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Pierre Bérard, Philippe Castillon

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REMARKS ON J. ESPINAR'S "FINITE INDEX OPERATORS ON SURFACES"

PIERRE BÉRARD AND PHILIPPE CASTILLON

ABSTRACT. In this paper, we make some remarks on José Espinar's paper "Finite index operators on surfaces" [[arXiv:0911.3767](#), to appear in *Journal of Geometric Analysis* (2011)].

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1. INTRODUCTION

In [1], we considered operators of the form $J = \Delta + aK - q$ on a complete non-compact Riemannian surface (M, g) , where Δ is the non-negative Laplacian, and K the Gaussian curvature associated with the metric g . The parameter a is some positive constant, and q is a non-negative locally integrable function on M . More precisely, we studied the consequences, for the geometry of the triple $(M, g; q)$, of the fact that the operator J is non-negative (in the sense of quadratic forms).

Motivated by applications to minimal and CMC surfaces, J. Espinar [3] considers a different framework (see also [4]). More precisely, he considers a Riemannian surface (M, g) , possibly with boundary ∂M and not necessarily complete, and operators of the form $\Delta + aK - c + P$, where the parameters a, c are positive constants, and P is a non-negative integrable function.

In this note, we consider complete surfaces without boundary, and prove results similar to those in [3, 4], under weaker assumptions. For this purpose, we apply the methods of [1].

2. GENERAL FRAMEWORK

Generally speaking, we will use the same notations as in [1], (M, g) will denote a complete (possibly compact) surface without boundary.

2.1. The operators. In this paper, we consider operators of the form,

$$(1) \quad J = \Delta + aK - q + P.$$

Here Δ is the non-negative Laplacian, and K the Gaussian curvature associated with the metric g . We let μ denote the Riemannian measure associated with g .

◊ We make the following assumptions on the operator J ,

$$(2) \quad \begin{cases} a & \text{is a positive constant,} \\ q & \text{is a non-negative, locally integrable function on } M, \\ & \text{and we let } c = \inf_M q \geq 0, \\ P & \text{is an integrable function on } M, \\ & \text{and we let } \|P\|_1 = \int_M |P| d\mu. \end{cases}$$

Note that we do not impose any sign condition on the function P .

◊ We say that the open geodesic ball $B(x_0, R)$ is *J-stable* if the operator J is non-negative in the sense of quadratic forms,

$$(3) \quad 0 \leq Q_J(\phi) = \int_M \{|d\phi|^2 + (aK - q + P)\phi^2\} d\mu$$

for all ϕ in $\text{Lip}_0(B(x_0, R))$, the Lipschitz functions with compact support inside the ball.

2.2. Volume growth assumptions. Fix a reference point x_0 in M . We consider the following assumptions on the volume growth on (M, g) .

◊ We say that (M, g) has *polynomial volume growth of order at most k* if there exists a constant C_k such that,

$$(4) \quad V(B(x_0, R)) \leq C_k(1 + R)^k,$$

for all $R > 0$.

◊ We say that (M, g) has *k -subpolynomial volume growth* if

$$(5) \quad \limsup_{R \rightarrow \infty} \frac{V(B(x_0, R))}{R^k} = 0.$$

◊ We say that (M, g) has *subexponential volume growth* if

$$(6) \quad \limsup_{R \rightarrow \infty} \frac{\ln(V(B(x_0, R)))}{R} = 0.$$

For a complete surface without boundary, these definitions do not depend on the choice of the reference point x_0 , although the constant C_k a priori does.

2.3. Fundamental inequalities. We briefly recall the notations of [1], Section 2. Given a reference point $x_0 \in M$, we consider the open geodesic balls $B(x_0, t)$, and their Euler-Poincaré characteristics $\chi(B(x_0, t))$. More precisely, we introduce the function,

$$\hat{\chi}(s) = \sup \{\chi(B(x_0, t)) \mid t \geq s\}.$$

This is a non-increasing function with a sequence of discontinuities, finite possibly empty, or infinite, $\{t_j\}_{j=1}^{\bar{N}}$, with $\bar{N} \in \mathbb{N} \cup \{\infty\}$. Note that this sequence depends on the choice of the reference point x_0 . We call ω_j the jump of the function $\hat{\chi}$ at the discontinuity t_j .

