Remarks on J. Espinar’s "Finite index operators on surfaces"
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1. Introduction

In [1], we considered operators of the form $J = \Delta + aK - q$ on a complete non-compact Riemannian surface $(M, g)$, where $\Delta$ is the non-negative Laplacian, and $K$ the Gaussian curvature associated with the metric $g$. The parameter $a$ is some positive constant, and $q$ is a non-negative locally integrable function on $M$. More precisely, we studied the consequences, for the geometry of the triple $(M, g; q)$, of the fact that the operator $J$ is non-negative (in the sense of quadratic forms).

Motivated by applications to minimal and cmc surfaces, J. Espinar [3] considers a different framework (see also [4]). More precisely, he considers a Riemannian surface $(M, g)$, possibly with boundary $\partial M$ and not necessarily complete, and operators of the form $\Delta + aK - c + P$, where the parameters $a, c$ are positive constants, and $P$ is a non-negative integrable function.

In this note, we consider complete surfaces without boundary, and prove results similar to those in [3,4], under weaker assumptions. For this purpose, we apply the methods of [1].

2. General framework

Generally speaking, we will use the same notations as in [1], $(M, g)$ will denote a complete (possibly compact) surface without boundary.


Keywords: Spectral theory, positivity, minimal surface, constant mean curvature surface.
2.1. The operators. In this paper, we consider operators of the form,
\begin{equation}
J = \Delta + aK - q + P.
\end{equation}
Here \( \Delta \) is the non-negative Laplacian, and \( K \) the Gaussian curvature associated with the metric \( g \). We let \( \mu \) denote the Riemannian measure associated with \( g \).

⋆ We make the following assumptions on the operator \( J \),
\begin{align}
& a \quad \text{is a positive constant}, \\
& q \quad \text{is a non-negative, locally integrable function on } M, \\
& P \quad \text{is an integrable function on } M,
\end{align}
and we let \( c = \inf_M q \geq 0 \), and we let \( \|P\|_1 = \int_M |P|d\mu \).

Note that we do not impose any sign condition on the function \( P \).

⋆ We say that the open geodesic ball \( B(x_0, R) \) is \( J \)-stable if the operator \( J \) is non-negative in the sense of quadratic forms,
\begin{equation}
0 \leq Q_J(\phi) = \int_M \{|d\phi|^2 + (aK - q + P)\phi^2\} d\mu
\end{equation}
for all \( \phi \) in \( \text{Lip}_0(B(x_0, R)) \), the Lipschitz functions with compact support inside the ball.

2.2. Volume growth assumptions. Fix a reference point \( x_0 \) in \( M \). We consider the following assumptions on the volume growth on \( (M,g) \).

⋆ We say that \( (M,g) \) has polynomial volume growth of order at most \( k \) if there exists a constant \( C_k \) such that,
\begin{equation}
V(B(x_0, R)) \leq C_k(1 + R)^k,
\end{equation}
for all \( R > 0 \).

⋆ We say that \( (M,g) \) has \( k \)-subpolynomial volume growth if
\begin{equation}
\limsup_{R \to \infty} \frac{V(B(x_0, R))}{R^k} = 0.
\end{equation}

⋆ We say that \( (M,g) \) has subexponential volume growth if
\begin{equation}
\limsup_{R \to \infty} \frac{\ln \left( V(B(x_0, R)) \right)}{R} = 0.
\end{equation}

For a complete surface without boundary, these definitions do not depend on the choice of the reference point \( x_0 \), although the constant \( C_k \) a priori does.

