf{G_n}] - [\underline{G}_n]\|_F^2\}}{\|[\underline{G}_n]\|_F^2} \right\}^{1/2} = \frac{1}{n} E\{\|[\mathbf{G_n}] - [I_n]\|_F^2\}^{1/2}, \tag{18}$$

where $\|\cdot\|_F$ is the Frobenius norm (see Sec. 2). The dispersion parameter should be chosen independent of n and such that

$$0 < \delta < \sqrt{\frac{n+1}{n+5}}, \tag{19}$$

to ensure that the fourth condition in the definition of SG^+ is verified.

Several procedures for the estimation of δ have been derived, depending on the type of information available:

(1) when no objective information is known about δ, a sensitivity analysis must be performed with δ as the parameter and its value estimated depending on the level of stochastic fluctuations (level of uncertainty);
(2) when a sufficient amount of experimental data is available, δ can be estimated using statistics;
(3) when a parametric model has been constructed in the low-frequency range, where data uncertainties are, in general, more important than model uncertainties, δ can be estimated through statistics on the first eigenfrequency;
(4) when the uncertain system pertains to a class of systems for which δ has already been studied, the same value can be re-used.

This estimation will not be considered here, and will be left for future study. In the numerical application that will be considered at the end of this paper, we will consider the fourth possibility and choose the value $\delta = 0.1$ for all the matrices.

4.3 The probability density function of a random matrix $[\mathbf{G_n}]$ in SG^+

The probability density function of a matrix in SG^+ is then computed using the entropy maximization principle with the constraints defined in Sec. 4.1. It is defined with respect to the measure $\widetilde{d}G_n$ on the set $\mathbb{M}_n^S(\mathbb{R})$ of $n \times n$ real symmetric matrices, where $\widetilde{d}G_n$ is such that

$$\widetilde{d}G_n = 2^{n(n-1)/4} \prod_{1 \leq i \leq j \leq n} d[G_n]_{ij} \tag{20}$$

with $dG_n = \prod_{1\leq i,j\leq n} d[G_n]_{ij}$ the Lebesgue measure on $\mathbb{R}^n$. With the usual normalization condition on the probability density function, it is

$$p_{[\mathbf{G_n}]}([G_n]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G_n]) \times C_{[\mathbf{G_n}]} \times (\det[G_n])^{(n+1)(1-\delta^2)/(2\delta^2)} \times \exp\left\{-\frac{n+1}{2\delta^2}\,\mathrm{tr}[G_n]\right\}, \quad (21)$$

in which $[G_n] \mapsto \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G_n])$ is a function from $\mathbb{M}_n(\mathbb{R})$ into $\{0,1\}$ that is equal to 1 when $[G_n]$ is in $\mathbb{M}_n^+(\mathbb{R})$ and 0 otherwise, and where constant $C_{[\mathbf{G_n}]}$ is equal to

$$C_{[\mathbf{G_n}]} = \frac{(2\pi)^{-n(n-1)/4}\left(\frac{n+1}{2\delta^2}\right)^{n(n+1)/(2\delta^2)}}{\prod_{j=1}^{n}\Gamma\left(\frac{n+1}{2\delta^2}+\frac{1-j}{2}\right)}, \quad (22)$$

where $z \mapsto \Gamma(z)$ is the gamma function defined for $z > 0$ by $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$.

4.4 Monte-Carlo simulation of a random matrix $[\mathbf{G_n}]$ in SG^+

Having values in $\mathbb{M}_n^+(\mathbb{R})$, $[\mathbf{G_n}]$ can be written $[\mathbf{G_n}] = [\mathbf{L_n}]^T[\mathbf{L_n}]$, where $[\mathbf{L_n}]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$. Let us introduce $\sigma_n = \delta(n+1)^{-1/2}$. It can been shown that

(1) random variables $([\mathbf{L_n}]_{ij})_{1\leq i\leq j\leq n}$ are independent;
(2) for $i < j$, $[\mathbf{L_n}]_{ij}$ can be written $[\mathbf{L_n}]_{ij} = \sigma_n\mathbf{U}_{ij}$, where $\mathbf{U}_{ij}$ is a Gaussian random variable with real values, zero mean and unit variance.
(3) for $i = j$, $[\mathbf{L_n}]_{ii}$ can be written $[\mathbf{L_n}]_{ii} = \sigma_n\sqrt{2\mathbf{V}_i}$, where $\mathbf{V}_i$ is a gamma random variable with positive real values and a probability density function $p_{\mathbf{V}_i}(v)$ (with respect to dv) in the form

$$p_{\mathbf{V}_i}(v) = \mathbb{1}_{\mathbb{R}^+}(v)\frac{1}{\Gamma(\frac{n+1}{2\delta^2}+\frac{1-i}{2})}v^{\frac{n+1}{2\delta^2}-\frac{1+i}{2}}e^{-v} \quad (23)$$

This algebraic structure of $[\mathbf{G_n}]$, allows an efficient procedure to be defined for the Monte Carlo numerical simulation of random matrix $[\mathbf{G_n}]$.

4.5 The positive-definite ensemble of random matrices

The normalized positive-definite ensemble of random matrices, denoted SE^+, was developed simultaneously with SG^+, the ensemble of normalized positive-definite ensemble of random matrices. The matrices $[\mathbf{A_n}]$ in SE^+ verify properties similar to those in SG^+, with any given matrix for the mean:

(1) $[\mathbf{A_n}] \in \mathbb{M}_n^+(\mathbb{R})$, almost surely;

(2) Matrix $[\mathbf{A_n}]$ is a second-order random variable: $E\{\|[\mathbf{A_n}]\|_F^2\} < +\infty$;
(3) The mean value $[\underline{A}_n]$ of $[\mathbf{A_n}]$ is a given matrix in $\mathbb{M}_n^+(\mathbb{R})$: $E\{[\mathbf{A_n}]\} = [\underline{A}_n]$;
(4) $[\mathbf{A_n}]$ is such that $E\{\ln(\det[\mathbf{A_n}])\} = \nu_{A_n}$, $|\nu_{A_n}| < +\infty$.

Since $[\underline{A}_n]$ is positive definite, there is an upper triangular matrix $[\underline{L}_n]$ in $\mathbb{M}_n(\mathbb{R})$ such that

$$[\underline{A}_n] = [\underline{L}_n]^T[\underline{L}_n], \tag{24}$$

and the ensemble SE^+ can be defined as the set of random matrices $[\mathbf{A_n}]$ which are written as

$$[\mathbf{A_n}] = [\underline{L}_n]^T[\mathbf{G_n}][\underline{L}_n], \tag{25}$$

in which $[\mathbf{G_n}]$ is in SG^+.

4.6 Probability model of a set of random matrices in SE^+

Let us consider a set of m random matrices $[\mathbf{A_n^1}], .., [\mathbf{A_n^m}]$ in SE^+, for which the mean values are given, but no information is available concerning the correlation tensor between any two of the random matrices. Applying the maximum entropy principle, it can be proved that the probability density function $([A_n^1], .., [A_n^m]) \mapsto p_{[\mathbf{A_n^1}],..,[\mathbf{A_n^m}]}([A_n^1], .., [A_n^m])$ from $(\mathbb{M}_n^+(\mathbb{R}))^m$ into $\mathbb{R}_+$ with respect to the measure $\widetilde{d}A_n^1 \times ... \times \widetilde{d}A_n^m$ on $(\mathbb{M}_n^S(\mathbb{R}))^m$ is written as

$$p_{[\mathbf{A_n^1}],..,[\mathbf{A_n^m}]}([A_n^1], .., [A_n^m]) = p_{[\mathbf{A_n^1}]}([A_n^1]) \times ... \times p_{[\mathbf{A_n^m}]}([A_n^m]), \tag{26}$$

which means that the $[\mathbf{A_n^1}], .., [\mathbf{A_n^m}]$ are independent random matrices.

4.7 Nonparametric model of random uncertainties for the impedance matrix

The principle of construction of the nonparametric probabilistic model of random uncertainties for the impedance matrix consists in replacing the matrices of mass, damping and stiffness of the mean model defined in Sec. 3.6.3 by random matrices $[\mathbf{M}]$, $[\mathbf{D}]$, $[\mathbf{K}]$ in SE^+. The mean value and a dispersion parameter have to be known for each of them. Using Monte Carlo techniques, independent samples (Sec. 4.6) of these matrices can be drawn and the corresponding dynamic stiffness matrix computed for all frequencies. The realizations of the impedance matrix $\{[Z]\}$ corresponding to these triplets of realizations of $\{([M], [D], [K])\}$ are then computed by condensation, for all frequencies, of the dynamic stiffness matrix on the DOFs of the boundary, using Eq. (13).

