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ON THE GEOMETRY UNDERLYING SUPERSYMMETRIC FLUX VACUA WITH INTERMEDIATE $SU(2)$ STRUCTURE

ANNA FINO AND LUIS UGARTE

ABSTRACT. We show that supersymmetric flux vacua with constant intermediate $SU(2)$ structure are closely related to some special classes of half-flat structures. More concretely, solutions of the SUSY equations IIA possess a symplectic half-flat structure, whereas solutions of the SUSY equations IIB admit a half-flat structure which is in certain sense near to the balanced condition. Using this result we show that compact simply connected manifolds do not admit type IIB solutions. New solutions of the SUSY equations IIA and IIB are constructed from hyperkähler 4-manifolds, special hypo 5-manifolds and 6-dimensional solvmanifolds.

1. INTRODUCTION

In [1] new supersymmetric four-dimensional Minkowski flux vacua of type II string theory with at least $N = 1$ supersymmetry (SUSY) on nilmanifolds and solvmanifolds have been found, by extending previous results by [19].

In [17, 18] it was shown that these supersymmetric conditions can be written in terms of the so-called generalized complex geometry [20, 25] and it was proved that the internal manifold has to be a (twisted) generalized Calabi-Yau manifold. An $N = 1$ supergravity vacuum implies the existence of a pair of spinors on the internal manifold. In dimension six the pair of spinors defines an $SU(3)$ structure, a static $SU(2)$ structure or an intermediate $SU(2)$ structure [1], which correspond respectively to the condition that the two spinors are parallel, orthogonal or between the two. These different cases are encoded in the context of the generalized geometry into an $SU(3) \times SU(3)$ structure on the bundle $TM \oplus T^*M$. The $SU(3) \times SU(3)$ structure can be encoded in a pair of compatible pure spinors, which are objects defined in generalized complex geometry on the generalized tangent bundle $TM \oplus T^*M$. When one of the two pure spinors is closed, the manifold is called generalized Calabi-Yau.

An interesting question is then to look for new explicit examples and a natural class is the one of the nilmanifolds, since by [5] they admit generalized Calabi-Yau structures. In [19] the authors only look for $SU(3)$ and static $SU(2)$ structures, since only these ones seemed to be compatible with the orientifold projections. But in [26] it was shown that intermediate $SU(2)$ structures are also possible if one allows a mixing of the usual $SU(2)$ structure forms under the projection conditions.

In [1] Andriot rewrote the projection conditions imposed by the orientifold for intermediate $SU(2)$ structures by introducing the “projection (eigen)basis”, i.e. the set of structure forms which are eigenvectors for the projection. These forms define a new $SU(2)$ structure, obtained by a rotation from the usual one and the

new $SU(2)$ structure coincides (modulo a rescaling) with the one appearing with dielectric pure spinors, introduced in [22, 27] in the ADS/CFT context. Since the pure spinors become simpler to study if they are written in terms of the projection basis variables, the supersymmetry (SUSY) conditions become simple and in this way Andriot found for constant intermediate $SU(2)$ structures new four dimensional (Minkowski) flux vacua of type II string theory with at least $N = 1$.

Inside the class of $SU(3)$ structures there is a special one which is strictly related to the construction of metrics with holonomy G_2 . An $SU(3)$ structure defines a non-degenerate 2-form F , an almost-complex structure J , and a complex volume form Ψ ; the $SU(3)$ structure is called *half-flat* if $F \wedge F$ and the real part of Ψ are closed [7]. Hypersurfaces in 7-dimensional manifolds with holonomy G_2 have a natural half-flat structure, given by the restriction of the holonomy group representation. In [24] Hitchin showed that, starting with a half-flat manifold (M, F, Ψ) , if certain evolution equations have a solution coinciding with (F, Ψ) at time zero then (M, F, Ψ) can be embedded isometrically as a hypersurface in a manifold with holonomy contained in G_2 .

If in addition F is closed the half-flat structure is called symplectic, and a half-flat structure with closed complex volume form Ψ is known as Hermitian balanced. Nilmanifolds of dimension 6 admitting invariant symplectic, resp. Hermitian balanced, half-flat structures have been classified in [11], resp. [28]. Recently 6-dimensional nilmanifolds carrying an invariant half-flat structure have been classified by Conti in [10], extending previous partial results [8, 11, 28, 6].

In this paper we show that supersymmetric flux vacua with constant intermediate $SU(2)$ structure are closely related to some special classes of half-flat structures. More concretely, in Section 3 we show that solutions of the SUSY equations IIA have a symplectic half-flat structure, whereas solutions of the SUSY equations IIB admit a half-flat structure which is in certain sense near to the Hermitian balanced condition. In particular, we prove that compact simply connected manifolds do not admit type IIB solutions. Solutions of the SUSY equations IIA and IIB are constructed from hyperkähler 4-manifolds and, more generally, from special hypo 5-manifolds, where by hypo we mean the natural $SU(2)$ structure induced on hypersurfaces in 6-dimensional manifolds with holonomy $SU(3)$ given by the restriction of the holonomy group representation [12]. In the last section we consider 6-dimensional solvmanifolds having both symplectic and Hermitian balanced half-flat structures, and using them we find new solutions of the SUSY equations IIA and IIB. The nilmanifolds considered in this paper have appeared previously in [19], where solutions with $SU(3)$ or static $SU(2)$ structure were found. On the other hand, on the solvmanifold of Example 4.2.1 $SU(3)$ structure solutions were given in [4, 19] (see also [2]), however to our knowledge the solvmanifold of Example 4.2.2 has not appeared previously in relation to the SUSY equations and provides a new class of solutions. In Section 4 it is also proved that in general the solutions of equations IIA or IIB are not stable by small deformations inside the class of half-flat structures (see Proposition 4.1).

2. INTERMEDIATE $SU(2)$ STRUCTURES

In this section we follow the conventions of [1], and recall the four-dimensional Minkowski flux vacua conditions of type II string theory with at least $N = 1$ supersymmetry as well as their relation to the structure group of the internal manifold.

As in [1] we consider type II supergravity (SUGRA) backgrounds, which are warped products of the Minkowski space $\mathbb{R}^{3,1}$ and of a 6-dimensional compact manifold M^6 . These warped products have a metric of the form

$$(1) \quad ds_{(10)}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{\mu\nu}(y) dy^\mu dy^\nu,$$

where η is the diagonal Minkowski metric with signature $(3, 1)$. The solutions will also have non zero background values for some of the RR and NS fluxes. Let $vol_{(4)}$ denote the warped 4-dimensional volume form. Poincaré invariance in dimension 4 requires the fluxes living on Minkowski space to be proportional to $vol_{(4)}$, so we will focus on non trivial fluxes living on the internal manifold M^6 .

As in [19] the total internal RR field F is given by

$$\text{IIA} : F = F_0 + F_2 + F_4 + F_6,$$

$$\text{IIB} : F = F_1 + F_3 + F_5,$$

where F_k is the internal k -form RR field, and it is related to the total 10-dimensional RR field-strength $F^{(10)}$ by

$$F^{(10)} = F + vol_{(4)} \wedge \lambda(*F).$$

Here $*$ denotes the Hodge star operator on (M^6, g) and λ is given by

$$\lambda(A_p) = (-1)^{\frac{p(p-1)}{2}} A_p,$$

for every p -form A_p .

In order to find such solutions one has to solve the equations of motion and the Bianchi identities for the fluxes, however, for the class of supergravity backgrounds we are interested in, the equations of motion for the metric and the dilaton ϕ are implied by the Bianchi identities and the 10-dimensional supersymmetry conditions [26], so one can solve the latter. These conditions are the annihilation of the supersymmetry variations of the gravitino ψ_μ and the dilatino λ given by (see [17])

$$\delta\psi_\mu = D_\mu \epsilon + \frac{1}{4} H_\mu \mathcal{P} \epsilon + \frac{1}{16} e^\phi \sum_n \mathcal{F}_{2n} \gamma_\mu \mathcal{P}_n \epsilon,$$

$$\delta\lambda = (\not{\partial}\phi + \frac{1}{2} \not{H} \mathcal{P}) \epsilon + \frac{1}{8} e^\phi \sum_n (-1)^{2n} (5 - 2n) \mathcal{F}_{2n} \mathcal{P}_n \epsilon,$$

with $H_\mu = \frac{1}{2} H_{\mu\nu\rho} \gamma^{\nu\rho}$, H being the NSNS flux. For IIA, $\mathcal{P} = \gamma_{11}$ and $\mathcal{P}_n = \gamma_{11}^n \sigma^1$ for $n = 0, \dots, 5$, while for IIB, $\mathcal{P} = -\sigma^3$, $\mathcal{P}_n = \sigma^1$ for $n = \frac{3}{2}, \frac{7}{2}$ and $\mathcal{P}_n = i\sigma^2$ for $n = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}$.

The 10-dimensional supersymmetry parameter ϵ can be written as a pair (ϵ^1, ϵ^2) of two Majorana-Weyl supersymmetry parameters and, because of the product structure of the solution (1), there should exist independent globally defined and non-vanishing spinors η^j on M^6 such that each ϵ^j is given as

$$\epsilon^j = \zeta^j \otimes \sum_a f_a^j \eta_a^j + c.c., \quad j = 1, 2,$$

where ζ^1 and ζ^2 are the 4-dimensional supersymmetry parameters.

In order to get (at least) $N = 1$ supersymmetry, it is required the existence of (at least) a pair (η^1, η^2) of globally defined and non-vanishing spinors on the internal manifold M^6 satisfying the SUSY conditions. The existence of this pair of internal spinors generically implies that the structure group of the tangent bundle over the internal manifold M^6 is reduced to a subgroup $G \subset SO(6)$. This is due to the fact that the spinors which are globally defined must not transform under G and therefore are singlets under the $SO(6) \rightarrow G$ decomposition. The pair (η^1, η^2)

can be parametrized and different types of G -structures are defined on the internal manifold depending on the values of the parameters. $SU(3)$ and intermediate $SU(2)$ structures on 6-manifolds arise naturally in this context as we recall next.

