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Approximate twistors and positive mass

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Abstract

In this paper the problem of comparing initial data to a reference solution for the vacuum Einstein field equations is considered. This is not done in a coordinate sense, but through quantification of the deviation from a specific symmetry. In a recent paper [T. Bäckdahl, J.A. Valiente Kroon, Phys. Rev. Lett. **104**, 231102 (2010)] this problem was studied with the Kerr solution as a reference solution. This analysis was based on valence 2 Killing spinors. In order to better understand this construction, in the present article we analyse the analogous construction for valence 1 spinors solving the twistor equation. This yields an invariant that measures how much the initial data deviates from Minkowski data. Furthermore, we prove that this invariant vanishes if and only if the mass vanishes. Hence, we get a proof of the positivity of mass.

1 Introduction

The idea that constructions involving spinor fields are a valuable tool for the analysis of global properties of 3-manifolds was brought to the fore by Witten's proof of the positivity of the mass [17]. Witten's ideas were then used to prove the positivity of other notions of mass like Bondi's —see e.g. [11, 10].

In a recent paper [1] spinorial equations have motivated the construction of a new geometric invariant for initial data sets for the vacuum Einstein field equations —see also [2] for a more detailed exposition. The motivation behind this construction is to develop a method to compare spacetimes (and initial data sets thereof) in a gauge-independent, coordinate-free manner. The key idea is to carry out this comparison by quantifying how much the spacetime (or its initial data) fail to have a particular symmetry. Following this strategy, it is possible to obtain a geometric invariant with the property of vanishing if and only if the initial data set corresponds to initial data for the Kerr spacetime —thus, it measures the *non-Kerrness* of the data.

The starting point of the non-Kerrness is the notion of valence 2 Killing spinors. These are spinorial fields $\kappa_{AB} = \kappa_{(AB)}$ satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0.$$

The existence of such spinorial fields in the development of initial data sets for the vacuum Einstein field equations can be encoded at the level of the data via the so-called *Killing spinor initial data equations* —see [1, 2, 8]. These equations include among others the so-called *spatial Killing spinor equation* (an explanation of the notation is given in the sequel)

$$\nabla_{(AB}\kappa_{CD)} = 0. \tag{1}$$

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This equation admits, for a generic initial data set for the Einstein field equations, only the trivial solution. The key insight of [1, 2] was that if one composes the differential operator of this equation with its formal adjoint one obtains an elliptic equation which with the appropriate boundary conditions can be shown to always have a solution. If one evaluates the resulting spinor in the Killing spinor initial data equations one obtains a quantitative measure of the deviation from the existence of a symmetry in the data. The existence of a Killing spinor is a strong condition to be imposed on a vacuum spacetime. As a consequence, it turns out that the construction described in this paragraph can be used to provide a characterisation of data for the Kerr spacetime.

The purpose of the present article is to shed light into the construction of [1, 2] by analysing an analogous construction motivated by the *twistor equation* (or *valence 1 Killing spinor equation*):

$$\nabla_{A'}(A\kappa_B) = 0.$$

In this case, the analogue of equation (1) is the *spatial twistor equation*

$$\nabla_{(AB}\kappa_{C)} = 0.$$

This equations has been used to provide conditions on a 3-manifold to be embeddable in a conformally flat spacetime —see [16]. The analogous of the procedure of [1, 2] then produces an invariant that turns out to be related to the mass.

It should be emphasised that in contrast to the analysis of [1, 2] which could be performed, to some greater length, using tensorial methods, the present discussion is intrinsically spinorial.

Outline of the article

Section 2 provides a brief discussion of the theory of spacetimes with solutions to the so-called twistor equation. In particular, it provides a characterisation of the Minkowski spacetime in terms of the existence of a specific solution to this equation. Section 3 provides a short overview of the space spinor formalism to be used in our analysis. Section 4 is concerned with the question of how to encode in an initial data set that its development will have a solution to the twistor equation. Section 5 introduces the approximate twistor equation: an elliptic equation which with suitable conditions always admits a unique solution for asymptotically Euclidean initial data sets —see Theorem 2. Whereas Section 5 is concerned with formal elliptic properties of the equation, Section 6 discusses its solvability for asymptotically Euclidean manifolds. Section 7 presents a characterisation of Minkowski initial data by means of a geometric invariant constructed out of the solution to the approximate twistor equation provided by Theorem 2. Finally, in Section 8 we discuss the connection between our invariant and the mass of the data.

General notation and conventions

All throughout, $(\mathcal{M}, g_{\mu\nu})$ will be an orientable and time orientable globally hyperbolic vacuum spacetime. Here, and in what follows, μ, ν, \dots denote abstract 4-dimensional tensor indices. The metric $g_{\mu\nu}$ will be taken to have signature $(+, -, -, -)$. Let ∇_μ denote the Levi-Civita connection of $g_{\mu\nu}$. The sign of the Riemann tensor will be given by the equation

$$\nabla_\mu \nabla_\nu \xi_\zeta - \nabla_\nu \nabla_\mu \xi_\zeta = R_{\nu\mu\zeta}{}^\eta \xi_\eta$$

The triple $(\mathcal{S}, h_{ab}, K_{ab})$ will denote initial data on a hypersurface of the spacetime $(\mathcal{M}, g_{\mu\nu})$. The symmetric tensors h_{ab}, K_{ab} will denote, respectively, the 3-metric and the extrinsic curvature of the 3-manifold \mathcal{S} . The metric h_{ab} will be taken to be negative definite —that is, of signature $(-, -, -)$. The indices a, b, \dots will denote abstract 3-dimensional tensor indices, while i, j, \dots will denote 3-dimensional tensor coordinate indices. Let D_a denote the Levi-Civita covariant derivative of h_{ab} .

Spinors will be used systematically. We follow the conventions of [13]. In particular, A, B, \dots will denote abstract spinorial indices, while $\mathbf{A}, \mathbf{B}, \dots$ will be indices with respect to a specific

frame. Tensors and their spinorial counterparts are related by means of the solder form $\sigma_\mu^{AA'}$ satisfying $g_{\mu\nu} = \sigma_\mu^{AA'} \sigma_\nu^{BB'} \epsilon_{AB} \epsilon_{A'B'}$, where ϵ_{AB} is the antisymmetric spinor and $\epsilon_{A'B'}$ its complex conjugate copy. One has, for example, that $\xi_\mu = \sigma_\mu^{AA'} \xi_{AA'}$. Let $\nabla_{AA'}$ denote the spinorial counterpart of the spacetime connection ∇_μ . Besides the connection $\nabla_{AA'}$, two other spinorial connections will be used: D_{AB} , the spinorial counterpart of the Levi-Civita connection D_a and ∇_{AB} , the Sen connection of $(\mathcal{M}, g_{\mu\nu})$. Apart from these derivatives we will also use the normal derivative $\nabla \equiv \tau^\mu \nabla_\mu$. Full details will be given in Section 3.

2 The twistor equation: general theory

A *valence 1 Killing spinor* is a spinor κ_A satisfying the *twistor equation*

$$\nabla_{A'}(A\kappa_B) = 0. \quad (2)$$

Taking a further derivative of equation (2), antisymmetrising and commuting the covariant derivatives one finds the integrability condition

$$\Psi_{ABCD}\kappa^D = 0, \quad (3)$$

where Ψ_{ABCD} denotes the self-dual Weyl spinor. The above integrability imposes strong restrictions on the algebraic type of the Weyl spinor as it states that the vacuum has to be of Petrov type N or O.

2.1 A characterisation of the Minkowski spacetime

We will use valence 1 Killing spinors to formulate a characterisation of the Minkowski spacetime.

