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# Conformally related non-vacuum spacetimes

J Hruška and M Žofka

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, 180 00 Prague 8 - Holešovice, Czech Republic

E-mail: bukikekh@seznam.cz, zofka@mbox.troja.mff.cuni.cz

**Abstract.** We present an example of two non-trivially conformally related solutions of Einstein-Maxwell equations. To our knowledge, this is the first case of non-vacuum spacetimes related through a non-trivial conformal transformation representing thus an extension of the Brinkmann's and following theorems.

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## 1. General Background

Finding exact solutions to Einstein equations is very difficult. Therefore, much attention has been paid to methods of generating new solutions from existing ones. One of them is to use a conformal transformation.

Conformal transformations play an important role in higher-derivative theories of gravity where they serve to compare these theories to the classical GR [1, 2, 3]. They also arise naturally in quantum fields on a curved background and are discussed in connection with the AdS/CFT correspondence [4, 5, 6].

Our approach in this article is very simple—we start from a seed metric, which does not even have to represent a physically viable solution to Einstein equations. We define a non-negative scalar conformal factor varying across the manifold and produce the resulting metric by multiplying the original metric by the conformal factor. We now require the new metric to satisfy Einstein equations. In our case, we actually used a solution of the Einstein-Maxwell equations (with a non-zero electromagnetic field) as our seed. This has the advantage that the resulting metric also represents an electrovacuum solution as Maxwell equations are conformally invariant. Therefore, we do not need to deal with the electromagnetic part of Einstein-Maxwell equations. However, it may still happen that we actually do not obtain a new solution but the seed spacetime in a new coordinate system. Therefore, we must carefully check our solutions to make sure they cannot be coordinate-transformed back into the original spacetime.

## 2. Existing Theorems

There are a number of theorems on conformally related spacetimes. Brinkmann [7] dealt with conformally related Einstein spaces  $R_{\mu\nu} = ag_{\mu\nu}$  with  $a$  a constant. He showed that any two distinct (properly) conformally related Einstein spaces are either two vacuum  $pp$ -waves, or Minkowski and (anti) de Sitter. His theorems were later generalized by Daftardar-Gejji [8] who extended this theorem to include the cases where the two Einstein tensors are equal and where they differ by a cosmological constant term. In the former case, both spaces are (not necessarily vacuum)  $pp$ -waves; in the latter, for perfect fluids with  $\mu = 0$ , both spaces are Robertson-Walker with equations of state  $\mu + 3p = 0$  or  $\mu = p$ . Van den Bergh [9] established that the only null Einstein-Maxwell fields obtainable by a conformal transformation of a Ricci-flat solution are  $pp$ -waves. In all cases, the seed metric is Ricci-flat. We are interested in a case where the two spacetimes are solutions of full Einstein-Maxwell equations. Physically, this means that the causal structure of both spacetimes is identical in the corresponding regions. Yet, in general, it turned out that generating new solutions via a conformal transformation does not produce any interesting, non-trivial results. In our paper, we present the first non-trivial explicit example of two, non-vacuum solutions to Einstein equations that are conformally related. Moreover, we show that the two spacetimes are not isometric.

## 3. The generating method

We exploit the fact that, in a 4-dimensional spacetime, the action of a source-free Maxwell field is conformally invariant and we thus generate solutions of Einstein-Maxwell equations. The invariance tells us that if  $F_{\mu\nu}$  is a solution of source-free Maxwell equations on a seed spacetime  $(M, g_{\mu\nu})$ , then it is also a solution on  $(M, \Omega^2 g_{\mu\nu})$ . It implies that if we begin with a solution of Einstein-Maxwell equation, then after applying a conformal transformation, it is only Einstein equations we have to worry about, Maxwell equations are satisfied automatically.

### 3.1. Apparatus

Let us consider a conformal transformation of the seed metric  $g_{\mu\nu}$  ‡

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} , \quad (1)$$

where  $\Omega$  is a function on  $M$ . Then the following equation holds for conformally rescaled Ricci tensor  $\tilde{R}_{\mu\nu}$  computed from the new metric  $\tilde{g}_{\mu\nu}$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{2}{\Omega} \Omega_{;\mu\nu} + \frac{4}{\Omega^2} \Omega_{;\mu} \Omega_{;\nu} - \frac{1}{\Omega} (\square\Omega) g_{\mu\nu} - \frac{1}{\Omega^2} \|d\Omega\|^2 g_{\mu\nu} , \quad (2)$$

where all covariant derivatives are taken with respect to the seed metric,  $\square\Omega := g^{\mu\nu} \Omega_{;\mu\nu}$  and  $\|d\Omega\|^2 := g^{\mu\nu} \Omega_{;\mu} \Omega_{;\nu}$ .

‡ Our signature is  $(-, +, +, +)$ .