The existence of a globally defined non-vanishing spinor η_+ on a 6-dimensional manifold M^6 defines a reduction of the structure group of the tangent bundle over M^6 to $SU(3)$. Therefore, on the internal manifold we have an almost Hermitian structure (J, g) and a $(3, 0)$ -form Ψ such that

$$F \wedge \Psi = 0, \quad \frac{4}{3}F^3 = i\Psi \wedge \bar{\Psi} \neq 0,$$

where F is the fundamental 2-form associated to (J, g) . The spinor η_+ is a Weyl spinor and it is supposed to have positive chirality and unit norm. Complex conjugation acts sending η_+ in η_- . The forms (F, Ψ) can be obtained as bilinears of the globally defined spinor. Indeed:

$$F_{\mu\nu} = -i\eta_+^\dagger \gamma_{\mu\nu} \eta_+, \quad \Psi_{\mu\nu\rho} = -i\eta_-^\dagger \gamma_{\mu\nu\rho} \eta_+.$$

An $SU(2)$ structure on a 6-dimensional manifold M^6 is defined by two orthogonal globally defined spinors η_+ and χ_+ , which we can suppose of unit norm, or equivalently by an almost Hermitian structure (J, g) , a $(1, 0)$ -form α , a real 2-form ω and a $(2, 0)$ -form Ω satisfying the following conditions

$$\begin{aligned} \omega^2 &= \frac{1}{2}\Omega \wedge \bar{\Omega} \neq 0, \\ \omega \wedge \Omega &= 0, \quad \Omega \wedge \Omega = 0, \\ i_\alpha \Omega &= 0, \quad i_\alpha \omega = 0, \end{aligned}$$

where by i_α we denote the contraction by the vector field dual to α and we take α such that $\|\alpha\|^2 = i_{\bar{\alpha}}\alpha = \bar{\alpha}_{\bar{a}}g^{\bar{a}b}\alpha_b = 2$.

The forms (α, ω, Ω) are related to the globally defined spinors (η_+, χ_+) by the relations

$$\begin{aligned} \alpha_\mu &= \eta_-^\dagger \gamma_\mu \chi_+, \\ \omega_{\mu\nu} &= -i\eta_+^\dagger \gamma_{\mu\nu} \eta_+ + i\chi_+^\dagger \gamma_{\mu\nu} \chi_+, \\ \Omega_{\mu\nu} &= \eta_-^\dagger \gamma_{\mu\nu} \chi_-. \end{aligned}$$

The spinor χ_+ can be rewritten in terms of η_- as $\chi_+ = \frac{1}{2}\alpha\eta_-$.

The $SU(2)$ structure is naturally embedded in the $SU(3)$ structure defined by η_+ by:

$$(2) \quad F = \omega + \frac{i}{2}\alpha \wedge \bar{\alpha}, \quad \Psi = \alpha \wedge \Omega.$$

Conversely, if one has an $SU(3)$ structure (F, Ψ) on M^6 and a $(1, 0)$ -form α of norm $\sqrt{2}$, then it has been proved in [1, Appendix A2] that ω and Ω defined by these formulas

$$\omega = F - \frac{i}{2}\alpha \wedge \bar{\alpha}, \quad \Omega = \frac{1}{2}i_{\bar{\alpha}}\Psi$$

provide an $SU(2)$ structure.

Given a pair (η_+^1, η_+^2) of globally defined non-vanishing internal spinors corresponding to the internal components of the supersymmetry parameters, one can parametrize them as

$$(3) \quad \eta_+^1 = a\eta_+, \quad \eta_+^2 = b(k_{\parallel}\eta_+ + k_{\perp}\frac{1}{2}\alpha\eta_-),$$

with $0 \leq k_{\parallel} \leq 1$, $k_{\perp} = \sqrt{1 - k_{\parallel}^2}$ and a, b non-zero complex numbers such that $a = \|\eta_+^1\|$ and $b = \|\eta_+^2\|$. As in [1], we consider $b = \bar{a}$ so that the relative phases of the spinors are fixed by $|a|$ and θ , the latter given by $e^{i\theta} = \bar{a}/a$.

Now depending on the values of k_{\parallel} and k_{\perp} one can define starting from the spinors different type of G structures. Indeed, if $k_{\perp} = 0$, or equivalently if η_+^1 and η_+^2 are parallel, then one has an $SU(3)$ structure. If $k_{\perp} \neq 0$ one has an $SU(2)$ structure and in the particular case when $k_{\perp} = 1$ and $k_{\parallel} = 0$ one gets the so called static $SU(2)$ structure. But one can consider as in [1] the intermediate case $k_{\perp} \neq 0$ and $k_{\parallel} \neq 0$. The two orthogonal spinors η_+ and $\chi_+ = \frac{1}{2}\alpha\eta_-$ define an $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ on the internal manifold M^6 as above and, relating the numbers k_{\parallel} and k_{\perp} to the angle $\phi \in [0, \frac{\pi}{2}]$ between the spinors by

$$k_{\parallel} = \cos(\phi), \quad k_{\perp} = \sin(\phi),$$

one obtains (see [13]) the family of $SU(3)$ structures on M^6 given by

$$\begin{aligned} \tilde{F}_{\phi} &= \cos(2\phi)\omega + \frac{i}{2}\alpha \wedge \bar{\alpha} + \sin(2\phi)Re(\Omega), \\ \tilde{\Psi}_{\phi} &= \alpha \wedge (-\sin(2\phi)\omega + \cos(2\phi)Re(\Omega) + iIm(\Omega)), \end{aligned}$$

or equivalently the family of $SU(2)$ structures

$$(4) \quad \begin{aligned} \tilde{\omega}_{\phi} &= \cos(2\phi)\omega + \sin(2\phi)Re(\Omega), \\ \tilde{\Omega}_{\phi} &= -\sin(2\phi)\omega + \cos(2\phi)Re(\Omega) + iIm(\Omega). \end{aligned}$$

Definition 2.1. [1] The $SU(2)$ structure on $(M^6, J, g, \alpha, \omega, \Omega)$ defined by (4) is called *intermediate* if k_{\parallel} and k_{\perp} are both different from zero. It is called *static* (or *orthogonal*) if $k_{\perp} = 1$ and $k_{\parallel} = 0$.

We recall that an $SU(3)$ structure (F, Ψ) is said to be *half-flat* if $d(F \wedge F) = 0$ and $d(Re(\Psi)) = 0$. If in addition $dF = 0$, then the $SU(3)$ structure is said to be *symplectic half-flat*.

Definition 2.2. An $SU(3)$ structure (F, Ψ) on M^6 is called *Hermitian balanced* if $d(F \wedge F) = 0$ and $d\Psi = 0$.

From now on by a *symplectic half-flat*, resp. *Hermitian balanced*, $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ we mean that the associated $SU(3)$ structure given by (2) is symplectic half-flat, resp. Hermitian balanced.

3. SUSY EQUATIONS

In this section we show that supersymmetric flux vacua with intermediate $SU(2)$ structure are closely related to the existence of special classes of half-flat structures on the internal manifold. We begin by recalling the SUSY conditions derived by Andriot in [1].

To solve the SUSY conditions, rather than using Killing spinors methods or G -structures tools, in [1] it was used the formalism of generalized complex geometry [20, 25, 19]. In generalized complex geometry for a d -dimensional manifold M , one considers the bundle $TM \oplus T^*M$, whose sections are generalized vectors (sums of a vector and a 1-form). The spinors on $TM \oplus T^*M$ are Majorana-Weyl $Cliff(d, d)$ spinors, and locally they can be seen as polyforms, i.e. sums of even/odd differential forms, which correspond to positive/negative chirality spinors. A $Cliff(d, d)$ spinor

is pure if it is annihilated by half of the $\text{Cliff}(d, d)$ gamma matrices. Such pure spinors can be obtained also as tensor products of $\text{Cliff}(d)$ spinors.

In the supergravity context, the $\text{Cliff}(6, 6)$ pure spinors are defined as a biproduct $\Phi_{\pm} = \eta_{\pm}^1 \otimes \eta_{\pm}^{2\dagger}$ of the internal supersymmetry parameters and, via the Fierz identity, they can be seen as polyforms

$$\Phi_{\pm} = \eta_{\pm}^1 \otimes \eta_{\pm}^{2\dagger} = \frac{1}{8} \sum_{k=0}^6 \frac{1}{k!} (\eta_{\pm}^{2\dagger} \gamma_{\mu_k \dots \mu_1} \eta_{\pm}^1) \gamma^{\mu_1 \dots \mu_k}.$$

The explicit expressions of the two pure spinors in terms of the forms (α, ω, Ω) are then

$$\begin{aligned} \Phi_+ &= \frac{|a|^2}{8} e^{-i\theta} e^{\frac{1}{2}\alpha \wedge \bar{\alpha}} (k_{||} e^{-i\omega} - i k_{\perp} \Omega), \\ \Phi_- &= -\frac{|a|^2}{8} \alpha \wedge (k_{\perp} e^{-i\omega} + i k_{||} \Omega). \end{aligned}$$

In general, by [20] a pure spinor Φ can be written as

$$\Phi = \Omega_k \wedge e^{B+iK},$$

where Ω_k is a holomorphic k -form, and B and K are real 2-forms. The rank k of the form Ω_k is the type of the spinor. For the intermediate $SU(2)$ structure where both $k_{||}$ and k_{\perp} are different from zero, by [1], the two pure spinors Φ_+ and Φ_- can be rewritten as

$$\Phi_+ = \frac{|a|^2}{8} e^{-i\theta} k_{||} e^{\frac{1}{2}\alpha \wedge \bar{\alpha} - i\omega - i \frac{k_{\perp}}{k_{||}} \Omega}, \quad \Phi_- = -\frac{|a|^2}{8} k_{\perp} \alpha \wedge e^{-i\omega + i \frac{k_{||}}{k_{\perp}} \Omega}$$

and thus Φ_+ and Φ_- have respectively type 0 and 1. In the case of the $SU(3)$ structure (limit $k_{\perp} = 0$),

$$\Phi_+ = \frac{|a|^2}{8} e^{-i\theta} e^{-iF}, \quad \Phi_- = -i \frac{|a|^2}{8} \Psi$$

and the two pure spinors are of type 0 and 3, respectively. In the case of a static $SU(2)$ structure (the other limit $k_{||} = 0$) one has

$$\Phi_+ = -i \frac{|a|^2}{8} e^{-i\theta} \Omega \wedge e^{\frac{1}{2}\alpha \wedge \bar{\alpha}}, \quad \Phi_- = -\frac{|a|^2}{8} \alpha \wedge e^{-i\omega}$$

and the two pure spinors are of type 2 and 1, respectively.

