Experimental identification in the ultrasonic range of a mechanical model for cortical bones

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Abstract
This paper deals with the construction of a simplified elastoacoustic model which allows the ultrasonic wave propagation to be simulated in a complex biomechanical system. This simplified model consists in a fluid-solid multilayer system. In this simplified model, the main source of uncertainties is due to the constitutive equation for the solid layer which is chosen as a homogeneous transverse isotropic elastic medium. In order to improve this simplified model, a probabilistic model of the effective elasticity tensor of the solid medium is developped. A method is presented for the experimental identification in a statistical sense of the model parameters using the ultrasonic transmission technique. A complete application is presented for the human cortical bone for which an experimental database is available.

1 Introduction

Biomechanical systems are often very complex to be modeled in regard to the complexity level of their constitutive material at the microscopic scale. Very often, these biomechanical systems are modeled using a mechanical model which can be more or less sophisticated using or not a multiscale approach. Nevertheless, assumptions yielding modeling simplifications and approximations are introduced and therefore the complexity level of these biomechanical systems is almost always greater than the complexity level of the mechanical model. Since the model developed is always a rough approximation of the real biomechanical system, it is interesting to model uncertainties in order to extend the domain of validy of the simplified model. Nevertheless, it is important to give an experimental validation of such simplified models. Thus, the purpose of this paper is the experimental identification in a probabilistic sense of the mechanical properties of the cortical bone. The main objective is firstly to propose a simplified model adapted to this technique in order to perform such an identification and secondly to present an experimental validation of this simplified model. In this paper, the biomechanical system under consideration is the human cortical bone, with the skin and the marrow. This system is submitted to an acoustical impulse in the ultrasonic range and the velocity of the first arriving signal is experimentally measured. The simplified mechanical model presented in [1,2,3] is used to predict its transient response in the ultrasonic range. Thus, the simplified mechanical model is a fluid-solid semi-infinite multilayer system in which the solid layer (the cortical bone) is a homogeneous anisotropic elastic material and the two others semi-infinite layers are fluids (skin, muscle and marrow). The uncertainties introduced in the construction of this simplified model are taken into account by a parametric probabilistic approach for the elasticity tensor. The construction of this probabilistic model is carried out using the information theory with the available information on the mechanical and probabilistic properties of the random effective elasticity tensor. The identification of the parameters of this probabilistic model is carried out by solving an inverse stochastic problem related to the simplified model and using an experimental database obtained by in vivo ultrasonic axial transmission on cortical bones of a given set of patients. A computational optimization problem is then introduced, consisting in minimizing a cost function with respect to the stochastic simplified model parameters. This cost function is defined by taking into account the type of experimental observations obtained by using the axial transmission technique. The simplex algorithm is used to solve the optimization problem. At each iteration of the simplex algorithm, the value of this cost function is calculated using a stochastic solver based on the Monte Carlo method. Thus, for each realization of the random effective elastic tensor, it is necessary to predict a transient elastic wave response of a fluid-solid semi-infinite multilayer system submitted to an acoustical impulse. Nevertheless, the numerical cost for constructing such a transient elastic wave can be prohibitive for the sotchastic inverse problem if usual computational methods are used. Consequently, in order to decrease the computational cost of the optimization problem, a new fast, hybrid numerical method developped in [2] is used. This mechanical solver is based on a time-domain formulation associated with a space Fourier transform for the infinite dimensions and a finite element approximation for the finite dimension. The complete stochastic model is presented with its experimental validation.