We call *admissible* a function $\xi : [0, Q] \rightarrow \mathbb{R}$, which is C^1 and piecewise C^2 , with $\xi, \xi'' \geq 0$ and $\xi' \leq 0$. Let $N(Q)$ be the largest integer n such that $t_n \leq Q$.

We now recall two key results from [1].

◊ The topology of M is controlled by the function $\widehat{\chi}$. More precisely, we have the inequality (see [1], Lemma 2.1),

$$(7) \quad 1 - \sum_{n=1}^{\overline{N}} \omega_n \leq \chi(M).$$

◊ Assume that the operator J satisfies the assumptions (2), and let $B(x_0, Q)$ be some J -stable ball in M . Let ξ be any admissible function on $[0, Q]$, with $\xi(Q) = 0$, and let r denote the distance function to the center x_0 of the ball. Plugging the function $\xi(r)$ into the quadratic form for J and applying [1], Lemma 2.3, we obtain the inequality

$$(8) \quad \left\{ \begin{array}{l} \int_{B(x_0, Q)} q \xi^2(r) d\mu \leq 2\pi a \xi^2(0) - 2\pi a \sum_{j=1}^{N(Q)} \omega_j \xi^2(t_j) \\ \quad + \int_{B(x_0, Q)} P \xi^2(r) \\ \quad + \int_{B(x_0, Q)} [(1 - 2a)(\xi')^2 - 2a\xi\xi''] (r) d\mu, \end{array} \right.$$

which yields the weaker inequality,

$$(9) \quad \left\{ \begin{array}{l} c \int_{B(x_0, Q)} \xi^2(r) d\mu \leq 2\pi a \xi^2(0) + \|\xi\|_\infty^2 \|P\|_1 \\ \quad + \int_{B(x_0, Q)} [(1 - 2a)(\xi')^2 - 2a\xi\xi''] (r) d\mu. \end{array} \right.$$

3. STATEMENTS

Inequality (8) shows that the case in which the operator $J = \Delta + aK - q + P$ is non-negative –under the assumptions (2)– is similar to the case in which the operator $\Delta + aK - q$ has finite index, as treated in [1], Theorem 4.1. More precisely, we have the following result.

Theorem 3.1. *Let (M, g) be a complete Riemannian surface without boundary, and let J be the operator,*

$$J = \Delta + aK - q + P,$$

with $q \geq 0$ locally integrable and P an integrable function. Assume that $J \geq 0$ on $\text{Lip}_0(M)$, and that either of the following conditions holds,

- (i) $a > \frac{1}{4}$, or
- (ii) $a = \frac{1}{4}$, and (M, g) has subexponential volume growth, or
- (iii) $a \in (0, \frac{1}{4})$, and (M, g) has k_a -subpolynomial volume growth, with $k_a = 2 + \frac{4a}{1-4a}$.

Then, either M is closed, or (M, g) is non-compact with finite topology and at most quadratic area growth. In particular, (M, g) is conformally equivalent to a closed Riemannian surface with at most finitely many points removed. Furthermore, q is integrable on (M, g) , and we have,

$$(10) \quad \int_M q d\mu \leq 2\pi a \chi(M) + \int_M P d\mu.$$

Remark. When considering an operator of the form $J = \Delta + aK + W$, taking $q = W_-$ and $P = W_+$, the previous result gives the following. If either of the conditions (i), (ii) or (iii) holds, and if W_+ is integrable, then $W \in L^1(M, \mu)$, M has finite conformal type, and

$$0 \leq 2\pi a \chi(M) + \int_M W d\mu.$$

The interesting case, in the present framework, is the case in which the infimum c of the function q is positive. We have the following result.