Two pure spinors are said to be compatible if they have three common annihilators. A pair of compatible pure spinors defines an $SU(3) \times SU(3)$ structure on $TM \oplus T^*M$. Depending on the relation between the spinors $\eta_{\pm}^{1,2}$, this translates on TM into the $SU(3)$, static $SU(2)$ or intermediate $SU(2)$ structures discussed above. So the formalism of generalized complex geometry allows to give a unified characterization of the topological properties a $N = 1$ vacuum has to satisfy: it must admit an $SU(3) \times SU(3)$ structure on $TM \oplus T^*M$. And so to satisfy this condition, one may verify that our vacua admit a pair of compatible pure spinors.

An $N = 1$ vacuum should satisfy the SUSY conditions, the equations of motion and the Bianchi identities for the fluxes. By [17, 18] the SUSY equations can be then written in terms of pure spinors by

$$\begin{aligned} (d - H \wedge)(e^{2A-\phi} \Phi_1) &= 0, \\ (d - H \wedge)(e^{A-\phi} \text{Re}(\Phi_2)) &= 0, \\ (d - H \wedge)(e^{3A-\phi} \text{Im}(\Phi_2)) &= \frac{e^{4A}}{8} * \lambda(F), \end{aligned}$$

with $\Phi_1 = \Phi_{\pm}$, $\Phi_2 = \Phi_{\mp}$ for IIA/IIB (upper/lower), following the conventions of [19]. The first of these equations implies that one of the two pure spinors (the one with the same parity as the RR fields) must be twisted (because of the $-H\wedge$) conformally closed. A manifold admitting a twisted closed pure spinor is a twisted Generalized Calabi-Yau (see [24, 19]) and one looks for vacua on such manifolds.

The equations of motion of the fluxes are

$$(d + H\wedge)(e^{4A} * F) = 0, \quad d(e^{4A-2\phi} * H) = \mp e^{4A} \sum_p F_p \wedge *F_{p+2},$$

with the upper/lower sign for IIA/IIB.

The Bianchi identities (we assume no NS source) are

$$(d - H\wedge)F = \delta(\text{source}), \quad dH = 0,$$

where $\delta(\text{source})$ is the charge density of the allowed sources: space-filling D -branes or orientifold planes (O -planes). In compactification to 4-dimensional Minkowski, the trace of the energy momentum tensor must be zero. Then O -planes are needed since they are the only known sources with a negative charge, that can thus cancel the flux contribution to this trace. As in [1] the RR Bianchi identities are then assumed to be

$$(d - H\wedge)F = \sum_i Q^i V^i,$$

where Q^i is the source charge and V^i is (up to a sign) its internal co-volume (the co-volume of the cycle wrapped by the source). The sign of the Q^i indicates whether the source is a D -brane ($Q^i > 0$) or an O -plane ($Q^i < 0$).

For intermediate $SU(2)$ structures (for which $\frac{k_{\perp}}{k_{\parallel}}$ is constant) in the large volume limit from the SUSY conditions one gets that the H Bianchi identities is automatically satisfied. Furthermore, for this class of compactifications, it was shown in [19] that the equations of motion for the RR fluxes are implied by the SUSY conditions. And it was shown in [26] that the equation of motion of H is implied by the SUSY conditions and the Bianchi identities. Then, in order to find a solution, having a pair of compatible pure spinors on an twisted generalized Calabi-Yau manifold with at least one O -plane, as in [1] one has to verify that the SUSY conditions and the RR Bianchi identities are satisfied.

The presence of O -planes implies that the solution has to be invariant under the action of the orientifold. As shown in [26] the first step to derive the orientifold projection on the pure spinors is to compute those for the internal SUSY parameters. This can be done starting from the projection on the 10-dimensional SUSY spinorial parameters ϵ^i and then reducing to the internal spinors η_{\pm}^i . In our conventions, we have

$$\begin{aligned} O5 : \sigma(\eta_{\pm}^1) &= \eta_{\pm}^2, & \sigma(\eta_{\pm}^2) &= \eta_{\pm}^1, \\ O6 : \sigma(\eta_{\pm}^1) &= \eta_{\mp}^2, & \sigma(\eta_{\pm}^2) &= \eta_{\mp}^1, \end{aligned}$$

where σ is the target space reflection in the directions transverse to the O -plane. Using the expressions (3) for the internal spinors, one gets as in [1] the following projection conditions at the orientifold plane:

$$\begin{aligned} O5 : e^{i\theta} &= \pm 1, & \alpha &\perp O5, \\ O6 : e^{i\theta} &\text{free}, & \text{Re}(\alpha) &\parallel O6, & \text{Im}(\alpha) &\perp O6. \end{aligned}$$

The previous conditions can be expressed on α as

$$O5 : \sigma(\alpha) = -\alpha, \quad O6 : \sigma(\alpha) = \bar{\alpha}.$$

By [26] if the G -structures are constant (as the one which are considering), and if we work on nil/solvmanifolds (which will be our case), these conditions are valid everywhere (not only at the orientifold plane). Starting from the projections on the η_{\pm}^i , as in [26, 1] one may derive the projections of the pure spinors Φ_{\pm} and from them those for the $SU(2)$ structure forms. In particular one has

$$\begin{aligned} O5 : \sigma(\omega) &= (k_{\parallel}^2 - k_{\perp}^2)\omega + 2k_{\parallel}k_{\perp}Re(\Omega), & \sigma(\Omega) &= -k_{\parallel}^2\Omega + k_{\perp}^2\bar{\Omega} + 2k_{\parallel}k_{\perp}\omega, \\ O6 : \sigma(\omega) &= (k_{\perp}^2 - k_{\parallel}^2)\omega - 2k_{\parallel}k_{\perp}Re(\Omega), & \sigma(\Omega) &= -k_{\perp}^2\Omega + k_{\parallel}^2\bar{\Omega} - 2k_{\parallel}k_{\perp}\omega. \end{aligned}$$

By introducing as in [26]

$$O5 : k_{\parallel} = \cos(\phi), \quad k_{\perp} = \sin(\phi), \quad 0 \leq \phi \leq \frac{\pi}{2},$$

$$O6 : k_{\parallel} = \cos(\phi + \frac{\pi}{2}) = -\sin(\phi), \quad k_{\perp} = \sin(\phi + \frac{\pi}{2}) = \cos(\phi), \quad -\frac{\pi}{2} \leq \phi \leq 0,$$

one gets in both cases the following formulas:

$$\begin{aligned} \sigma(\omega) &= \cos(2\phi)\omega + \sin(2\phi)Re(\Omega), \\ \sigma(Re(\Omega)) &= \sin(2\phi)\omega - \cos(2\phi)Re(\Omega), \\ \sigma(Im(\Omega)) &= -Im(\Omega). \end{aligned}$$

Since the previous projection conditions are not very tractable, in [1] he worked in the projection (eigen)basis

$$(5) \quad \begin{aligned} \omega_{\parallel} &= \frac{1}{2}(\omega + \sigma(\omega)), & \omega_{\perp} &= \frac{1}{2}(\omega - \sigma(\omega)), \\ Re(\Omega)_{\parallel} &= \frac{1}{2}(Re(\Omega) + \sigma(Re(\Omega))), & Re(\Omega)_{\perp} &= \frac{1}{2}(Re(\Omega) - \sigma(Re(\Omega))), \end{aligned}$$

which can then be expressed in terms of the original $SU(2)$ -structure as

$$\begin{aligned} \omega_{\parallel} &= \frac{1}{2}((1 + \cos(2\phi))\omega + \sin(2\phi)Re(\Omega)), \\ \omega_{\perp} &= \frac{1}{2}((1 - \cos(2\phi))\omega - \sin(2\phi)Re(\Omega)), \\ Re(\Omega)_{\parallel} &= \frac{1}{2}((1 - \cos(2\phi))Re(\Omega) + \sin(2\phi)\omega), \\ Re(\Omega)_{\perp} &= \frac{1}{2}((1 + \cos(2\phi))Re(\Omega) - \sin(2\phi)\omega). \end{aligned}$$

As in [1] one takes $e^A = |a|^2 = 1$ and go to the large volume limit, i.e. $A = 0$ and $e^{\phi} = g_s$ constant. This is indeed the regime in which one will look for solutions. The only remaining freedom is θ that we do not really need to fix. Moreover, we choose to look only for intermediate $SU(2)$ structure, i.e. with $k_{\parallel} \neq 0$ and $k_{\perp} \neq 0$ constant. Taking the coefficients constant is important because it simplifies drastically the search for solutions and the SUSY conditions are much simpler. In fact, by using the projection (eigen)basis (5) and the results in [17, 18], together with further simplifications as explained in [1], one can rewrite the SUSY equations in the following form:

• *SUSY equations IIA:*

$$(6) \quad \begin{cases} d(Re(\alpha)) = 0, \\ k_{\parallel}H = k_{\perp}d(Im(\Omega)), \\ d(Re(\Omega)_{\perp}) = k_{\parallel}k_{\perp}Re(\alpha) \wedge d(Im(\alpha)), \\ H \wedge Re(\alpha) = -\frac{k_{\perp}}{k_{\parallel}}d(Im(\alpha) \wedge Re(\Omega)_{\parallel}), \end{cases}$$

together with F_0, F_2 and F_4 given by

$$\begin{aligned} g_s * F_0 &= \frac{1}{2}k_\perp d(Im(\alpha)) \wedge (Im(\Omega))^2 + \frac{1}{k_\parallel} H \wedge Re(\alpha) \wedge Re(\Omega)_\parallel, \\ g_s * F_2 &= -k_\parallel d(Im(\alpha)) \wedge Im(\Omega) + \frac{1}{k_\parallel} d(Re(\Omega)_\parallel) \wedge Re(\alpha), \\ g_s * F_4 &= -k_\perp d(Im(\alpha)). \end{aligned}$$

• *SUSY equations IIB:*

$$(7) \quad \begin{cases} d(Re(\alpha)) = 0, \\ d(Im(\alpha)) = 0, \\ k_\parallel H = k_\perp d(Im(\Omega)), \\ Re(\alpha) \wedge H = -\frac{k_\perp}{k_\parallel} Im(\alpha) \wedge d(Re(\Omega)_\perp), \\ Im(\alpha) \wedge H = \frac{k_\perp}{k_\parallel} Re(\alpha) \wedge d(Re(\Omega)_\perp), \\ Re(\alpha) \wedge Im(\alpha) \wedge d(Re(\Omega)_\parallel) = -H \wedge Im(\Omega), \end{cases}$$

together with F_1 and F_3 given by

$$\begin{aligned} k_\perp e^{i\theta} g_s * F_1 &= H \wedge Re(\Omega)_\parallel, \\ k_\perp e^{i\theta} g_s * F_3 &= d(Re(\Omega)_\parallel). \end{aligned}$$