2.3. Fundamental inequalities. We briefly recall the notations of [1], Section 2. Given a reference point \( x_0 \in M \), we consider the open geodesic balls \( B(x_0, t) \), and their Euler-Poincaré characteristics \( \chi(B(x_0, t)) \). More precisely, we introduce the function,
\begin{equation}
\hat{\chi}(s) = \sup \{ \chi(B(x_0, t)) \mid t \geq s \}.
\end{equation}

This is a non-increasing function with a sequence of discontinuities, finite possibly empty, or infinite, \( \{t_j\}_{j=1}^N \), with \( N \in \mathbb{N} \cup \{\infty\} \). Note that this sequence depends on the choice of the reference point \( x_0 \). We call \( \omega_j \) the jump of the function \( \hat{\chi} \) at the discontinuity \( t_j \).
We call *admissible* a function \( \xi : [0, Q] \to \mathbb{R} \), which is \( C^1 \) and piecewise \( C^2 \), with \( \xi, \xi'' \geq 0 \) and \( \xi' \leq 0 \). Let \( N(Q) \) be the largest integer \( n \) such that \( t_n \leq Q \).

We now recall two key results from [1].

- The topology of \( M \) is controlled by the function \( \hat{\chi} \). More precisely, we have the inequality (see [1], Lemma 2.1),

\[
1 - \sum_{n=1}^{N} \omega_n \leq \chi(M).
\]

- Assume that the operator \( J \) satisfies the assumptions (2), and let \( B(x_0, Q) \) be some \( J \)-stable ball in \( M \). Let \( \xi \) be any admissible function on \([0, Q]\), with \( \xi(Q) = 0 \), and let \( r \) denote the distance function to the center \( x_0 \) of the ball. Plugging the function \( \xi(r) \) into the quadratic form for \( J \) and applying [1], Lemma 2.3, we obtain the inequality

\[
\int_{B(x_0, Q)} q \xi^2(r) \, d\mu \leq 2\pi a \xi^2(0) - 2\pi a \sum_{j=1}^{N(Q)} \omega_n \xi^2(t_n) + \int_{B(x_0, Q)} P \xi^2(r) \, d\mu,
\]

which yields the weaker inequality,

\[
\left\{ \begin{array}{l}
\int_{B(x_0, Q)} c \xi^2(r) \, d\mu \\
\leq 2\pi a \xi^2(0) + \|\xi\|_{\infty}^2 \|P\|_1 + \int_{B(x_0, Q)} [(1 - 2a)(\xi')^2 - 2a \xi \xi''] \, d\mu.
\end{array} \right.
\]

### 3. Statements

Inequality (8) shows that the case in which the operator \( J = \Delta + aK - q + P \) is non-negative –under the assumptions (2)– is similar to the case in which the operator \( \Delta + aK - q \) has finite index, as treated in [1], Theorem 4.1. More precisely, we have the following result.

**Theorem 3.1.** Let \((M, g)\) be a complete Riemannian surface without boundary, and let \( J \) be the operator,

\[
J = \Delta + aK - q + P,
\]

with \( q \geq 0 \) locally integrable and \( P \) an integrable function. Assume that \( J \geq 0 \) on \( \text{Lip}_0(M) \), and that either of the following conditions holds,

(i) \( a > \frac{1}{4} \), or  
(ii) \( a = \frac{1}{4} \), and \((M, g)\) has subexponential volume growth, or  
(iii) \( a \in (0, \frac{1}{4}) \), and \((M, g)\) has \( k_a \)-subpolynomial volume growth, with \( k_a = 2 + \frac{1}{1 - 4a} \).

Then, either \( M \) is closed, or \((M, g)\) is non-compact with finite topology and at most quadratic area growth. In particular, \((M, g)\) is conformally equivalent to a closed Riemannian surface with at most finitely many points removed. Furthermore, \( q \) is integrable on \((M, g)\), and we have,

\[
\int_M q \, d\mu \leq 2\pi a \chi(M) + \int_M P \, d\mu.
\]
Remark. When considering an operator of the form $J = \Delta + aK + W$, taking $q = W_-$ and $P = W_+$, the previous result gives the following. If either of the conditions (i), (ii) or (iii) holds, and if $W_+$ is integrable, then $W \in L^1(M, \mu)$, $M$ has finite conformal type, and

$$0 \leq 2\pi a \chi(M) + \int_M W d\mu.$$

The interesting case, in the present framework, is the case in which the infimum $c$ of the function $q$ is positive. We have the following result.

