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Robust parametric updating of uncertain finite element models from experimental modal analysis

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Abstract

In this paper, a methodology is presented to perform the robust updating of complex uncertain dynamical systems with respect to modal experimental data in the context of structural dynamics. Since both model uncertainties and data uncertainties must be considered in the computational model, then the uncertain computational model is constructed by using the nonparametric probabilistic approach. We present an extension to the probabilistic case of the input error methodology for modal analysis adapted to the deterministic updating problem. It is shown that such an extension to the robust updating context induces some conceptual difficulties and is not straightforward. The robust updating formulation leads us to solve a mono-objective optimization problem in presence of inequality probabilistic constraints. A numerical application is presented in order to show the efficiency of the proposed method.

1 Introduction

The updating of computational models using experimental data is currently a challenge of interest in structural dynamics. The updating formulation involves an optimization problem for which the cost function can be defined from the operator of the computational model (input error formulation) or from the inverse of the operator of the computational model (output error formulation). These last decades, such an updating has been carried out using deterministic computational models (see for instance, [1] for the input error formulations and [2, 3, 4] for the output error formulations). It is well known that deterministic computational models are not sufficient to accurately predict the dynamical behaviour of complex structures. The uncertainties have then to be taken into account in the computational models by using probabilistic models as soon as the probability theory can be used. More recently, the terminology of robust updating has been introduced in order to define updating optimization problems using uncertain computational models. We then can distinguish robust updating with respect to parameter uncertainties from robust updating with respect to both model uncertainties and parameter uncertainties. Parameter uncertainties are taken into account by using parametric probabilistic approaches, see for instance [5, 6, 7] and [8] for rotating structures.

. Both model uncertainties and parameter uncertainties can be taken into account by using the nonparametric probabilistic approach recently introduced [9, 10, 11]. The recent works related to the robust updating formulations concern robust updating with respect to parameter uncertainties [12, 13, 14] and robust updating with respect to both model uncertainties and parameter uncertainties [15, 16] . Until now, all these robust updating formulations belong to the class of the output error formulations . The motivation of this paper is to propose a robust updating methodology with respect to both model uncertainties and parameter uncertainties using modal experimental data by constructing a formulation based on the input errors. Such a methodology is based on the deterministic updating formulation [1] . This paper proposes to extend such a deterministic updating formulation to the probabilistic case. Note that this extension is not trivial. The paper is organized as follows. Section 1 is devoted to the description of the available experimental data (experimental eigenfrequencies and experimental eigenmodes). In Section 2, the deterministic updating methodology [1] is summarized. This deals with the updating of a mean computational model for which the updating parameters are called the mean updating parameters. The main idea is to modify the generalized eigenvalue problem of the mean computational model in order to calculate a residue which characterizes the good matching between the mean computational model and the available experimental data. The cost function is defined from this residue and is then optimized with respect to the admissible set of the mean updating parameters. Section 3 deals with the robust updating formulation. In this robust updating context, there are model uncertainties which are such that the available experimental data can not exactly be reproduced by any computational model. This context does not allow the strategy of deterministic updating to be effective. The main idea is thus to implement the nonparametric probabilistic approach in a mean computational model in order to take into account both model uncertainties and parameter uncertainties. First of all, a modified Craig and Bampton dynamical substructuring method [17, 18, 19, 20] is introduced in order to construct a mean reduced matrix equation allowing the deterministic residue to be calculated. In a second step, the generalized matrices of this mean reduced equation are replaced by random matrices for which the probability model is explicitly constructed. With such an approach, the uncertainty level of each random matrix is controlled by a dispersion parameter. We then obtain a random residue which is defined as a function of the updating parameters which are the updating mean parameters related to the mean computational model and the dispersion parameters which allow the uncertainty level in the computational model to be controlled. In a third step, the cost function is defined as the second-order moment of the norm of the random residue. Difficulties arise from a conceptual point of view. A straightforward generalization of the deterministic optimization problem which would consist in optimizing the cost function with respect to the admissible set of the updating parameters would yield a deterministic updated computational model which would not be compatible with the existence of model uncertainties in the computational model. The formulation is then modified by adding probabilistic constraints related to the nonreducible gap between the uncertain computational model and the experiments due to the presence of model uncertainties. In Section 4, a numerical example is presented in order to validate the methodology proposed.

2 Description of the experimental data

In this Section, the assumptions concerning the available experimental data are given. It is assumed that experimental modal analysis has been carried out on one manufactured dynamical system with free free boundary conditions. Consequently, there are $m = 6$ rigid-body modes associated with 6 zero eigenvalues which are not taken into account in the analysis. The experimental data consists in r experimental elastic eigenvalues denoted by $0 < \underline{\lambda}_1^{exp} < \ldots < \underline{\lambda}_r^{exp}$ which are stored in the $(r \times r)$ real diagonal matrix $[\underline{\Lambda}^{exp}]$ for which $[\underline{\Lambda}^{exp}]_{\alpha\beta} = \underline{\lambda}_\alpha^{exp}\, \delta_{\alpha\beta}$ for α and β in $\{1, \cdots, r\}$ and where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if not. The corresponding experimental eigenmodes are measured at n_{obs} observation points. One then denotes by $[\underline{\Phi}^{exp}]$ the $(n_{obs} \times r)$ real modal matrix whose columns are the $\mathbb{C}^{n_{obs}}$-vector $\underline{\varphi}_\alpha^{exp}$ corresponding to the experimental eigenmode measured at each of the n_{obs} observation points and related to experimental eigenvalue $\underline{\lambda}_\alpha$ with $\alpha \in \{1, \ldots, r\}$.

3 Summarizing the input error methodology for experimental modal data analysis

In this Section, it is assumed that the manufactured dynamical system can be modeled by a deterministic computational model which is called the mean computational model. The usual methodology for the updating of a deterministic computational model using modal analysis is the output error method (see for instance [4]) which consists in solving a multi-objective optimization problem in order to simultaneously minimize the distance between each experimental eigenvalue / eigenvector and between each eigenvalue / eigenvector of the deterministic computational model to be updated. The alternative formulation used in this Section belongs to the input error methodologies. This means that the cost function which quantifies the gap between the mean computational model and the experimental data is directly defined from the operators of the mean computational model so that the eigenfrequencies and the eigenmodes are simultaneously treated with a coherent way. This deterministic updating yields to solve a mono-objective optimization problem with respect to the admissible set of the updating parameters of the mean computational model. Since the robust updating proposed for modal analysis is based on the method proposed in [1], we briefly summarize it below in order to improve the readability of the manuscript.

3.1 Residue in the mean computational model

The mean computational model of the dynamical system is constructed using the finite element method and has n DOF (degrees of freedom). It is assumed that the finite element mesh is compatible with the n_{obs} experimental measurement points. Let $\mathbf{s}$ be the $\mathbb{R}^s$-vector of the updating parameters of the mean computational model called the updating mean parameters. Vector $\mathbf{s}$ belongs to an admissible set $\mathcal{S}$ corresponding to a given family of mean computational models. Assuming the dynamical system to be linear, for fixed $\mathbf{s}$ in $\mathcal{S}$, the generalized eigenvalue problem related to the conservative dynamical system is written as: find $(\underline{\lambda}_\alpha, \underline{\boldsymbol{\varphi}}_\alpha)$ belonging to $\mathbb{R}^+ \times \mathbb{R}^n$ such that

$$\mathbf{0} = \left([\underline{K}(\mathbf{s})] - \underline{\lambda}_\alpha\, [\underline{M}(\mathbf{s})]\right) \underline{\boldsymbol{\varphi}}_\alpha \quad , \quad \alpha = 1, \ldots, r \quad , \tag{1}$$

in which the matrices $[\underline{M}(\mathbf{s})]$ and $[\underline{K}(\mathbf{s})]$ are the finite element mass and stiffness matrices. Since the dynamical system has free free boundary conditions, matrices $[\underline{M}(\mathbf{s})]$ and $[\underline{K}(\mathbf{s})]$ are $(n \times n)$ positive-definite and positive symmetric real matrices whose bloc decomposition with respect to the n_{obs} experimental measurement DOF and the $n_2 = n - n_{obs}$ non measured DOF is written as

$$[\underline{M}(\mathbf{s})] = \begin{bmatrix} [\underline{M}_{11}(\mathbf{s})] & [\underline{M}_{12}(\mathbf{s})] \\ [\underline{M}_{12}(\mathbf{s})]^T & [\underline{M}_{22}(\mathbf{s})] \end{bmatrix} \quad , \quad [\underline{K}(\mathbf{s})] = \begin{bmatrix} [\underline{K}_{11}(\mathbf{s})] & [\underline{K}_{12}(\mathbf{s})] \\ [\underline{K}_{12}(\mathbf{s})]^T & [\underline{K}_{22}(\mathbf{s})] \end{bmatrix} \quad . \tag{2}$$