Proposition 1. *Assume that κ_A is a solution to $\nabla_{A'}(A\kappa_B) = 0$ in an asymptotically flat vacuum spacetime. Let $\eta_A \equiv \nabla_A{}^{A'}\bar{\kappa}_{A'}$. If $\eta_A \neq 0$ at some point, then the spacetime is isomorphic to the Minkowski spacetime.*

Proof. As the twistor equation holds, then the integrability condition (3) is satisfied. Similar calculations show that $\square\kappa_A = 0$ and $\nabla_{A'}(A\eta_B) = 0$. Therefore we also have $\Psi_{ABCD}\eta^D = 0$. Furthermore we find $\nabla^A{}_{A'}\eta_A = \frac{1}{2}\square\bar{\kappa}_{A'} = 0$. This means that $\nabla_{AA'}\eta_A = 0$. Assume now that $\eta_A \neq 0$ at some point. Then one has that $\eta_A \neq 0$ everywhere.

We will now prove that the set $\eta^A\kappa_A = 0$ does not have any interior points. We do this by contradiction. Assume $\eta^A\kappa_A = 0$ in a neighbourhood of a point p . By the relation $\nabla_{AA'}(\eta^B\kappa_B) = \frac{1}{2}\eta_A\bar{\eta}_{A'}$ we then see that also $\eta_A\bar{\eta}_{A'} = 0$ in a neighbourhood of p . This contradicts $\eta_A \neq 0$. Hence, the set $\eta^A\kappa_A = 0$ does not have any interior points, thus this set has measure zero. This means that we can use $\{\eta_A, \kappa_A\}$ locally as a dyad on some neighbourhood around almost every point in the manifold. From the conditions $\Psi_{ABCD}\kappa^D = 0$, $\Psi_{ABCD}\eta^D = 0$ we conclude that $\Psi_{ABCD} = 0$ at almost every point. By continuity we get $\Psi_{ABCD} = 0$ at every point on the manifold. Hence, the spacetime is conformally flat. Together with asymptotic flatness and the vacuum field equations, we get that the spacetime is flat, i.e. isomorphic to the Minkowski spacetime. \square

3 Space spinors: general theory

The analysis in this article is based on an analysis of the space spinor split of equation (2). Here we follow the conventions and notations introduced in [1, 2].

3.1 Basic definitions

Let τ^ν be the future pointing vector tangent to a congruence of timelike curves and let $\tau^{AA'}$ denote its spinorial counterpart. We will use the normalization $\tau_{AA'}\tau^{AA'} = 2$. Also, let K_{ABCD} denote the spinorial counterpart of the second fundamental form K_{ab} . Furthermore, let

$$\Omega_{ABCD} \equiv K_{(ABCD)}, \quad K = K_{PQ}{}^{PQ}.$$

The Sen connection associated to the congruence defined by $\tau^{AA'}$ is given by

$$\nabla_{AB} \equiv \tau_{(A}{}^{A'} \nabla_{B)A'}.$$

The latter can be written in terms of the intrinsic spinorial Levi-Civita covariant derivative D_{AB} and the spinor K_{ABCD} . For example, given a valence 1 spinor π_A one has that

$$\nabla_{AB}\pi_C = D_{AB}\pi_C + \frac{1}{2}K_{ABC}{}^Q\pi_Q,$$

with the obvious generalisations to higher valence spinors. Furthermore we denote the normal derivative with $\nabla \equiv \tau^{AA'}\nabla_{AA'}$. Observe that one can tell the different derivatives apart by their indices. The spacetime derivative $\nabla_{AA'}$ has one unprimed index and one primed index, whereas the Sen connection ∇_{AB} has two unprimed indices. The normal derivative ∇ has no indices.

Essential for our analysis is the notion of Hermitian conjugation. Again, given the spinor π_A , we define its *Hermitian conjugate* via

$$\hat{\pi}_A \equiv \tau_A{}^{B'} \bar{\pi}_{B'}.$$

The Hermitian conjugate can be extended to higher valence symmetric spinors in the obvious way. The symmetric spinors ν_{AB} and ξ_{ABCD} are said to be real if

$$\hat{\nu}_{AB} = -\nu_{AB}, \quad \hat{\xi}_{ABCD} = \xi_{ABCD}.$$

It can be verified that $\nu_{AB}\hat{\nu}^{AB}$, $\xi_{ABCD}\hat{\xi}^{ABCD} \geq 0$. If the spinors are real, then there exist real spatial tensors ν_a , ξ_{ab} such that ν_{AB} and ξ_{ABCD} are their spinorial counterparts. For symmetric spinors with an odd number of indices like κ_A , ξ_{ABC} there is no corresponding notion of reality. However, it can still be shown that $\kappa_A\hat{\kappa}^A$, $\xi_{ABC}\hat{\xi}^{ABC} \geq 0$. The differential operator D_{AB} is real in the sense that

$$\widehat{D_{AB}\pi_C} = -D_{AB}\hat{\pi}_C.$$

However, for the Sen covariant derivative one has that

$$\widehat{\nabla_{AB}\pi_C} = -\nabla_{AB}\hat{\pi}_C + \frac{1}{2}K_{ABC}{}^D\hat{\pi}_D.$$

The restriction to \mathcal{S} of an arbitrary spinor expression with only unprimed indices can be treated as a spinor expression intrinsic to \mathcal{S} . From Section 4.4 to the end of the paper all expressions will be treated as intrinsic to \mathcal{S} . Before that it will be clear from the context if an expression is valid only on the slice \mathcal{S} or on the entire spacetime.

3.2 Commutators

Let

$$\square_{AB} \equiv \nabla_{C'(A}\nabla_{B)}{}^{C'}, \quad \hat{\square}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\square_{A'B'} = \tau_A{}^{A'}\tau_B{}^{B'}\nabla_{C(A'}\nabla_{B')}{}^C.$$

In vacuum, the action of these operators on a spinor π_A is given by

$$\square_{AB}\pi_C = \Psi_{ABCQ}\pi^Q, \quad \hat{\square}_{AB}\pi_C = 0.$$

In terms of \square_{AB} and $\hat{\square}_{AB}$, the commutators of ∇ and ∇_{AB} read

$$[\nabla, \nabla_{AB}] = \hat{\square}_{AB} - \square_{AB} - \frac{1}{2}K_{AB}\nabla + K^D{}_{(A}\nabla_{B)D} - K_{ABCD}\nabla^{CD}, \quad (4a)$$

$$\begin{aligned} [\nabla_{AB}, \nabla_{CD}] = & \frac{1}{2}(\epsilon_{A(C}\square_{D)B} + \epsilon_{B(C}\square_{D)A}) + \frac{1}{2}(\epsilon_{A(C}\hat{\square}_{D)B} + \epsilon_{B(C}\hat{\square}_{D)A}) \\ & + \frac{1}{2}(K_{CDAB}\nabla - K_{ABCD}\nabla) + K_{CDQ(A}\nabla_{B)}{}^Q - K_{ABQ(C}\nabla_{D)}{}^Q. \end{aligned} \quad (4b)$$

3.3 Space spinor expressions in Cartesian coordinates

In the sequel it will be sometimes necessary to give spinorial expressions in terms of Cartesian or asymptotically Cartesian frames and coordinates. For this we make use of the spatial Infeld-van der Waerden symbols $\sigma^i_{\mathbf{AB}}, \sigma_i^{\mathbf{AB}}$. Given $x^i, x_i \in \mathbb{R}^3$ we shall follow the convention that

$$x^{\mathbf{AB}} \equiv \sigma_i^{\mathbf{AB}} x^i, \quad x_{\mathbf{AB}} \equiv \sigma_i^{\mathbf{AB}} x_i,$$

with

$$x^{\mathbf{AB}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -x^1 + ix^2 & x^3 \\ x^3 & x^1 + ix^2 \end{pmatrix}, \quad x_{\mathbf{AB}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -x_1 - ix_2 & x_3 \\ x_3 & x_1 - ix_2 \end{pmatrix}.$$

4 Twistor initial data

In this section we review some aspects of the space spinor decomposition of the twistor equation (2). A first analysis along these lines was first carried out in [8].