2 Experimental database

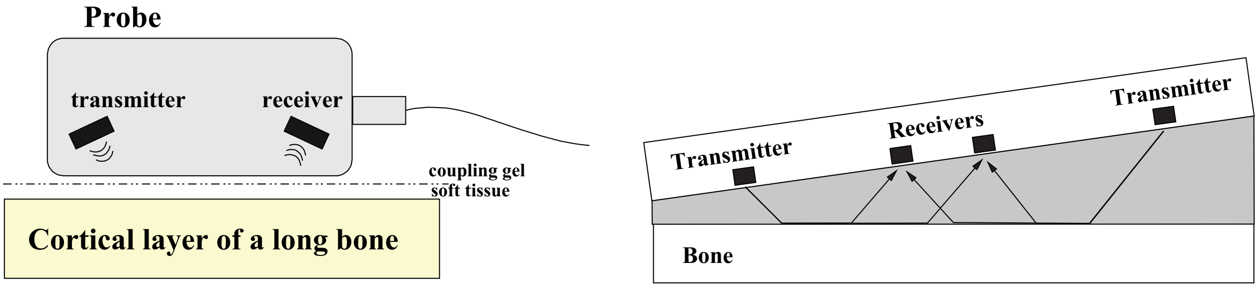

Figure 1: Experimental configuration

The ultrasonic axial transmission technique is used to construct an experimental database (see for instance [4-6]). The experimental configuration is described by Fig. 1. A device has been designed and is made up of several receivers and transmitters. A coupling gel is applied at the interface between the device and the skin of a patient. Each transmitter generates an acoustical impulse in the ultrasonic range that propagates in the coupling gel, the skin, the muscle, the cortical bone and the marrow. The axial transmission technique consists in recording these signals at several receivers located. The first arriving contribution of the signal (FAS) is considered. Following the signal processing method used with the experimental device the velocity of FAS is determined from the time of flight of the first extremum of the contribution. Figure 2 shows a part of a simulated signal and the FAS.

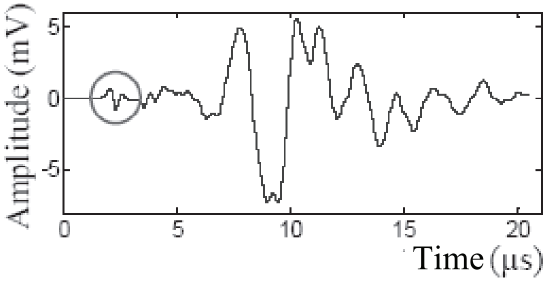

Figure 2: First arriving signal

These *in vivo* measurements were previously performed on a population of 168 subjects examined at the third distal radius. This group is a subset of a larger group of patients who participated to a clinical evaluation of the bidirectional axial transmission device. The multi-element probe operating at a center frequency of 1 MHz recorded twenty series of axially transmitted signals without particular angular scanning protocol except natural micro-movements of the operator. The experimental database finally consisted of 2018 measurements of FAS velocity. Each velocity measurement is considered as a realization of a random variable $V^{\exp}$. Thus, the database is made up of $N = 2747$ statistical independent realizations $V^{\exp}(\hat{\theta}_1), \ldots, V^{\exp}(\hat{\theta}_N)$ of random variable $V^{\exp}$. The mean value of $V^{\exp}$ is $v^{\exp} = E\{V^{\exp}\}$ and its coefficient of variation $\Delta^{\exp}$ is defined by $(\Delta^{\exp})^2 = E\{(V^{\exp})^2\}/(v^{\exp})^2 - 1$ in which $E\{\cdot\}$ is the mathematical expectation. Accordingly, the database consists of $N = 2018$ statistically independent realizations $V^{\exp}(\hat{\theta}_1), \ldots, V^{\exp}(\hat{\theta}_N)$ of random variable $V^{\exp}$. Using the usual statistical estimators and since N is sufficiently large, $v^{\exp}$ and $\Delta^{\exp}$ can reasonably be estimated by

$$v^{\exp} = \frac{1}{N}\sum_{k=1}^{N} V^{\exp}(\hat{\theta}_k) \quad , \quad \Delta^{\exp} = \frac{1}{v^{\exp}}\sqrt{\frac{1}{N}\sum_{k=1}^{N} V^{\exp}(\hat{\theta}_k)^2 - (v^{\exp})^2}\ .$$