5 Methodology for the construction of the probabilistic model of an impedance matrix

In the previous sections, the theoretical material needed for the construction of a probabilistic model of the impedance matrix was presented. In this section, the practical methodology for that construction is introduced. Three main steps are identified:

(1) The impedance of the mean model is computed. The mean model is the usual model which is used for deterministic mechanics studies. Therefore, among the very numerous models that exist in the literature (some of them were briefly recalled in the introduction of this paper), any can be chosen, usually depending on the specific field of study. For the computation of the soil impedance matrix in soil-structure interaction problems, for example, a BE model is often chosen, or sometimes a mixed BE-FE model, as will be the case in the application that is presented at the end of this paper. More specifically, a set of values $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\leq\ell\leq L}$ of the mean impedance matrix at a finite number of frequencies $(\omega_\ell)_{1\leq\ell\leq L}$ is computed.

(2) The set of values $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\leq\ell\leq L}$ is interpolated to yield a matrix-valued rational function in the form $\omega \mapsto [\underline{N}(\omega)]/\underline{q}(\omega)$, which approximates the behavior of the impedance matrix $[\underline{\widetilde{Z}}(\omega)]$ of the mean model. The function $\omega \mapsto [\underline{N}(\omega)]$ is a matrix-valued polynomial in $(i\omega)$, and the function $\omega \mapsto \underline{q}(\omega)$ is a scalar polynomial in $(i\omega)$. This interpolation problem is very general and has been treated extensively in the literature [34–36]. Many methods can be applied, of which one was chosen and will be described in this section.

(3) The identification of the matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$ from the polynomials $\omega \mapsto [\underline{N}(\omega)]$ and $\omega \mapsto \underline{q}(\omega)$ is then presented. This step does not involve any approximation, but it will be shown that multiple solutions can arise, which means that several sets of matrices $\{[\underline{M}], [\underline{D}], [\underline{K}]\}$ may correspond to the same rational function $\omega \mapsto [\underline{N}(\omega)]/\underline{q}(\omega)$.

(4) Finally, once the matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$ have been computed, and supposing the values of the dispersion parameters given (see Sec. 4.2), the results of the previous section can be used to compute the probabilistic model of the random matrices $[\mathbf{M}]$, $[\mathbf{D}]$ and $[\mathbf{K}]$, and, by condensation, that of the random impedance matrix $[\mathbf{Z}(\omega)]$.

The first step consists in the usual computation of the impedance matrix of a deterministic dynamical system, and will not be described any further. The user of the methodology described herein is supposed to have computed the set of values $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\leq\ell\leq L}$ of the mean impedance matrix using methods appropriate for his specific mechanical problem. The fourth step was extensively described in the previous section, and will neither be further commented. The remainder of this methodological section will therefore concentrate on the second and third steps: the computation of the mean matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$ from the knowledge of the set of values $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\leq\ell\leq L}$.

5.1 *Interpolation of the set of values* $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\leq\ell\leq L}$

Considering the hidden state variables model that was chosen in Sec. 3.6.3, its block-decomposition, and the corresponding projection on the n_Σ DOFs of the boundary, we have, for any frequency ω_ℓ in $\mathbb{R}$,

$$[\underline{Z}(\omega_\ell)] = [\underline{S}_\Sigma(\omega_\ell)] - [\underline{S}_c(\omega_\ell)][\underline{S}_H(\omega_\ell)]^{-1}[\underline{S}_c(\omega_\ell)]^T = \frac{[\underline{N}(\omega_\ell)]}{\underline{q}(\omega_\ell)}, \tag{27}$$

where $\omega \mapsto [\underline{N}(\omega)]$ is a polynomial in $(i\omega)$ with real matrix-valued coefficients, of even degree n_N, and $\omega \mapsto \underline{q}(\omega)$ is a polynomial in $(i\omega)$ with real scalar coefficients, of even degree n_q. We have $n_N = n_q + 2$. We suppose here that we have a set of values of the impedance matrix $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\leq\ell\leq L}$, evaluated at a finite set of frequencies, and consider the interpolation of that data set to yield the polynomials $[\underline{N}]$ and $\underline{q}$.

5.1.1 *Choice of the number of hidden variables*

The method is presented assuming that the degrees of $[\underline{N}]$ and $\underline{q}$ are known. In the course of an interpolation process, iterations are performed on the number n_H of hidden variables in the model until a given level of accuracy is reached. n_N and n_q are chosen such that $n_N = 2n_H + 2$ and $n_q = 2n_H$.

5.1.2 *Choice of the cost function and of the resolution scheme*

This interpolation problem requires the choice of a cost function to be minimized, and of a minimization scheme. Virtually hundreds of these exist in the literature [34–36], even if we restrict ourselves to the multidimensional (MIMO) case. We chose here to set the problem in terms of a linear least square cost function, minimizing it by projection on an orthonormal basis of functions [37–39]. This should in no way be seen as a limitation for the method presented in this article: any cost function associated with any resolution scheme would be appropriate as long as the interpolation is performed in terms of a numerator and denominator functions with $n_N = n_q + 2$, and n_N even. It appears rather to be a nice asset of the method to allow for the choice of the cost function and of the resolution scheme to be performed depending on the type of mean impedance matrix that is available: computed by a given numerical method, measured experimentally, or even read from charts. For example, a given cost function might be more adapted for the extraction of a function measured in presence of noise and another one more appropriate for a noise-free function.

5.1.3 Weighted discrete linear least square rational approximation using orthogonal vectors of polynomials

The polynomials $[\underline{N}]$ and $\underline{q}$ will be chosen so as to minimize the cost function

$$\epsilon = \sum_{\ell=1}^{L} w(\omega_\ell)^2 \| \underline{N}(\omega_\ell) - \underline{q}(\omega_\ell)[\underline{\widetilde{Z}}(\omega_\ell)] \|_F^2, \tag{28}$$

where the weight function $\omega \mapsto w(\omega)$ can be used to control numerical problems that appear when one of the frequencies ω_ℓ is close to a pole of $\underline{q}$ [40]. The interpolation method that we consider here consists in expanding $\omega \mapsto [\underline{N}(\omega)]$ and $\omega \mapsto \underline{q}(\omega)$ on a basis of vectors of polynomials, orthonormal with respect to a discrete scalar product depending on the set of values $\{[\underline{\widetilde{Z}}(\omega_\ell)]\}_{1\le\ell\le L}$ of the target matrix function $\omega \mapsto [\underline{\widetilde{Z}}(\omega)]$. The minimization of the weighted least squares cost function in Eq. (28) will be seen to reduce to a simple algebraic relation. All computational activity will in fact go into the construction of the orthonormal basis of vectors of polynomials, which is a far easier problem in terms of condition number.

For a given m in $\mathbb{N}$, let $\mathbb{C}[\omega]^m$ be the set of all the m-vectors whose elements are constituted of polynomials of any degree with complex coefficients. Let Δ be a vector of $\mathbb{N}^m$ allowing the degree of each of the elements of the vector to be specified, and let $\mathbb{C}[\omega]^m_\Delta \subset \mathbb{C}[\omega]^m$ be the set of all m-vectors of polynomials with complex coefficients and degrees $\Delta(1), ..., \Delta(m)$. For example, the vector

$$[P(\omega)] = \begin{bmatrix} (i\omega)^2 - 1 \\ (i\omega) + 2 \\ (i\omega)^7 \end{bmatrix} \tag{29}$$

is in $\mathbb{C}[\omega]^3_{(2,1,7)}$. For the remainder of the section, we will work with vectors of polynomials of length $m_1 = n_\Sigma^2 + 1$, and we choose Δ in $\mathbb{N}^{m_1}$ such that $\Delta = (n_N, n_N, ..., n_N, n_q)$. We now introduce an operator allowing a square matrix to be transformed into a vector, the operator $[A] \mapsto \text{vec}([A])$ from $\mathbb{M}_{n_\Sigma}(\mathbb{C})$ into $\mathbb{C}^{n_\Sigma^2}$ which stacks the columns of matrix $[A]$ on top of each other. The vector of polynomials P is then defined in $\mathbb{C}[\omega]^{m_1}_\Delta$ by

$$[P(\omega)] = \begin{bmatrix} \text{vec}([\underline{N}(\omega)]) \\ \underline{q}(\omega) \end{bmatrix} = \sum_{k=1}^{m_2} \alpha_k [e_k(\omega)], \tag{30}$$