4.1 General observations

Given a spinor κ_A (not necessarily a solution to the twistor equation), it will be convenient to define the following spinors:

$$\begin{aligned} \xi_A &\equiv \frac{2}{3} \nabla_A^B \kappa_B, \\ \xi_{ABC} &\equiv \nabla_{(AB} \kappa_{C)}, \\ H_{A'AB} &\equiv \nabla_{A'} (A \kappa_B). \end{aligned}$$

We will use this notation throughout the rest of the paper. Clearly, for a solution to the twistor equation one has

$$H_{A'AB} = 0.$$

The spinors ξ_A and ξ_{ABC} arise in the space spinor decomposition of the spinor $H_{A'AB}$. Furthermore, the spinors ξ_A and ξ_{ABC} correspond to the irreducible components of $\nabla_{AB} \kappa_C$ so that one can write

$$\nabla_{AB} \kappa_C = \xi_{ABC} - \xi_{(A} \epsilon_{B)C}. \quad (5)$$

The irreducible components of the derivative $\nabla_{AB} \xi_C$ in vacuum are given by:

$$\nabla_{AB} \xi^B = \frac{1}{2} \nabla^{BC} \xi_{ABC} + \frac{1}{2} K \xi_A + \frac{1}{2} \Omega_{ABCD} \xi^{BCD}, \quad (6a)$$

$$\nabla_{(AB} \xi_{C)} = 2 \nabla_{(A}^D \xi_{BC)D} + \Psi_{ABCD} \kappa^D + \frac{2}{3} K \xi_{ABC} - \Omega_{ABCD} \xi^D - \xi_{(A}^{DF} \Omega_{BC)DF}. \quad (6b)$$

4.2 Propagation of the twistor equation

A straightforward consequence of the twistor equation (2) in a vacuum spacetime is that:

$$\square \kappa_A = 0, \quad (7)$$

where $\square \equiv \nabla^{AA'} \nabla_{AA'}$. The latter equation is obtained by applying the differential operator $\nabla^{AA'}$ to equation (2) and then using the vacuum commutator relation for the spacetime Levi-Civita connection. The wave equation (7) plays a role in the discussion of the *propagation* of the Killing spinor equation —cfr. [8].

Lemma 1. *Let κ_A be a solution to equation (7). Then the spinor field $H_{A'AB}$ will satisfy the wave equation*

$$\square H_{A'AB} = 2 H_{A'}^{CD} \Psi_{ABCD}. \quad (8)$$

The crucial observation is that the right hand side of equation (8) is a homogeneous expression of the unknown. The hyperbolicity of equation (8) implies the following result:

Proposition 2. *The development $(\mathcal{M}, g_{\mu\nu})$ of an initial data set for the vacuum Einstein field equations, $(\mathcal{S}, h_{ab}, K_{ab})$, has a solution to the twistor equation in the domain of dependence of $\mathcal{U} \subset \mathcal{S}$ if and only if the following equations are satisfied on \mathcal{U} .*

$$H_{A'AB} = 0, \quad (9a)$$

$$\nabla H_{A'AB} = 0. \quad (9b)$$

4.3 The twistor initial data equations

The *twistor initial data* conditions of Proposition 2 can be reexpressed in terms of conditions on the spinor κ_A which are intrinsic to the hypersurface \mathcal{S} . Extensive computations using the `xAct` suite for `Mathematica` render the following result:

Theorem 1. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an initial data set for the vacuum Einstein field equations, where \mathcal{S} is a Cauchy hypersurface. Let $\mathcal{U} \subset \mathcal{S}$ be an open set. The development of the initial data set will then have a solution to the twistor equation in the domain of dependence of \mathcal{U} if and only if*

$$\xi_{ABC} = 0, \quad (10a)$$

$$\Psi_{ABCD}\kappa^D = 0, \quad (10b)$$

are satisfied on \mathcal{U} . The valence 1 Killing spinor is obtained by evolving (7) with initial data satisfying conditions (10a)-(10b) and

$$\nabla\kappa_A = -\xi_A, \quad (11)$$

on \mathcal{U} .

Remark 1. Conditions (10a)-(10b) are intrinsic to $\mathcal{U} \subset \mathcal{S}$ and will be referred to as the *twistor initial data equations*. In particular, equation (10a), which can be written as

$$\nabla_{(AB}\kappa_{C)} = 0, \quad (12)$$

will be called the *spatial twistor equation*, whereas (10b) will be known as the *algebraic condition*. The self-dual Weyl spinor Ψ_{ABCD} can be written in terms of quantities intrinsic to the initial hypersurface \mathcal{S} using

$$\Psi_{ABCD} = E_{ABCD} + iB_{ABCD},$$

with

$$\begin{aligned} E_{ABCD} &= -r_{(ABCD)} + \frac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - \frac{1}{6}\Omega_{ABCD}K, \\ B_{ABCD} &= -i D^Q{}_{(A}\Omega_{BCD)Q}, \end{aligned}$$

and where the spinor r_{ABCD} is the spinorial counterpart of the Ricci tensor, r_{ab} , of the 3-metric h_{ab} .

Proof. The proof of Theorem 1 consists of a space spinor decomposition of the conditions (9a)-(9b) and of an analysis of the dependencies of the resulting conditions. All calculations are made on $\mathcal{U} \subset \mathcal{S}$. The equation $H_{A'AB} = 0$ is equivalent to

$$\begin{aligned} \xi_{ABC} &= 0, \\ \nabla\kappa_A &= -\xi_A. \end{aligned}$$

The wave equation $\square\kappa_A = 0$ renders

$$\nabla\nabla\kappa_A = -K\nabla\kappa_A - 2\nabla^{BC}\xi_{ABC} - 2\nabla_{AB}\xi^B + K_{AB}\xi^B + K^{BC}\xi_{ABC}.$$

Using the above equations, the equation $\nabla H_{A'AB} = 0$ on \mathcal{S} is seen to be equivalent to

$$\begin{aligned} \nabla_{AB}\xi^B &= \frac{1}{2}K\xi_A, \\ \nabla_{(AB}\xi_{C)} &= -\Psi_{ABCD}\kappa^D - \Omega_{ABCD}\xi^D. \end{aligned}$$

Also using the equations (6a)-(6b), one will see that

$$\begin{aligned}\xi_{ABC} &= 0, \\ \Psi_{ABCD}\kappa^D &= 0, \\ \nabla\kappa_A &= -\xi_A,\end{aligned}$$

on \mathcal{U} is enough to guarantee $H_{A'AB} = 0$ everywhere in the domain of dependence of \mathcal{U} , if we evolve κ_A by $\square\kappa_A = 0$. This completes the proof. \square

4.4 The integrability conditions of the spatial twistor equation

The condition $\xi_{ABC} \equiv \nabla_{(AB}\kappa_{C)} = 0$ does not immediately give information about the other irreducible component of $\nabla_{AB}\kappa_C$, namely ξ_A . However, using $\xi_{ABC} = 0$ in the relations (6a)-(6b) one finds that $\nabla_{AB}\xi_C$ can be written in terms of ξ_A , κ_A and curvature spinors. We get:

$$\begin{aligned}\nabla_{AB}\xi^B &= \frac{1}{2}K\xi_A, \\ \nabla_{(AB}\xi_{C)} &= \Psi_{ABCD}\kappa^D - \Omega_{ABCD}\xi^D.\end{aligned}$$

From these we can make the observation

Lemma 2. *Assume that $\nabla_{(AB}\kappa_{C)} = 0$, then*

$$\nabla_{AB}\nabla_{CD}\kappa_E = H_{ABCDE},$$

where H_{ABCDE} is a linear combination of κ_A and $\nabla_{AB}\kappa_C$ with coefficients depending on Ψ_{ABCD} , $\hat{\Psi}_{ABCD}$ and K_{ABCD} .

Remark. It is important to point out that the assertion of the Lemma 2 is false if $\nabla_{(AB}\kappa_{C)} \neq 0$.