The stress-energy tensor of a source-free Maxwell field has the form

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \quad (3)$$

and so it transforms as

$$\tilde{T}_{\mu\nu} = \Omega^{-2}T_{\mu\nu} . \quad (4)$$

Suppose we have a triplet  $(M, g_{\mu\nu}, F_{\mu\nu})$  which is a solution of Einstein-Maxwell equations. We can now write Einstein equations<sup>§</sup> for  $(M, \tilde{g}_{\mu\nu}, F_{\mu\nu})$ :

$$\tilde{R}_{\mu\nu} = 8\pi\tilde{T}_{\mu\nu} = 8\pi\Omega^{-2}T_{\mu\nu} = \Omega^{-2}R_{\mu\nu} , \quad (5)$$

so the equations we will solve are  $\Omega^2\tilde{R}_{\mu\nu} - R_{\mu\nu} = 0$  or

$$(\Omega^2 - 1)R_{\mu\nu} - 2\Omega\Omega_{;\mu\nu} + 4\Omega_{;\mu}\Omega_{;\nu} - \Omega(\square\Omega)g_{\mu\nu} - g_{\mu\nu}\|d\Omega\|^2 = 0 \quad (6)$$

in detail. The trace of (6) yields a necessary condition  $\square\Omega = 0$ , so  $\Omega$  has to be a harmonic function and (6) simplifies to the final form

$$(\Omega^2 - 1)R_{\mu\nu} - 2\Omega\Omega_{;\mu\nu} + 4\Omega_{;\mu}\Omega_{;\nu} - g_{\mu\nu}\|d\Omega\|^2 = 0. \quad (7)$$

This system is generally overdetermined, we have 10 equations for a single function  $\Omega$ , however, in a special case, we might (and will) be able to solve it<sup>||</sup>.

### 3.2. Application

As could be expected, equation (7) does not allow for a non-trivial solution in most cases. We explicitly checked the Reissner-Nordström, Bonnor-Melvin, Bertotti-Robinson, Tariq-Tupper, and Ozsváth solutions and showed that none of these spacetimes are suitable seeds. However, let us now turn our attention to *pp*-waves, which are spacetimes admitting a covariantly constant null vector field  $k^\mu$  [10, 11] (p. 383 and 323, respectively). The metric can be written in the form

$$ds^2 = -2H(u, \xi, \bar{\xi})du^2 - 2dudv + 2d\xi d\bar{\xi} . \quad (8)$$

where  $\xi = \frac{1}{\sqrt{2}}(x + iy)$  is a complex coordinate. In these coordinates, the covariantly constant null vector field is simply  $\partial/\partial_v$ . Not all *pp*-waves carry a non-zero Maxwell field. In order for them to do so, the function  $H$  has to satisfy

$$H = f(\xi, u) + \bar{f}(\bar{\xi}, u) + 8\pi F(\xi, u)\bar{F}(\bar{\xi}, u) \quad (9)$$

and the Maxwell field is then  $F_{\mu\nu} = k_{[\mu}F_{;\nu]} + c.c.$  The Ricci tensor of a *pp*-wave is given by  $R_{\mu\nu} = 2H_{;\xi\bar{\xi}}k_\mu k_\nu$ . Having all the necessary ingredients, we can try to solve (7) for  $\Omega$ . It turns out that this generally overdetermined system has a nontrivial solution if and only if  $H$  can be written as

$$H = \xi\bar{\xi}\Phi^2(u) + (\xi + \bar{\xi})h(u) + g(u) , \quad (10)$$

where  $\Phi^2$ ,  $h$ , and  $g$  are arbitrary real functions of  $u$  only. Thus, except for the flat case, the spacetimes we are interested in have a non-zero Ricci tensor. There is a

<sup>§</sup> Note that the stress-energy tensor of a Maxwell field is traceless, so the Ricci scalar vanishes.

<sup>||</sup> Actually, there is always the solution  $\Omega^2 \equiv 1$ , but then the transformation is trivial.

transformation of coordinates [10] (p. 383) that preserves the metric in the form (8) and allows us to get rid of the last two terms. A  $pp$ -wave with such a form of  $H$  is called a plane wave and is conformally flat. In fact, McLenaghan *et al.* [12] proved that such a  $pp$ -wave is the only conformally flat null Einstein-Maxwell field. Then,  $\Omega$  has to be a function of  $u$  only, too, and has to satisfy a second order ODE

$$\Omega\ddot{\Omega} - 2\dot{\Omega}^2 + \Phi^2(1 - \Omega^2) = 0, \quad (11)$$

where  $\dot{\Omega} := d\Omega/du$ . With  $\Omega$  depending only on  $u$ , it can be shown, that the vector field  $\partial/\partial_v$  is covariantly constant in the new spacetime  $(M, \tilde{g}_{\mu\nu})$  as well, so the generated spacetime is again a  $pp$ -wave, undoubtedly a conformally flat one, in agreement with McLenaghan's results. A question naturally arises, whether the generated wave is generally different from (non-isometric to) the original one or we just generate different coordinate expressions of the same wave. To answer this question, let us examine the equivalence problem for an explicit pair of the seed and the generated wave. A good choice, that will allow us to solve (11) particularly easily is to start with the following  $pp$ -wave