**Theorem 3.2.** Let $(M, g)$ be a complete Riemannian surface without boundary, and let $J$ be the operator,

$$J = \Delta + aK - q + P,$$

with $q \geq c > 0$ locally integrable, and $P$ an integrable function on $(M, g)$. Assume that $J \geq 0$ on $\text{Lip}_0(M)$, and that either of the following conditions holds,

(i) $a > \frac{1}{4}$, or
(ii) $a = \frac{1}{4}$, and $(M, g)$ has subexponential volume growth, or
(iii) $a \in (0, \frac{1}{4})$, and $(M, g)$ has polynomial volume growth of degree at most $k$, for some $k$.

Then, either $M$ is closed, or $(M, g)$ is non-compact with finite topology and finite volume. In particular, $(M, g)$ is conformally equivalent to a closed Riemannian surface with at most finitely many points removed. In both case, $M$ compact or non-compact,

$$cV(M, g) \leq \int_M q d\mu \leq 2\pi a \chi(M) + \int_M P d\mu,$$

where $V(M, g)$ is the volume of $(M, g)$.

Remark. Under conditions (i) and (ii), this result is a direct consequence of Theorem 3.1. Note however that we only need a polynomial volume growth condition in (iii), without any bound on the degree (compare with Theorem 3.1). This is so because the condition that $J \geq 0$, with $c > 0$, is quite strong. One might wonder whether it is possible to weaken the growth condition in (ii).

Theorems 3.1 and 3.2 have their counterparts with the assumption that the operator $J$ is non-negative replaced by the assumption that the operator $J$ has finite index. As a matter of fact, one can immediately reduce the former case to the latter by using the following proposition of independent interest.

**Proposition 3.3.** Let $(M, g)$ be a complete Riemannian manifold and let $W$ be a locally integrable function on $M$. Then the operator $\Delta + W$ has finite index if and only if there exists a locally integrable function $P$ with compact support such that the operator $\Delta + W + P$ is non-negative.
4. Proofs

4.1. Proof of Theorem 3.1. Let us first deal with the case in which $M$ is closed. In this case, we can use the constant function $1$ in the quadratic form associated with the operator $J$,

$$Q_J(f) = \int_M ((df)^2 + (aK - q + P)f^2) \, d\mu$$

and (10) follows immediately from the Gauss-Bonnet theorem.

From now on, we assume that $(M, g)$ is complete, non-compact.

$\diamond$ Case (i). Assume that $B(x_0, Q)$ is a $J$-stable ball for some $Q$. Let $\xi(t) = (1 - t/Q)^\alpha$, for some $\alpha \geq 1$. Then,

$$\frac{1}{Q^2}(\xi')^2 - 2a\xi\xi'' = -\alpha[(4a - 1)\alpha - 2a](1 - t/Q)^{2\alpha - 2}.$$ 

Choose $\alpha = \frac{2a}{4a - 1}$. Apply (8) with these choices of $\xi$ and $\alpha$. Then,

$$\int_M q(1 - r/Q)^{2\alpha} + 2\pi a \sum_{n=1}^{N(Q)} \omega_n (1 - t_n/Q)^{2\alpha} \leq 2\pi a + \int_M P(1 - r/Q)^{2\alpha}.$$ 