Note that $(\frac{1}{\cos \phi} Re(\Omega)_\perp, \frac{1}{\sin \phi} Re(\Omega)_\parallel, Im(\Omega), \alpha)$ define a new $SU(2)$ structure $(\alpha, \omega', \Omega')$ on M with

$$\omega' = \frac{1}{\sin \phi} Re(\Omega)_\parallel, \quad \Omega' = \frac{1}{\cos \phi} Re(\Omega)_\perp + i Im(\Omega),$$

and then a new $SU(3)$ structure (F', Ψ') . Moreover, since we will consider only O6 planes in IIA and O5 planes in IIB, by using a local adapted basis for this $SU(2)$ -structure (see [12]), one has that $(\hat{\alpha}, \hat{\omega}, \hat{\Omega})$ given by

$$\hat{\alpha} = Re(\alpha) + i k_\parallel Im(\alpha), \quad \hat{\omega} = \frac{1}{k_\perp} Re(\Omega)_\perp, \quad \hat{\Omega} = Im(\Omega) - i \frac{1}{k_\parallel} Re(\Omega)_\parallel,$$

is also an $SU(2)$ structure on M in the IIA case, and

$$\hat{\alpha}_1 = k_\parallel Re(\alpha) + i Im(\alpha), \quad \hat{\omega} = \frac{1}{k_\perp} Re(\Omega)_\parallel, \quad \hat{\Omega} = \frac{1}{k_\parallel} Re(\Omega)_\perp + i Im(\Omega),$$

$$\hat{\alpha}_2 = k_\parallel Im(\alpha) - i Re(\alpha), \quad \hat{\omega} = \frac{1}{k_\perp} Re(\Omega)_\parallel, \quad \hat{\Omega} = \frac{1}{k_\parallel} Re(\Omega)_\perp + i Im(\Omega),$$

are $SU(2)$ structures on M in the IIB case. We will use these structures in the next theorems, and the corresponding $SU(3)$ structures will be denoted by $(\hat{F}, \hat{\Psi})$. Notice that the almost complex structure \hat{J} and the metric \hat{g} change with respect to those given by $(\alpha, \omega', \Omega')$.

Theorem 3.1. *Let $(M^6, J, g, \alpha, \omega, \Omega)$ be a 6-dimensional manifold endowed with an $SU(2)$ structure such that the 2-forms $Re(\Omega)_\parallel, Re(\Omega)_\perp, Im(\Omega)$ satisfy the equations (6), then M^6 admits a symplectic half-flat $SU(2)$ structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ with $d(Re(\hat{\alpha})) = 0$. Conversely, if M^6 has a symplectic half-flat $SU(2)$ structure*

$(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ such that $d(\text{Re}(\hat{\alpha})) = 0$, then the forms $(\text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega), \alpha)$ defined by

$$\frac{1}{k_{\perp}} \text{Re}(\Omega)_{\perp} = \hat{\omega}, \quad \text{Im}(\Omega) - i \frac{1}{k_{||}} \text{Re}(\Omega)_{||} = \hat{\Omega}, \quad \text{Re}(\alpha) + ik_{||} \text{Im}(\alpha) = \hat{\alpha}$$

are a solution of the equations (6).

Proof. For type IIA the 2-forms $\frac{1}{k_{\perp}} \text{Re}(\Omega)_{\perp}, \frac{1}{k_{||}} \text{Re}(\Omega)_{||}, \text{Im}(\Omega)$ together with the complex 1-form $\text{Re}(\alpha) + ik_{||} \text{Im}(\alpha)$ define a new $SU(2)$ structure with

$$\hat{\omega} = \frac{1}{k_{\perp}} \text{Re}(\Omega)_{\perp}, \quad \hat{\Omega} = \text{Im}(\Omega) - i \frac{1}{k_{||}} \text{Re}(\Omega)_{||}, \quad \hat{\alpha} = \text{Re}(\alpha) + ik_{||} \text{Im}(\alpha)$$

and then a new $SU(3)$ structure $(\hat{J}, \hat{F}, \hat{\Psi})$, with

$$\hat{F} = \hat{\omega} + k_{||} \text{Re}(\alpha) \wedge \text{Im}(\alpha), \quad \hat{\Psi} = \hat{\alpha} \wedge \hat{\Omega}.$$

By the second equation of (6) we have

$$H = \frac{k_{\perp}}{k_{||}} d(\text{Im}(\Omega)).$$

Then by the last equation of (6) we obtain

$$d(\text{Im}(\Omega)) \wedge \text{Re}(\alpha) = -d(\text{Im}(\alpha) \wedge \text{Re}(\Omega)_{||}),$$

i.e. that the real part of the form $\hat{\Psi} = \hat{\alpha} \wedge \hat{\Omega}$ is closed.

Moreover, by

$$d(\text{Re}(\Omega)_{\perp}) = k_{||} k_{\perp} \text{Re}(\alpha) \wedge d(\text{Im}(\alpha)),$$

it follows that

$$d(\text{Re}(\Omega)_{\perp} + k_{||} k_{\perp} \text{Re}(\alpha) \wedge \text{Im}(\alpha)) = k_{\perp} d(\hat{F}) = 0,$$

and so we have a symplectic half-flat $SU(2)$ structure on M^6 . Conversely, if M^6 has a symplectic half-flat $SU(2)$ structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ such that $d(\text{Re}(\hat{\alpha})) = 0$, we have that the fundamental form

$$\hat{F} = \hat{\omega} + \text{Re}(\hat{\alpha}) \wedge \text{Im}(\hat{\alpha}) = \frac{1}{k_{\perp}} \text{Re}(\Omega)_{\perp} + k_{||} \text{Re}(\alpha) \wedge \text{Im}(\alpha)$$

is closed and thus the equation

$$d(\text{Re}(\Omega)_{\perp}) = k_{||} k_{\perp} \text{Re}(\alpha) \wedge d(\text{Im}(\alpha))$$

in (6) holds. By the closedness of the real part of the $(3, 0)$ -form $\hat{\Psi} = \hat{\alpha} \wedge \hat{\Omega}$ we have that also the last equation in (6) is satisfied for $H = \frac{k_{\perp}}{k_{||}} d(\text{Im}(\Omega))$. \square

Theorem 3.2. *Let $(M^6, J, g, \alpha, \omega, \Omega)$ be a 6-dimensional manifold endowed with an $SU(2)$ structure such that the 2-forms $\text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega)$ satisfy the equations (7), then M^6 admits two half-flat $SU(2)$ structures $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ and $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2, \hat{\omega}, \hat{\Omega})$ such that $d(\hat{\alpha}_1) = 0$ and $\hat{\alpha}_2 = k_{||} \text{Im}(\hat{\alpha}_1) - i \frac{\text{Re}(\hat{\alpha}_1)}{k_{||}}$.*

Conversely, let M^6 be a 6-dimensional manifold endowed with a half-flat $SU(2)$ structure $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ satisfying $d(\hat{\alpha}_1) = 0$; if $k_{||} \in (0, 1)$ is such that the

$SU(2)$ structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_{||} Im(\hat{\alpha}_1) - i \frac{Re(\hat{\alpha}_1)}{k_{||}}, \hat{\omega}, \hat{\Omega})$ is half-flat, then the forms $(Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega), \alpha)$ defined by

$$\frac{1}{k_{\perp}} Re(\Omega)_{||} = \hat{\omega}, \quad \frac{1}{k_{||}} Re(\Omega)_{\perp} + i Im(\Omega) = \hat{\Omega}, \quad k_{||} Re(\alpha) + i Im(\alpha) = \hat{\alpha}_1,$$

where $k_{\perp} = \sqrt{1 - k_{||}^2}$, are a solution of the equations (7).

Proof. As we already remarked previously for type IIB the 2-forms $\frac{1}{k_{||}} Re(\Omega)_{\perp}, \frac{1}{k_{\perp}} Re(\Omega)_{||}, Im(\Omega), \alpha$ define a new $SU(2)$ structure with

$$\hat{\omega} = \frac{1}{k_{\perp}} Re(\Omega)_{||}, \quad \hat{\Omega} = \frac{1}{k_{||}} Re(\Omega)_{\perp} + i Im(\Omega), \quad \hat{\alpha}_1 = k_{||} Re(\alpha) + i Im(\alpha)$$

and then a new $SU(3)$ structure $(\hat{J}_1, \hat{F}, \hat{\Psi}_1)$, with

$$\hat{F} = \hat{\omega} + k_{||} Re(\alpha) \wedge Im(\alpha), \quad \hat{\Psi}_1 = \hat{\alpha}_1 \wedge \hat{\Omega}.$$

Suppose that the 2-forms $Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega)$ satisfy the equations (7), then by the first five equations we have

$$d(\alpha \wedge (Re(\Omega)_{\perp} + i Im(\Omega))) = 0.$$

Then, $d(Re(\hat{\alpha}_1 \wedge \hat{\Omega})) = 0$.

Since $(\hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ is an $SU(2)$ structure, the condition

$$Im(\Omega)^2 = \frac{1}{k_{\perp}^2} Re(\Omega)_{||}^2$$

is satisfied [12] and by the last equation of (7) we get

$$\begin{aligned} Re(\alpha) \wedge Im(\alpha) \wedge d(Re(\Omega)_{||}) &= -\frac{k_{\perp}}{k_{||}} d(Im(\Omega)) \wedge Im(\Omega) \\ &= -\frac{1}{k_{\perp} k_{||}} d(Re(\Omega)_{||}) \wedge Re(\Omega)_{||}. \end{aligned}$$

Therefore

$$d((k_{\perp} k_{||} Re(\alpha) \wedge Im(\alpha) + Re(\Omega)_{||})^2) = 0,$$

i.e. $d(\hat{F} \wedge \hat{F}) = 0$. Then we have a half-flat $SU(2)$ structure.

Consider

$$\hat{\alpha}_2 = k_{||} Im(\alpha) - i Re(\alpha)$$

and define $\hat{\Psi}_2 = \hat{\alpha}_2 \wedge \hat{\Omega}$. We have two $SU(3)$ structures $(\hat{F}, \hat{\Psi}_1)$ and $(\hat{F}, \hat{\Psi}_2)$. Indeed,

$$Re(\hat{\alpha}_1) \wedge Im(\hat{\alpha}_1) = Re(\hat{\alpha}_2) \wedge Im(\hat{\alpha}_2) = k_{||} Re(\alpha) \wedge Im(\alpha),$$

so \hat{F} is the same in both cases.