5 The approximate twistor equation

The spatial twistor equation (10a) is an overdetermined condition for the spinor κ_A , so we can not expect that a generic initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ admits a solution. One would therefore like to weaken the equation so that it always admits a unique solution if one specifies the asymptotic behaviour in a specific way. The strategy will be to compose the spatial twistor operator with its formal adjoint. In this section we will follow this idea and construct the approximate twistor equation. The existence and uniqueness of solutions will be proved in Section 6.

5.1 The approximate twistor operator

Let \mathfrak{S}_1 and \mathfrak{S}_3 denote, respectively, the spaces of totally symmetric valence 1 and valence 3 spinors. Given ζ_{ABC} , $\chi_{ABC} \in \mathfrak{S}_3$, we introduce an inner product in \mathfrak{S}_3 via:

$$\langle \zeta_{ABC}, \chi_{DEF} \rangle \equiv \int_S \zeta_{ABC} \hat{\chi}^{ABC} d\mu,$$

where $d\mu$ denotes the volume form of the 3-metric h_{ab} . We introduce the spatial twistor operator Φ via

$$\Phi : \mathfrak{S}_1 \rightarrow \mathfrak{S}_3, \quad \Phi(\kappa)_{ABC} = \nabla_{(AB}\kappa_{C)}.$$

Now, consider the pairing

$$\langle \nabla_{(AB}\kappa_{C)}, \zeta_{DEF} \rangle = \int_S \nabla_{(AB}\kappa_{C)} \hat{\zeta}^{ABC} d\mu = \int_S \nabla_{AB}\kappa_C \hat{\zeta}^{ABC} d\mu.$$

The formal adjoint of the spatial Killing operator, Φ^* , can be obtained from the latter expression by integration by parts. To this end we note the identity

$$\int_{\mathcal{U}} \nabla_{AB}\kappa_C \hat{\zeta}^{ABC} d\mu = \int_{\partial\mathcal{U}} n_{AB}\kappa_C \hat{\zeta}^{ABC} dS + \int_{\mathcal{U}} \kappa^C (\Omega_C \widehat{ABD} \zeta_{ABD} - \nabla^{\widehat{AB}} \zeta_{ABC}) d\mu, \quad (13)$$

with $\mathcal{U} \subset \mathcal{S}$, and where dS denotes the area element of $\partial\mathcal{U}$, n_{AB} is the spinorial counterpart of its outward pointing normal, and ζ_{ABC} is a symmetric spinor. From (13) it follows that

$$\Phi^* : \mathfrak{S}_3 \rightarrow \mathfrak{S}_1, \quad \Phi^*(\zeta)_A = \nabla^{BC} \zeta_{ABC} - \Omega_A{}^{BCD} \zeta_{BCD}. \quad (14)$$

We shall call the composition operator $\mathbf{L} \equiv \Phi^* \circ \Phi : \mathfrak{S}_1 \rightarrow \mathfrak{S}_1$ given by

$$\mathbf{L}(\kappa_A) \equiv \nabla^{BC} \nabla_{(AB} \kappa_{C)} - \Omega_A{}^{BCD} \nabla_{BC} \kappa_D = 0, \quad (15)$$

the *approximate twistor operator*, and equation (15) the *approximate twistor equation*. Note that every solution to the spatial twistor equation (12) is also a solution to equation (15). Furthermore, as it will be discussed in the proof of Proposition 6, if a solution to the approximate twistor equation has a sufficiently fast decay at infinity, then it follows from (13) that it is also a solution to the spatial twistor equation.

5.2 Ellipticity of the approximate twistor operator

As a prior step to the analysis of the solutions to the approximate twistor equation (15), we look first at its ellipticity properties.

Lemma 3. *The operator \mathbf{L} defined by equation (15) is a formally self-adjoint elliptic operator.*

Proof. The operator is by construction formally self-adjoint as it is given by the composition of an operator and its formal adjoint. In order to verify ellipticity, we will use the fact that commuting derivatives does not change the principal symbol of an operator. Therefore, we use the vacuum commutators to get

$$\nabla^{AB} \nabla_{BC} \kappa_A = \frac{1}{2} \nabla^{AB} \nabla_{AB} \kappa_C - \frac{1}{2} \Omega_{CABD} \nabla^{BD} \kappa^A + \frac{1}{3} K \nabla_{CA} \kappa^A.$$

From this we see that

$$\mathbf{L}(\kappa_A) \equiv \frac{2}{3} \nabla^{BC} \nabla_{BC} \kappa_A + \frac{2}{3} \Omega_{ABCD} \nabla^{CD} \kappa^B + \frac{2}{9} K \nabla_{AB} \kappa^B,$$

which is manifestly elliptic. \square

We note that the approximate twistor equation (15) arises naturally from a variational principle. More precisely, it is the Euler-Lagrange equation of the functional

$$J = \int_{\mathcal{S}} \nabla_{(AB} \kappa_{C)} \widehat{\nabla^{AB} \kappa^C} d\mu. \quad (16)$$

6 The approximate twistor equation in asymptotically Euclidean manifolds

After having studied some formal properties of the twistor initial data equations (10a)-(10b),(11), and the approximate twistor equation (15), we proceed to analyse their solvability on asymptotically Euclidean manifolds.

6.1 Asymptotic flatness assumptions

In what follows, we will be concerned with vacuum spacetimes arising as the development of asymptotically Euclidean data sets. Let $(\mathcal{S}, h_{ab}, K_{ab})$, denote a smooth initial data set for the vacuum Einstein field equations. For an *asymptotic end* of \mathcal{S} it will be understood an open set diffeomorphic to the complement of a closed ball in \mathbb{R}^3 . The 3-manifold \mathcal{S} will be assumed to have an arbitrary number ($N \geq 1$) of ends. Besides paracompactness and orientability, no further topological restrictions will be made. Hence, the 3-manifold could have an arbitrary number of handles.

On each asymptotic end it will be assumed that it is possible to introduce asymptotically Cartesian coordinates x^i with $r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$, such that the intrinsic metric and extrinsic curvature of \mathcal{S} satisfy in the asymptotic end

$$h_{ij} = -(1 + 2mr^{-1})\delta_{ij} + o_\infty(r^{-3/2}), \quad (17a)$$

$$K_{ij} = o_\infty(r^{-5/2}). \quad (17b)$$

This class of data can be described as *asymptotically Schwarzschildian*. Here, and in what follows, the fall off conditions of the various fields will be expressed in terms of weighted Sobolev spaces H_β^s , where s is a non-negative integer and β is a real number. Here we use the conventions for these spaces given in [3] —see also [2]. We say that $\eta \in H_\beta^\infty$ if $\eta \in H_\beta^s$ for all s . Thus, the functions in H_β^∞ are smooth over \mathcal{S} and have a fall off at infinity such that $\partial^l \eta = o(r^{\beta-|l|})$. We will often write $\eta = o_\infty(r^\beta)$ for $\eta \in H_\beta^\infty$ at the asymptotic end.

6.2 Asymptotic form of solutions to the spatial twistor equation

In the sequel, given an initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the decay conditions (17a)-(17b), it will be necessary to show that it is always possible to solve the equation

$$\nabla_{(AB}\kappa_{C)} = o_\infty(r^{-3/2}), \quad (18)$$

order by order without making any further assumptions on the data. For this we use asymptotically Cartesian coordinates and consider a normalised dyad $\{o_A, \iota_A\}$ such that $o_A \iota^A = 1$. A direct calculation allows us to verify that:

Proposition 3. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ denote an initial data set for the vacuum Einstein field equations satisfying at each asymptotic end the decay conditions (17a)-(17b). Let m denote the ADM mass of one of these ends. Then on this end*

$$\kappa_A = (1 + \frac{1}{2}mr^{-1})x_{AB}o^B + o_\infty(r^{-1/2}), \quad (19)$$

satisfies equation (18).