where $m_2 = n_\Sigma^2(n_N+1)+n_q+1$, $([e_k(\omega)])_{1\leq k\leq m_2}$ is a basis of $\mathbb{C}[\omega]_\Delta^{m_1}$, that has to be chosen, and $(\alpha_k)_{1\leq k\leq m_2}$ the coordinates of P in that basis. Eq. (28) can then be rewritten

$$\epsilon = \sum_{\ell=1}^{L}[P(\omega_\ell)]^*[W(\omega_\ell)][P(\omega_\ell)] = \sum_{j=1}^{m_2}\sum_{k=1}^{m_2}\sum_{\ell=1}^{L}\alpha_j\alpha_k[e_j(\omega_\ell)]^*[W(\omega_\ell)][e_k(\omega_\ell)], \tag{31}$$

where the star denotes the conjugate transpose of a matrix, and the weight matrix $\omega\mapsto[W(\omega)]$, with values in $\mathbb{M}_{m_1}^{0+}(\mathbb{C})$, the set of $m_1\times m_1$ complex positive matrices, is defined, for $\omega\in\mathbb{R}$, by

$$[W(\omega)] = w^2(\omega)\begin{bmatrix}[I_{n_\Sigma^2}] & -\operatorname{vec}([\underline{\widetilde{Z}}(\omega)]) \\ -\operatorname{vec}([\underline{\widetilde{Z}}(\omega)])^* & \|[\underline{\widetilde{Z}}(\omega)])\|_F^2\end{bmatrix}. \tag{32}$$

The hermitian sesquilinear form

$$([P_1],[P_2])_W = \Re\left\{\sum_{\ell=1}^{L}[P_1(\omega_\ell)]^*[W(\omega_\ell)][P_2(\omega_\ell)]\right\} \tag{33}$$

will be called a "semi-inner product", in the sense that the corresponding linear form $\|[P_1]\|_W = ([P_1],[P_1])_W$ is a semi-norm on $\mathbb{C}[\omega]_\Delta^{m_1}$. An algebraic basis $([e_k(\omega)])_{1\leq k\leq m_2}$ of $\mathbb{C}[\omega]_\Delta^{m_1}$ can then be constructed for that semi-inner product using the Gram-Schmidt orthonormalization procedure [41], such that $([e_j(\omega)],[e_k(\omega)])_W = \delta_{jk}$. Using that basis for the expansion of $[P(\omega)]$ in Eq. (30), Eq. (31) yields

$$\epsilon = \sum_{j=1}^{m_2}\sum_{k=1}^{m_2}\sum_{\ell=1}^{L}\alpha_j\alpha_k([e_j(\omega_\ell)],[e_k(\omega_\ell)])_W = \sum_{k=1}^{m_2}\alpha_k^2. \tag{34}$$

The problem of interpolating an impedance matrix $\omega\mapsto[\underline{\widetilde{Z}}(\omega)]$ by a matrix-valued rational function $\omega\mapsto[\underline{N}(\omega)]/\underline{q}(\omega)$, which was originally presented as the minimization of the cost function ϵ in Eq. (28), was recast into a problem of minimizing independently each coordinate α_k of a vector of polynomials $[P(\omega)]$ of $\mathbb{C}[\omega]_\Delta^{m_1}$ in a basis orthonormal for a certain semi-inner product $(\cdot,\cdot)_W$. To avoid the trivial solution $[P(\omega)]=0$, $\underline{q}$ or $[\underline{N}]$ should be constrained, for example by imposing that the coefficient of highest degree of $\underline{q}$ be 1. The only computational costs therefore lie in the construction of the orthonormal basis $(e_k)_{1\leq k\leq m_2}$. The Gram-Schmidt orthonormalization procedure is an appropriate method for that purpose.

5.1.4 Optimization of the algorithm

This algorithm has reached a more mature level of development than what is presented here. All issues are not discussed as it is not the purpose of this paper, and the reader is refered to the literature [42,43] on the subject. Among the issues concerning the optimization of this method, the parallelization of the process is treated in [37] and the construction of the orthonormal basis using a recurrence relation between the elements of the basis is presented in [38], both allowing for greater accuracy and speed. Finally it should be observed that all the elements of the matrix-valued polynomial $\underline{N}$ have been chosen of equal degree when each could be chosen independently in Δ, also diminishing unnecessary computational costs.

5.2 *Identification of the matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$*

Once the coefficients of the polynomials $\omega \mapsto [\underline{N}(\omega)]$ and $\omega \mapsto \underline{q}(\omega)$ have been determined for a given number n_H of hidden variables, matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$ have to be constructed in $\mathbb{M}_n^+(\mathbb{R})$ such that Eq. (17) and (27) are verified, for all ω in $\mathbb{R}$. This identification problem is not uniquely defined. In a first section, an equivalence relation will be introduced, allowing for a simplification of the form of the matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$ that are sought. Then the actual identification is described.

5.2.1 *Non-unicity of the solution*

By definition, two triplets $\{[\underline{M}_1], [\underline{D}_1], [\underline{K}_1]\}$ and $\{[\underline{M}_2], [\underline{D}_2], [\underline{K}_2]\}$ in $(\mathbb{M}_n^+(\mathbb{R}))^3$ are said by definition to be equivalent if the corresponding impedance matrices $\omega \mapsto [Z_1(\omega)]$ and $\omega \mapsto [Z_2(\omega)]$ are equal for all frequencies $\omega \in \mathbb{R}$. This relation is denoted

$$\{[\underline{M}_1], [\underline{D}_1], [\underline{K}_1]\} \equiv \{[\underline{M}_2], [\underline{D}_2], [\underline{K}_2]\}. \tag{35}$$

Since there are many such triplets that correspond to the same impedance matrix, we can choose, in some limited manner, the form of the matrices of the triplet that we try to identify, without restricting the generality of the identification process. More specifically, it is shown in App. B that, rather than looking for the matrices $[\underline{M}]$, $[\underline{D}]$, and $[\underline{K}]$ in the entire $\mathbb{M}_n^+(\mathbb{R})$, we can seek them in subspaces where they have the following form:

$$[\underline{M}] = \begin{bmatrix} [\underline{M}_\Sigma] & [0_{n_\Sigma n_H}] \\ [0_{n_H n_\Sigma}] & [I_{n_\Sigma}] \end{bmatrix}, [\underline{D}] = \begin{bmatrix} [\underline{D}_\Sigma] & [\underline{D}_c] \\ [\underline{D}_c]^T & [d_H] \end{bmatrix}, [\underline{K}] = \begin{bmatrix} [\underline{K}_\Sigma] & [\underline{K}_c] \\ [\underline{K}_c]^T & [k_H] \end{bmatrix}, \tag{36}$$

where $[\underline{M}_\Sigma]$, $[\underline{D}_\Sigma]$ and $[\underline{K}_\Sigma]$ are in $\mathbb{M}_{n_\Sigma}^+(\mathbb{R})$, $[\underline{D}_c]$ and $[\underline{K}_c]$ are in $\mathbb{M}_{n_\Sigma n_H}(\mathbb{R})$, and $[d_H]$ and $[k_H]$ are diagonal matrices of $\mathbb{M}_{n_H}^+(\mathbb{R})$. In general, $[d_H]$ is not a diagonal matrix, but we will assume it, as is usually done in structural dynamics (see App. B for details). We introduce the parameters $(d_\ell)_{1\leq\ell\leq n_H}$ and $(k_\ell)_{1\leq\ell\leq n_H}$ such that $[d_H] = \mathrm{diag}\,(d_\ell)_{1\leq\ell\leq n_H}$ and $[k_H] = \mathrm{diag}\,(k_\ell)_{1\leq\ell\leq n_H}$.