Theorem 3.2. *Let (M, g) be a complete Riemannian surface without boundary, and let J be the operator,*

$$J = \Delta + aK - q + P,$$

with $q \geq c > 0$ locally integrable, and P an integrable function on (M, g) . Assume that $J \geq 0$ on $\text{Lip}_0(M)$, and that either of the following conditions holds,

- (i) $a > \frac{1}{4}$, or
- (ii) $a = \frac{1}{4}$, and (M, g) has subexponential volume growth, or
- (iii) $a \in (0, \frac{1}{4})$, and (M, g) has polynomial volume growth of degree at most k , for some k .

Then, either M is closed, or (M, g) is non-compact with finite topology and finite volume. In particular, (M, g) is conformally equivalent to a closed Riemannian surface with at most finitely many points removed. In both case, M compact or non-compact,

$$(11) \quad cV(M, g) \leq \int_M q d\mu \leq 2\pi a \chi(M) + \int_M P d\mu,$$

where $V(M, g)$ is the volume of (M, g) .

Remark. Under conditions (i) and (ii), this result is a direct consequence of Theorem 3.1. Note however that we only need a polynomial volume growth condition in (iii), without any bound on the degree (compare with Theorem 3.1). This is so because the condition that $J \geq 0$, with $c > 0$, is quite strong. One might wonder whether it is possible to weaken the growth condition in (ii).

Theorems 3.1 and 3.2 have their counterparts with the assumption that the operator J is *non-negative* replaced by the assumption that the operator J has *finite index*. As a matter of fact, one can immediately reduce the former case to the latter by using the following proposition of independent interest.

Proposition 3.3. *Let (M, g) be a complete Riemannian manifold and let W be a locally integrable function on M . Then the operator $\Delta + W$ has finite index if and only if there exists a locally integrable function P with compact support such that the operator $\Delta + W + P$ is non-negative.*

4. PROOFS

4.1. Proof of Theorem 3.1. Let us first deal with the case in which M is closed. In this case, we can use the constant function $\mathbf{1}$ in the quadratic form associated with the operator J ,

$$Q_J(f) = \int_M (|df|^2 + (aK - q + P)f^2) d\mu$$

and (10) follows immediately from the Gauss-Bonnet theorem.

From now on, we assume that (M, g) is complete, non-compact.

◇ *Case (i).* Assume that $B(x_0, Q)$ is a J -stable ball for some Q . Let $\xi(t) = (1 - t/Q)_+^\alpha$, for some $\alpha \geq 1$. Then,

$$(12) \quad (1 - 2a)(\xi')^2 - 2a\xi\xi'' = -\frac{\alpha[(4a - 1)\alpha - 2a]}{Q^2} \left(1 - \frac{t}{Q}\right)_+^{2\alpha-2}.$$

Choose $\alpha = \frac{2a}{4a-1}$. Apply (8) with these choices of ξ and α . Then,

$$(13) \quad \int_M q \left(1 - \frac{r}{Q}\right)_+^{2\alpha} + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \leq 2\pi a + \int_M P \left(1 - \frac{r}{Q}\right)_+^{2\alpha}.$$

Since M is complete non-compact, and under the assumption of the theorem, inequality (13) holds for all $Q > 0$, and we can let Q tend to infinity. Using the monotone convergence theorem for the left-hand side and the dominated convergence theorem for the right-hand side, we get

$$\int_M q d\mu \leq 2\pi a \left(1 - \sum_1^{\bar{N}} \omega_n\right) + \int_M P d\mu,$$

and inequality (10) follows from Lemma 2.1 in [1]. This inequality implies that the topology is finite (with a lower bound for the Euler characteristic), and that q is integrable. To show that the surface is parabolic, we prove that the volume growth is at most quadratic. To do so, we proceed as in [1]. From (9) and (12), choosing α large enough, we conclude that there exists a positive constant C_α such that

$$\frac{C_\alpha}{2^{2\alpha-2}Q^2} V(B(x_0, \frac{Q}{2})) \leq \frac{C_\alpha}{Q^2} \int_M \left(1 - \frac{t}{Q}\right)_+^{2\alpha-2} d\mu \leq 2\pi a + \|P\|_1,$$

which concludes the proof.