Now, equation

$$Im(\alpha) \wedge H = \frac{k_{\perp}}{k_{||}} Re(\alpha) \wedge d(Re(\Omega)_{\perp})$$

implies that $d(Re \hat{\Psi}_1) = 0$, whereas equation

$$Re(\alpha) \wedge H = -\frac{k_{\perp}}{k_{||}} Im(\alpha) \wedge d(Re(\Omega)_{\perp})$$

implies that $d(Re \hat{\Psi}_2) = 0$. In conclusion we have that $(\hat{F}, \hat{\Psi}_1)$ and $(\hat{F}, \hat{\Psi}_2)$ are half-flat.

To prove the converse, we first notice that the fundamental form \hat{F}_1 is given by $\hat{F}_1 = \hat{\omega} + Re(\hat{\alpha}_1) \wedge Im(\hat{\alpha}_1) = \frac{1}{k_{\perp}} Re(\Omega)_{||} + k_{||} Re(\alpha) \wedge Im(\alpha)$, and therefore

the closedness of the 4-form $\hat{F}_1 \wedge \hat{F}_1$ implies the last equation of (7) for $H = \frac{k_{\perp}}{k_{\parallel}} d(Im(\Omega))$.

Let us consider the complex 3-form $\hat{\Psi}_j = \hat{\alpha}_j \wedge \hat{\Omega}$, $j = 1, 2$. Since $Re(\hat{\Psi}_j) = Re(\hat{\alpha}_j) \wedge Re(\hat{\Omega}) - Im(\hat{\alpha}_j) \wedge Im(\hat{\Omega})$, we get that

$$d(Re(\hat{\Psi}_j)) = \frac{k_{\parallel}}{k_{\perp}} Im(\hat{\alpha}_j) \wedge H - \frac{1}{k_{\parallel}} Re(\hat{\alpha}_j) \wedge d(Re(\Omega)_{\perp}).$$

For $j = 1$ we get the equation $Im(\alpha) \wedge H = \frac{k_{\perp}}{k_{\parallel}} Re(\alpha) \wedge d(Re(\Omega)_{\perp})$, and for $j = 2$ we get $Re(\alpha) \wedge H = -\frac{k_{\perp}}{k_{\parallel}} Im(\alpha) \wedge d(Re(\Omega)_{\perp})$, so the equations (7) are satisfied. \square

Remark 3.3. Notice that in the conditions of Theorem 3.2 we can define a 1-parametric family of half-flat $SU(2)$ structures connecting the two given structures. In fact, by considering the usual rotation

$$\hat{\alpha}_t = k_{\parallel} \sin t Re(\alpha) + k_{\perp} \cos t Im(\alpha) + i \sin t Im(\alpha) - i \cos t Re(\alpha),$$

we have that $(\hat{F}, \hat{\alpha}_t \wedge \hat{\Omega})$ is half-flat for any value of t . Notice that the fundamental 2-form \hat{F} does not depend on t because $Re(\hat{\alpha}_t) \wedge Im(\hat{\alpha}_t) = k_{\parallel} Re(\alpha) \wedge Im(\alpha)$. The almost complex structure \hat{J}_t is given with respect to the basis $\{Re(\alpha), Im(\alpha)\}$ by

$$\hat{J}_t = \begin{pmatrix} \sin t \cos t (\frac{1}{k_{\parallel}} - k_{\parallel}) & k_{\parallel} \sin^2 t + \frac{\cos^2 t}{k_{\parallel}} \\ -\frac{\sin^2 t}{k_{\parallel}} - k_{\parallel} \cos^2 t & -\sin t \cos t (\frac{1}{k_{\parallel}} - k_{\parallel}) \end{pmatrix}$$

Next, using the characterization given in Theorem 3.2, we show that if there is solution of the SUSY equations IIB for any $k_{\parallel} \in (0, 1)$, then the manifold must be Hermitian balanced.

Proposition 3.4. *Let $(J, g, \alpha, \omega, \Omega)$ be a half-flat $SU(2)$ structure on a 6-manifold M^6 such that $d(\alpha) = 0$ and for each $\lambda \in (0, 1)$ the $SU(2)$ structure $(J_{\lambda}, g_{\lambda}, \beta_{\lambda} = \lambda Im(\alpha) - i \frac{Re(\alpha)}{\lambda}, \omega, \Omega)$ is half-flat. Then, $(J, g, \alpha, \omega, \Omega)$ is Hermitian balanced.*

Proof. Let us consider the $SU(3)$ structure (F, Ψ) given by $F = \omega + \frac{i}{2} \alpha \wedge \bar{\alpha}$ and $\Psi = \alpha \wedge \Omega$. According to Definition 2.2 we have to prove that $Im(\Psi)$ is a closed form. Let (F, Φ_{λ}) be the $SU(3)$ structure associated to $(J_{\lambda}, g_{\lambda}, \beta_{\lambda}, \omega, \Omega)$. Then, the real and imaginary parts of the forms Ψ and Φ_{λ} are given by

$$\begin{aligned} Re(\Psi) &= Re(\alpha) \wedge Re(\Omega) - Im(\alpha) \wedge Im(\Omega), \\ Im(\Psi) &= Re(\alpha) \wedge Im(\Omega) + Im(\alpha) \wedge Re(\Omega), \\ Re(\Phi_{\lambda}) &= \frac{Re(\alpha)}{\lambda} \wedge Im(\Omega) + \lambda Im(\alpha) \wedge Re(\Omega), \\ Im(\Phi_{\lambda}) &= -\frac{Re(\alpha)}{\lambda} \wedge Re(\Omega) + \lambda Im(\alpha) \wedge Im(\Omega). \end{aligned}$$

The limit of the 3-form Φ_{λ} exists when $\lambda \rightarrow 1$ and equals $-i\Psi$. Since $Re(\Phi_{\lambda})$ is closed for any $\lambda \in (0, 1)$, we conclude that $Im(\Psi)$ is closed. Therefore, the $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ is Hermitian balanced. \square

Given a Hermitian balanced $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ such that $d(\alpha) = 0$ and $Re(\alpha) \wedge d(Re(\Omega)) = 0$ one can construct a solution of the SUSY equations IIB for any $k_{\parallel} \in (0, 1)$.

Proposition 3.5. *Let $(J, g, \alpha, \omega, \Omega)$ be a Hermitian balanced $SU(2)$ structure on a 6-manifold M^6 such that $d(\alpha) = 0$ and $Re(\alpha) \wedge d(Re(\Omega)) = 0$, then for each $\lambda \in (0, 1)$ the $SU(2)$ structure $(J_\lambda, g_\lambda, \beta_\lambda = \lambda Im(\alpha) - i \frac{Re(\alpha)}{\lambda}, \omega, i\Omega)$ is half-flat.*

Proof. We have $\Psi_\lambda = (\lambda Im(\alpha) - i \frac{Re(\alpha)}{\lambda}) \wedge (iRe(\Omega) - Im(\Omega))$ and thus

$$\begin{aligned} Re(\Psi_\lambda) &= -\lambda Im(\alpha) \wedge Im(\Omega) + \frac{Re(\alpha)}{\lambda} \wedge Re(\Omega), \\ Im(\Psi_\lambda) &= \lambda Im(\alpha) \wedge Re(\Omega) + \frac{Re(\alpha)}{\lambda} \wedge Im(\Omega). \end{aligned}$$

By the assumptions on the $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ we get in particular that

$$Re(\alpha) \wedge d(Re(\Omega)) = Im(\alpha) \wedge d(Im(\Omega)) = 0$$

and therefore that $d(Re(\Psi_\lambda)) = 0$. \square

Example 3.6. Let us consider a nilmanifold corresponding to the nilpotent Lie algebra $\mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23)$, that is, there is a basis $\{e^1, \dots, e^6\}$ satisfying

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12} \quad de^6 = e^{14} + e^{23}.$$

We consider the structure (F, Ψ) given by the 2-form $F = e^{13} + e^{24} - e^{56}$ and the (3,0)-form $\Psi = (e^1 + i e^3)(e^2 + i e^4)(e^6 + i e^5)$. Although \mathfrak{h}_4 admits Hermitian balanced structures, the previous structure is only half-flat. In fact, F^2 and $Re(\Psi)$ are closed, but $d(Im(\Psi)) = -e^{1234}$.

For the complex 3-form $\Phi_\lambda = (\lambda e^3 - i \frac{e^1}{\lambda})(e^2 + i e^4)(e^6 + i e^5)$, a direct calculation shows that

$$d(Re(\Phi_\lambda)) = \frac{1}{\lambda} d(e^{125} + e^{146}) + \lambda d(e^{326} - e^{345}) = \frac{2\lambda^2 - 1}{\lambda} e^{1234},$$

which implies that $Re(\Phi_\lambda)$ is closed only for $\lambda = \pm \frac{1}{\sqrt{2}}$. Notice that $Im(\Phi_\lambda)$ is closed for any λ , so $(F, \Phi_{\pm \frac{1}{\sqrt{2}}})$ are Hermitian balanced and $(F, \Phi_{\frac{1}{\sqrt{2}}})$ provides a solution to equations (7). In fact, by Theorem 3.2 we have the following explicit solution $(\alpha, Re(\Omega)_\parallel, Re(\Omega)_\perp, Im(\Omega))$ of the SUSY equations IIB for $k_\parallel = k_\perp = \frac{1}{\sqrt{2}}$:

$$Re(\alpha) = \sqrt{2} e^1, \quad Im(\alpha) = e^3, \quad Re(\Omega)_\parallel = \frac{1}{\sqrt{2}}(e^{24} - e^{56}),$$

$$Re(\Omega)_\perp = \frac{1}{\sqrt{2}}(e^{26} - e^{45}), \quad Im(\Omega) = e^{25} + e^{46}.$$

Notice that the fluxes are:

$$H = -e^{234}, \quad e^{i\theta} g_s * F_3 = -e^{126} + e^{145} + e^{235}, \quad e^{i\theta} g_s * F_1 = e^{23456}.$$

Now, let us start from the (Hermitian balanced) structure Φ_λ for $\lambda = -\frac{1}{\sqrt{2}}$, that is,

$$\Phi_{-\frac{1}{\sqrt{2}}} = \left(-\frac{e^3}{\sqrt{2}} + i\sqrt{2}e^1\right)(e^2 + i e^4)(e^6 + i e^5),$$

and consider the complex 3-form Θ_μ given by

$$\Theta_\mu = \left(\sqrt{2}\mu e^1 + i \frac{e^3}{\sqrt{2}\mu}\right)(e^2 + i e^4)(e^6 + i e^5).$$