Since $M$ is complete non-compact, and under the assumption of the theorem, inequality (13) holds for all $Q > 0$, and we can let $Q$ tend to infinity. Using the monotone convergence theorem for the left-hand side and the dominated convergence theorem for the right-hand side, we get

$$\int_M q \, d\mu \leq 2\pi a(1 - \bar{N}) + \int_M P \, d\mu,$$

and inequality (10) follows from Lemma 2.1 in [1]. This inequality implies that the topology is finite (with a lower bound for the Euler characteristic), and that $q$ is integrable. To show that the surface is parabolic, we prove that the volume growth is at most quadratic. To do so, we proceed as in [1]. From (9) and (12), choosing $\alpha$ large enough, we conclude that there exists a positive constant $C_\alpha$ such that

$$\frac{C_\alpha}{2^{2\alpha - 2}Q^2} V(B(x_0, (Q/2))) \leq \frac{C_\alpha}{Q^2} \int_M (1 - t/Q)^{2\alpha - 2} \, d\mu \leq 2\pi a + \|P\|_1,$$

which concludes the proof.

$\diamond$ Case (ii). Assume that $B(x_0, Q)$ is a $J$-stable ball. Take $\xi(t) = e^{-\alpha t}e^{-\alpha Q}$ for some $\alpha > 0$. Then,

$$\xi'(t)^2 - \xi''(t) = \alpha^2 e^{-\alpha t}e^{-\alpha Q}.$$ 

Applying (8) with $a = \frac{1}{4}$ and $\xi$ as above, gives

$$\left\{ \begin{array}{l} \int_{B(x_0, Q)} q\xi^2(r) \, d\mu + \frac{\alpha^2}{2} \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n) \leq \xi^2(0) + \int_M P\xi^2(r) \, d\mu + \frac{\alpha^2}{2} e^{-\alpha Q} \int_{B(x_0, Q)} e^{-\alpha r} \, d\mu. \end{array} \right.$$ 

Since $M$ is complete non-compact, inequality (14) holds for all $Q > 0$, and we can let $Q$ tend to infinity and argue as in [1]. The point is that the last term in the right-hand side of (14) goes to zero when $Q$ tends to infinity.
for any fixed $\alpha > 0$, because $M$ has subexponential area growth. Using monotone and dominated convergence theorems, it follows that
\[
\int_M q e^{-2\alpha r} \, d\mu + \frac{\pi}{2} \sum_{n=1}^{N} \omega_n e^{-2\alpha t_n} \leq \frac{\pi}{2} + \int_M P e^{-\alpha r} \, d\mu.
\]
Letting $\alpha$ tend to zero, and using Lemma 2.1, we get inequality (10). In particular, $M$ has finite topology and $q$ is integrable. To get quadratic area growth, we use inequality (9) with the test function $\xi$ given in Lemma 2.4. We get the inequality
\[
\left\{\begin{array}{l}
\frac{1}{4R^2} \int_{B(R)} e^{2(1-\frac{2\alpha}{\pi})^2} \, d\mu \leq \frac{\pi}{2} e^2 + \|P\|_1 \\
+ \frac{1}{2} \alpha^2 \beta^2 e^{-\alpha Q} \int_{C(R,Q)} e^{-\alpha r} \, d\mu,
\end{array}\right.
\]
and we let $Q$ tend to infinity to finish the proof.