The matrix formulation allowing the deterministic updating to be solved consists in modifying the generalized eigenvalue problem (1) by introducing the $(n \times 1)$ residue vector $\underline{\mathbb{r}}_\alpha(\mathbf{s})$ defined, for $\alpha = 1, \ldots, r$, by

$$\underline{\mathbb{r}}_\alpha(\mathbf{s}) = \left(\begin{bmatrix} [\underline{K}_{11}(\mathbf{s})] & [\underline{K}_{12}(\mathbf{s})] \\ [\underline{K}_{12}(\mathbf{s})]^T & [\underline{K}_{22}(\mathbf{s})] \end{bmatrix} - \underline{\lambda}_\alpha^{exp} \begin{bmatrix} [\underline{M}_{11}(\mathbf{s})] & [\underline{M}_{12}(\mathbf{s})] \\ [\underline{M}_{12}(\mathbf{s})]^T & [\underline{M}_{22}(\mathbf{s})] \end{bmatrix} \right) \begin{bmatrix} \underline{\boldsymbol{\varphi}}_\alpha^{exp} \\ \underline{\boldsymbol{\varphi}}_{2,\alpha}(\mathbf{s}) \end{bmatrix} \tag{3}$$

This residue represents the dynamic reaction forces induced by the mean computational model for the experimental measurements. Note that the residue is zero for an exact model. In Eq. (3), for a given updating mean parameter $\mathbf{s}$ belonging to $\mathcal{S}$, the component $\underline{r}_{\alpha,k}(\mathbf{s})$ of vector $\underline{\mathbb{r}}_\alpha(\mathbf{s})$ quantifies the residue with respect to the mean computational model which is induced by the experimental elastic eigenvalue and by the eigenmode number α for the DOF number k. Since the information concerning the experimental eigenmodes is only available on a restricted number of DOF corresponding to the experimental measurement points, it is assumed that no errors are induced on the nonmeasured DOF. With such an assumption, the residue vector $\underline{\mathbb{r}}_\alpha(\mathbf{s})$ is then written as

$$\underline{\mathbb{r}}_\alpha(\mathbf{s}) = \begin{bmatrix} \underline{\mathbf{r}}_\alpha(\mathbf{s}) \\ \mathbf{0} \end{bmatrix} \quad . \tag{4}$$

Equations (3) and (4) allow the unknown quantities $\underline{\boldsymbol{\varphi}}_{2,\alpha}(\mathbf{s})$ and $\underline{\mathbf{r}}_\alpha(\mathbf{s})$ to be calculated. It should be noted that this calculation requires the inversion of the matrix $[\underline{B}_\alpha(\mathbf{s})]$ defined by

$$[\underline{B}_\alpha(\mathbf{s})] = [\underline{K}_{22}(\mathbf{s})] - \underline{\lambda}_\alpha^{exp}[\underline{M}_{22}(\mathbf{s})] \quad , \quad \alpha = 1, \ldots, r \quad . \tag{5}$$

This means that for all $\mathbf{s}$ belonging to $\mathcal{S}$ and for all α belonging to $\{1,\ldots,r\}$, matrix $[\underline{B}_\alpha(\mathbf{s})]$ must be invertible. Note that this condition is practically verified if we are interested in the low eigenfrequencies and if the measured DOF are regularly distributed on the whole structure. For instance, a sufficient condition could consist in considering the first eigenfrequency of the structure fixed at each measured DOF which should be much larger than the frequency band of analysis. It is assumed that the number r of experimental eigenvalue/eigenmodes which has to be considered in this deterministic updating is chosen in order to fulfill this condition.

3.2 Formulation of the deterministic updating

The deterministic updating is solved by simultaneously minimizing the residue vectors $\underline{\mathbf{r}}_\alpha(\mathbf{s})$ for all α belonging to $\{1,\ldots,r\}$. The cost function is defined as a function of the updating mean parameters $\mathbf{s}$ by

$$\underline{j}(\mathbf{s}) = ||[\underline{\mathcal{R}}(\mathbf{s})]||_F^2 \quad , \tag{6}$$

in which the $(r \times r)$ real matrix $[\underline{\mathcal{R}}(\mathbf{s})]$ is defined by $[\underline{\mathcal{R}}(\mathbf{s})]_{\alpha\beta} = \underline{\boldsymbol{\varphi}}_\alpha^{exp,T}\, \underline{\mathbf{r}}_\beta(\mathbf{s})$. In Eq. (6), $||[\underline{X}]||_F^2 = tr([\underline{X}]\,[\underline{X}]^T)$. The solution of this deterministic updating problem is then given by

$$\underline{\mathbf{s}}^{opt} = arg \min_{\mathbf{s}\in\mathcal{S}} \underline{j}(\mathbf{s}) \quad . \tag{7}$$

4 Robust input error methodology for experimental modal data analysis

In this Section, it is assumed that the computational model used for modeling the manufactured dynamical system for which experimental modal data are available contains significant model uncertainties. Consequently, the deterministic updating formulation presented in Section III can be improved in taking in to account the presence of model uncertainties. It should be noted that in general, the optimization of a deterministic computational model can produce a non optimal result with respect to the robust optimization of an uncertain computational model as it is shown for instance in [21] . We then propose to adapt the deterministic updating formulation presented in Section III to the robust updating context as explained in Section I. The nonparametric probabilistic approach is then implemented in the mean reduced matrix model. The formulation of the optimization problem is then discussed in order to capture the largest possible class of uncertain computational models. ed in order to capture the largest possible class of uncertain computational models.

4.1 Mean reduced computational model

The proposed dynamical substructuring method is based on the Craig and Bampton method [17, 18, 19, 20] for which the coupling interface is constituted of the n_{obs} measurements DOF. The modifications consist in replacing the usual static liftings by the "modal liftings" related to each experimental eigenvalue. For a given α belonging to $\{1,\ldots,r\}$, the projection basis is given by

$$\begin{bmatrix} \underline{\boldsymbol{\varphi}}_\alpha^{exp} \\ \underline{\boldsymbol{\varphi}}_{2,\alpha}(\mathbf{s}) \end{bmatrix} = [\underline{H}_\alpha(\mathbf{s})] \begin{bmatrix} \underline{\boldsymbol{\varphi}}_\alpha^{exp} \\ \underline{\mathbf{q}}_\alpha(\mathbf{s}) \end{bmatrix} \quad , \quad [\underline{H}_\alpha(\mathbf{s})] = \begin{bmatrix} [I] & [\mathbb{0}] \\ [\underline{S}_\alpha(\mathbf{s})] & [\underline{\Psi}(\mathbf{s})] \end{bmatrix} . \tag{8}$$

In Eq. (8), $\underline{\mathbf{q}}_\alpha(\mathbf{s})$ is the $\mathbb{R}^N$-vector of the generalized coordinates. The matrix $[\underline{\Psi}(\mathbf{s})]$ is the $(n_2 \times N)$ real modal matrix whose columns are the eigenvectors $\underline{\boldsymbol{\psi}}_1(\mathbf{s}),\ldots,\underline{\boldsymbol{\psi}}_N(\mathbf{s})$ corresponding to the lowest eigenvalues of the generalized eigenvalue problem related to the mean computational model with fixed measured DOF: find $(\underline{\lambda}_\alpha^0, \underline{\boldsymbol{\psi}}_\alpha)$ belonging to $\mathbb{R}^+ \times \mathbb{R}^{n_2}$ such that

$$\mathbf{0} = \left([\underline{K}_{22}(\mathbf{s})] - \underline{\lambda}_\alpha^0(\mathbf{s})\,[\underline{M}_{22}(\mathbf{s})]\right)\underline{\boldsymbol{\psi}}_\alpha(\mathbf{s}) \tag{9}$$

.