It is easy to check that $Re(\Theta_\mu)$ is closed for any value of μ . Notice that in particular $\Theta_{\frac{1}{\sqrt{2}}} = \Psi$, i.e. one member in the family is precisely the half-flat structure (F, Ψ)

given at the beginning. Again, by Theorem 3.2 we have the following solution $(\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))$ of the SUSY equations IIB for any $k_{||} = \mu \in (0, 1)$:

$$Re(\alpha) = -\frac{1}{\sqrt{2}k_{||}}e^3, \quad Im(\alpha) = \sqrt{2}e^1, \quad Re(\Omega)_{||} = k_{\perp}(e^{24} - e^{56}),$$

$$Re(\Omega)_{\perp} = k_{||}(e^{26} - e^{45}), \quad Im(\Omega) = e^{25} + e^{46},$$

where $k_{\perp} = \sqrt{1 - k_{||}^2}$. The fluxes are:

$$H = -\frac{k_{\perp}}{k_{||}}e^{234}, \quad e^{i\theta}g_s * F_3 = -e^{126} + e^{145} + e^{235}, \quad e^{i\theta}g_s * F_1 = \frac{k_{\perp}}{k_{||}}e^{23456}.$$

From Theorem 3.1 it follows that a compact 6-manifold M^6 admitting a solution to equations (6) satisfies those topological restrictions imposed by the existence of a symplectic form, in particular the Betti numbers $b_2(M^6)$ and $b_4(M^6)$ do not vanish. Next we prove that $b_1(M^6) \geq 2$ for any compact manifold M^6 admitting solution to (7). In Examples 4.2.1 and 4.2.2 we show that this lower bound can be attained.

Proposition 3.7. *Let $(M^6, J, g, \alpha, \omega, \Omega)$ be a 6-dimensional compact manifold endowed with an $SU(2)$ structure such that the 2-forms $Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega)$ satisfy the equations (7), then M^6 has first Betti number ≥ 2 . In particular, there is no solution on compact simply connected 6-dimensional manifolds.*

Proof. From equations (7) we can prove that the closed 1-forms $Re(\alpha)$ and $Im(\alpha)$ are harmonic with respect to g . In fact, the 5-form $*Re(\alpha)$ is closed because it is a (constant) multiple of $Im(\alpha) \wedge (Im(\Omega))^2$, which is closed by the last equation of (7), taking into account the value of H . Similarly, the 5-form $*Im(\alpha)$ is also closed. \square

Let us remind that a Riemannian manifold (N, g) is called *hyperkähler* if there are three complex structures, I, J, K on N satisfying the quaternion relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = K = -JI,$$

and such that I, J, K are parallel. In particular, we have three Kähler forms ω_I, ω_J and ω_K on N .

Proposition 3.8. *Let (N^4, I, J, K) be a compact 4-dimensional hyperkähler manifold. Then, on the 6-dimensional manifold $M^6 = N^4 \times \mathbb{T}^2$ there exist solutions to the SUSY equations IIA and IIB. More precisely, if β^1, β^2 is a basis of 1-forms on the torus \mathbb{T}^2 , then $(\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))$ given by*

$$Re(\alpha) = \beta^1, \quad Im(\alpha) = \frac{1}{k_{||}}\beta^2, \quad Re(\Omega)_{||} = -k_{||}\omega_J, \quad Re(\Omega)_{\perp} = k_{\perp}\omega_K, \quad Im(\Omega) = \omega_I,$$

solves equations (6), and $(\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))$ given by

$$Re(\alpha) = \frac{1}{k_{||}}\beta^1, \quad Im(\alpha) = \beta^2, \quad Re(\Omega)_{||} = k_{\perp}\omega_K, \quad Re(\Omega)_{\perp} = k_{||}\omega_I, \quad Im(\Omega) = \omega_J,$$

are solutions to equations (7).

Proof. Let us consider $\hat{\alpha}_1 = \beta^1 + i\beta^2$, $\hat{\omega} = \omega_K$ and $\hat{\Omega} = \omega_I + i\omega_J$. Since the $SU(3)$ structure $(\hat{F} = \beta^{12} + \omega_K, \hat{\Psi} = \hat{\alpha}_1 \wedge \hat{\Omega})$ is integrable, the $SU(2)$ structure $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ is obviously symplectic half-flat. Moreover, for any $k_{||} \in (0, 1)$ the

$SU(2)$ structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_{||} \beta^2 - i \frac{\beta^1}{k_{||}}, \hat{\omega}, \hat{\Omega})$ is also half-flat and then the result follows from Theorems 3.1 and 3.2. \square

Notice that any compact hyperkähler surface is either a complex torus with a flat metric or a K3-surface with Calabi-Yau metric [3]. Also observe that for the solutions given in this proposition all the fluxes vanish.

Next we generalize the previous proposition by means of hypo structures on 5-manifolds. We recall that an $SU(2)$ structure on a 5-manifold P^5 is an $SU(2)$ -reduction of the principal bundle of linear frames on P , equivalently a triple (η, ω_1, Φ) , where η is a 1-form, ω_1 is a 2-form and $\Phi = \omega_2 + i\omega_3$ is a complex 2-form on P such that

$$\eta \wedge \omega_1 \wedge \omega_1 \neq 0, \quad \Phi^2 = 0, \quad \omega_1 \wedge \Phi = 0, \quad \Phi \wedge \bar{\Phi} = 2\omega_1 \wedge \omega_1,$$

and Φ is of type $(2,0)$ with respect to ω_1 . Following [12], a $SU(2)$ structure on a 5-manifold P^5 is said to be *hypo* if $d\omega_1 = d(\omega_2 \wedge \eta) = d(\omega_3 \wedge \eta) = 0$.

Proposition 3.9. *Let $(P^5, \eta, \omega_1, \omega_2, \omega_3)$ be a compact 5-dimensional manifold endowed with a hypo $SU(2)$ structure such that $d\eta = 0 = d\omega_2$. Then, on the 6-dimensional manifold $M^6 = P^5 \times S^1$ there exist solutions to the SUSY equations IIA and IIB. More precisely, if β is a global nonvanishing 1-form on S^1 then $(\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))$ given by*

$$Re(\alpha) = \beta, \quad Im(\alpha) = \frac{1}{k_{||}} \eta, \quad Re(\Omega)_{||} = -k_{||} \omega_3, \quad Re(\Omega)_{\perp} = k_{\perp} \omega_1, \quad Im(\Omega) = \omega_2,$$

solves equations (6), and $(\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))$ given by

$$Re(\alpha) = \frac{1}{k_{||}} \beta, \quad Im(\alpha) = \eta, \quad Re(\Omega)_{||} = k_{\perp} \omega_3, \quad Re(\Omega)_{\perp} = k_{||} \omega_1, \quad Im(\Omega) = \omega_2,$$

are solutions to equations (7).

Proof. It is clear that the $SU(2)$ structure $(\hat{J}, \hat{g}, \hat{\alpha} = \beta + i\eta, \hat{\omega} = \omega_1, \hat{\Omega} = \omega_2 + i\omega_3)$ on M is symplectic half-flat. On the other hand, the $SU(2)$ structure $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1 = \beta + i\eta, \hat{\omega} = \omega_3, \hat{\Omega} = \omega_1 + i\omega_2)$ on M is half-flat and, for any $k_{||} \in (0, 1)$, the $SU(2)$ structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_{||} \eta - i \frac{\beta}{k_{||}}, \hat{\omega}, \hat{\Omega})$ is also half-flat. Therefore, the result follows from Theorems 3.1 and 3.2. \square

It is obvious that given a compact hyperkähler 4-manifold N^4 we can consider $P^5 = N^4 \times S^1$, but there are other manifolds to which this result can be applied. For example, a nilmanifold corresponding to the Lie algebra $(0, 0, 0, 12, 13)$ with the hypo structure $\eta = e^1$, $\omega_1 = e^{24} - e^{35}$, $\omega_2 = e^{25} + e^{34}$ and $\omega_3 = e^{23} + e^{45}$. We will treat this example in more detail in Section 4.1.

4. NEW EXPLICIT SOLUTIONS OF THE SUSY EQUATIONS IIA AND IIB

In this section we show compact solvmanifolds admitting structures solving the SUSY equations IIA and IIB. From Theorem 3.1 and Proposition 3.5 we consider compact 6-solvmanifolds admitting both symplectic half-flat and Hermitian balanced $SU(3)$ structures.

4.1. Nilmanifolds. Conti and Tomassini classified [12] the nilmanifolds admitting invariant symplectic half-flat structures. It turns out that the underlying nilpotent Lie algebra must be isomorphic to the abelian Lie algebra, $\mathfrak{h}_6 = (0, 0, 0, 0, 12, 13)$ or $(0, 0, 0, 12, 13, 23)$. Apart from the abelian Lie algebra, only \mathfrak{h}_6 admits Hermitian balanced structure [28]. In fact, up to equivalence, there is a 1-parametric family of Hermitian balanced structures, which are described as follows (see [15] for details). The complex equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} - \omega^{2\bar{1}},$$

define a complex structure J on the Lie algebra \mathfrak{h}_6 and any complex structure on \mathfrak{h}_6 is equivalent to J . With respect to J , any Hermitian balanced structure is equivalent to one and only one of the form

$$F_t = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + t^2 \omega^{3\bar{3}}),$$

for some $t \neq 0$.