5.2.2 Identification

On the one hand, with the triplet $\{[\underline{M}], [\underline{D}], [\underline{K}]\}$ in the form of Eq. (36), the impedance matrix is

$$[\underline{Z}(\omega)] = [\underline{S}_\Sigma(\omega)] - \sum_{\ell=1}^{n_H} \frac{(i\omega[\underline{D}_c]^\ell + [\underline{K}_c]^\ell)(i\omega[\underline{D}_c]^\ell + [\underline{K}_c]^\ell)^T}{-\omega^2 + i\omega d_\ell + k_\ell}, \tag{37}$$

where $[\underline{D}_c]^\ell$ and $[\underline{K}_c]^\ell$ represent the ℓ^{th} columns of $[\underline{D}_c]$ and $[K_c]$. Expanding this equation, we get

$$\begin{aligned}[\underline{Z}(\omega)] = -\omega^2[\underline{M}_\Sigma] + i\omega[\underline{D}_\Sigma] + \left([\underline{K}_\Sigma] - \sum_{\ell=1}^{n_H} [\underline{D}_c]^\ell[\underline{D}_c]^{\ell T}\right) \\ - \sum_{\ell=1}^{n_H} i\omega \frac{[\underline{D}_c]^\ell[\underline{K}_c]^{\ell T} + [\underline{K}_c]^\ell[\underline{D}_c]^{\ell T} - d_\ell[\underline{D}_c]^\ell[\underline{D}_c]^{\ell T}}{-\omega^2 + i\omega d_\ell + k_\ell} \\ - \sum_{\ell=1}^{n_H} \frac{[\underline{K}_c]^\ell[\underline{K}_c]^{\ell T} - k_\ell[\underline{D}_c]^\ell[\underline{D}_c]^{\ell T}}{-\omega^2 + i\omega d_\ell + k_\ell}\end{aligned} \tag{38}$$

On the other hand, assuming that there are no real poles nor repeated poles, the matrix-valued rational function computed in Sec. 5.1 can be expanded in an unique pole-residue expansion:

$$\frac{[\underline{N}(\omega)]}{\underline{q}(\omega)} = -\omega^2[R_{-2}] + i\omega[R_{-1}] + [R_0] + \sum_{\ell=1}^{2n_H} \frac{[R_\ell]}{i\omega - p_\ell}, \tag{39}$$

where the $([R_\ell])_{-2\leq\ell\leq0}$ are in $\mathbb{M}^S_{n_\Sigma}(\mathbb{R})$ (by construction), the $([R_\ell])_{1\leq\ell\leq 2n_H}$ are in $\mathbb{M}^S_{n_\Sigma}(\mathbb{C})$ and the poles $(p_\ell)_{1\leq\ell\leq 2n_H}$ are complex. Noting that each pair $(p_\ell, [R_\ell])$ is associated with its associated complex conjugate pair $(\overline{p_\ell}, \overline{[R_\ell]})$, we reorder and regroup the terms of Eq. (39) as

$$\frac{[\underline{N}(\omega)]}{\underline{q}(\omega)} = -\omega^2[R_{-2}] + i\omega[R_{-1}] + [R_0] + \sum_{\ell=1}^{n_H} \frac{2i\omega\Re\{[R_\ell]\} - 2\Re\{[R_\ell]\overline{p_\ell}\}}{-\omega^2 - 2i\omega\Re\{p_\ell\} + \|p_\ell\|^2}. \tag{40}$$

This pole-residue expansion is known to present computational issues, particularly when two poles of the expansion are close to each other. Special care should therefore be put in this operation. Comparing Eq. (38) and Eq. (40), that hold for all frequencies in $\mathbb{R}$, we get three sets of equations, to be solved one after the other:

(1) the first one is an uncoupled system of $2n_H + 2$ equations

$$\begin{cases} [\underline{M}_\Sigma] = [R_{-2}], \\ [\underline{D}_\Sigma] = [R_{-1}], \\ d_\ell = -2\Re\{p_\ell\}, \quad 1 \leq \ell \leq n_c, \\ k_\ell = \|p_\ell\|^2, \qquad 1 \leq \ell \leq n_c, \end{cases} \tag{41}$$

that gives directly the values of $[\underline{M}_\Sigma]$, $[\underline{D}_\Sigma]$ and d_ℓ and k_ℓ, for $1 \leq \ell \leq n_H$;

(2) the second one is a coupled system of $2n_H$ equations

$$\begin{cases} [\underline{K}_c]^\ell [\underline{K}_c]^{\ell T} - k_\ell [\underline{D}_c]^\ell [\underline{D}_c]^{\ell T} = 2\Re\{[R_\ell]\overline{p_\ell}\}, & 1 \leq \ell \leq n_c, \\ [\underline{D}_c]^\ell [\underline{K}_c]^{\ell T} + [\underline{K}_c]^\ell [\underline{D}_c]^{\ell T} - d_\ell [\underline{D}_c]^\ell [\underline{D}_c]^{\ell T} = -2\Re\{[R_\ell]\}, & 1 \leq \ell \leq n_c, \end{cases} \tag{42}$$

setting the values of the $[\underline{D}_c]^\ell$ and $[\underline{K}_c]^\ell$, given those of d_ℓ and k_ℓ, for $1 \leq \ell \leq n_H$;

(3) finally, the third one gives directly the value of $[\underline{K}_\Sigma]$ when the second system has been solved,

$$[\underline{K}_\Sigma] - \sum_{\ell=1}^{n_H} [\underline{D}_c]^\ell [\underline{D}_c]^{\ell T} = [R_0]. \tag{43}$$

.

The only system that has got to be explicited is the second one. It is in fact partially decoupled, as the equations are coupled only in pairs for each ℓ, for $1 \leq \ell \leq n_H$. The n_H systems of equations are all in the form

$$\begin{cases} [X][X]^T - \alpha\overline{\alpha}[Y][Y]^T = \overline{\alpha}[A] + \alpha\overline{A}, \\ [X][Y]^T + [Y][X]^T + (\alpha + \overline{\alpha})[Y][Y]^T = -(A + \overline{A}), \end{cases} \tag{44}$$

where the $[X]$ and $[Y]$ are the unknowns, in $\mathbb{M}_{n_\Sigma}(\mathbb{R})$, α is a given complex scalar, and $[A]$ is a given matrix in $\mathbb{M}^S_{n_\Sigma}(\mathbb{C})$. In App. C, a solution $([X],[Y])$ to the system of Eq. (44) is described.

This final step allows us to identify completely the matrices $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$ representing an approximate hidden state variables model corresponding to an impedance matrix function given at a discrete set of frequencies. The number of hidden variables is chosen in order to provide sufficient accuracy between the impedance matrix computed from the approximate hidden state variables model and the impedance matrix that was given as input.

6 Numerical Applications

In this section, two applications of the methods described in this paper are presented. The first one is a simple problem that illustrates the process of identification of the hidden variables model, as described in Sec. 5. The second one describes the entire process of constructing the probabilistic model of the impedance matrix of a circular rigid foundation on piles in a layered soil. subsectionIdentification of the hidden variables model This first example is a very simple one for which we choose a mechanical model in terms of mass, damping and stiffness matrices, compute the corresponding impedance matrix $[\underline{\widetilde{Z}}(\omega)]$, and try to recover the matrices $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$ of the corresponding mean hidden state variables model. These last matrices are then compared to the original ones. The probabilistic model of the random impedance matrix $[\mathbf{Z}(\omega)]$ is not constructed here as we just aim at showing the accuracy of the identification process of the mean hidden variables model, as described in Sec. 5.

Consider therefore a mechanical system with a boundary constrained to a single-DOF and 10 internal DOFs. The resonance frequencies of the internal DOFs are $f_0 = \{1, 2, 4, 4.5, 5, 7, 7, 10, 12, 15\}$Hz. We suppose that the resonance modes at 1Hz and 5Hz are not coupled to the DOF of the boundary and therefore should not appear in the impedance function. The real and imaginary parts of the impedance function are drawn (dashed line) on Fig. C.2.

[Fig. 2 about here.]

In order to illustrate the process of identification of the hidden state variables model, we perform it using two different impedance inputs: the first one is the impedance given on the frequency range $B_1 = [0, 10]$Hz, and the second one is the same impedance but given on the entire range $B_2 = [0, 17]$Hz. The graphs of the approximate impedance corresponding to each of the two hidden variables model, identified for the input ranges B_1 and B_2, are represented in Fig. C.2, respectively in solid and dash-dotted line. In the first case, only the first resonance frequencies are identified, while in the second there are all perfectly computed. In that last cases, all the dynamic characteristics of the original system are almost perfectly identified, with the exception of those corresponding to the resonance frequencies at 1Hz and 5Hz, which are completely unseen. This shows that the method is only able to identify dynamic characteristics that were present in the input impedance matrix. To identify the two frequencies that are lacking in our hidden variables model, we would need to add more information in the input impedance matrix, for example, the impedance function for another DOF, that these resonance frequencies actually impact.

The CPU time necessary to compute the hidden variables model is completely negligible (15s and 60s with a 1Ghz-processor, for the first and second cases respectively). This time increases with the number of frequency points (1000 and 1700 here), the number of hidden variables (5 and 7 here), and the number of elements in the impedance matrix (1 in both cases here).

6.1 Impedance of a FE-BE model

We now consider the construction of the probabilistic model of the random impedance matrix $[\mathbf{Z}(\omega)]$ of a rigid circular foundation on piles in a layered soil (Fig. C.3).

[Fig. 3 about here.]