In Eq. (8), the matrix $[\underline{S}_\alpha(\mathbf{s})]$ is the $(n_2 \times n_{obs})$ real matrix of the "modal" boundary functions defined by

$$[\underline{S}_\alpha(\mathbf{s})] = -[\underline{B}_\alpha(\mathbf{s})]^{-1}\,([\underline{K}_{12}(\mathbf{s})]^T - \underline{\lambda}_\alpha^{exp}\,[\underline{M}_{12}(\mathbf{s})]^T) \quad , \quad \alpha = 1, \ldots, r \quad , \tag{10}$$

in which $[\underline{B}_\alpha(\mathbf{s})]$ is defined by Eq. (5) It should be noted that the usual Craig and Bampton method corresponds to Eq. (10) for which the mass dynamic term is not taken into account. Let $\mathbb{m} = N + n_{obs}$. The mean reduced matrix equation which allows $\underline{\mathbf{r}}_\alpha(\mathbf{s})$ and $\underline{\mathbf{q}}_\alpha(\mathbf{s})$ to be calculated is then written as

$$\begin{bmatrix}\underline{\mathbf{r}}_\alpha(\mathbf{s})\\ \mathbf{0}\end{bmatrix} = \left([\underline{K}_{red,\alpha}(\mathbf{s})] - \underline{\lambda}_\alpha^{exp}\,[\underline{M}_{red,\alpha}(\mathbf{s})]\right)\begin{bmatrix}\boldsymbol{\varphi}_\alpha^{exp}\\ \underline{\mathbf{q}}_\alpha(\mathbf{s})\end{bmatrix} \quad , \tag{11}$$

in which the matrices $[\underline{M}_{red,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})]$ are the $(\mathbb{m} \times \mathbb{m})$ positive-definite and positive symmetric real mass and stiffness matrices defined by $[\underline{M}_{red,\alpha}(\mathbf{s})] = [\underline{H}_\alpha(\mathbf{s})]^T\,[\underline{M}(\mathbf{s})]\,[\underline{H}_\alpha(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})] = [\underline{H}_\alpha(\mathbf{s})]^T\,[\underline{K}(\mathbf{s})]\,[\underline{H}_\alpha(\mathbf{s})]$.

4.2 Stochastic computational model

The nonparametric probabilistic approach [9, 10, 11] recently introduced is used to model uncertainties in Eq. (11). The method consists in replacing the deterministic matrices $[\underline{M}_{red,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})]$ by random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ for which the probability distribution is known and such that $\mathcal{E}\{[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]\} = [\underline{M}_{red,\alpha}(\mathbf{s})]$ and $\mathcal{E}\{[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]\} = [\underline{K}_{red,\alpha}(\mathbf{s})]$ in which $\mathcal{E}$ is the mathematical expectation. The scalar parameters δ_M and δ_K are the dispersion parameters which allow the amount of uncertainty to be quantified. The random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ are written as $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)] = [\underline{L}_{M,\alpha}(\mathbf{s})]^T\,[\mathbf{G}_M(\delta_M)]\,[\underline{L}_{M,\alpha}(\mathbf{s})]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)] = [\underline{L}_{K,\alpha}(\mathbf{s})]^T\,[\mathbf{G}_K(\delta_K)]\,[\underline{L}_{K,\alpha}(\mathbf{s})]$ in which the matrices $[\underline{L}_{M,\alpha}(\mathbf{s})]$ and $[\underline{L}_{K,\alpha}(\mathbf{s})]$ are $\left(\mathbb{m} \times \mathbb{m}\right)$ and $\left((\mathbb{m} - m) \times \mathbb{m}\right)$ real matrices such that $[\underline{M}_{red,\alpha}(\mathbf{s})] = [\underline{L}_{M,\alpha}(\mathbf{s})]^T\,[\underline{L}_{M,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})] = [\underline{L}_{K,\alpha}(\mathbf{s})]^T\,[\underline{L}_{K,\alpha}(\mathbf{s})]$ and where the matrices $[\mathbf{G}_M(\delta_M)]$ and $[\mathbf{G}_K(\delta_K)]$ are full $\left(\mathbb{m} \times \mathbb{m}\right)$ and $\left((\mathbb{m} - m) \times (\mathbb{m} - m)\right)$ random matrices with values in the set of all the positive-definite symmetric real matrices. All the details concerning the construction of the probability model of these random matrices can be found in [9, 10, 11]. Let $\boldsymbol{\delta} = (\delta_M, \delta_K)$ be the vector of the dispersion parameters which have to be updated. It can be shown from the construction of the probability model that dispersion parameter $\boldsymbol{\delta}$ must belong to the admissible set $\Delta = \left\{[0\,,\,\sqrt{\frac{\mathbb{m}+1}{\mathbb{m}+5}}] \times [0\,,\,\sqrt{\frac{\mathbb{m}-m+1}{\mathbb{m}-m+5}}]\right\}$. It should be noted that there exists an algebraic representation useful to the Monte Carlo numerical simulation. It should also be noted that the same random matrices $[\mathbf{G}_M(\delta_M)]$ and $[\mathbf{G}_K(\delta_K)]$ are used to construct the random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ for all α belonging to $\{1, \ldots, r\}$. The stochastic matrix equation whose unknowns are the random residue vector $\mathbf{R}_\alpha(\mathbf{s}, \boldsymbol{\delta})$ and the random vector $\mathbf{Q}_\alpha(\mathbf{s})$ of the random generalized coordinates is written as

$$\begin{bmatrix}\mathbf{R}_\alpha(\mathbf{s}, \boldsymbol{\delta})\\ \mathbf{0}\end{bmatrix} = \left([\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)] - \underline{\lambda}_\alpha^{exp}\,[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]\right)\begin{bmatrix}\boldsymbol{\varphi}_\alpha^{exp}\\ \mathbf{Q}_\alpha(\mathbf{s}, \boldsymbol{\delta})\end{bmatrix} \quad , \tag{12}$$

4.3 Estimation of $\mathbf{Q}_\alpha(\mathbf{s}, \boldsymbol{\delta})$

The matrices $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ and $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ are block decomposed with respect to the number of experimental measured DOF and with respect to the number of generalized coordinates N such that

$$[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)] = \begin{bmatrix}[\boldsymbol{\mathcal{K}}_{1,\alpha}(\mathbf{s}, \delta_K)] & [\mathbf{K}_{c,\alpha}(\mathbf{s}, \delta_K)]\\ [\mathbf{K}_{c,\alpha}(\mathbf{s}, \delta_K)]^T & [\boldsymbol{\mathcal{K}}_{2,\alpha}(\mathbf{s}, \delta_K)]\end{bmatrix} , \; [\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)] = \begin{bmatrix}[\boldsymbol{\mathcal{M}}_{1,\alpha}(\mathbf{s}, \delta_M)] & [\mathbf{M}_{c,\alpha}(\mathbf{s}, \delta_M)]\\ [\mathbf{M}_{c,\alpha}(\mathbf{s}, \delta_M)]^T & [\boldsymbol{\mathcal{M}}_{2,\alpha}(\mathbf{s}, \delta_M)]\end{bmatrix} . \tag{13}$$