Remark. Formula (19) implies the following expansion for ξ_A :

$$\xi_A = (1 - mr^{-1})o_A + o_\infty(r^{-3/2}). \quad (20)$$

For later reference we notice that

$$\nabla_{AB}r = -x_{AB}r^{-1} + o_\infty(r^{-1/2}), \quad (21a)$$

$$\nabla_{(AB}\xi_{C)} = -mr^{-3}x_{(AB}o_{C)} + o_\infty(r^{-5/2}). \quad (21b)$$

6.3 Existence and uniqueness of spinors with twistor asymptotics

In this section we prove that given a spinor κ_A satisfying equations (18) and $\xi_A = o_A + o_\infty(r^{-1/2})$, the asymptotic expansion (19) is unique up to a translation.

Proposition 4. *Given an asymptotic end for which (17a)-(17b) hold, there exists*

$$\kappa_A = o_\infty(r^{3/2}), \quad (22)$$

such that

$$\xi_{ABC} = o_\infty(r^{-3/2}), \quad \xi_A = o_A + o_\infty(r^{-1/2}). \quad (23)$$

The spinor κ_A is unique up to order $o_\infty(r^{-1/2})$, apart from a (complex) constant term.

Remark 1. The complex constant term arising in Proposition 4 contains four real parameters. In the sequel, given a particular choice of asymptotically Cartesian coordinates and frame, we will set this constant term to zero. For any other choice of coordinates and frames, the constant can be transformed away by a translation.

Remark 2. The condition $\xi_{\mathbf{ABC}} = o_\infty(r^{-3/2})$ implies $\xi_{ABC} \in L^2$. Furthermore the conditions in Proposition 4 are coordinate independent.

Proof. A direct calculation shows that the expansion (19) yields (20) and $\xi_{\mathbf{ABC}} = o_\infty(r^{-3/2})$. Hence, (19) gives a solution of the desired form. In order to prove uniqueness we make use of the linearity of the integrability conditions (6a)-(6b). Note that the translational freedom gives an ambiguity of a constant term in $\kappa_{\mathbf{A}}$. Let

$$\mathring{\kappa}_{\mathbf{A}} \equiv \left(1 + \frac{1}{2}mr^{-1}\right) x_{\mathbf{AB}} o^{\mathbf{B}} \quad (24)$$

Let $\check{\kappa}_{\mathbf{A}}$, be an arbitrary solution to the system (22), (23). Furthermore, let $\kappa_{\mathbf{A}} \equiv \check{\kappa}_{\mathbf{A}} - \mathring{\kappa}_{\mathbf{A}}$. We then have

$$\xi_{ABC} = o_\infty(r^{-3/2}), \quad \xi_A = o_\infty(r^{-1/2}), \quad \kappa_A = o_\infty(r^{3/2}).$$

To obtain the desired conclusion we only need to prove that $\kappa_{\mathbf{A}} = C_{\mathbf{A}} + o_\infty(r^{-1/2})$, where $C_{\mathbf{A}}$ is a constant. This is equivalent to $D_{AB}\kappa_C = o_\infty(r^{-3/2})$. Note that we now have coordinate independent statements to prove.

We note that from (17a)-(17b) it follows that

$$K_{ABCD} = o_\infty(r^{-5/2}), \quad \Psi_{ABCD} = o_\infty(r^{-3+\varepsilon}),$$

with $\varepsilon > 0$. From (5) and a sharp multiplication result for weighted Sobolev spaces given in Lemma 2.4 in [12] one will get

$$D_{AB}\kappa_C = \xi_{ABC} - \xi_{(A}\epsilon_{B)C} - \frac{1}{2}K_{ABC}{}^D\kappa_D = o_\infty(r^{-1/2}).$$

Integrating the latter yields

$$\kappa_A = o_\infty(r^{1/2}).$$

The constant of integration is incorporated in the remainder term. Estimating all terms in (6a) and (6b) gives

$$\nabla_{AB}\xi^B = o_\infty(r^{-5/2}), \quad (25a)$$

$$\nabla_{(AB}\xi_{C)} = o_\infty(r^{-5/2}). \quad (25b)$$

Hence, $\nabla_{AB}\xi_C = o_\infty(r^{-5/2})$, and therefore $D_{AB}\xi_C = o_\infty(r^{-5/2})$. Integrating this yields $\xi_A = o_\infty(r^{-3/2})$. Here the constants of integration are forced to vanish by the condition $\xi_A = o_\infty(r^{-1/2})$. Hence,

$$D_{AB}\kappa_C = \xi_{ABC} - \xi_{(A}\epsilon_{B)C} - \frac{1}{2}K_{ABC}{}^D\kappa_D = o_\infty(r^{-3/2}).$$

from where the result follows. \square

From the asymptotic solutions we can obtain a globally defined spinor $\mathring{\kappa}_A$ on \mathcal{S} that will act as a seed for our approximate twistor.

Corollary 1. *There are spinors $\mathring{\kappa}_A$, defined everywhere on \mathcal{S} , such that the asymptotics at each end is given by (19). Different choices of $\mathring{\kappa}_A$ can only differ by a spinor in $H_{-1/2}^\infty$.*

Proof. Proposition 4 gives the existence at each end. Smoothly cut off these functions, and paste them together. This gives a smooth spinor $\mathring{\kappa}_A$ defined everywhere on \mathcal{S} . Furthermore $\nabla_{(AB}\mathring{\kappa}_{C)} \in H_{-3/2}^\infty$. \square

6.4 Fredholm properties

In this section we study the invertibility properties of the approximate twistor operator \mathbf{L} given by equation (15) on asymptotically Euclidean manifolds. The necessary elliptic theory for this analysis has been developed in e.g. [5, 6, 7, 9], and has been adapted to our context in [2].

The decay assumptions (17a)-(17b) imply that \mathbf{L} is an *asymptotically homogeneous elliptic operator* —see e.g. [5, 9]. This is the standard assumption on elliptic operators on asymptotically Euclidean manifolds. It follows from [5], Theorem 6.3 that:

Lemma 4. *The elliptic operator*

$$\mathbf{L} : H_\delta^2 \rightarrow H_{\delta-2}^0,$$

where δ is not a non-negative integer is a linear operator with finite dimensional Kernel and closed range.

We will also need the following ancillary result —cfr. [7] for an analogous result for Killing vectors.

Proposition 5. *Let $\nu_A \in H_{-1/2}^\infty$ such that $\nabla_{(AB}\nu_{C)} = 0$. Then $\nu_A = 0$ on \mathcal{S} .*

Proof. We will use Theorem 20 of [2], which is an adaptation of a result in [7]. From Lemma 2 it follows that $\nabla_{AB}\nabla_{CD}\nu_E$ can be expressed as a linear combination of lower order derivatives with smooth coefficients with the proper decay. Thus, Theorem 20 of [2] applies with $m = 1$ and one obtains the desired result. \square

We are now in the position to discuss the Kernel of the approximate twistor operator in the case of spinor fields that decay at infinity.

Proposition 6. *Let $\nu_A \in H_{-1/2}^\infty$. If $\mathbf{L}(\nu_A) = 0$, then $\nu_A = 0$.*

Proof. Using the identity (13) with $\zeta_{ABC} = \nabla_{(AB}\nu_{C)}$ and assuming that $\mathbf{L}(\nu_C) = 0$, one obtains

$$\int_S \nabla^{AB}\nu^C \widehat{\nabla_{(AB}\nu_{C)}} d\mu = \int_{\partial S_\infty} n^{AB}\nu^C \widehat{\nabla_{(AB}\nu_{C)}} dS,$$

where ∂S_∞ denotes the sphere at infinity. Assume now, that $\nu_A \in H_{-1/2}^\infty$. It follows that $\nabla_{(AB}\nu_{C)} \in H_{-3/2}^\infty$ and furthermore, using the finer multiplication Lemma 15 of [2] that

$$n^{AB}\nu^C \widehat{\nabla_{(AB}\nu_{C)}} = o(r^{-2}).$$

The integration of the latter over a finite sphere of sufficiently large radius is of type $o(1)$. Thus one has that

$$\int_{\partial S_\infty} n^{AB}\nu^C \widehat{\nabla_{(AB}\nu_{C)}} dS = 0,$$

from where

$$\int_S \nabla^{AB}\nu^C \widehat{\nabla_{(AB}\nu_{C)}} d\mu = 0.$$

Therefore, one concludes that

$$\nabla_{(AB}\nu_{C)} = 0.$$

That is, ν_A has to be a solution to the spatial twistor equation. Using Proposition 5 it follows that $\nu_A = 0$ on \mathcal{S} . \square

6.5 Existence of approximate twistors

We are now in the position of providing an existence proof to solutions to equation (15) with the asymptotic behaviour discussed in section 6.2.