3 Simplified model

A simplified model of the biomechanical system made up of the coupling gel, the skin, the cortical bone and the marrow has been developped in [2, 3]. This simplifed model is composed of an elastic solid semi-infinite layer between two acoustic fluid semi-infinite layers (see Fig. 3). Let $\mathbf{R}(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the reference

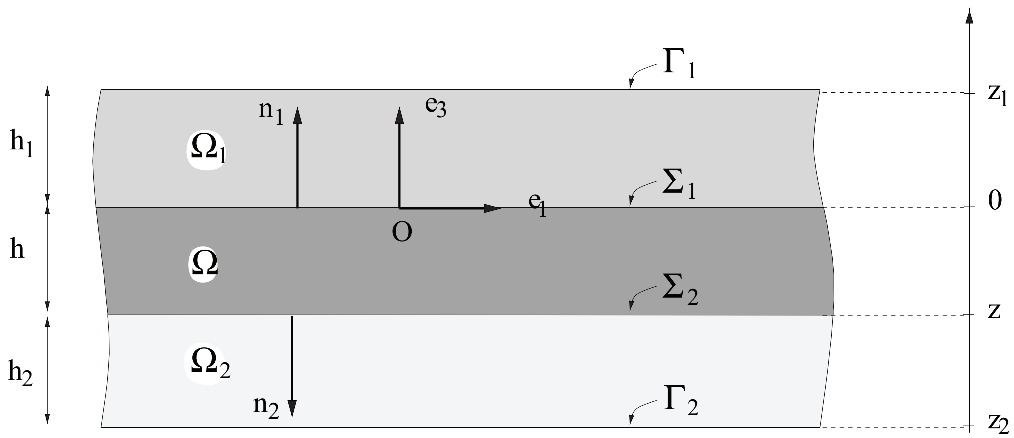

Figure 3: Geometry of the multilayer system

Cartesian frame where O is the origin of the space and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis for this space. The coordinate of the generic point $\mathbf{x}$ in $\mathbb{R}^3$ is (x_1, x_2, x_3). The thicknesses of the layers are denoted by h_1, h and h_2. The first acoustic fluid layer occupies the open unbounded domain Ω_1 , the second acoustic fluid layer occupies the open unbounded domain Ω_2 and the elastic solid layer occupies the open unbounded domain Ω. Let $\partial\Omega_1 = \Gamma_1 \cup \Sigma_1$, $\partial\Omega = \Sigma_1 \cup \Sigma_2$ and $\partial\Omega_2 = \Sigma_2 \cup \Gamma_2$ (see Fig. 3) be respectively the boundaries of Ω_1, Ω and Ω_2 in which $\Gamma_1, \Sigma_1, \Sigma_2$ and Γ_2 are the planes defined by

$$\begin{aligned}
\Gamma_1 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_1\} \\
\Sigma_1 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = 0\} \\
\Sigma_2 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z\} \\
\Gamma_2 &= \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_2\}
\end{aligned}$$

in which $z_1 = h_1$, $z = -h$ and $z_2 = -(h + h_2)$. Therefore, the domains Ω_1, Ω and Ω_2 are unbounded along the transversal directions $\mathbf{e}_1$ and $\mathbf{e}_2$ whereas they are bounded along the vertical direction $\mathbf{e}_3$. Let $\mathbf{u}$ be the displacement field of the solid elastic and, for $k = 1, 2$, let p_k be the pressure field in the fluid occupying the domain Ω_k. We then have (see, for instance [7,8])