The random residue vector $\mathbf{R}_\alpha(\mathbf{s},\boldsymbol{\delta})$ and the random vector $\mathbf{Q}_\alpha(\mathbf{s})$ of the random generalized coordinates solution of the random matrix equation (12) are then given by

$$\mathbf{R}_\alpha(\mathbf{s},\boldsymbol{\delta}) = [\boldsymbol{\mathcal{B}}_{1,\alpha}(\mathbf{s},\boldsymbol{\delta})]\,\underline{\boldsymbol{\varphi}}_\alpha^{exp} + [\mathbf{B}_{c,\alpha}(\mathbf{s},\boldsymbol{\delta})]\,\mathbf{Q}_\alpha(\mathbf{s},\boldsymbol{\delta}) \tag{14}$$

$$\mathbf{Q}_\alpha(\mathbf{s},\boldsymbol{\delta}) = -[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s},\boldsymbol{\delta})]^{-1}\,[\mathbf{B}_{c,\alpha}(\mathbf{s},\boldsymbol{\delta})]^T\,\underline{\boldsymbol{\varphi}}_\alpha^{exp} \quad , \tag{15}$$

in which $[\boldsymbol{\mathcal{B}}_{1,\alpha}(\mathbf{s},\boldsymbol{\delta})] = [\boldsymbol{\mathcal{K}}_{1,\alpha}(\mathbf{s},\boldsymbol{\delta})] - \underline{\lambda}_\alpha^{exp}\,[\boldsymbol{\mathcal{M}}_{1,\alpha}(\mathbf{s},\boldsymbol{\delta})]$, $[\mathbf{B}_{c,\alpha}(\mathbf{s},\boldsymbol{\delta})] = [\mathbf{K}_{c,\alpha}(\mathbf{s},\boldsymbol{\delta})] - \underline{\lambda}_\alpha^{exp}\,[\mathbf{M}_{c,\alpha}(\mathbf{s},\boldsymbol{\delta})]$ and $[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s},\boldsymbol{\delta})] = [\boldsymbol{\mathcal{K}}_{2,\alpha}(\mathbf{s},\boldsymbol{\delta})] - \underline{\lambda}_\alpha^{exp}\,[\boldsymbol{\mathcal{M}}_{2,\alpha}(\mathbf{s},\boldsymbol{\delta})]$. The calculation of random vector $\mathbf{Q}_\alpha$ requires the inversion of the random matrix $[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s},\boldsymbol{\delta})]$ for all α belonging to $\{1,\ldots,r\}$. It is assumed that the number r of experimental eigenvalues is chosen under the assumption that random matrix $[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s},\boldsymbol{\delta})]$ is invertible almost surely.

4.4 Robust updating formulation

The robust updating formulation requires to define the cost function from the uncertain computational model as a function of the updating mean parameter $\mathbf{s}$ and of the dispersion parameter $\boldsymbol{\delta}$. In coherence with Eq. (6), the cost function denoted by $j(\mathbf{s},\boldsymbol{\delta})$ is written as

$$j(\mathbf{s},\boldsymbol{\delta}) = \mathcal{E}\{||[\boldsymbol{\mathcal{R}}(\mathbf{s},\boldsymbol{\delta})]||_F^2\} \quad , \tag{16}$$

in which the $(r \times r)$ real matrix $[\boldsymbol{\mathcal{R}}(\mathbf{s},\boldsymbol{\delta})]$ is defined by

$$[\boldsymbol{\mathcal{R}}(\mathbf{s},\boldsymbol{\delta})]_{\alpha\beta} = \underline{\boldsymbol{\varphi}}_\alpha^{exp,T}\,\mathbf{R}_\beta(\mathbf{s},\boldsymbol{\delta}) \quad . \tag{17}$$

Note that the cost function $j(\mathbf{s},\boldsymbol{\delta})$ tends to the cost function $\underline{j}(\mathbf{s})$ as δ_M and δ_K go to zero, which means as the structure tends to be deterministic. The straightforward generalization of Eq. (7) to the random case yields the solution $(\mathbf{s}^{opt},\boldsymbol{\delta}^{opt}) = arg\,\min_{\mathbf{s}\in\mathcal{S}}\,j(\mathbf{s},\boldsymbol{\delta})$. The following comment shows that this formulation is not adapted to the robust updating context. If the deterministic updating context assumed that there were no model uncertainties and no parameter uncertainties, then it would mean that the family of deterministic models would be able to exactly reproduce the experimental data. In that case, the deterministic cost function would be zero for the updated solution. In the present context of robust updating, there are model uncertainties which are then taken into account by a class of computational model generated with the nonparametric probabilistic approach. The above formulation for robust updating tends to minimize the model uncertainties ($\boldsymbol{\delta} \rightarrow 0$) which means that this formulation is equivalent to the deterministic updating formulation. However, since it is assumed that there are significant model uncertainties, the class of deterministic computational models is not able to reproduce the experiments. Consequently, the cost function is doubtlessly minimized but is nonzero and there still exists an irreducible distance between each eigenvalue /eigenvector of the updated computational model and each experimental eigenvalue / eigenvector. The above formulation for robust updating is then not correct. In order to generate a larger class of uncertain computational models, additional probabilistic constraints involving these distances are added in the formulation of the robust updating optimization problem. Let $\Delta\Lambda$ and $\Delta\widetilde{\Phi}$ be the positive-valued random variables defined by

$$\Delta\Lambda(\mathbf{s},\boldsymbol{\delta}) = \sqrt{\frac{1}{r}\sum_{\alpha=1}^{r}\{\Delta\Lambda_\alpha(\mathbf{s},\boldsymbol{\delta})\}^2} \quad , \quad \Delta\Lambda_\alpha(\mathbf{s},\boldsymbol{\delta}) = \frac{|\Lambda_\alpha(\mathbf{s},\boldsymbol{\delta}) - \underline{\lambda}_\alpha^{exp}|}{\underline{\lambda}_\alpha^{exp}} \quad , \tag{18}$$

$$\Delta\widetilde{\Phi}(\mathbf{s},\boldsymbol{\delta}) = \sqrt{\frac{1}{r}\sum_{\alpha=1}^{r}\{\Delta\widetilde{\Phi}_\alpha(\mathbf{s},\boldsymbol{\delta})\}^2} \quad , \quad \Delta\widetilde{\Phi}_\alpha(\mathbf{s},\boldsymbol{\delta}) = \frac{||\widetilde{\boldsymbol{\Phi}}_\alpha(\mathbf{s},\boldsymbol{\delta}) - \underline{\boldsymbol{\varphi}}_\alpha^{exp}||}{||\underline{\boldsymbol{\varphi}}_\alpha^{exp}||} \quad . \tag{19}$$

In Eqs. (18) and (19), for each α belonging to $\{1,\ldots,r\}$, the positive-valued random eigenvalue $\Lambda_\alpha(\mathbf{s},\boldsymbol{\delta})$ and the $\mathbb{R}^{n_{obs}}$-valued random eigenvector $\widetilde{\boldsymbol{\Phi}}_\alpha(\mathbf{s},\boldsymbol{\delta})$ restricted to the measurement DOF are defined by the generalized eigenvalue problem related to the uncertain computational model which is written as: find $(\Lambda_\alpha(\mathbf{s},\boldsymbol{\delta}), \boldsymbol{\Psi}_\alpha(\mathbf{s},\boldsymbol{\delta}))$

$$\mathbf{0} = \Big([\mathbf{K}_{red,\alpha}(\mathbf{s},\delta_K)] - \Lambda_\alpha(\mathbf{s},\boldsymbol{\delta})\,[\mathbf{M}_{red,\alpha}(\mathbf{s},\delta_M)]\Big)\boldsymbol{\Psi}_\alpha(\mathbf{s},\boldsymbol{\delta}) \quad ,\alpha = 1,\ldots,r \quad , \tag{20}$$

for which random eigenvector $\widetilde{\boldsymbol{\Phi}}_\alpha(\mathbf{s},\boldsymbol{\delta})$ is reconstructed by

$$\widetilde{\boldsymbol{\Phi}}_\alpha(\mathbf{s},\boldsymbol{\delta}) = [\underline{H}_1]\,\boldsymbol{\Psi}_\alpha(\mathbf{s},\boldsymbol{\delta}) \quad , \tag{21}$$

where $[\underline{H}_1] = [[I] \quad [\mathbb{0}]]$ is the first row bloc of matrix $[\underline{H}_\alpha(\mathbf{s})]$. We now introduce the probabilistic constraints. Let $g_\Lambda(\mathbf{s},\boldsymbol{\delta};\beta_\Lambda,\varepsilon_\Lambda)$ and $g_{\widetilde{\Phi}}(\mathbf{s},\boldsymbol{\delta};\beta_\Phi,\varepsilon_\Phi)$ be the functions defined by

$$g_\Lambda(\mathbf{s},\boldsymbol{\delta};\beta_\Lambda,\varepsilon_\Lambda) = \beta_\Lambda - Proba\left(\Delta\Lambda(\mathbf{s},\boldsymbol{\delta}) < \varepsilon_\Lambda\right) \tag{22}$$