\[\diamond\text{ Case (iii).}\] Assume that $B(x_0, Q)$ is a $J$-stable ball. Take $\xi(t) = (1 + \epsilon t)^{-\alpha} - (1 + \epsilon Q)^{-\alpha}$ with $\epsilon > 0$ and $\alpha = \frac{2a}{1-4a}$. Then,
\[
(1 - 2a)(\xi')^2 - 2a \xi \xi'' = 2a e^2 \alpha (\alpha + 1)(1 + \epsilon Q)^{-\alpha}(1 + \epsilon t)^{-\alpha - 2}.
\]
Applying (8) to $\xi$ we find,
\[
\left\{\begin{array}{l}
\int_{B(x_0, Q)} q \xi^2(r) \, d\mu + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n) \leq \\
2\pi a \xi^2(0) + \int_M P \xi^2(r) \, d\mu \\
+ 2a e^2 \alpha (\alpha + 1)(1 + \epsilon Q)^{-\alpha} \int_{B(x_0, Q)} (1 + \epsilon t)^{-\alpha - 2} \, d\mu.
\end{array}\right.
\]
Since $M$ is complete non-compact, inequality (15) holds for all $Q > 0$, we can let $Q$ tend to infinity, and argue as in [1]. The point is that the last term in the right-hand side of (15) goes to zero when $Q$ tends to infinity for any fixed $\epsilon > 0$, because of the assumption on the area growth of $M$. It follows that
\[
\int_M q(1 + \epsilon t)^{-\alpha} \, d\mu + 2\pi a \sum_{n=1}^{N} \omega_n (1 + \epsilon t_n)^{-\alpha} \leq 2\pi a + \int_M P(1 + \epsilon t)^{-\alpha} \, d\mu.
\]
Letting $\epsilon$ tend to zero and using Lemma 2.1, we get (10). In particular, $M$ has finite topology and $q$ is integrable. To get the quadratic area growth, we use inequality (9) and the test function $\xi$ given in Lemma 2.5. We get the inequality,
\[
\left\{\begin{array}{l}
\frac{\alpha \beta}{R^2} \int_{B(R)} (1 + \frac{1}{\pi})^{-2\beta - 2} \, d\mu \leq 2\pi a + \|P\|_1 \\
+ 2a e^2 \alpha (\alpha + 1)(1 + \epsilon Q)^{-\alpha} \int_{B(x_0, Q)} (1 + \epsilon r)^{-\alpha - 2} \, d\mu.
\end{array}\right.
\]
We can conclude the proof by letting $Q$ tend to infinity. \(\square\)

4.2. Proof of Theorem 3.2. Cases (i) and (ii) are direct consequences of Theorem 3.1, applying inequality (10) to the function $q \geq c > 0$. In case (iii), we first prove that $(M, g)$ has in fact polynomial volume growth of degree $k$ less than $2 + \frac{1}{1-4a}$, this follows from the assumption $c > 0$. 

\[\square\]
\(\diamond \) Case (iii), Preliminaries. Assume that \(B(x_0, Q)\) is a \(J\)-stable ball. Take \(\xi(t) = (1 + \epsilon t)^{-\alpha} - (1 + \epsilon Q)^{-\alpha}\) for \(\epsilon, \alpha > 0\). Then,
\[
\begin{align*}
(1 - 2a)(\xi')^2 &- 2a\xi'' = 2\epsilon^2[(1 - 4a)\alpha - 2a](1 + \epsilon t)^{-2\alpha - 2} \\
+ 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha}(1 + \epsilon t)^{-\alpha - 2}.
\end{align*}
\]
Applying (9) to (16) we find,
\[
\begin{align*}
c\int_{B(x_0, Q)} \xi^2(r) d\mu &\leq (2\pi a + ||P||_1)\xi^2(0) \\
+ \epsilon^2\alpha[(1 - 4a)\alpha - 2a] &\int_{B(x_0, Q)} (1 + \epsilon r)^{-2\alpha - 2} d\mu \\
+ 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha} &\int_{B(x_0, Q)} (1 + \epsilon r)^{-\alpha - 2} d\mu.
\end{align*}
\]
Call respectively \(A_2\) and \(A_3\) the last two terms in the right-hand side of the preceding inequality.

Assume that there exists a positive constant \(C_k\) such that \(V(B(x_0, t)) \leq C_k(1 + t)^k\), for all \(t > 0\). Then,
\[
\begin{align*}
\int_{B(x_0, Q)} & (1 + \epsilon r)^{-\beta} d\mu \leq C_k(1 + \epsilon Q)^{-\beta}(1 + Q)^k \\
+ \beta &\epsilon C_k \int_{0}^{Q} (1 + \epsilon t)^{-\beta - 1}(1 + t)^k dt.
\end{align*}
\]
Since \((M, g)\) is complete non-compact, we can let \(Q\) tend to infinity in (16).