Theorem 2. *Given an asymptotically Euclidean initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the asymptotic conditions (17a)-(17b), there exists a smooth unique solution to equation (15) with asymptotic behaviour at each end given by (19).*

Proof. We consider the Ansatz

$$\kappa_A = \dot{\kappa}_A + \theta_A, \quad \theta_A \in H_{-1/2}^2,$$

with $\dot{\kappa}$ given by Corollary 1. Substitution into equation (15) renders the following equation for the spinor θ_A :

$$\mathbf{L}(\theta_C) = -\mathbf{L}(\dot{\kappa}_C). \quad (26)$$

By construction it follows that $\nabla_{(AB}\dot{\kappa}_C) \in H_{-3/2}^\infty$, so that $F_C \equiv -\mathbf{L}(\dot{\kappa}_C) \in H_{-5/2}^\infty$. Using the Fredholm Alternative for second order elliptic systems (cfr. Theorem 23 in [2]), one concludes that equation (26) has a unique solution if F_A is orthogonal to all $\nu_A \in H_{-1/2}^0$ in the Kernel of $\mathbf{L}^* = \mathbf{L}$. Proposition 6 states that this Kernel is trivial. Thus, there are no restrictions on F_A and equation (26) has a unique solution as desired. Due to elliptic regularity, any $H_{-1/2}^2$ solution to the previous equation is in fact a $H_{-1/2}^\infty$ solution —cfr. Theorem 24 in [2]. Thus, θ_A is smooth. To see that κ_A does not depend on the particular choice of $\dot{\kappa}_A$, let $\dot{\kappa}'_A$ be another choice. Let κ'_A be the corresponding solution to (26). Due to Corollary 1, we have $\dot{\kappa}_A - \dot{\kappa}'_A \in H_{-1/2}^\infty$. Hence, we have $\kappa_A - \kappa'_A \in H_{-1/2}^\infty$ and $\mathbf{L}(\kappa_A - \kappa'_A) = 0$. According to Proposition 6, $\kappa_A - \kappa'_A = 0$, and the proof is complete. \square

The following is a direct consequence of Theorem 2, and will be crucial for the construction of our geometric invariant.

Corollary 2. *A solution, κ_A , to equation (15) with asymptotic behaviour given by (19) satisfies $J < \infty$ where J is the functional given by equation (16).*

Proof. The functional J given by equation (16) is the L^2 norm of $\nabla_{(AB}\kappa_C)$. Now, if κ_A is the solution given by Theorem 2, one has that $\nabla_{(AB}\kappa_C) \in H_{-3/2}^0$. In our conventions this reads

$$J = \|\nabla_{(AB}\kappa_C)\|_{L^2} = \|\nabla_{(AB}\kappa_C)\|_{H_{-3/2}^0} < \infty.$$

The result follows. \square

Remark. Again, let κ_A be the solution to equation (15) given by Theorem 2. Using the identity (13) with $\zeta_{ABC} = \nabla_{(AB}\kappa_C)$ one obtains

$$J = \int_{\partial S_\infty} n^{AB} \kappa^C \widehat{\nabla_{(AB}\kappa_C)} dS < \infty.$$

Thus, the invariant J evaluated at the solution κ_A given by Theorem 2 can be expressed as a boundary integral at infinity. A crude estimation of the integrand of the boundary integral does not allow us to directly establish its boundedness. In any case this follows from Corollary 2.

7 The geometric invariant

In this section we use the functional given by (16) and the algebraic condition (10b) to construct the desired geometric invariant measuring the deviation of $(\mathcal{S}, h_{ab}, K_{ab})$ from Minkowski initial data. For this purpose, let κ_A be a solution to equation (15) as given by Theorem 2. Define

$$I' \equiv \int_S \Psi_{ABCD} \kappa^D \hat{\Psi}^{ABC} {}_F \hat{\kappa}^F d\mu. \quad (27)$$

The geometric invariant is then defined by

$$I \equiv J + I'. \quad (28)$$

Remark. It should be stressed that by construction I is coordinate independent and that $I \geq 0$. We also have the following lemma.

Lemma 5. *The geometric invariant given by (28) is finite for an initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the decay conditions (17a)-(17b).*

Proof. From Corollary 2 we already have $J < \infty$. From the form of the decay assumptions (17a)-(17b) we have $\Psi_{ABCD} \in H_{-3+\varepsilon}^\infty$, $\varepsilon > 0$. Using the multiplication rule for weighted Sobolev spaces—see e.g. Theorem 23 in [2]—together with $\kappa_A \in H_{1+\varepsilon}^\infty$ we obtain

$$\Psi_{ABCD}\kappa^D \in H_{-3/2}^\infty.$$

Thus, again one finds that $I' < \infty$. Hence, the invariant (28) is finite and well defined. \square

The invariant I can be used to provide a global characterisation of Minkowski initial data. More precisely, one has that:

Theorem 3. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an asymptotically Euclidean initial data set for the vacuum Einstein field equations satisfying on each of its asymptotic ends the decay conditions (17a)-(17b). Let I be the invariant defined by equations (16), (27) and (28), where κ_A is given as the only solution to equation (15) with asymptotic behaviour on each end given by (19). The invariant I vanishes if and only if $(\mathcal{S}, h_{ab}, K_{ab})$ is an initial data set for the Minkowski spacetime.*

Proof. Due to our smoothness assumptions, if $I = 0$ it follows that equations (10a)-(10b) are satisfied on the whole of \mathcal{S} . Thus, the development of $(\mathcal{S}, h_{ab}, K_{ab})$ will have, at least in a slab, a solution to the twistor equation (2). Now, because of the asymptotic behaviour (19) one can find points in the development of the data for which $\eta_A = \nabla_A{}^{A'}\bar{\kappa}_{A'} \neq 0$. Thus, in view of Proposition 1 one has that the development of the data is isomorphic to the Minkowski spacetime. \square

8 Connection to the mass

As a consequence of the Theorem of the Positivity of the Mass [14, 15, 17] one knows that if the ADM mass of a regular initial data set for the vacuum Einstein field equations $(\mathcal{S}, h_{ab}, K_{ab})$ vanishes, then the initial data must be data for the Minkowski spacetime. As a consequence, the mass provides a characterisation of Minkowskian data. This suggests that the geometric invariant given by Theorem 3 must be related to the mass of the initial data set. In this section we show that this is indeed the case. More precisely, our methods provide a proof of the following positivity of mass result:

Theorem 4. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an initial data set for the vacuum Einstein field equations satisfying the decay conditions (17a)-(17b). Furthermore assume that \mathcal{S} has the topology of \mathbb{R}^3 . Let m be the ADM mass of the asymptotic end of the initial data set. Then m is non-negative. Moreover, if $m = 0$, then $(\mathcal{S}, h_{ab}, K_{ab})$ is initial data for the Minkowski spacetime.*

The proof of this theorem, together with some other relevant observations, will be given in the following subsections.