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\sigma = \mathbf{0} \quad , \mathbf{x} \in \Omega \quad , t > 0 \quad , \tag{1}$$

$$\frac{1}{c_k^2} \frac{\partial^2 p_k}{\partial t^2} - \Delta p_k = \frac{\partial Q_k}{\partial t} \quad , \mathbf{x} \in \Omega_k \quad , t > 0 \quad , \tag{2}$$

where ρ and σ are the mass density and the Cauchy stress tensor field of the solid; div is the divergence operator with respect to $\mathbf{x}$; c_k is the wave velocity of the fluid occupying domain Ω_k; Δ is the Laplacian operator with respect to $\mathbf{x}$; Q_k is the source density applied in domain Ω_k such that

$$\frac{\partial Q_1}{\partial t}(\mathbf{x}, t) = \rho_1 \, F(t) \delta_0(x_1 - x_1^S) \delta_0(x_3 - x_3^S) \quad , \tag{3}$$

$$Q_2(\mathbf{x}, t) = 0 \quad , \tag{4}$$

in which $F(t) = F_1 \sin(2\pi f_c t) e^{-4(t\, f_c - 1)^2}$ where $f_c = 1\,\mathrm{MHz}$ is the center frequency and $F_1 = 100\,\mathrm{N}$; ρ_k is the mass density of domain Ω_k; δ_0 is the Dirac function at the origin and x_1^S and x_3^S are the coordinates of a line source modeling the acoustical impulse. The boundary conditions at time $t > 0$ are written as $\sigma\,\mathbf{n}_1 = -p_1\,\mathbf{n}_1$ in Σ_1, $\sigma\,\mathbf{n}_2 = -p_2\,\mathbf{n}_2$ in Σ_2, $p_k = 0$ in Γ_k and $\boldsymbol{\nabla} p_k \cdot \mathbf{n}_k = -\rho_k\,\ddot{\mathbf{u}} \cdot \mathbf{n}_k$ in Σ_k with $\mathbf{n}_1 = (0, 0, 1)$ and $\mathbf{n}_2 = (0, 0, -1)$. The initial conditions at time $t = 0$ are written as $\mathbf{u} = 0$ and $\dot{\mathbf{u}}$ in $\Omega \cup \Sigma_1 \cup \Sigma_2$ for the displacement field and $p_k = 0$ and $\dot{p}_k$ in $\Omega_k \cup \Gamma_k \cup \Sigma_k$ for the pressure fields. The constitutive equation of the solid elastic medium is written as

$$\sigma(\mathbf{x}, t) = \sum_{i,j,k,h=1}^{3} c_{ijkh}\, \varepsilon_{kh}(\mathbf{x}, t)\, \mathbf{e}_i \otimes \mathbf{e}_j \tag{5}$$

in which $\sum_{i,j,k,h=1}^{3} c_{ijkh} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_h$ is the effective elasticity tensor of the medium and $\varepsilon_{kh} = \frac{1}{2}(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k})$ are the components of the linearized strain tensor on basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Let $[C]$ be the effective elasticity matrix such that

$$[C] = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} & \sqrt{2}c_{1123} & \sqrt{2}c_{1131} & \sqrt{2}c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & \sqrt{2}c_{2223} & \sqrt{2}c_{2231} & \sqrt{2}c_{2213} \\ c_{3311} & c_{3322} & c_{3333} & \sqrt{2}c_{3323} & \sqrt{2}c_{3331} & \sqrt{2}c_{3312} \\ \sqrt{2}c_{2311} & \sqrt{2}c_{2322} & \sqrt{2}c_{2333} & 2c_{2323} & 2c_{2331} & 2c_{2312} \\ \sqrt{2}c_{3111} & \sqrt{2}c_{3122} & \sqrt{2}c_{3133} & 2c_{3123} & 2c_{3131} & 2c_{3112} \\ \sqrt{2}c_{1211} & \sqrt{2}c_{1222} & \sqrt{2}c_{1233} & 2c_{1223} & 2c_{1231} & 2c_{1212} \end{pmatrix} . \tag{6}$$