$$g_{\widetilde{\Phi}}(\mathbf{s},\boldsymbol{\delta};\beta_\Phi,\varepsilon_\Phi) = \beta_\Phi - Proba\left(\Delta\Phi(\mathbf{s},\boldsymbol{\delta}) < \varepsilon_\Phi\right) \quad , \tag{23}$$

in which $Proba$ denotes the probability and where ε_Λ, ε_Φ and β_Λ, β_Φ denote a given error level and a given probability level respectively. The robust updating formulation consists in defining, for a given $\boldsymbol{\beta} = (\beta_\Lambda,\beta_\Phi)$ belonging to $[0\,,\,1[\times[0\,,\,1[$ and for a given $\boldsymbol{\varepsilon} = (\epsilon_\Lambda,\epsilon_\Phi)$ belonging to $]0,+\infty[\times]0,+\infty[$, the solution $(\mathbf{s}^{opt},\boldsymbol{\delta}^{opt})$ as

$$(\mathbf{s}^{opt},\boldsymbol{\delta}^{opt}) = arg \min_{\substack{(\mathbf{s},\boldsymbol{\delta}) \in \{\mathcal{S}\times\Delta\} \\ \mathbf{g}(\mathbf{s},\boldsymbol{\delta};\boldsymbol{\beta},\boldsymbol{\varepsilon}) < \mathbf{0}}} j(\mathbf{s},\boldsymbol{\delta}) \quad , \tag{24}$$

in which $\mathbf{g}(\mathbf{s},\boldsymbol{\delta};\boldsymbol{\beta},\boldsymbol{\varepsilon}) = (g_\Lambda(\mathbf{s},\boldsymbol{\delta};\beta_\Lambda,\varepsilon_\Lambda), g_{\widetilde{\Phi}}(\mathbf{s},\boldsymbol{\delta};\beta_\Phi,\varepsilon_\Phi))$. The existence of a solution for this optimization problem cannot be proven in the general case. A specific analysis must be carried out for every application (see Section V).

5 Numerical Validation

5.1 Description of the mean finite element model

The numerical validation is carried out using the truss system presented in [1]. This structure is located in the plane $(OX\,,\,OY)$ of a Cartesian coordinate system. The truss is constituted of 4 vertical bars, 4 diagonal bars and 2 horizontal beams. For the non updated truss, all the bars and beams are made up of a homogeneous isotropic elastic material with mass density $\rho_0 = 2\,800\,kg \times m^{-3}$, Poisson ratio $\nu_0 = 0.3$ and Young modulus $E_0 = 0.75 \times 10^{11}\,N \times m^{-2}$. The vertical bars have a constant cross-section of $0.6 \times 10^{-2}\,m^2$ and a length of $3\,m$. The diagonal bars have a constant cross-section of $0.3 \times 10^{-2}\,m^2$ and a length of $5.83\,m$. The horizontal beams have a constant cross-section of $S_0 = 0.4 \times 10^{-2}\,m^2$, a constant beam inertia of $0.756 \times 10^{-1}\,m^4$ and a length of $15\,m$. The truss has free-free boundary conditions. The mean finite element model of this truss is constituted of 41 bar elements (with two nodes) and 42 beam elements (with two nodes) yielding $n = 166$ DOF (see Fig. 1). There is only one updating parameter $s = \rho\,S_0$ with ρ the mass density of the upper beam which has to be updated. It should be noted that for this non updated truss, $s_0 = 11.2\,kg/m$. The admissible set $\mathcal{S}$ for the updating parameter s of the mean computational model is taken as $\mathcal{S} = [10\,,\,40]\,kg/m$.

5.2 Description of an data basis

Since no experiment has been carried out on this truss, a numerical experiment is generated to represent the experimental data basis. The experimental data are simulated as follows. We consider the stochastic computational model corresponding to the mean computational model with uncertainties and defined by Eqs. (20) and (21). For $s = s_0$ and $\delta_K = \delta_M = \delta_0$ with $\delta_0 = 0.3$, one realization $\Lambda_\alpha(s_0,\delta_0;\theta)$ of the random eigenvalues $\Lambda_\alpha(s_0, \delta_0)$, and the corresponding realization $\widetilde{\mathbf{\Phi}}_\alpha(s_0, \delta_0; \theta)$ of the random eigenvectors $\widetilde{\mathbf{\Phi}}_\alpha(s_0, \delta_0)$ are calculated using the stochastic computational model. Then, an arbitrary finite perturbation is applied to every eigenvalues $\Lambda_\alpha(s_0, \delta_0; \theta)$ without modifying the eigenmodes $\widetilde{\mathbf{\Phi}}_\alpha(s_0, \delta_0; \theta)$ and thus defining the experimental data. Consequently, this experimental data cannot be obtained with a deterministic updating of the truss ($\delta_M = \delta_K = 0$) for which the mass density ρ of the upper beam is the updating parameter. The experimental data basis is thus constituted of (1) $r = 3$ elastic experimental eigenfrequencies $\underline{\nu}_1^{exp} = 93\, Hz$, $\underline{\nu}_2^{exp} = 110\, Hz$ and $\underline{\nu}_3^{exp} = 170\, Hz$ and (2) the translational components corresponding to $n_{obs} = 28$ translational measured DOF and representing the corresponding experimental eigenmodes (see Fig. 1).

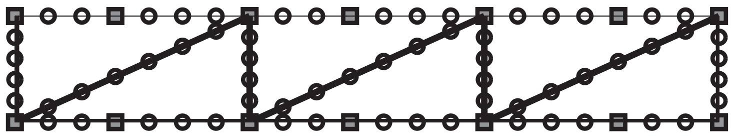

Figure 1: Finite element mesh of the truss. Symbol $\circ$: nodes of the mesh, symbol $\Box$ measured nodes, thick solid line: elements with fixed properties, thin solid line: elements whose properties have to be updated.

5.3 Deterministic updating

The results concerning the deterministic updating formulation (see Section 2) are presented in order to construct a reference solution. The deterministic updating optimization problem yields $s^{opt,det} = 31\, kg/m$ for which cost function $\underline{j}(s^{opt,det})$ is normalized to 1. Figure 2 and Table 1 quantify the differences with respect to each eigenfrequency and with respect to each eigenmode for the non updated mean computational model and for the updated mean computational model. For a given α belonging to $\{1, \ldots, r\}$, we introduce $\Delta\underline{\lambda}_\alpha(s)\, \underline{\lambda}_\alpha^{exp} = |\underline{\lambda}_\alpha(s) - \underline{\lambda}_\alpha^{exp}|$ and $\Delta\tilde{\underline{\phi}}_\alpha(s)\, ||\underline{\boldsymbol{\varphi}}_\alpha^{exp}|| = ||\tilde{\underline{\boldsymbol{\varphi}}}_\alpha(s) - \underline{\boldsymbol{\varphi}}_\alpha^{exp}||$. Let $\Delta\underline{\lambda}_\alpha^{ini} = \Delta\underline{\lambda}_\alpha(s_0)$, $\Delta\underline{\lambda}_\alpha^{opt,det} = \Delta\underline{\lambda}_\alpha(s^{opt,det})$, $\Delta\tilde{\underline{\phi}}_\alpha^{ini} = \Delta\tilde{\underline{\phi}}_\alpha(s_0)$, $\Delta\tilde{\underline{\phi}}_\alpha^{opt,det} = \Delta\tilde{\underline{\phi}}_\alpha(s^{opt,det})$ the similar quantities to those defined in Eq. (12) but for the deterministic case. Figure 2 shows the graphs $\alpha \mapsto \Delta\underline{\lambda}_\alpha^{ini}$, $\alpha \mapsto \Delta\underline{\lambda}_\alpha^{opt,det}$, $\alpha \mapsto \Delta\tilde{\underline{\phi}}_\alpha^{ini}$ and $\alpha \mapsto \Delta\tilde{\underline{\phi}}_\alpha^{opt,det}$. The results show the efficiency of the deterministic updating formulation to reduce the gap between the experiments and between the computational model. Nevertheless, the cost function is not zero which means that model uncertainties have to be taken into account in the modeling of the computational model which has to be updated.