The first step in that construction consists in computing a mean impedance matrix $[\underline{\widetilde{Z}}(\omega)]$ by classical means. Here, the piles and a bounded volume of soil around the piles are modeled using a FE method, and the rest of the soil is modeled using a BE method. The stiffness matrix of the bounded FE model is computed using 6 rigid body modes on the foundation interface, 2000 DOFs for the interface with the external soil, and 200 internal modes (highest resonance frequency at 19.2Hz). That stiffness matrix is coupled to the impedance matrix of the external soil computed using the BE method, and the impedance matrix of the entire soil-pile system with respect to the foundation interface is computed by condensation of that coupled BE-FE stiffness matrix on the 6 rigid body modes of the foundation. The FE model is a 19 m high, 40 m radius cylinder, with volumic elements for the soil and beam elements for the piles. The soil is composed of one layer for the first 15 m (Young's modulus $E = 150 \times 10^6$ N/m^2, Poisson coefficient $\nu = 0.48$, unit mass $\rho = 1900$ kg/m^3 and viscoelastic damping $\zeta = 2 \times 10^{-5}$) and another one for the last 4 m ($E = 2500 \times 10^6$ N/m^2, $\nu = .44$, $\rho = 2100$ kg/m^3, $\zeta = 2 \times 10^{-5}$). The damping matrix is taken proportional to the stiffness matrix ($C = 2\omega_0 \zeta K$, with the first eigenfrequency $\omega_0 = 38.7$ rad/s), and hysteretic damping is also considered in the soil ($\beta = 0.1$ in the first layer and $\beta = 0.06$ in the second).

Considering the symmetry of the problem, only eight terms of the impedance matrix $[\underline{\widetilde{Z}}(\omega)]$ have to be computed. They are the shaking terms in both horizontal directions, the pumping in the vertical direction, the rocking terms around the two horizontal directions, the torsion around the vertical and the coupling terms between the shaking in one horizontal direction and the rocking in the other one. Also the two shaking, rocking and coupling terms are the same. For computational efficiency, the identification is performed independently on the torsion and the pumping and jointly on the shaking, rocking and coupling.

In Fig. C.4-C.9, are represented the real and imaginary parts of these five terms of the impedance matrix, as well as those of the shaking-pumping coupling term.

[Fig. 4 about here.]

[Fig. 5 about here.]

[Fig. 6 about here.]

[Fig. 7 about here.]

[Fig. 8 about here.]

[Fig. 9 about here.]

The impedance computed from the FE-BE model $[\underline{\widetilde{Z}}(\omega)]$ is drawn in solid line, that of the approximate corresponding hidden variables model $[\underline{Z}(\omega)]$ in dashed line. The approximation is very accurate. 4 hidden variables only were introduced in the hidden variables model (1 for the pumping element, 1 for the torsion, and 2 for the shaking-rocking submatrix). As before, the computational time required to compute the hidden variables model is very small, and negligible when compared to the time necessary to compute the mean model. It is interesting to note that the hysteretic damping, which corresponds to a non-causal behavior of the model, induced a non-zero value of the imaginary part of the diagonal terms of the impedance. In the hidden variables model, this does not appear since the model imposed the causality (compare, for example, the solid line and the dotted line in Fig. C.7).

With this hidden variable model, the probabilistic model of the impedance matrix $[\mathbf{Z}(\omega)]$ can be constructed, and realizations drawn. In Fig. C.4-C.9, the 0.95-confidence bounds obtained from the computation of 1000 Monte Carlo trials are painted in grey, and one particular realization is drawn in dotted line. Although not drawn here, the convergence of the first two moments is verified for 1000 trials.

It should be noted that although we started from a mean model where symmetry of the problem induced a relative independence of the terms of the impedance matrix, this no longer holds for the realizations of the random matrix. Therefore, in Fig. C.9, we see that in the mean model, the shaking-pumping term is evaluated at zero for all frequencies, but that it can reach important values when uncertainty is considered.

7 Conclusions

In this paper, a method is presented that allows for a nonparametric probabilistic model of an impedance matrix to be constructed, from the knowledge only of the values of a mean impedance matrix at a discrete set of frequencies and of a small set of dispersion parameters. This mean impedance matrix can be computed by any available computational method. The methodology presented here does not rely on the knowledge of a previously built FE model of the domain Ω, and therefore is appropriate even for impedances of unbounded domains. A hidden state variables model is extracted from the values of the mean impedance matrix, and the probabilistic model is constructed around the random mass, damping and stiffness matrices, with given means.

An interesting direction for future work is the extension of the ideas presented in this paper to the construction of nonparametric probabilistic models of impedances measured experimentally. The main difference with what was presented here lies in the choice of the cost function (and consequently of the resolution scheme) for the interpolation problem. Indeed, noise has to be taken into account and the weighted least squares cost function is not necessarily the best fitted for that matter.

Deeply connected with this first question, the identification of the set of dispersion parameters has been left aside in this paper, and also represents an interesting challenge. Some work has already been done in that field, and the methodologies developed in these papers should be applied to the case of impedance matrices. In [16], the coupling of two substructures with different levels of uncertainty is considered. It is of a primary interest to consider this as, e.g. between a soil and a structure, the expected levels of uncertainty are very different. Previous work on uncertain soil-structure interaction problems [44,11], where the uncertainty was considered only on bounded regions of the entire domain, could then be re-examined.

Finally, the construction of a probabilistic model from charted normalized impedance functions [45] would also be interesting as it would open ways for the comparison of the levels of uncertainty between impedances of different domains.

Acknowledgements

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A Causality of the impedance matrix in the frequency domain

In order to show that $[Z(\omega)]$, defined by Eq. (13) is causal, with no explicit reference to the time domain, it is recalled [22] that a function $p \mapsto \hat{f}(p)$ with values in a Hilbert space $\mathcal{H}$ is the Laplace transform of a causal function f if and only if there exists a $\xi_0 \in \mathbb{R}$ such that $\hat{f}$ is holomorphic in $\mathbb{C}_{\xi_0} = \{p = \xi + i\eta, \xi \in]\xi_0, +\infty[, \eta \in \mathbb{R}\}$ and there exists a polynomial of $|p|$, $\mathrm{Pol}(|p|)$, such that $\|f(p)\|_{\mathcal{H}} \leq \mathrm{Pol}(|p|)$, where $\|\cdot\|_{\mathcal{H}}$ is the norm associated with $\mathcal{H}$, and the coefficients of Pol may depend on ξ_0. The definitions of $[S(\omega)]$ and $[Z(\omega)]$ in the frequency domain are extended to the domain of the complex frequencies with $\omega = -ip$, $p \in \mathbb{C}$, which leads to the classical definition of the Laplace transforms of the dynamic stiffness matrix $[\hat{S}(p)]$ and the impedance matrix $[\hat{Z}(p)]$. Being a polynomial in p, $p \mapsto [\hat{S}(p)]$ is holomorphic on the entire complex plane, and $p \mapsto \|[\hat{S}(p)]\|_F$ is a polynomial in p, ensuring that $[\hat{S}(p)]$ is the Laplace transform of a causal function in the time domain. The positive definiteness of $[M]$, $[D]$ and $[K]$ ensures that of $[M_H]$, $[D_H]$ and $[K_H]$, and therefore $p \mapsto [\hat{S}_H(p)]$ is invertible on $\mathbb{C}_{\xi_0}$. For $\xi_0 > 0$, $p \mapsto [\hat{Z}(p)]$ is holomorphic in $\mathbb{C}_{\xi_0}$, and the inverse of the determinant of $[\hat{S}_H(p)]$ can be bounded so that $[\hat{Z}(p)]$ corresponds in the time domain to a causal function.

B Choice of the form of the matrices $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$

We first try to show in this appendix, that, given

(1) any triplet $\{[\underline{M}], [\underline{D}], [\underline{K}]\}$ of $(\mathbb{M}_n^+(\mathbb{R}))^3$, such that, for all ω in $\mathbb{R}$, the matrix $-\omega^2[\underline{M}_H] + i\omega[\underline{D}_H] + [\underline{K}_H]$ is invertible, where the matrices $[\underline{M}_H]$, $[\underline{D}_H]$ and $[\underline{K}_H]$ are the "hidden parts" of the matrices $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$, defined by Eq. (12),
(2) any matrices $[F]$ and $[G]$, respectively in $\mathbb{M}_{n_\Sigma n_H}(\mathbb{R})$ and $\mathbb{M}_{n_H}^+(\mathbb{R})$,
(3) and $[U]$ in $\mathbb{M}_n(\mathbb{R})$, such that

$$[U] = \begin{bmatrix} [I_{n_\Sigma}] & [F] \\ [0_{n_H n_\Sigma}] & [G] \end{bmatrix}, \tag{B.1}$$

then

$$\{[\underline{M}], [\underline{D}], [\underline{K}]\} \equiv \{[U][\underline{M}][U]^T, [U][\underline{D}][U]^T, [U][\underline{K}][U]^T\}, \tag{B.2}$$

where the equivalence sign $\equiv$ is defined in Sec. 5.2.1.