For a transverse isotropic homogeneous medium, all the components $[C]_{ij}$ are zeros except the following

$$[C]_{11} = \frac{e_L^2(1 - \nu_T)}{(e_L - e_L\nu_T - 2e_T\nu_L^2)} \quad , \quad [C]_{22} = \frac{e_T(e_L - e_T\nu_L^2)}{(1 + \nu_T)(e_L - e_L\nu_T - 2e_T\nu_L^2)} \, , \tag{7}$$

$$[C]_{12} = \frac{e_T e_L \nu_L}{(e_L - e_L\nu_T - 2e_T\nu_L^2)} \quad , \quad [C]_{23} = \frac{e_T(e_L\nu_T + e_T\nu_L^2)}{(1 + \nu_T)(e_L - e_L\nu_T - 2e_T\nu_L^2)} \quad , \tag{8}$$

$$[C]_{44} = g_T \quad , \quad [C]_{55} = g_L \quad , \tag{9}$$

with $[C]_{22} = [C]_{33}$, $[C]_{12} = [C]_{13} = [C]_{21} = [C]_{31}$, $[C]_{23} = [C]_{32}$ and $[C]_{55} = [C]_{66}$ and where (1) e_L and e_T are the longitudinal and transversal Young moduli, (2) g_L and g_T are the longitudinal and transversal shear moduli and (3) ν_L and ν_T are the longitudinal and transversal Poisson coefficients such that $g_T = e_T/2(1 + \nu_T)$. For a given effective elasticity matrix $[C]$, the displacement field $\mathbf{u}$ and the pressure fields p_1 and p_2 are calculated using the fast and efficient hybrid solver presented in [2]. The symmetry of the problem allowed the previous problem to be reformulated in terms of the spatial coordinates x_1 and x_3 yielding a 2D-space boundary value problem. The solver is based on a time-domain formulation associated with a 1D-space Fourier transform for the infinite layer dimension (along the x_1 direction) and uses a finite element approximation in the direction perpendicular to the layers (along the x_3 direction). For a given mean elasticity matrix $[C]$, this solver allows the displacement field $\mathbf{u}$ in Ω and the pressure fields p_1 and p_2 in Ω_1 and Ω_2 respectively, to be calculated. Then, the velocity v^{velo} of the first arriving signal is deduced. Consequently, there exists a mapping g_{velo} such that

$$v^{\text{mod}} = g_{\text{velo}}([C]) \quad . \tag{10}$$

4 Stochastic simplified model

It is assumed that uncertainties are only related to the components c_{ijkh} of the effective elasticity tensor. The construction of the probabilistic model consists in substituting $[C]$ by a random matrix $[\mathbf{C}]$ for which the probability density function is constructed using the information theory (see [9, 10]) with the available information defined as follows: (1) the random matrix $[\mathbf{C}]$ is a second-order random variable with values in the set $\mathbb{M}^+(\mathbb{R})$ of all the (6×6) real symmetric positive-definite matrices; (2) the mean value of random matrix $[\mathbf{C}]$ is the mean elasticity matrix $[C]$; (3) the norm of the inverse matrix of $[\mathbf{C}]$ is a second-order random variable. It has been shown in [11, 12] that the random matrix $[\mathbf{C}]$ can then be written as

$$[\mathbf{C}] = [L]^T [\mathbf{G}][L] \quad , \tag{11}$$

in which the (6×6) upper triangular matrix $[L]$ corresponds to the Cholesky factorization $[C] = [L]^T[L]$ and where the probability density function $p_{[\mathbf{G}]}$ of random matrix $[\mathbf{G}]$ is written as

$$p_{[\mathbf{G}]}([G]) = \mathbb{1}_{\mathbb{M}^+(\mathbb{R})}([G])\, c\, (\det[G])^b \, \exp\{-a \text{tr}[G]\} \, , \tag{12}$$

where $a = 7/(2\delta^2)$, $b = a(1-\delta^2)$, $\mathbb{1}_{\mathbb{M}^+(\mathbb{R})}([G])$ is equal to 1 if $[G]$ belongs to $\mathbb{M}^+(\mathbb{R})$ and is equal to zero if $[G]$ does not belong to $\mathbb{M}^+(\mathbb{R})$, $\text{tr}[G]$ is the trace of matrix $[G]$ and where positive constant c is such that