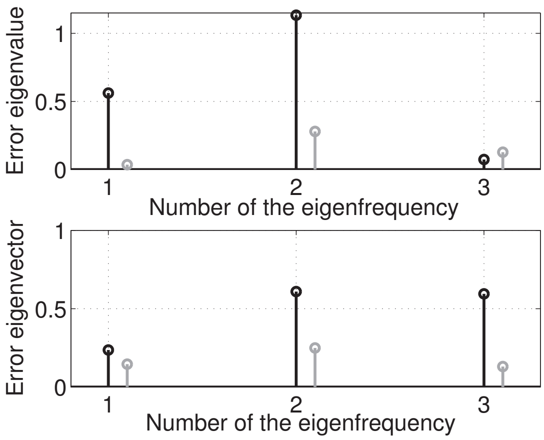

Figure 2: Quantification of the errors between the non updated and the updated mean computational model with the experimental data. Upper graph : graph of $\alpha \mapsto \Delta\underline{\lambda}_\alpha^{ini}$ (black line), $\alpha \mapsto \Delta\underline{\lambda}_\alpha^{opt,det}$ (gray line). Lower graph : graph of $\alpha \mapsto \Delta\underline{\tilde{\phi}}_\alpha^{ini}$ (black line), $\alpha \mapsto \Delta\underline{\tilde{\phi}}_\alpha^{opt,det}$ (gray line).

	$\Delta\underline{\lambda}_1$	$\Delta\underline{\lambda}_2$	$\Delta\underline{\lambda}_3$	$\Delta\underline{\tilde{\phi}}_1$	$\Delta\underline{\tilde{\phi}}_2$	$\Delta\underline{\tilde{\phi}}_3$
non updated	56.1%	113%	7.3%	23.6%	60.9%	59.4%
deterministic updating	3.6%	27.9%	12.7%	14.6%	24.9%	13%

Table 1: Quantification of the errors between the non updated and the updated mean computational model with the experimental data.

5.4 Convergence analysis with respect to the numerical parameters

In the context of the robust updating, the stochastic equations of the uncertain computational model are solved by using the Monte Carlo numerical simulation. In order to simplify the calculations, the same level of uncertainties is considered for the mass and for the stiffness terms, that is to say $\delta = \delta_M = \delta_K$. A convergence analysis is carried out in order to calculate the number N of eigenmodes to be kept in the modal reduction and the number n_s of realizations. The mean square convergence is analyzed by studying the function $(N, n_s) \mapsto Conv(N, n_s)$ defined by

$$Conv^2(N, n_s) = \frac{1}{n_s} \sum_{i=1}^{n_s} ||[\boldsymbol{\mathcal{R}}(s, \delta; \theta_i)]||_F^2 \quad , \tag{25}$$

in which $[\boldsymbol{\mathcal{R}}(s, \delta; \theta_i)]$ is the realization number i of random matrix $[\boldsymbol{\mathcal{R}}(s, \delta)]$ given by Eq. (17). The convergence analysis is carried out with $s = 11.2\, kg/m$ and with $\delta = 0.3$. Figure 3 shows the graph $n_s \mapsto Conv(N, n_s)$ for different values of N. It can be seen that a reasonable convergence is reached for $N = 110$ ans $n_s = 600$. From now on, the numerical calculations are carried out with the numerical parameters $N = 110$ ans $n_s = 600$.

5.5 Robust updating formulation without inequality constraints

As we have explained in Section 4, the robust updating formulation without inequality constraints does not allow the updating to be improved with respect to the presence of model uncertainties. In this subsection, we prove this result by using the numerical example. First, the case for which the level of uncertainty in the structure is assumed to be known is considered with $\delta = \delta^{fix} = 0.3$. The updated uncertain computational

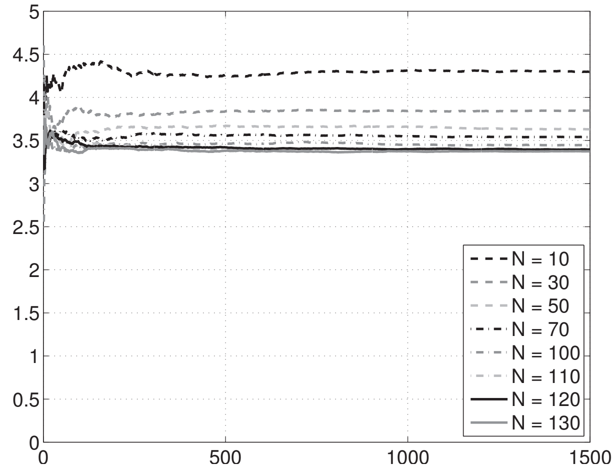

Figure 3: Convergence analysis : graph of function $n_s \mapsto Conv(N, n_s)$ for the truss structure with updating parameter $s = 11.2\, Kg/m$ and $\delta = 0.3$.

model is characterized by updating parameters $(s^{opt}, \delta^{fix}) = (26.2, 0.3)$ for which $j(s^{opt}, \delta^{fix}) = 1.18$. The generalized eigenvalue problem related to the updated uncertain computational model is then solved by using $n_s = 10\,000$ realizations in order to characterize, for each α belonging to $\{1, 2, 3\}$ the probability density functions of the random variables $\Delta\Lambda_\alpha^{opt} = \Delta\Lambda(s^{opt}, \delta^{fix})$ and $\Delta\widetilde{\Phi}_\alpha^{opt} = \Delta\widetilde{\Phi}(s^{opt}, \delta^{fix})$. For each α belonging to $\{1, 2, 3\}$, Table 2 shows the mean values $\mu_{\Delta\lambda_\alpha}$ and $\mu_{\Delta\tilde{\Phi}_\alpha}$, and the standard deviations $\sigma_{\Delta\lambda_\alpha}$ and $\sigma_{\Delta\tilde{\Phi}_\alpha}$ of the random variables $\Delta\Lambda_\alpha^{opt}$ and $\Delta\widetilde{\Phi}_\alpha^{opt}$. Figures 4 shows the probability density functions of the random variables $\Delta\Lambda_\alpha^{opt}$ and $\Delta\widetilde{\Phi}_\alpha^{opt}$. It can be seen that the mean error committed on each eigenvalue is lower than 29% and the mean error committed on each eigenvector is lower than 19%. Figure 5 shows the family of graphs corresponding to the function $\delta \mapsto j(s, \delta)$ for the admissible set $\mathcal{S}$. Clearly, it can be seen that if the uncertainty level is unknown, then the robust updating optimization problem goes to the deterministic solution presented in subsection C.

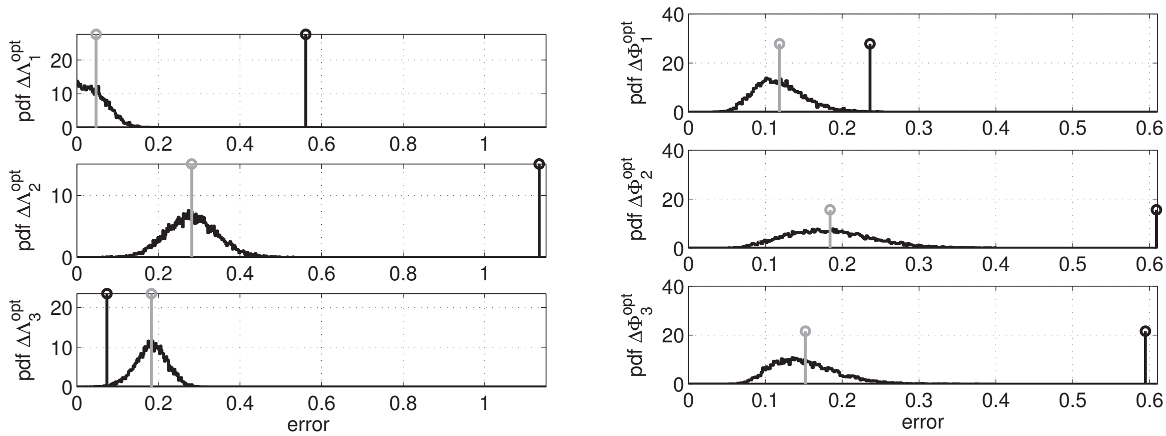

Figure 4: Updated uncertain computational model corresponding to $(s^{opt}, \delta^{fix}) = (26.2, 0.3)$. Left Figure : graph of the probability density functions $\Delta\Lambda_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\lambda}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph). Right Figure : graph of the probability density functions $\Delta\widetilde{\Phi}_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\tilde{\phi}}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph).