Let us therefore start from a triplet $\{[\underline{M}], [\underline{D}], [\underline{K}]\}$ such that, for all ω in $\mathbb{R}$, the matrix $-\omega^2[\underline{M}_H] + i\omega[\underline{D}_H] + [\underline{K}_H]$ is invertible, and consider the dynamic stiffness matrix $\omega \mapsto [\underline{S}(\omega)]$ corresponding to that triplet.

$$[\underline{S}(\omega)] = -\omega^2[\underline{M}] + i\omega[\underline{D}] + [\underline{K}] = \begin{bmatrix} [\underline{S}_\Sigma(\omega)] & [\underline{S}_c(\omega)] \\ [\underline{S}_c(\omega)]^T & [\underline{S}_H(\omega)] \end{bmatrix}. \tag{B.3}$$

The corresponding impedance matrix with respect to the first n_Σ DOFs is denoted $[\underline{Z}(\omega)]$ and is given by Eq. (27). We denote $[\underline{S}'(\omega)]$ and $[\underline{Z}'(\omega)]$ the dynamic stiffness matrix and the impedance matrix corresponding to the triplet $\{[U][\underline{M}][U]^T, [U][\underline{D}][U]^T, [U][\underline{K}][U]^T\}$. Dropping the dependency in ω for readability, we have

$$\begin{aligned}[\underline{S}'(\omega)] &= [U][\underline{S}(\omega)][U]^T \\ &= \begin{bmatrix} [\underline{S}_\Sigma(\omega)] + [\underline{S}_c(\omega)][F]^T + [F][\underline{S}_c(\omega)]^T + [F][\underline{S}_H(\omega)][F]^T & [\underline{S}_c(\omega)][G]^T + [F][\underline{S}_H(\omega)][G]^T \\ [G][\underline{S}_c(\omega)]^T + [G][\underline{S}_H(\omega)][F]^T & [G][\underline{S}_H(\omega)][G]^T \end{bmatrix}.\end{aligned} \tag{B.4}$$

Since $[\underline{S}_H(\omega)]$ and $[G]$ are invertible, $([G][\underline{S}_H(\omega)][G]^T)^{-1}$ exists and is equal to $[G]^{-T}[\underline{S}_H(\omega)]^{-1}[G]^{-1}$. The corresponding impedance is then

$$[\underline{Z}'(\omega)] = [\underline{S}_\Sigma(\omega)] - [\underline{S}_c(\omega)][\underline{S}_H(\omega)]^{-1}[\underline{S}_c(\omega)]^T = [\underline{Z}(\omega)], \tag{B.5}$$

and the result proposed in Eq. (B.1-B.2) is proved.

The choice of $[F]$ and $[G]$ is not important here if the only information available is that of the impedance ($[G]$ must be invertible). Let us therefore introduce $[\Phi]$ the (invertible) matrix whose columns are the eigenvectors ϕ solution of the generalized eigenvalues problem

$$[\underline{K}_H]\phi = \lambda[\underline{M}_H]\phi, \tag{B.6}$$

normalized with respect to the mass matrix $[\underline{M}_H]$, and choose $[G] = [\Phi]$. The matrices $[\Phi][\underline{M}_H][\Phi]^T$ and $[\Phi][\underline{K}_H][\Phi]^T$ in $\mathbb{M}^+_{n_H}(\mathbb{R})$ are then diagonal, and we will suppose, as is usually done in structural dynamics, that $[\underline{D}_H]$ is also diagonalized by $[\Phi]$. With such a choice of $[G]$, we get that the "hidden" parts of matrices $[U][\underline{M}][U]^T$, $[U][\underline{D}][U]^T$ and $[U][\underline{K}][U]^T$ are diagonal. Let us then choose $[F] = -[\underline{M}_c][\underline{M}_H]^{-1}$. This ensures that the "coupling" part of $[U][\underline{M}][U]^T$ is null.

We have therefore shown that, for any triplet of matrices, there exists an equivalent one such that

$$[\underline{M}] = \begin{bmatrix} [\underline{M}_\Sigma] & [0_{n_\Sigma n_H}] \\ [0_{n_H n_\Sigma}] & [I_{n_\Sigma}] \end{bmatrix}, [\underline{D}] = \begin{bmatrix} [\underline{D}_\Sigma] & [\underline{D}_c] \\ [\underline{D}_c]^T & [d_H] \end{bmatrix}, [\underline{K}] = \begin{bmatrix} [\underline{K}_\Sigma] & [\underline{K}_c] \\ [\underline{K}_c]^T & [k_H] \end{bmatrix}, \tag{B.7}$$

where $[\underline{M}_\Sigma]$, $[\underline{D}_\Sigma]$ and $[\underline{K}_\Sigma]$ are in $\mathbb{M}^+_{n_\Sigma}(\mathbb{R})$, $[\underline{D}_c]$ and $[\underline{K}_c]$ are in $\mathbb{M}_{n_\Sigma n_H}(\mathbb{R})$, and $[d_H]$ and $[k_H]$ are diagonal matrices of $\mathbb{M}^+_{n_H}(\mathbb{R})$.

C Solution of the coupled system in $[\underline{D}_c]^\ell$ and $[\underline{K}_c]^\ell$

We concentrate in this appendix on the resolution of a coupled system of matrix equations in the following form

$$\begin{cases} [X][X]^T - \alpha\overline{\alpha}[Y][Y]^T = \overline{\alpha}[A] + \alpha[\overline{A}] \\ [X][Y]^T + [Y][X]^T + (\alpha+\overline{\alpha})[Y][Y]^T = -([A]+[\overline{A}]) \end{cases} \tag{C.1}$$

where the $[X]$ and $[Y]$ are the unknowns, in $\mathbb{M}_{n_\Sigma}(\mathbb{R})$, α is a given complex scalar (with nonvanishing imaginary part), and $[A]$ is a given matrix in $\mathbb{M}^S_{n_\Sigma}(\mathbb{C})$.

For any square complex symmetric (non-hermitian) matrix $[U]$, the Takagi factorization [46] is a special case of the singular value decomposition, stating that there always exists a unitary matrix $[Q]$ such that

$$[U] = [Q][\Sigma][Q]^T, \tag{C.2}$$

where $[\Sigma]$ is the diagonal matrix of the singular values of $[U]$. Denoting $[\sqrt{\Sigma}]$ the diagonal matrix of the square roots of these singular values, and $[L_u] = [Q][\sqrt{\Sigma}]$, we have that

$$[L_u][L_u]^T = [U]. \tag{C.3}$$

Going back to the problem of Eq. (C.1), and since the decomposition of Eq. (C.3) is true for any matrix in $\mathbb{M}^S_n(\mathbb{C})$, it is always possible to find $[L]$ in $\mathbb{M}_n(\mathbb{C})$ such that

$$[L][L]^T = 2i\frac{[A]}{\Im\{\alpha\}}. \tag{C.4}$$

It is then easy to check that the pair $([X],[Y])$ defined by

$$\begin{cases} [X] = \Im\{\alpha\}\Re\{[L]\} - \Re\{\alpha\}\Im\{[L]\} \\ [Y] = \Im\{[L]\} \end{cases}, \tag{C.5}$$

is a solution of the system of Eq. (C.1).