$$c = \frac{(2\pi)^{-15/2} a^{6\,a}}{\prod_{j=1}^{6} \Gamma(\alpha_j)} \, ,$$

in which $\alpha_j = 7/(2\delta^2) + (1-j)/2$ and where Γ is the Gamma function. The parameter δ allows the dispersion of the random matrix $[\mathbf{C}]$ to be controlled. Thus, the parameters of the probabilistic model of uncertainties for the elasticity matrix are the components of $[C]$ and the coefficient δ. The velocity of the FAS constructed using this stochastic simplified model is a random variable denoted by V^{mod} that corresponds to the random experimental velocity V^{exp} introduced in Section 2 and we have (see Eq. (10))

$$V^{\text{mod}} = g_{\text{velo}}([\mathbf{C}]) \quad . \tag{13}$$

5 Optimization problem for the identification

The stochastic simplified model parameters that have to be identified are the coefficients e_L, ν_L, g_L, e_T and ν_T relative to $[C]$, the mass density ρ and the coefficient δ. Let $\mathbf{a}$ be the vector such that $\mathbf{a} = (\rho, e_L, \nu_L, g_L, e_T, \nu_T)$. The identifiction problem consists in finding vector $\mathbf{a}$ and coefficient δ such that the stochastic model can represent the experimental database in a statistical sense.. The optimal values $(\mathbf{a}^{\text{opt}}, \delta^{\text{opt}})$ for $(\mathbf{a}, \delta)$ is given by solving the following optimization problem

$$(\mathbf{a}^{\text{opt}}, \delta^{\text{opt}}) = \arg\min_{(\mathbf{a},\delta)} F^{\text{cost}}(\mathbf{a}, \delta) \quad , \tag{14}$$

in which $F^{\text{cost}}(\mathbf{a},\delta)$ is a cost function which has to be defined. The cost fuction F^{cost} adapted to the optimization problem is written as

$$F^{\text{cost}}(\mathbf{a},\delta) = \frac{(v^{\exp} - v^{\text{mod}}(\mathbf{a},\delta))^2}{(v^{\exp})^2} + \frac{(\Delta^{\exp} - \Delta^{\text{mod}}(\mathbf{a},\delta))^2}{(\Delta^{\exp})^2} \,,$$

in which

$$\Delta^{\text{mod}} = \sqrt{\frac{E\{(V^{\text{mod}}(\mathbf{a},\delta)^2\}}{v^{\text{mod}}(\mathbf{a},\delta))^2} - 1} \,.$$

The optimization problem defined by Eq. (14) is solved by the simplex algorithm. For each iteration of the simplex algorithm, the cost function has to be calculated which requires to solve the stochastic equations with an appropriate method such as the Monte Carlo method.

6 Experimental validation of the stochastic simplified model

This section is devoted to the experimental validation of the stochastic simplified model. The stochastic simplified model must be able to simulate the experimental database in a statistical sense. The experi-