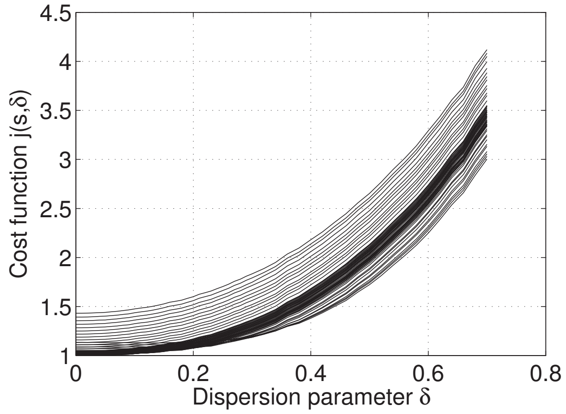

Figure 5: Family of graphs $\delta \mapsto j(s, \delta)$ for s belonging to $\mathcal{S}$.

5.6 Robust updating formulation with inequality constraints

We now present the results concerning the robust updating formulation in presence of inequality constraints obtained with Eq. (24). The updated mean parameter s^{opt} and the updated parameter δ^{opt} are analyzed as a function of the probability level and of the error level. Two cases are considered : (1) the case for which there is only one probabilistic constraint for the eigenvalue corresponding to $\beta_\Phi = 0$ and $\varepsilon_\Phi = +\infty$. We then study the function $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto \delta^{opt}$ defined from the domain $\mathcal{D}_{\Lambda,\delta}$ into the set $\mathcal{F}_{\Lambda,\delta}$ and the function $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto s^{opt}$ defined from the domain $\mathcal{D}_{\Lambda,s}$ into the set $\mathcal{F}_{\Lambda,s}$; (2) the case for which there is one probabilistic constraint for the eigenvector corresponding to $\beta_\Lambda = 0$ and $\varepsilon_\Lambda = +\infty$. We then study the function $(\beta_\Phi, \varepsilon_\Phi) \mapsto \delta^{opt}$ defined from the domain $\mathcal{D}_{\Phi,\delta}$ into the set $\mathcal{F}_{\Phi,\delta}$ and the function $(\beta_\Phi, \varepsilon_\Phi) \mapsto s^{opt}$ defined from the domain $\mathcal{D}_{\Phi,s}$ into the set $\mathcal{F}_{\Phi,s}$. The Figure 6 shows a bi-dimensional representation of the graph of the functions $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto \delta^{opt}$ and $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto s^{opt}$ (case 1). Figure 7 shows the graph of the functions $(\beta_\Phi, \varepsilon_\Phi) \mapsto \delta^{opt}$ and $(\beta_\Phi, \varepsilon_\Phi) \mapsto s^{opt}$ (case 2). In these figures, the blank zone corresponds to the values of the probability level and of the error level for which the optimization problem defined by Eq. (24) has no solution. By comparing figures 6 and 7, it can be seen that $\mathcal{D}_{\Lambda,\delta} \subset \mathcal{D}_{\Phi,\delta}$ and that $\mathcal{D}_{\Lambda,s} \subset \mathcal{D}_{\Phi,s}$ which means that the robust updating methodology allows the random eigenvectors to be better updated than the random eigenvalues. In addition, Figure 6 shows that significant model uncertainties ($\delta^{opt} > 0.1$) are obtained for small values of probability level ($\beta < 0.2$). In opposite, Figure 7 shows that significant model uncertainties on the eigenvectors ($\delta^{opt} > 0.1$) are obtained for large values of the probability level ($\beta < 0.6$). These results are coherent because we have introduced in the experimental data model errors only on the eigenvalues. The figures 6 to 7 show that $\mathcal{F}_{\Lambda,\delta} = [0\,,\,0.25]$, $\mathcal{F}_{\Phi,\delta} = [0\,,\,0.18]$, and $\mathcal{F}_{\Lambda,s} = [31\,,\,36.4]$, $\mathcal{F}_{\Phi,s} = [22.4\,,\,31.1]$. Clearly, the sets $\mathcal{F}_{\Lambda,s}$ and $\mathcal{F}_{\Phi,s}$ are almost disjoint which means that the optimal uncertain computational model strongly depends on the nature of the constraints used in the robust updating formulation. It can also be seen that the updated uncertain computational model related to the eigenvector probabilistic constraint is more sensitive to the updated mean parameter s^{opt} than to the updated dispersion parameter δ^{opt} whereas the contrary is observed when using the robust updating formulation related to the eigenvalue probabilistic constraint. Moreover, it can be seen that $\mathcal{F}_{\Phi,\delta} \subset \mathcal{F}_{\Lambda,\delta}$.

In order to analyze more precisely the results presented in the Fig. 6 to 7, we reanalyze the three cases for an error level equal to 0.25 with a probability level equal to 0.1. For α belonging to $\{1, 2, 3\}$, let $\mu_{\Delta\Lambda_\alpha}$, $\mu_{\Delta\tilde{\Phi}_\alpha}$ and $\sigma_{\Delta\Lambda_\alpha}$, $\sigma_{\Delta\tilde{\Phi}_\alpha}$ be the mean value and the standard deviation of random variable $\Delta\Lambda_\alpha$ and $\Delta\widetilde{\Phi}_\alpha$ defined by Eqs. (18) and (19). For each case, the main characteristics of the updated uncertain computational model are summarized in Tables 2 and 3. In order to characterize the efficiency of the proposed robust updating methodology, Figs. 8 to 9 show the probability density functions of the random variables $\Delta\Lambda_\alpha^{opt}$ and $\Delta\widetilde{\Phi}_\alpha^{opt}$ for the two cases. These figures show that the updating is improved in the probabilistic context because

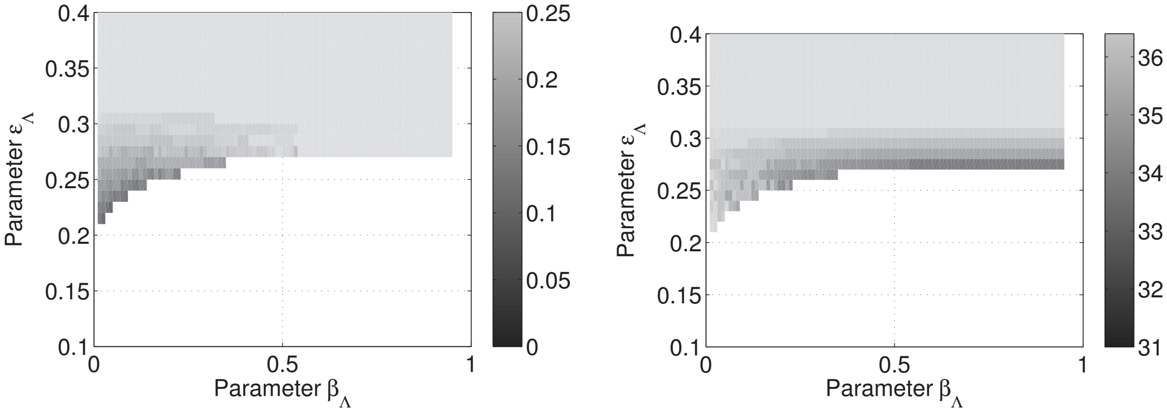

Figure 6: Left figure : graph of δ^{opt} with respect to β_Λ and ε_Λ for $\beta_\Phi = 0$, $\varepsilon_\Phi = +\infty$. Right figure : graph of s^{opt} with respect to β_Λ and ε_Λ for $\beta_\Phi = 0$, $\varepsilon_\Phi = +\infty$.