8.1 An expression for the mass

The following analysis is valid for a slice \mathcal{S} with an arbitrary number of asymptotic ends, without any extra restrictions on the topology. Let κ_A be a solution to $\mathbf{L}(\kappa_A) = 0$. A calculation reveals that

$$\mathbf{L}(\xi_A) = -\frac{2}{3}\xi^B\nabla_{AB}K. \quad (29)$$

In general, if ζ_A is an arbitrary spinor, one has that

$$\begin{aligned}\widehat{\mathbf{L}(\zeta_A)} &= -\nabla^{BC}\widehat{\nabla_{(AB}\zeta_{C)}} \\ &= \mathbf{L}(\hat{\zeta}_A) + \frac{2}{3}\zeta^B\nabla_{AB}K.\end{aligned}$$

In particular, for ξ_A as given by equation (29) one has that that

$$\mathbf{L}(\hat{\xi}_A) = 0. \quad (30)$$

We will exploit this observation to obtain an alternative expression for the total mass of an initial data set $(\mathcal{S}, h_{ab}, K_{ab})$. Let

$$M' \equiv \int_{\mathcal{S}} \nabla_{(AB}\hat{\xi}_{C)}\widehat{\nabla^{(AB}\hat{\xi}^{C)}}d\mu. \quad (31)$$

Now, under the decay assumptions (17a)-(17b), the expansions (20), (21a)-(21b) and equation (30) render

$$\begin{aligned}M' &= \int_{\partial\mathcal{S}_\infty} n_{AB}\hat{\xi}_C\widehat{\nabla^{(AB}\hat{\xi}^{C)}}dS + \int_{\mathcal{S}} \hat{\xi}_A\widehat{\mathbf{L}(\hat{\xi}^A)}d\mu \\ &= \int_{\partial\mathcal{S}_\infty} n_{AB}\hat{\xi}_C\widehat{\nabla^{(AB}\hat{\xi}^{C)}}dS \\ &= 4\pi M,\end{aligned}$$

where M is the sum of the ADM masses of the asymptotic ends —the total mass. Notice that the integral M' is the L^2 norm of $\nabla_{(AB}\hat{\xi}_{C)}$ and hence $M' \geq 0$. That is $M \geq 0$.

8.2 An alternative expression for the total mass

The Hermitian conjugate of the symmetrized derivative of κ_A is

$$\nabla_{(AB}\hat{\kappa}_{C)} = -\hat{\xi}_{ABC} - \Omega_{ABCD}\hat{\kappa}^D. \quad (32)$$

Furthermore

$$\mathbf{L}(\hat{\kappa}_A) = -\frac{2}{3}\hat{\kappa}^B\nabla_{AB}K. \quad (33)$$

We would like to conclude that $\mathbf{L}(\hat{\kappa}_A) \in H_{-5/2}^0$. For that purpose we study the function $f \equiv \kappa_A\hat{\kappa}^A\sigma^{-2}$. Due to the specified asymptotics, this function is bounded at infinity. Furthermore it is continuous, and therefore bounded everywhere on \mathcal{S} . Let $C \equiv \sup_{\mathcal{S}} f < \infty$, then we have

$$\begin{aligned}\|\mathbf{L}(\hat{\kappa}_A)\|_{H_{-5/2}^0}^2 &= \frac{4}{9}\|\hat{\kappa}^B\nabla_{AB}K\|_{H_{-5/2}^0}^2 = \frac{2}{9}\int_{\mathcal{S}} \kappa_A\hat{\kappa}^A D_{BC}K\widehat{D^{BC}K}\sigma^2d\mu \\ &\leq \frac{2}{9}C\int_{\mathcal{S}} D_{BC}K\widehat{D^{BC}K}\sigma^4d\mu = \frac{2}{9}C\|D_{AB}K\|_{H_{-7/2}^0}^2 < \infty.\end{aligned} \quad (34)$$

Let $\theta_A \in H_{-1/2}^2$ be the unique solution to the elliptic equation

$$\mathbf{L}(\theta_A) = \mathbf{L}(\hat{\kappa}_A).$$

The existence and uniqueness is guaranteed by $\mathbf{L}(\hat{\kappa}_A) \in H_{-5/2}^0$ and the Fredholm alternative —see the proof of Theorem 2. Motivated by the previous equation one defines $\check{\kappa}_A \equiv \theta_A - \hat{\kappa}_A$. Clearly, one has that $\mathbf{L}(\check{\kappa}_A) = 0$. By elliptic regularity we have $\check{\kappa}_A \in H_{3/2}^\infty$. Let

$$\check{\xi}_A \equiv \frac{2}{3}\nabla_A{}^B\check{\kappa}_B.$$

Some further computations reveal that

$$\begin{aligned}\check{\xi}_A &= \frac{2}{3}\nabla_A{}^B\theta_B + \hat{\xi}_A + \frac{1}{3}K\hat{\kappa}_A, \\ \mathbf{L}(\check{\xi}_A) &= -\frac{2}{3}\check{\xi}^B\nabla_{AB}K, \\ \nabla^{BC}(\Omega_{ABCD}\check{\xi}^D) &= \Omega_{ABCD}\nabla^{BC}\check{\xi}^D - \frac{1}{2}\check{\xi}_A\Omega_{BCDF}\Omega^{BCDF} + \frac{2}{3}\check{\xi}^B\nabla_{AB}K.\end{aligned}$$

Hence,

$$\nabla^{BC}(\nabla_{(AB}\check{\xi}_C) + \Omega_{ABCD}\check{\xi}^D) = \Omega_{ABCD}\nabla^{BC}\check{\xi}^D - \frac{1}{2}\check{\xi}_A\Omega_{BCDF}\Omega^{BCDF}.$$

Furthermore, it can be seen that

$$\widehat{\nabla_{(AB}\check{\xi}_C)} = \nabla_{(AB}\check{\xi}_C) + \Omega_{ABCD}\check{\xi}^D.$$

Define

$$M'' \equiv \int_S \nabla_{(AB}\check{\xi}_C) \widehat{\nabla^{AB}\check{\xi}^C} d\mu. \quad (35)$$

Integration by parts and the expressions discussed in the previous paragraphs imply that:

$$\begin{aligned} M'' &= \int_S (\nabla_{AB}\check{\xi}_C + \Omega_{ABCD}\check{\xi}^D)(\widehat{\nabla^{AB}\check{\xi}^C} + \Omega^{ABC}{}_F\check{\xi}^F) d\mu \\ &= \int_{\partial S_\infty} n_{AB}\check{\xi}_C(\widehat{\nabla^{AB}\check{\xi}^C} + \Omega^{ABC}{}_F\check{\xi}^F) dS + \int_S \check{\xi}_A(\Omega^A{}_{BCD}\widehat{\nabla^{BC}\check{\xi}^D} - \frac{1}{2}\check{\xi}^A\Omega_{BCDF}\Omega^{BCDF}) d\mu \\ &\quad + \int_S \Omega_{ABCD}\check{\xi}^D\widehat{\nabla^{AB}\check{\xi}^C} + \frac{1}{2}\Omega_{BCDF}\Omega^{BCDF}\check{\xi}_A\check{\xi}^A d\mu \\ &= \int_{\partial S_\infty} n_{AB}\check{\xi}_C\widehat{\nabla^{AB}\check{\xi}^C} dS \\ &= \int_{\partial S_\infty} n_{AB}(\hat{\xi}_A + \frac{2}{3}\nabla_A{}^B\theta_B + \frac{1}{3}K\hat{\kappa}_A)(\widehat{\nabla^{AB}\check{\xi}^C} + \frac{2}{3}\nabla^{AB}\widehat{\nabla^C}\theta_D + \frac{1}{3}\nabla^{AB}(K\hat{\kappa}^C)) dS \\ &= \int_{\partial S_\infty} n_{AB}\hat{\xi}_A\widehat{\nabla^{AB}\check{\xi}^C} dS \\ &= M'. \end{aligned} \quad (36)$$

Hence, $M'' = M'$ also gives an expression for the total mass.