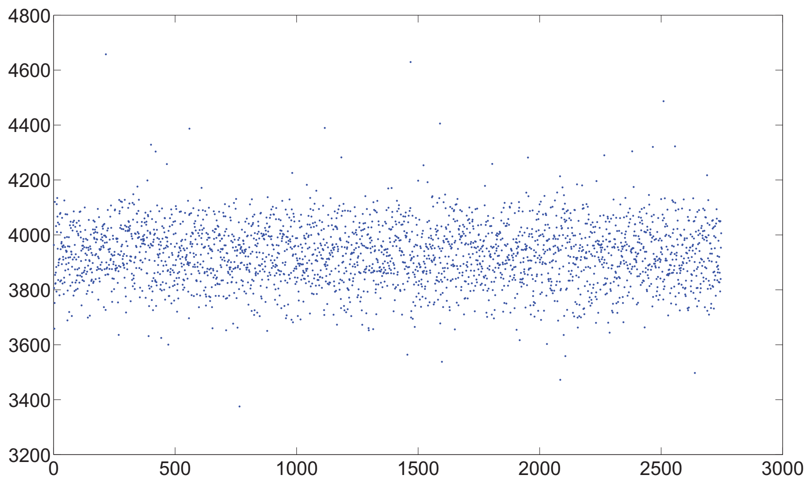

Figure 4: Graph of the realizations $V^{\exp}(\hat{\theta}_1), \ldots V^{\exp}(\hat{\theta}_N)$

mental validation is performed with the *in vivo* experimental database presented in Section 2 and is made up of $N = 2747$ mesurements $V^{\exp}(\hat{\theta}_1), \ldots, V^{\exp}(\hat{\theta}_N)$ plotted in Fig. 4. The probability density function $v \mapsto p_{V^{\exp}}(v)$ of the random variable $V^{\exp}$ estimated with the $N = 2747$ experimental realizations $V^{\exp}(\hat{\theta}_1), \ldots, V^{\exp}(\hat{\theta}_N)$ is shown in Fig. 5. The identification of the vector $\mathbf{a} = (\rho, e_L, \nu_L, g_L, e_T, \nu_T)$ and the coefficient δ is carried out using the method presented in Section 5 with $h_1 = 10^{-2}$m, $h = 4 \times 10^{-3}$m, $h_2 = 10^{-2}$m, $\rho_1 = \rho_2 = 1000$ kg.m^{-3} and $c_1 = c_2 = 1500$ m.s^{-1} The solution $\mathbf{a}^{\text{opt}} = \left(\rho^{\text{opt}}, e_L^{\text{opt}}, \nu_L^{\text{opt}}, g_L^{\text{opt}}, e_T^{\text{opt}}, \nu_T^{\text{opt}}\right)$ and δ^{opt} are such that $\rho^{\text{opt}} = 1598.8$ kg.m^{-3}, $e_L^{\text{opt}} = 17.717$ GPa, $\nu_L^{\text{opt}} = 0.3816$, $g_L^{\text{opt}} = 4.7950$ GPa, $e_T^{\text{opt}} = 9.8254$ GPa, $\nu_T^{\text{opt}} = 0.4495$ and $\delta^{\text{opt}} = 0.1029$. For $\mathbf{a} = \mathbf{a}^{\text{opt}}$ and $\delta = \delta^{\text{opt}}$, the realizations $V^{\text{mod}}(\theta_1), \ldots, V^{\text{mod}}(\theta_N)$ of random velocity V^{mod} are constructed with the stochastic simplified model and then, the probability density function $v \mapsto p_{V^{\text{mod}}}(v)$ of V^{mod} is estimated. Figure 6 shows the graphs of $v \mapsto p_{V^{\text{mod}}}(v)$. Figure 7 compares the graphs of $v \mapsto p_{V^{\text{mod}}}(v)$ and $v \mapsto p_{V^{\exp}}(v)$ in logarithm scale. This figure shows that the stochastic simplified model is able to predict in a statistical sense the velocity of the first arriving signal in a good accordance with the experimental tests.

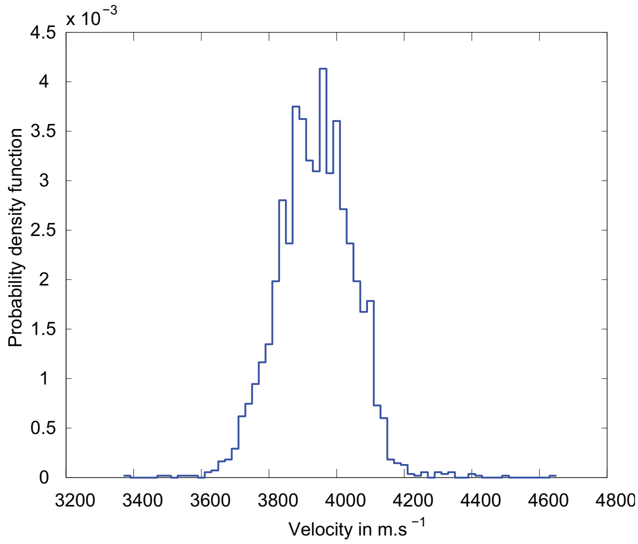

Figure 5: Graph of the probability density function $v \mapsto p_{V^{\text{exp}}}(v)$