$$ds^2 = -4\xi\bar{\xi}du^2 - 2dudv + 2d\xi d\bar{\xi}, \quad (12)$$

so  $H = 2\xi\bar{\xi}$  and  $\Phi^2 = 2$ . A particular solution of (11) is then  $\Omega = \tanh(u)$  and so the generated wave has the following line element

$$\tilde{ds}^2 = \tanh^2(u)(-4\xi\bar{\xi}du^2 - 2dudv + 2d\xi d\bar{\xi}). \quad (13)$$

In our case, the first covariant derivative of the Ricci tensor is sufficient to prove the non-equivalence. The original spacetime has  $R_{\mu\nu} = 4k_\mu k_\nu$  and its covariant derivative vanishes, i.e.  $R_{\mu\nu;\rho} = 0$ . However, the new spacetime has  $\tilde{R}_{\mu\nu} = 4\Omega^{-6}\tilde{k}_\mu\tilde{k}_\nu$  and the covariant derivative no longer vanishes because of the non-zero gradient of the conformal factor  $\Omega$ , which proves the non-equivalence of  $ds^2$  and  $\tilde{ds}^2$ .

Are we bringing any news or can one draw our conclusions from the results given in [12]? We can obviously conformally transform every conformally flat seed  $pp$ -wave directly to any other solution of the McLenaghan type and the question is: can we always keep the original Maxwell tensor or do we generally need a different Maxwell field to produce a solution of full Einstein-Maxwell equations? In the following paragraph, we will argue that the latter is the case.

Let us take a seed and multiply it by a general conformal factor depending on  $u$  only (see above) to get

$$\tilde{ds}^2 = \Omega^2(u)(-2\Phi^2(u)\xi\bar{\xi}du^2 - 2dudv + 2d\xi d\bar{\xi}). \quad (14)$$

Now apply the following coordinate transformation

$$\xi = \frac{\xi'}{\Omega(u)}, \quad (15)$$

$$v = v' - \frac{\xi'\bar{\xi}'\dot{\Omega}(u)}{\Omega^3(u)}, \quad (16)$$

$$\Omega^2(u)du = du', \quad (17)$$

This yields

$$\widetilde{ds}^2 = -2\Phi'^2(u')\xi'\bar{\xi}'du'^2 - 2du'dv' + 2d\xi'd\bar{\xi}', \quad (18)$$

with

$$\Phi'^2(u') = \frac{\Phi^2(u)}{\Omega^4(u)} - \frac{\ddot{\Omega}(u)}{\Omega^5(u)} + \frac{2\dot{\Omega}^2(u)}{\Omega^6(u)}, \quad (19)$$

where  $u$  has to be viewed as a function of  $u'$ , so we arrive at the canonical form of the metric again.

Now we apply the above transformation to the seed Maxwell tensor and see whether it necessarily corresponds to the canonical form of the Maxwell tensor as given in Stephani et al. The condition that the two fields be the same takes precisely the form (11) and it is obviously not satisfied identically. We thus established that, in general, we cannot proceed from one  $pp$ -wave of the form (8) with (10) to another keeping the same Maxwell field and, therefore, results in [12] do not imply ours. Additionally, we gave an explicit example of two conformally related Einstein-Maxwell fields. All the  $pp$ -waves of the above special form thus split into equivalence classes defined by their Maxwell fields.

Moreover, (19) implicitly gives us the conformal factor that transforms the wave back to Minkowski if we require  $\Phi'^2 = 0$ . It also implies that if we take Minkowski in null coordinates

$$ds^2 = -2dudv + 2d\xi d\bar{\xi}, \quad (20)$$

then *all* conformal transformations with an *arbitrary*  $\Omega = \Omega(u)$  result in  $pp$ -waves of the McLenaghan type.

#### 4. Conclusions and Outlook

To our knowledge, this is the only non-trivial example of applying a conformal transformation to a seed spacetime to produce a non-vacuum solution to Einstein equations. Furthermore, due to the properties of conformal transformations, it is clear that the source remains in the form of an (vacuum) electromagnetic field. The original and resulting spacetimes are explicitly shown not to be identical. Thus, our solutions form the only known pair of conformally related non-vacuum spacetimes. The prospects of using conformal transformations to produce new solutions to Einstein equations may be dim but at least we provided a specific example where this approach works although the resulting solution belongs to the same class as the seed spacetime.

It might be of interest to see whether there may be any other non-trivial example that would not involve  $pp$ -waves. Note that previous literature always relies exactly on this type of spacetimes. Are there any suitable seeds other than  $pp$ -waves? Can the method be generalized to be less restrictive?

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