\(\diamond\) Case (iii) continued. Define \(k_0\) by
\[
k_0 = \inf\{k \mid \exists C_k \text{ such that } V(B(x_0, t)) \leq C_k(1 + t)^k, \forall t > 0\}.
\]

Claim: \(k_0 < 2 + \frac{4\alpha}{1 - 2}\). Indeed if not, let \(k_1\) be such that \(k_0 < k_1 < k_0 + \frac{1}{2}\). Choose \(\alpha\) such that \(2\alpha + 2 = k_1 + \frac{1}{2}\), and \(\epsilon = 1\). Using (17), one finds that the term \(A_2\) in (16) is uniformly bounded when \(Q\) tends to infinity. Similarly, one sees that the term \(A_3\) tends to zero as \(Q\) tends to infinity. It follows that for any \(R > 0\), one has that
\[
c\int_{B(x_0, R)} (1 + r)^{-2\alpha} d\mu \leq C(k_1),
\]
which implies that \(cV(B(x_0, R)) \leq C(k_1)(1 + R)^{2\alpha} \leq C(k_1)(1 + R)^{k_0 - 1}\). This contradicts the definition of \(k_0\).

Since \(k_0 < 2 + \frac{4\alpha}{1 - 2}\), the assumption of Theorem 5.1 (iii) is satisfied and we can conclude. \(\square\)

4.3. Proof of Proposition 3.3.

\(\diamond\) Assume that \(\Delta + W\) has finite index on \(C_0^1(M)\). Then there exists a compact \(K \subset M\) such that \(\Delta + W\) is non-negative on \(C_0^1(M \setminus K)\). Take \(\phi\) to be a smooth function with compact support, such that \(0 \leq \phi \leq 1\) and \(\phi \equiv 1\) in a compact neighborhood of \(K\). Given any \(\psi \in C_0^1(M)\), write \(\psi\) as \(\psi = \phi \psi + (1 - \phi) \psi\). An easy computation gives,
\[
\begin{align*}
\int_{M} |d\psi|^2 + W |\psi|^2 &= \\
\int_{M} |d((1 - \phi) \psi)|^2 + W((1 - \phi) \psi)^2 &+ \int_{M} W(\phi^2 + 2\phi(1 - \phi)) |\psi|^2 \\
- \frac{1}{2} \int_{M} \psi^2 \Delta ((1 - \phi)^2) &- \int_{M} \psi^2 |d\phi|^2 \\
+ 2 \int_{M} \phi(1 - \frac{1}{2} \phi) |d\psi|^2.
\end{align*}
\]
Because $\Delta + W$ is non-negative in $M \setminus K$, and because of our choice of $\phi$, the first and fourth terms in the right-hand side of (20) are non-negative. The other terms can be written as
\begin{equation}
\int_M P \psi^2,
\end{equation}
where the function $P$ is defined by
\begin{equation}
P := |d\phi|^2 - \Delta (\phi (1 - \frac{1}{2} \phi)) - W \phi^2 - 2 \phi (1 - \phi) W.
\end{equation}
Recall that $W$ is locally integrable and that $\phi$ is smooth with compact support. It follows that $P$ is locally integrable, with compact support. By (20), the operator $\Delta + W + P$ is non-negative on $C^1_0(M)$, as stated.

Assume that there exists a function $P$, which is locally integrable with compact support, such that $\Delta + W + P$ is non-negative on $C^1_0(M)$. Let $K$ be a compact neighborhood of the support of $P$. Then,

$$0 \leq \int_M |d\psi|^2 + W \psi^2 + P \psi^2 = \int_M |d\psi|^2 + W \psi^2,$$

for any $\psi \in C^1_0(M \setminus K)$, and this means that $\Delta + W$ is non-negative on $C^1_0(M \setminus K)$. By a result of B. Devyver [2], this implies that $\Delta + W$ has finite index on $C^1_0(M)$. □

References