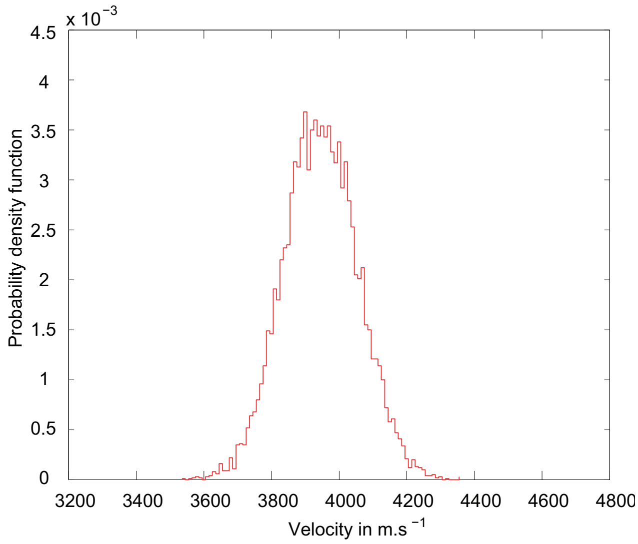

Figure 6: Graph of the probability density functions $v \mapsto p_{V^{\text{mod}}}(v; \mathbf{a}^{\text{opt}}, \delta^{\text{opt}})$ with $\mathbf{a} = \mathbf{a}^{\text{opt}}$ and $\delta = \delta^{\text{opt}}$

7 Conclusion

A simplified elastoacoustic model has been developped to simulate the ultrasonic wave propagation in a complex biomechanical system made up of multilayer media. In order to improve the simplified model, the uncertainties related to the solid layer have been taken into account using a probabilistic approach. A method has been presented to identify the parameters of the stochastic simplified model. The capability of the proposed stochastic simplified model to predict the velocity of the first arriving signal in the statistical sense has been demonstrated using a large experimental *in vivo* database.

8 Acknowledgments

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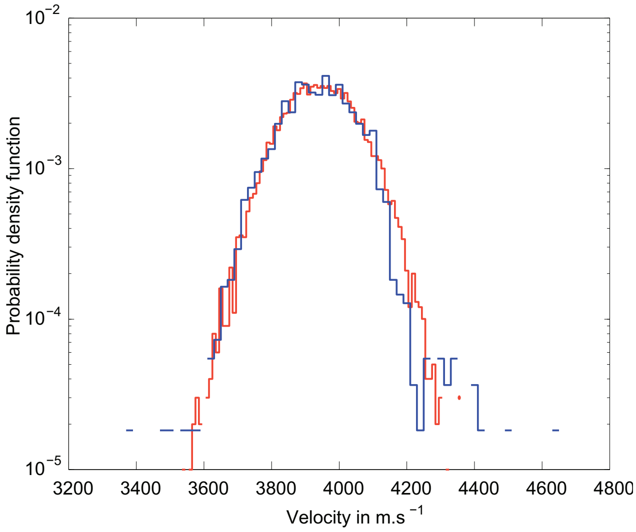

Figure 7: Graphs of $v \mapsto \log(p_{V^{\exp}}(v))$ and $v \mapsto \log(p_{V^{\exp}}(v; \mathbf{a}^{\mathrm{opt}}, \delta^{\mathrm{opt}}))$.

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