Indeed, we have

$$\begin{aligned}[X][X]^T = {} & \Im\{\alpha\}^2\Re\{[L]\}\Re\{[L]\}^T + \Re\{\alpha\}^2\Im\{[L]\}\Im\{[L]\}^T \\ & - \Re\{\alpha\}\Im\{\alpha\}(\Im\{[L]\}\Re\{[L]\}^T + \Re\{[L]\}\Im\{[L]\}^T),\end{aligned} \tag{C.6}$$

$$[Y][Y]^T = \Im\{[L]\}\Im\{[L]\}^T, \tag{C.7}$$

$$\begin{aligned}\overline{\alpha}[A] + \alpha[\overline{A}] = {} & \Im\{\alpha\}^2(\Re\{[L]\}\Re\{[L]\}^T - \Im\{[L]\}\Im\{[L]\}^T) \\ & - \Re\{\alpha\}\Im\{\alpha\}(\Im\{[L]\}\Re\{[L]\}^T + \Re\{[L]\}\Im\{[L]\}^T),\end{aligned} \tag{C.8}$$

$$\begin{aligned}[X][Y]^T + [Y][X]^T = {} & \Im\{[\alpha]\}(\Im\{[L]\}\Re\{[L]\}^T + \Re\{[L]\}\Im\{[L]\}^T) \\ & - 2\Re\{[\alpha]\}\Im\{[L]\}\Im\{[L]\}^T,\end{aligned} \tag{C.9}$$

and

$$-([A] + [\overline{A}]) = \Im\{[\alpha]\}(\Im\{[L]\}\Re\{[L]\}^T + \Re\{[L]\}\Im\{[L]\}^T). \tag{C.10}$$

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List of Figures

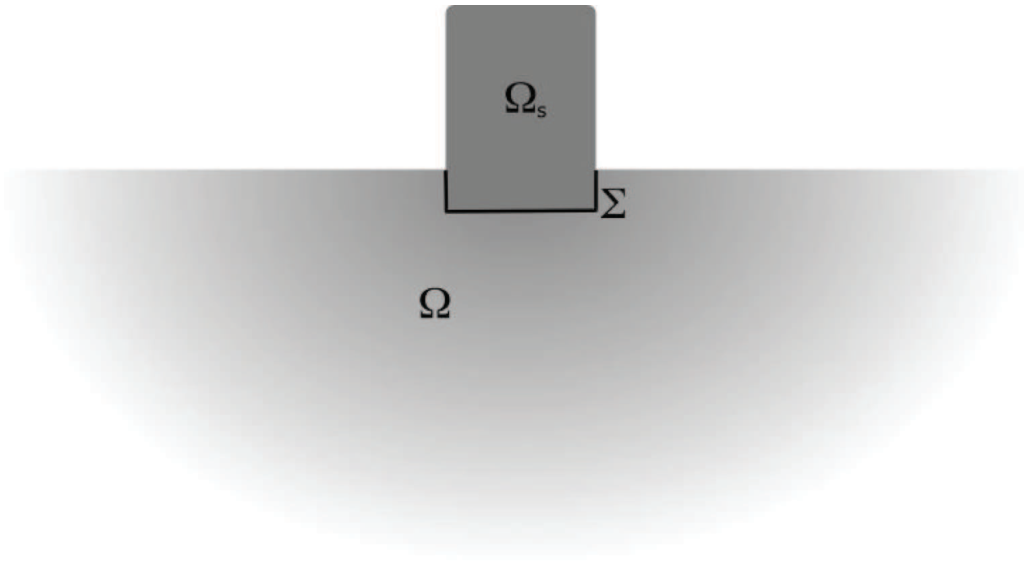
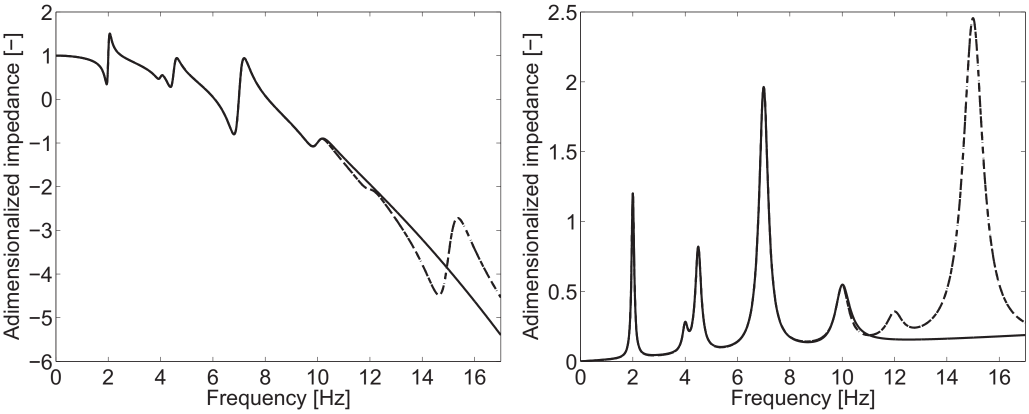


Fig. C.1. Setting of the problem : bounded structure Ω_s, unbounded domain Ω and coupling boundary Σ



(a) real part (b) imaginary part

Fig. C.2. Real and imaginary parts of the impedance (normalized by the static value) for the given input impedance (dash-dotted line), the impedance computed from the hidden variables model identified on B_1 (solid line) and the impedance computed from the hidden variables model identified on B_2 (dashed line, superimposed on the dash-dotted line).

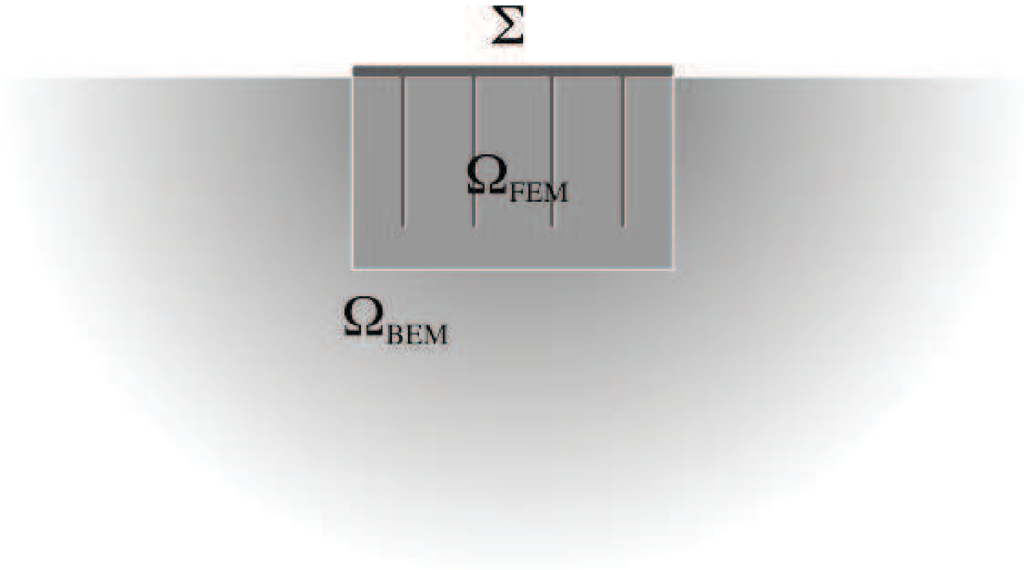
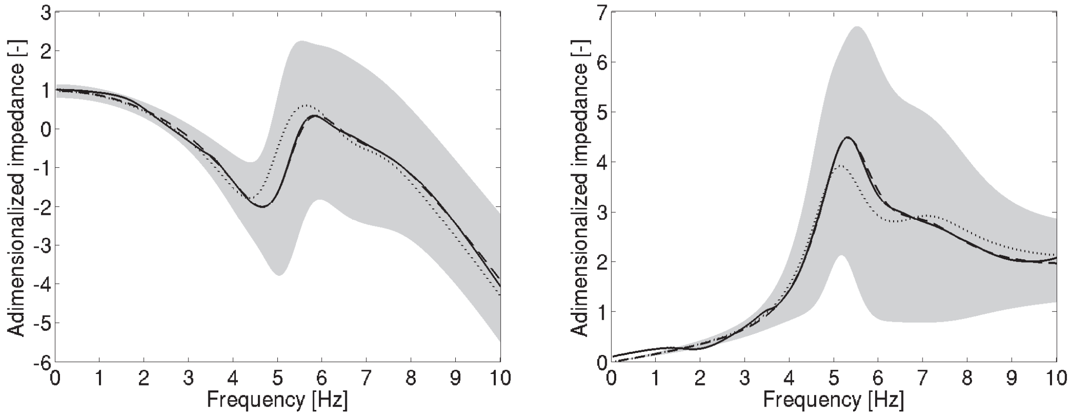


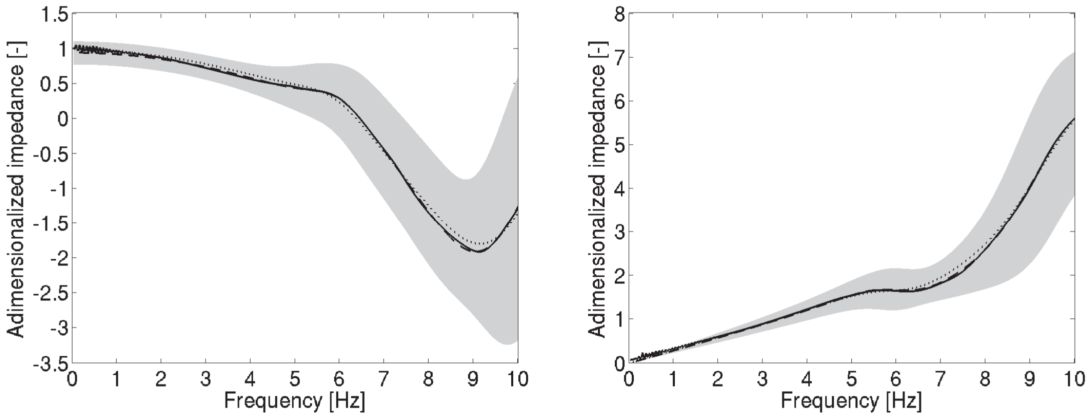
Fig. C.3. Model of the foundation on piles in a layered half space: the piles and a bounded volume of soil around the piles are modeled with FE, and the rest of the soil is modeled with BE. The impedance is considered with respect to displacement on Σ, on top of the foundation.