Let us consider the basis of 1-forms $\{\beta^1, \dots, \beta^6\}$ given by

$$\beta^1 + i\beta^4 = \omega^1, \quad \beta^2 + i\beta^3 = \omega^2, \quad \beta^5 + i\beta^6 = \frac{1}{2}\omega^3.$$

In terms of this basis, we have the structure equations

$$(8) \quad d\beta^1 = d\beta^2 = d\beta^3 = d\beta^4 = 0, \quad d\beta^5 = \beta^{12}, \quad d\beta^6 = \beta^{13},$$

and the complex structure J and the fundamental form F_t are given by

$$(9) \quad J\beta^1 = -\beta^4, J\beta^2 = -\beta^3, J\beta^5 = -\beta^6, \quad F_t = \beta^{14} + \beta^{23} + 4t^2 \beta^{56}.$$

Notice that the associated metric is $g_t = \beta^1 \otimes \beta^1 + \dots + \beta^4 \otimes \beta^4 + 4t^2 \beta^5 \otimes \beta^5 + 4t^2 \beta^6 \otimes \beta^6$. From now on we consider the Hermitian balanced $SU(3)$ structure (F_t, Ψ_t) on \mathfrak{h}_6 given by (9) and

$$(10) \quad \Psi_t = 2t(\beta^1 + i\beta^4) \wedge (\beta^2 + i\beta^3) \wedge (\beta^5 + i\beta^6).$$

• *Solutions to equations IIB arising from Hermitian balanced structures on \mathfrak{h}_6 :* For each $t \neq 0$, the structure (F_t, Ψ_t) provides solutions to the SUSY equations IIB. According to Theorem 3.2, let us consider the half-flat $SU(2)$ structure $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ given by

$$\hat{\alpha}_1 = \beta^1 + i\beta^4, \quad \hat{\omega} = \beta^{23} + 4t^2 \beta^{56}, \quad \hat{\Omega} = 2t(\beta^{25} - \beta^{36}) + 2t i(\beta^{26} + \beta^{35}).$$

By (8) the forms $\beta^{25} - \beta^{36}$ and $\beta^{26} + \beta^{35}$ are closed, therefore for any $k_{||} \in (0, 1)$ we conclude that the $SU(2)$ structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_{||} \text{Im}(\hat{\alpha}_1) - i \frac{\text{Re}(\hat{\alpha}_1)}{k_{||}}, \hat{\omega}, \hat{\Omega})$ is half-flat. Therefore, in terms of the basis $\{\beta^1, \dots, \beta^6\}$ we get the following explicit solutions $(\alpha, \text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ of the SUSY equations (7):

$$\text{Re}(\alpha) = \frac{1}{k_{||}}\beta^1, \quad \text{Im}(\alpha) = \beta^4, \quad \text{Re}(\Omega)_{||} = k_{\perp}(\beta^{23} + 4t^2 \beta^{56}),$$

$$\text{Re}(\Omega)_{\perp} = 2t k_{||}(\beta^{25} - \beta^{36}), \quad \text{Im}(\Omega) = 2t(\beta^{26} + \beta^{35}),$$

where $k_{\perp} = \sqrt{1 - k_{||}^2}$. Notice that the fluxes are $H = 0 = F_1$, and $e^{i\theta} g_s * F_3 = 4t^2(\beta^{126} - \beta^{135})$.

• *Solutions of the SUSY equations IIA on \mathfrak{h}_6* : For each $t \neq 0$, we consider the SU(2) structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ given by

$$(11) \quad \hat{\alpha} = \beta^1 + i\beta^4, \quad \hat{\omega} = 2t\beta^{25} - 2t\beta^{36}, \quad \hat{\Omega} = (\beta^2 + 2ti\beta^5) \wedge (-\beta^3 + 2ti\beta^6).$$

Since the forms β^{14} and $\beta^{25} - \beta^{36}$ are closed, and

$$d(\text{Re}(\hat{\alpha} \wedge \hat{\Omega})) = \beta^1 \wedge d(\beta^{23} + 4t^2\beta^{56}) + 2t\beta^4 \wedge d(\beta^{26} + \beta^{35}) = 0,$$

we have that the SU(2) structure is symplectic half-flat for any $t \neq 0$. According to Theorem 3.1, since $d(\text{Re}(\hat{\alpha})) = d\beta^1 = 0$, the forms $(\alpha, \text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ given by

$$\text{Re}(\alpha) = \beta^1, \quad \text{Im}(\alpha) = \frac{1}{k_{||}}\beta^4, \quad \text{Re}(\Omega)_{||} = -2tk_{||}(\beta^{26} + \beta^{35}),$$

$$\text{Re}(\Omega)_{\perp} = 2tk_{\perp}(\beta^{25} - \beta^{36}), \quad \text{Im}(\Omega) = -\beta^{23} - 4t^2\beta^{56},$$

provide solutions to the SUSY equations (6). Notice that the fluxes are $H = -4t^2\frac{k_{\perp}}{k_{||}}(\beta^{126} - \beta^{135})$, $F_0 = 0$, $g_s * F_2 = -4\frac{t^2}{k_{||}^2}(\beta^{1246} - \beta^{1345})$ and $F_4 = 0$.

Next we show that solutions to equations IIA (resp. IIB) in general are not stable by small deformations inside the class of half-flat structures. For that, we first show explicitly that any Hermitian balanced structure (F_t, Ψ_t) on \mathfrak{h}_6 given by (9)–(10) can be deformed into a symplectic half-flat structure (11) along a curve of half-flat structures. For each $\vartheta \in \mathbb{R}$, let us consider the SU(3) structure $(F_t^{\vartheta}, \Psi_t^{\vartheta})$ given by

$$F_t^{\vartheta} = \beta^{14} + \cos \vartheta \beta^{23} + 2t \sin \vartheta \beta^{25} - 2t \sin \vartheta \beta^{36} + 4t^2 \cos \vartheta \beta^{56},$$

and

$$\Psi_t^{\vartheta} = (\beta^1 + i\beta^4) \wedge (\beta^2 + i \cos \vartheta \beta^3 + 2ti \sin \vartheta \beta^5) \wedge (-\sin \vartheta \beta^3 + 2t \cos \vartheta \beta^5 + 2ti\beta^6).$$

A direct calculation shows that $\text{Re}(\Psi_t^{\vartheta})$ is closed and $dF_t^{\vartheta} = 4t^2 \cos \vartheta (\beta^{126} - \beta^{135})$, which implies that $F_t^{\vartheta} \wedge dF_t^{\vartheta} = 0$. Therefore, the structure is half-flat for any ϑ , and (F_t^0, Ψ_t^0) is the Hermitian balanced structure given by (9)–(10), and $(F_t^{\frac{\pi}{2}}, \Psi_t^{\frac{\pi}{2}})$ is the symplectic structure (11).

Since F_t^{ϑ} is symplectic if and only if $\cos \vartheta = 0$, by Theorem 3.1 we have that the half-flat structure $(F_t^{\vartheta}, \Psi_t^{\vartheta})$ does not solve equations (6) for $\vartheta \in (0, \frac{\pi}{2})$.

On the other hand, let us fix ϑ and consider the half-flat structure $(F_t^{\vartheta}, \Psi_t^{\vartheta})$. For any $\lambda \in (0, 1)$, a direct calculation shows that the structure $(F = F_t^{\vartheta}, \Phi_{\lambda})$ given by

$$\Phi_{\lambda} = (\lambda\beta^4 - i\frac{\beta^1}{\lambda}) \wedge (\beta^2 + i \cos \vartheta \beta^3 + 2ti \sin \vartheta \beta^5) \wedge (-\sin \vartheta \beta^3 + 2t \cos \vartheta \beta^5 + 2ti\beta^6).$$

is half-flat if and only if $\sin \vartheta = 0$. From Theorem 3.2 we conclude that the half-flat structure $(F_t^{\vartheta}, \Psi_t^{\vartheta})$ does not provide a solution to equations (7) for $\vartheta \in (0, \frac{\pi}{2})$.

Therefore, we have proved the following result:

Proposition 4.1. *The half-flat structure $(F_t^{\vartheta}, \Psi_t^{\vartheta})$ does not solve neither (6) nor (7) for any $\vartheta \in (0, \frac{\pi}{2})$. Therefore, solutions to the SUSY equations IIA or IIB in general are not stable by small deformations inside the class of half-flat structures.*

4.2. Compact solvmanifolds. In this section we describe in detail two compact solvmanifolds solving the SUSY equations IIA and IIB.

4.2.1. *Example.* Let us consider the 6-dimensional 2-step completely solvable Lie algebra $\mathfrak{s}_1 = (0, 0, 13, -14, 15, -16)$ with structure equations

$$(12) \quad d\beta^1 = d\beta^2 = 0, \quad d\beta^3 = \beta^{13}, \quad d\beta^4 = -\beta^{14}, \quad d\beta^5 = \beta^{15}, \quad d\beta^6 = -\beta^{16}.$$

The corresponding simply-connected Lie group S_1 is isomorphic to $\mathbb{R} \times (\mathbb{R} \times_{\phi} \mathbb{R}^4)$, where

$$\phi(t) = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R}.$$

Since $\phi(1) = \exp^{SL(4, \mathbb{R})}(\phi'(0)) \in SL(4, \mathbb{Z})$, by [16, Theorem 4] we have that $\Gamma = \mathbb{Z} \times_{\phi} \mathbb{Z}^4$ is a lattice in $\mathbb{R} \times_{\phi} \mathbb{R}^4$ and therefore $\mathbb{Z} \times \Gamma = \Gamma_1$ is a lattice of S_1 . By Hattori's Theorem [23] we have that the de Rham cohomology of the compact quotient S_1/Γ_1 is isomorphic to the Chevalley-Eilenberg cohomology $H^*(\mathfrak{s}_1)$ of \mathfrak{s}_1 and thus in particular $b_1(S_1/\Gamma_1) = 2$, $b_2(S_1/\Gamma_1) = 5$ and $b_3(S_1/\Gamma_1) = 8$.

Let us consider the almost complex structure

$$J\beta^1 = -\beta^2, \quad J\beta^3 = -\beta^5, \quad J\beta^4 = \beta^6.$$

The basis of (1,0)-forms $\omega^1 = \beta^1 + i\beta^2$, $\omega^2 = \beta^3 + i\beta^5$ and $\omega^3 = -\beta^4 + i\beta^6$ satisfies

$$d\omega^1 = 0, \quad d\omega^2 = \frac{1}{2}\omega^{12} + \frac{1}{2}\omega^{\bar{1}\bar{2}}, \quad d\omega^3 = -\frac{1}{2}\omega^{13} - \frac{1}{2}\omega^{\bar{1}\bar{3}}.$$

Therefore, the almost complex structure J is integrable. Since the 2-form $F = \beta^{12} + \beta^{35} - \beta^{46}$ satisfies that $F^2 = 2(\beta^{1235} - \beta^{1246} - \beta^{3546})$ is closed, we get a Hermitian balanced SU(2) structure.