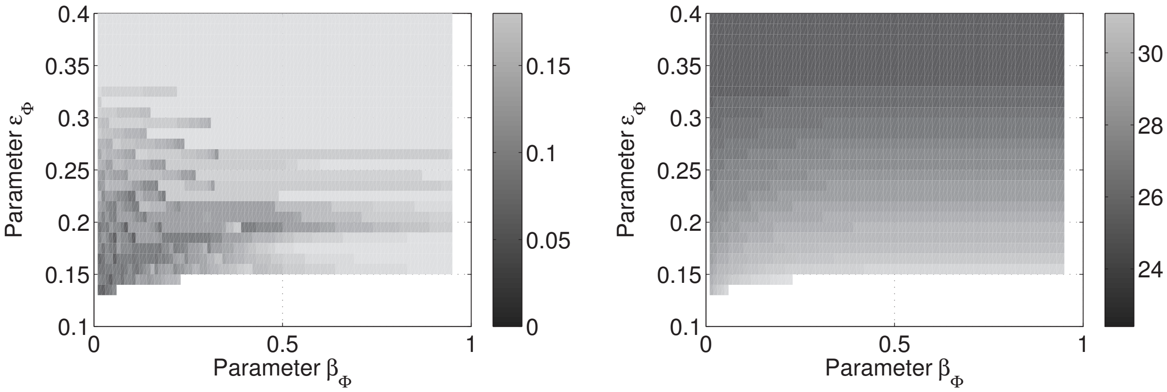

Figure 7: Left figure : graph of δ^{opt} with respect to β_Φ and ε_Φ for $\beta_\Lambda = 0$, $\varepsilon_\Lambda = +\infty$. Right figure : graph of s^{opt} with respect to β_Φ and ε_Φ for $\beta_\Lambda = 0$, $\varepsilon_\Lambda = +\infty$.

the value of the error is smaller than for the non updated mean computational model. It can be seen that if only one constraint is considered, then the other one is not verified which means that there can remain an important error (for instance $\mu_{\Delta\Lambda_\alpha} = 0.33$ for case 2 for which there is only one eigenvector probability constraint). Moreover, it should be noted that the robust updating using both constraints guarantees that the mean error committed for each eigenvalue and eigenvector with respect to the experimental data is lower than 23.5% (note that this result is not presented in the manuscript).

	$\mu_{\Delta\lambda_1}$	$\mu_{\Delta\lambda_2}$	$\mu_{\Delta\lambda_3}$	$\sigma_{\Delta\lambda_1}$	$\sigma_{\Delta\lambda_2}$	$\sigma_{\Delta\lambda_3}$
constraint on eigenvalue	7.7%	22.7%	15.2%	2.1%	2.8%	2%
constraint on eigenvector	1%	33%	11%	0.7%	1.2%	0.8%
no constraint , $\delta^{fix} = 0.3$	4.7%	28.2%	18.2%	3.3%	6.1%	3.8%
	$\mu_{\Delta\tilde{\Phi}_1}$	$\mu_{\Delta\tilde{\Phi}_2}$	$\mu_{\Delta\tilde{\Phi}_3}$	$\sigma_{\Delta\tilde{\Phi}_1}$	$\sigma_{\Delta\tilde{\Phi}_2}$	$\sigma_{\Delta\tilde{\Phi}_3}$
constraint on eigenvalue	16%	27.5%	15.1%	1.7%	3%	2.6%
constraint on eigenvector	12.2%	19.9%	11.3%	0.7%	1.2%	1%
no constraint , $\delta^{fix} = 0.3$	11.8%	18.4%	15.2%	3.2%	5.5%	4.2%

Table 2: Quantification of the errors induced by the updated computational model with respect to the experimental data.

	s^{opt}	δ^{opt}	$j(s^{opt}, \delta^{opt})$	$-g_\Lambda(s^{opt}, \delta^{opt}, 0.25, 0.1)$	$-g_{\tilde{\Phi}}(s^{opt}, \delta^{opt}, 0.25, 0.1)$
constraint on eigenvalue	32.2	0.15	1.06	0.014	< 0
constraint on eigenvector	28.6	0.06	1.03	< 0	0.024
no constraint , $\delta^{fix} = 0.3$	26.2	0.3	1.18	< 0	0.27

Table 3: Characteristics of the updated computational model for each case.

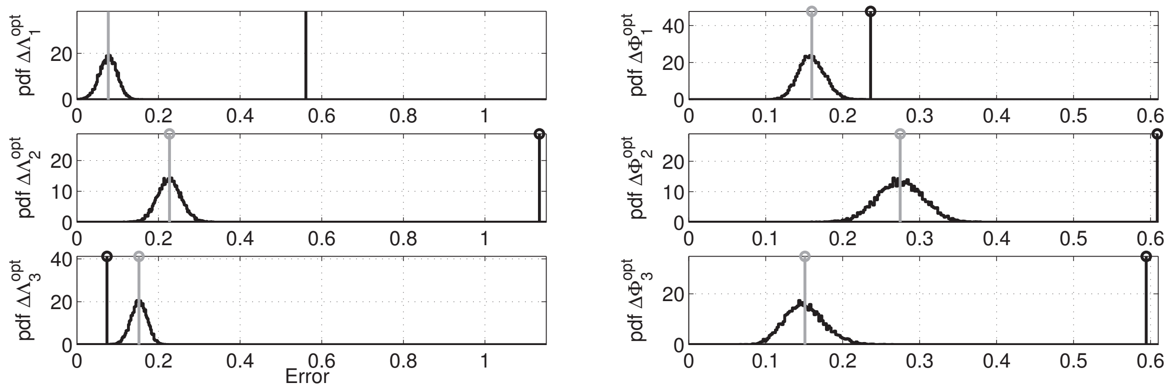

Figure 8: Updated uncertain computational model corresponding to $\beta_\Phi = 0$, $\epsilon_\Phi = +\infty$, $\beta_\Lambda = 0.1$, $\epsilon_\Lambda = 0.25$ and yielding $(s^{opt}, \delta^{opt}) = (32.2, 0.15)$. Left figure : graph of the probability density functions $\Delta\Lambda_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\lambda}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph). Right figure : graph of the probability density functions $\Delta\widetilde{\Phi}_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\tilde{\phi}}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph).

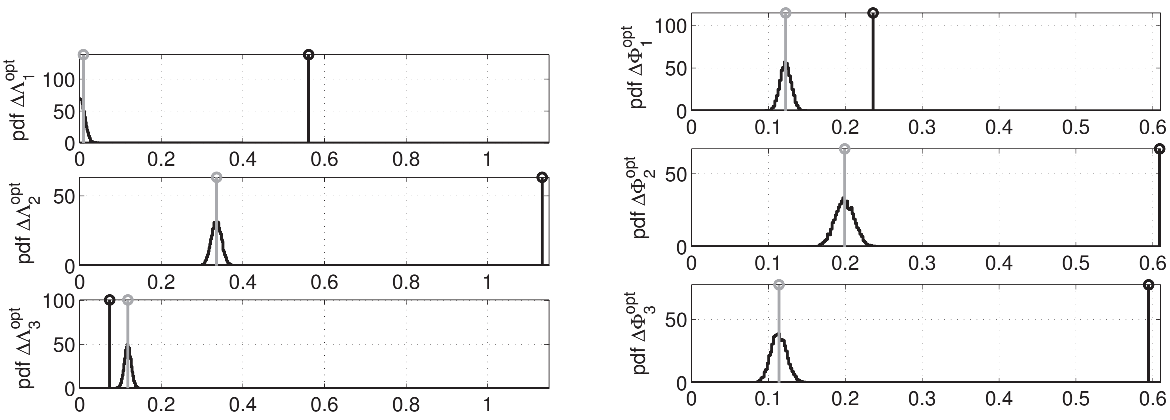

Figure 9: Updated uncertain computational model corresponding to $\beta_\Lambda = 0$, $\epsilon_\Lambda = +\infty$, $\beta_\Phi = 0.1$, $\epsilon_\Phi = 0.25$ and yielding $(s^{opt}, \delta^{opt}) = (28.6, 0.06)$. Left figure : graph of the probability density functions $\Delta\Lambda_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\lambda}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph). Right figure : graph of the probability density functions $\Delta\widetilde{\Phi}_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\widetilde{\phi}}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph).

6 Conclusions

A not straightforward methodology to perform the robust updating of complex uncertain dynamical systems with respect to modal experimental data in the context of structural dynamics has been presented. The present formulation based on an input error methodology adapted to the deterministic updating problem has been extended to the robust updating context required in presence of model uncertainties in the computational model. The robust updating formulation leads a mono-objective optimization problem to be solved in presence of inequality probabilistic constraints. An application is presented in order to validate the proposed approach.

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