(a) real part

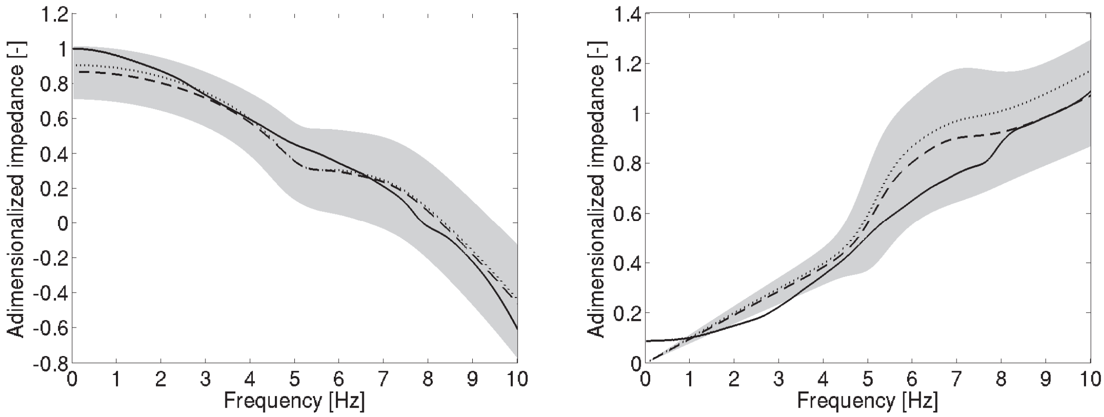
(b) imaginary part

Fig. C.4. Real and imaginary parts of the shaking element of the impedance matrix: condensation of the FEM-BEM model (solid line), condensation of the hidden variables model (dashed line), 0.95-confidence bounds for the 1000 Monte Carlo trials (grey patch), and value for one particular Monte Carlo trial (dotted line).



(a) real part (b) imaginary part

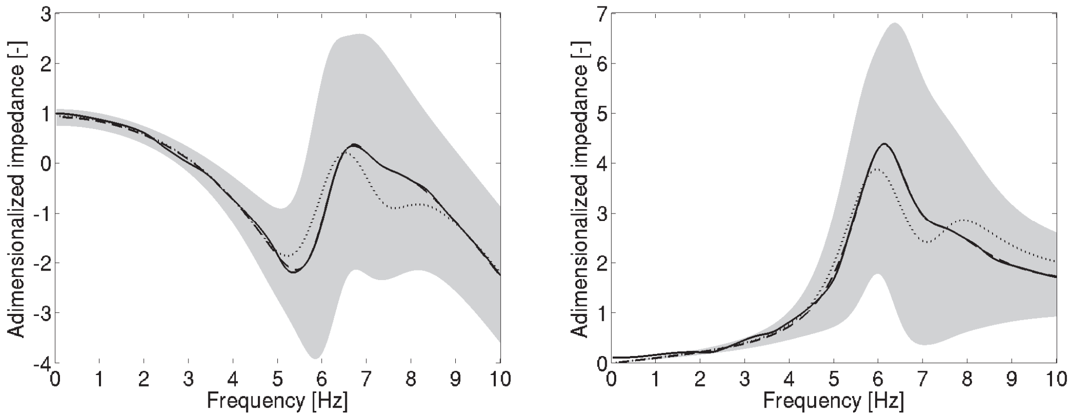
Fig. C.5. Real and imaginary parts of the pumping element of the impedance matrix: condensation of the FEM-BEM model (solid line), condensation of the hidden variables model (dashed line), 0.95-confidence bounds for the 1000 Monte Carlo trials (grey patch), and value for one particular Monte Carlo trial (dotted line).



(a) real part

(b) imaginary part

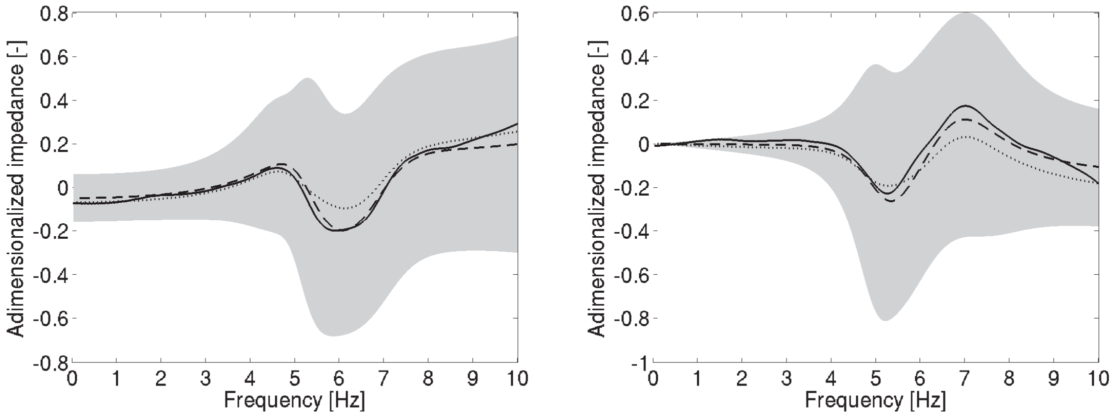
Fig. C.6. Real and imaginary parts of the rocking element of the impedance matrix: condensation of the FEM-BEM model (solid line), condensation of the hidden variables model (dashed line), 0.95-confidence bounds for the 1000 Monte Carlo trials (grey patch), and value for one particular Monte Carlo trial (dotted line).



(a) real part

(b) imaginary part

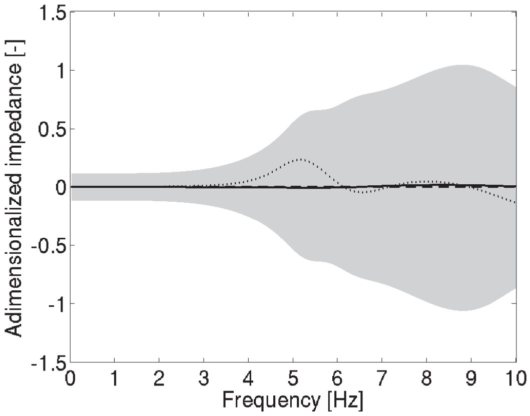
Fig. C.7. Real and imaginary parts of the torsion element of the impedance matrix: condensation of the FEM-BEM model (solid line), condensation of the hidden variables model (dashed line), 0.95-confidence bounds for the 1000 Monte Carlo trials (grey patch), and value for one particular Monte Carlo trial (dotted line).



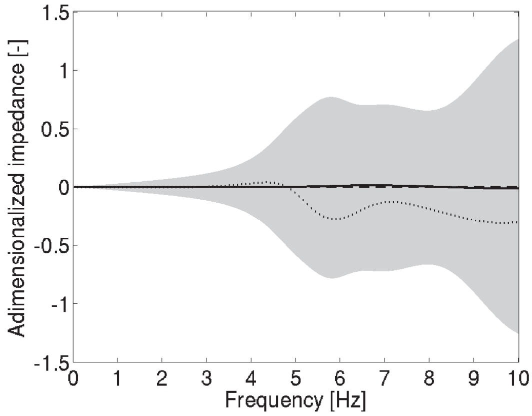
(a) real part

(b) imaginary part

Fig. C.8. Real and imaginary parts of the shaking-rocking coupling element of the impedance matrix: condensation of the FEM-BEM model (solid line), condensation of the hidden variables model (dashed line), 0.95-confidence bounds for the 1000 Monte Carlo trials (grey patch), and value for one particular Monte Carlo trial (dotted line).



(a) real part



(b) imaginary part

Fig. C.9. Real and imaginary parts of the shaking-pumping coupling element of the impedance matrix: condensation of the FEM-BEM model (solid line), condensation of the hidden variables model (dashed line), 0.95-confidence bounds for the 1000 Monte Carlo trials (grey patch), and value for one particular Monte Carlo trial (dotted line).