8.3 Initial data sets with vanishing mass

Assume now that the slice S has the topology of \mathbb{R}^3 . This implies that it has only one asymptotic end. Furthermore assume that $M' = 0$, that is $m = 0$. Then it follows that $M'' = 0$ and

$$\nabla_{(AB}\hat{\xi}_C) = 0, \quad \nabla_{(AB}\hat{\xi}_C) = 0.$$

Using $\mathbf{L}(\kappa_A) = 0$ in equation (6a) one obtains

$$\nabla_{AB}\xi^B = \frac{1}{2}K\xi_A \iff \nabla_{AB}\hat{\xi}^B = 0.$$

In the same way we get $\nabla_{AB}\hat{\xi}^B = 0$. These results can be combined to give

$$\nabla_{AB}\hat{\xi}_C = 0, \quad (37a)$$

$$\nabla_{AB}\hat{\xi}_C = 0. \quad (37b)$$

Hence, ξ_C and $\hat{\xi}$ are covariantly constant spinors. One can exploit this property by taking a derivative of (37a) and (37b) and using the commutators. This yields

$$0 = -2\nabla_{(A}{}^D\nabla_{B)D}\hat{\xi}_C = \Psi_{ABCD}\hat{\xi}^D, \quad (38a)$$

$$0 = -2\nabla_{(A}{}^D\nabla_{B)D}\hat{\xi}_C = \Psi_{ABCD}\hat{\xi}^D, \quad (38b)$$

Consequently, both $\hat{\xi}_A$ and $\hat{\xi}_A$ are principal spinors of Ψ_{ABCD} . Furthermore,

$$\hat{\xi}_A\hat{\xi}^A = \hat{\xi}_A\hat{\xi}^A + o_\infty(r^{-1/2}) = \xi_A\hat{\xi}^A + o_\infty(r^{-1/2}) = 1 + o_\infty(r^{-1/2}),$$

and also $\nabla_{BC}(\hat{\xi}_A \hat{\xi}^A) = 0$. So, one can conclude that $\hat{\xi}_A \hat{\xi}^A = 1$. We can therefore use $\{\hat{\xi}_A, \hat{\xi}^A\}$ as a dyad on the entire slice \mathcal{S} . The equations (38a)-(38b) then yield $\Psi_{ABCD} = 0$ on \mathcal{S} .

Remark 1. Given that $\Psi_{ABCD} = 0$ on \mathcal{S} , one can use known results on the causal propagation of the Weyl tensor in vacuum spacetimes to conclude that $\Psi_{ABCD} = 0$ on the development of \mathcal{S} —see e.g. [4]. The asymptotic conditions then imply that the development is the Minkowski spacetime. However, it is of interest to conclude the same result from arguments purely intrinsic to the hypersurface that imply that the invariant I as given by equation (28) has to satisfy $I = 0$.

To pursue the idea expressed in the previous paragraph we proceed as follows: From the equivalence

$$\nabla_{(AB} \hat{\xi}_{C)} = 0 \iff \nabla_{(AB} \xi_{C)} = -\Omega_{ABCQ} \xi^Q,$$

and equation (6b) it follows that

$$2\nabla_{(A} {}^D \xi_{BC)D} + \Psi_{ABCD} \kappa^D + \frac{2}{3} K \xi_{ABC} - \xi_{(A} {}^{DF} \Omega_{BC)DF} = 0 \quad (39)$$

Contracting (39) with $\hat{\xi}^C$ and $\hat{\xi}^C$ and using (38a)-(38b) yields

$$\hat{\xi}^C \nabla^D {}_{(A} \xi_{BC)D} = \frac{1}{3} \hat{\xi}^C \xi_{(A} {}^{DF} \Omega_{B)CDF} + \frac{1}{6} \hat{\xi}^C \xi_C {}^{DF} \Omega_{ABDF} - \frac{1}{3} K \hat{\xi}^C \xi_{ABC}, \quad (40a)$$

$$\hat{\xi}^C \nabla^D {}_{(A} \xi_{BC)D} = \frac{1}{3} \hat{\xi}^C \xi_{(A} {}^{DF} \Omega_{B)CDF} + \frac{1}{6} \hat{\xi}^C \xi_C {}^{DF} \Omega_{ABDF} - \frac{1}{3} K \hat{\xi}^C \xi_{ABC}. \quad (40b)$$

Define the spinors

$$\nu_{AB} \equiv \xi_{ABD} \hat{\xi}^D, \quad \check{\nu}_{AB} \equiv \xi_{ABD} \hat{\xi}^D.$$

The sharp multiplication result for weighted Sobolev spaces of Lemma 2.4 in [12] yields $\nu_{AB}, \check{\nu}_{AB} \in H_{-3/2}^\infty$. The equations (40a)-(40b) can be reexpressed as

$$\begin{aligned} \nabla_{(A} {}^C \nu_{B)C} &= \frac{1}{2} \Omega_{ABCD} \nu^{CD} - \frac{1}{3} K \nu_{AB}, \\ \nabla_{(A} {}^C \check{\nu}_{B)C} &= \frac{1}{2} \Omega_{ABCD} \check{\nu}^{CD} - \frac{1}{3} K \check{\nu}_{AB}. \end{aligned}$$

Which is equivalent to

$$D_{(A} {}^C \nu_{B)C} = 0, \quad D_{(A} {}^C \check{\nu}_{B)C} = 0.$$

From the trivial topology of the 3-dimensional manifold \mathcal{S} one concludes that there exist (globally) scalars $\nu, \check{\nu} \in H_{-1/2}^\infty$ such that

$$\nu_{AB} = D_{AB} \nu, \quad \check{\nu}_{AB} = D_{AB} \check{\nu}.$$

The normalisation condition $\hat{\xi}_A \hat{\xi}^A = 1$ then implies that

$$\xi_{ABC} = \xi_{ABD} \hat{\xi}^D \hat{\xi}_C - \xi_{ABD} \hat{\xi}^D \hat{\xi}_C = \check{\nu}_{AB} \hat{\xi}_C - \nu_{AB} \hat{\xi}_C.$$

By virtue of the equations (37a)-(37b) we have

$$\xi_{ABC} = \nabla_{(AB} \left(\check{\nu} \hat{\xi}_{C)} - \nu \hat{\xi}_{C)} \right).$$

Hence, $\mathbf{L}(\check{\nu} \hat{\xi}_C - \nu \hat{\xi}_C) = 0$. Note also that $\check{\nu} \hat{\xi}_C - \nu \hat{\xi}_C \in H_{-1/2}^\infty$, by the sharp multiplication result. Thus, the triviality of the Kernel of $\mathbf{L} : H_{-1/2}^\infty \rightarrow H_{-5/2}^\infty$ yields

$$\check{\nu} \hat{\xi}_C - \nu \hat{\xi}_C = 0,$$

and, moreover, $\xi_{ABC} = 0$. One concludes that $I = 0$. Therefore, the spacetime is isometric to the Minkowski spacetime.

Remark. It is of interest to note that the integral M' as given by expression (31) evaluated over a subset $\mathcal{U} \subset \mathcal{S}$ can be interpreted the mass of that portion of the slice; it is non-negative by construction and it tends to the complete ADM mass as \mathcal{U} grows to cover \mathcal{S} . The drawback of this construction is that one needs to solve $\mathbf{L}(\kappa_A)$ on the entire slice \mathcal{S} —that is, it is a construction that needs global information.

9 Conclusions

In this article we have used the idea of quantifying how much a spacetime fails to have a particular symmetry to construct a global geometric invariant characterising initial data sets for the Minkowski spacetime. Not surprisingly, this invariant turns out to be related to the ADM mass of the data. The approach advocated in this article provides expressions for the mass as L^2 norms of some auxiliary spinorial fields. This suggests that variations of our approach could be used to obtain estimates of parts of the initial data in terms of the mass. A property of potential relevance for the discussion of the time evolution of the invariants is the ability to switch, according to need, between expressions for the invariants given in terms of bulk or surface integrals. As pointed out in the introduction, the main motivation for the analysis presented in this article is to develop intuition and a mathematical toolkit for the analysis of similar questions for the more complicated geometric invariants of [1, 2]. These tantalising possibilities will be analysed elsewhere.

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