• *Solutions of equations IIB arising from the Hermitian balanced structure on \mathfrak{s}_1 :* The previous structure provides solutions to the SUSY equations IIB. Let $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ be the half-flat SU(2) structure given by

$$\hat{\alpha}_1 = \beta^1 + i\beta^2, \quad \hat{\omega} = \beta^{35} - \beta^{46}, \quad \hat{\Omega} = -\beta^{34} - \beta^{56} + i(\beta^{36} + \beta^{45}).$$

It follows from (12) that the 2-forms β^{34} , β^{56} and $\beta^{36} + \beta^{45}$ are closed, which implies that for any $k_{||} \in (0, 1)$ the SU(2) structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_{||} \text{Im}(\hat{\alpha}_1) - i\frac{\text{Re}(\hat{\alpha}_1)}{k_{||}}, \hat{\omega}, \hat{\Omega})$ is half-flat. By Theorem 3.2 we get the following solutions $(\alpha, \text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ of the SUSY equations (7):

$$\text{Re}(\alpha) = \frac{1}{k_{||}}\beta^1, \quad \text{Im}(\alpha) = \beta^2, \quad \text{Re}(\Omega)_{||} = k_{\perp}(\beta^{35} - \beta^{46}),$$

$$\text{Re}(\Omega)_{\perp} = -k_{||}(\beta^{34} + \beta^{56}), \quad \text{Im}(\Omega) = \beta^{36} + \beta^{45},$$

where $k_{\perp} = \sqrt{1 - k_{||}^2}$. Notice that the fluxes are $H = 0 = F_1$, and $e^{i\theta} g_s * F_3 = 2\beta^1 \wedge (\beta^{35} + \beta^{46})$.

• *Solutions of the SUSY equations IIA on \mathfrak{s}_1 :* Let us consider the SU(2) structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ given by

$$\hat{\alpha} = \beta^1 + i\beta^2, \quad \hat{\omega} = \beta^{34} + \beta^{56}, \quad \hat{\Omega} = (\beta^3 + i\beta^4) \wedge (\beta^5 + i\beta^6).$$

Since the forms β^{12} , β^{34} and β^{56} are closed, and

$$d(\text{Re}(\hat{\alpha} \wedge \hat{\Omega})) = \beta^1 \wedge d(\beta^{35} - \beta^{46}) - \beta^2 \wedge d(\beta^{36} + \beta^{45}) = 0,$$

we have that the $SU(2)$ structure is symplectic half-flat. By Theorem 3.1, since $d(\text{Re}(\hat{\alpha})) = d\beta^1 = 0$, the forms $(\alpha, \text{Re}(\Omega)_{\parallel}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ given by

$$\text{Re}(\alpha) = \beta^1, \quad \text{Im}(\alpha) = \frac{1}{k_{\parallel}}\beta^2, \quad \text{Re}(\Omega)_{\parallel} = -k_{\parallel}(\beta^{36} + \beta^{45}),$$

$$\text{Re}(\Omega)_{\perp} = k_{\perp}(\beta^{34} + \beta^{56}), \quad \text{Im}(\Omega) = \beta^{35} - \beta^{46},$$

provide solutions to the SUSY equations (6). The fluxes are $H = 2\frac{k_{\perp}}{k_{\parallel}}(\beta^{135} + \beta^{146})$, $F_0 = 0$, $g_s * F_2 = -\frac{2}{k_{\parallel}^2}(\beta^{1235} + \beta^{1246})$ and $F_4 = 0$.

As in the previous example, the particular solutions on \mathfrak{s}_1 to equations IIA and IIB given above are not stable by small deformations inside the class of half-flat structures. For each $\vartheta \in \mathbb{R}$, the $SU(2)$ structure $(F^{\vartheta}, \Psi^{\vartheta})$ given by

$$F^{\vartheta} = \beta^{12} + \cos \vartheta(\beta^{34} + \beta^{56}) + \sin \vartheta(\beta^{35} - \beta^{46})$$

and

$$\Psi^{\vartheta} = (\beta^1 + i\beta^2) \wedge (\beta^3 + i\cos \vartheta \beta^4 + i\sin \vartheta \beta^5) \wedge (-\sin \vartheta \beta^4 + \cos \vartheta \beta^5 + i\beta^6)$$

is half-flat, and for $\vartheta = 0$ (resp. $\vartheta = \frac{\pi}{2}$) we get the symplectic (resp. Hermitian balanced) half-flat structure given above. A direct calculation shows that the half-flat structure $(F^{\vartheta}, \Psi^{\vartheta})$ does not solve neither (6) nor (7) for any $\vartheta \in (0, \frac{\pi}{2})$.

4.2.2. *Example.* Let us consider the solvable Lie algebra $\mathfrak{s}_2 = (0, 0, -13 - 24, -14 + 23, 15 + 26, 16 - 25)$, that is, there is a basis of 1-forms $\{\beta^1, \dots, \beta^6\}$ satisfying

$$(13) \quad \begin{cases} d\beta^1 = d\beta^2 = 0, \\ d\beta^3 = -\beta^{13} - \beta^{24}, \\ d\beta^4 = -\beta^{14} + \beta^{23}, \\ d\beta^5 = \beta^{15} + \beta^{26}, \\ d\beta^6 = \beta^{16} - \beta^{25}. \end{cases}$$

The existence of a lattice Γ_2 of S_2 of the associated simply connected solvable Lie group S_2 was proved in [29] (see also [14]). The de Rham cohomology of the compact quotient S_2/Γ_2 (also known as Nakamura manifold) is not isomorphic to $H^*(\mathfrak{s}_2)$ (see [14, 29] and more recently [21, 9] for the cohomology of solvmanifolds). In particular $b_1(S_2/\Gamma_2) = 2$, $b_2(S_2/\Gamma_2) = 5$ and $b_3(S_2/\Gamma_2) = 8$.

Let us consider the almost complex structure

$$J\beta^1 = -\beta^2, \quad J\beta^3 = -\beta^4, \quad J\beta^5 = -\beta^6.$$

The basis of $(1,0)$ -forms $\omega^1 = \beta^1 + i\beta^2$, $\omega^2 = \beta^3 + i\beta^4$ and $\omega^3 = \beta^5 + i\beta^6$ satisfies

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{2\bar{1}}, \quad d\omega^3 = -\omega^{3\bar{1}},$$

that is, J is integrable.

For each $t \in \mathbb{R} - \{0\}$, the $SU(3)$ structure (F_t, Ψ_t) given by

$$F_t = t^2\beta^{12} + \beta^{34} + \beta^{56}, \quad \Psi_t = t(\beta^1 + i\beta^2) \wedge (\beta^3 + i\beta^4) \wedge (\beta^5 + i\beta^6),$$

defines a 1-parametric family of (non-equivalent) Hermitian balanced $SU(3)$ structures on \mathfrak{s}_2 and thus a 1-parametric family of (non-equivalent) Hermitian balanced $SU(2)$ structures.

Notice that the associated metric is $g_t = t^2 \beta^1 \otimes \beta^1 + t^2 \beta^2 \otimes \beta^2 + \beta^3 \otimes \beta^3 + \dots + \beta^6 \otimes \beta^6$.

• *Solutions to equations IIB arising from Hermitian balanced structures on \mathfrak{s}_2* : For each $t \neq 0$, the structure (F_t, Ψ_t) provides solutions to the SUSY equations IIB. According to Theorem 3.2, we consider the half-flat $SU(2)$ structure $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ given by

$$\hat{\alpha}_1 = t\beta^1 + it\beta^2, \quad \hat{\omega} = \beta^{34} + \beta^{56}, \quad \hat{\Omega} = \beta^{35} - \beta^{46} + i(\beta^{36} + \beta^{45}).$$

By (13) the forms $\beta^{35} - \beta^{46}$ and $\beta^{36} + \beta^{45}$ are closed, therefore for any $k_{||} \in (0, 1)$ we conclude that the $SU(2)$ structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_{||} \text{Im}(\hat{\alpha}_1) - i \frac{\text{Re}(\hat{\alpha}_1)}{k_{||}}, \hat{\omega}, \hat{\Omega})$ is half-flat. Therefore, in terms of the basis $\{\beta^1, \dots, \beta^6\}$ we get the following explicit solutions $(\alpha, \text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ of the SUSY equations (7):

$$\text{Re}(\alpha) = \frac{t}{k_{||}}\beta^1, \quad \text{Im}(\alpha) = t\beta^2, \quad \text{Re}(\Omega)_{||} = k_{\perp}(\beta^{34} + \beta^{56}),$$

$$\text{Re}(\Omega)_{\perp} = k_{||}(\beta^{35} - \beta^{46}), \quad \text{Im}(\Omega) = \beta^{36} + \beta^{45},$$

where $k_{\perp} = \sqrt{1 - k_{||}^2}$. Notice that the fluxes are $H = 0 = F_1$, and $e^{i\theta} g_s * F_3 = -2\beta^1 \wedge (\beta^{34} - \beta^{56})$.

• *Solutions of the SUSY equations IIA on \mathfrak{s}_2* : For each $t \neq 0$, we consider the $SU(2)$ structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ given by

$$\hat{\alpha} = t\beta^1 + it\beta^2, \quad \hat{\omega} = -\beta^{36} - \beta^{45}, \quad \hat{\Omega} = (\beta^6 + i\beta^3) \wedge (\beta^5 + i\beta^4).$$

Since the forms β^{12} and $\beta^{36} + \beta^{45}$ are closed, and

$$\frac{1}{t} d(\text{Re}(\hat{\alpha} \wedge \hat{\Omega})) = \beta^1 \wedge d(\beta^{34} + \beta^{56}) + \beta^2 \wedge d(\beta^{35} - \beta^{46}) = 0,$$

we have that the $SU(2)$ structure is symplectic half-flat for any $t \neq 0$. According to Theorem 3.1, since $d(\text{Re}(\hat{\alpha})) = t d\beta^1 = 0$, the forms $(\alpha, \text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ given by

$$\text{Re}(\alpha) = t\beta^1, \quad \text{Im}(\alpha) = \frac{t}{k_{||}}\beta^2, \quad \text{Re}(\Omega)_{||} = -k_{||}(\beta^{35} - \beta^{46}),$$

$$\text{Re}(\Omega)_{\perp} = -k_{\perp}(\beta^{36} + \beta^{45}), \quad \text{Im}(\Omega) = -\beta^{34} - \beta^{56},$$

provide solutions to the SUSY equations (6). Notice that the fluxes are $H = 2 \frac{k_{\perp}}{k_{||}}(\beta^{134} - \beta^{156})$, $F_0 = 0$, $g_s * F_2 = -2 \frac{t}{k_{||}^2}(\beta^{1234} - \beta^{1256})$ and $F_4 = 0$.

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