◇ *Case (ii).* Assume that $B(x_0, Q)$ is a J -stable ball. Take $\xi(t) = e^{-at} - e^{-\alpha Q}$ for some $\alpha > 0$. Then,

$$(\xi')^2 - \xi\xi'' = \alpha^2 e^{-at} e^{-\alpha Q}.$$

Applying (8) with $a = \frac{1}{4}$ and ξ as above, gives

$$(14) \quad \begin{cases} \int_{B(x_0, Q)} q \xi^2(r) d\mu + \frac{\pi}{2} \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n) \leq \\ \frac{\pi}{2} \xi^2(0) + \int_M P \xi^2(r) d\mu + \frac{\alpha^2}{2} e^{-\alpha Q} \int_{B(x_0, Q)} e^{-\alpha r} d\mu. \end{cases}$$

Since M is complete non-compact, inequality (14) holds for all $Q > 0$, and we can let Q tend to infinity and argue as in [1]. The point is that the last term in the right-hand side of (14) goes to zero when Q tends to infinity

for any fixed $\alpha > 0$, because M has subexponential area growth. Using monotone and dominated convergence theorems, it follows that

$$\int_M qe^{-2\alpha r} d\mu + \frac{\pi}{2} \sum_{n=1}^N \omega_n e^{-2\alpha t_n} \leq \frac{\pi}{2} + \int_M P e^{-\alpha r} d\mu.$$

Letting α tend to zero, and using [1] Lemma 2.1, we get inequality (10). In particular, M has finite topology and q is integrable. To get quadratic area growth, we use inequality (9) with the test function ξ given in [1] Lemma 2.4. We get the inequality

$$\begin{cases} \frac{1}{4R^2} \int_{B(R)} e^{2(1-\frac{r}{2R})^2} d\mu \leq \frac{\pi}{2} e^2 + \|P\|_1 \\ + \frac{1}{2} \alpha^2 \beta^2 e^{-\alpha Q} \int_{C(R,Q)} e^{-\alpha r} d\mu, \end{cases}$$

and we let Q tend to infinity to finish the proof.

◇ *Case (iii).* Assume that $B(x_0, Q)$ is a J -stable ball. Take $\xi(t) = (1 + \epsilon t)^{-\alpha} - (1 + \epsilon Q)^{-\alpha}$ with $\epsilon > 0$ and $\alpha = \frac{2a}{1-4a}$. Then,

$$(1 - 2a)(\xi')^2 - 2a\xi\xi'' = 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha}(1 + \epsilon t)^{-\alpha-2}.$$

Applying (8) to ξ we find,

$$(15) \quad \begin{cases} \int_{B(x_0, Q)} q\xi^2(r) d\mu + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n) \leq \\ 2\pi a \xi^2(0) + \int_M P \xi^2(r) d\mu \\ + 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha} \int_{B(x_0, Q)} (1 + \epsilon r)^{-\alpha-2} d\mu. \end{cases}$$

Since M is complete non-compact, inequality (15) holds for all $Q > 0$, we can let Q tend to infinity, and argue as in [1]. The point is that the last term in the right-hand side of (15) goes to zero when Q tends to infinity for any fixed $\epsilon > 0$, because of the assumption on the area growth of M . It follows that

$$\int_M q(1 + \epsilon t)^{-\alpha} d\mu + 2\pi a \sum_{n=1}^N \omega_n (1 + \epsilon t_n)^{-\alpha} \leq 2\pi a + \int_M P(1 + \epsilon t)^{-\alpha} d\mu.$$

Letting ϵ tend to zero and using [1] Lemma 2.1, we get (10). In particular, M has finite topology and q is integrable. To get the quadratic area growth, we use inequality (9) and the test function ξ given in [1] Lemma 2.5. We get the inequality,

$$\begin{cases} \frac{\alpha\beta}{R^2} \int_{B(R)} (1 + \frac{r}{R})^{-2\beta-2} d\mu \leq 2\pi a + \|P\|_1 \\ + 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha} \int_{B(x_0, Q)} (1 + \epsilon r)^{-\alpha-2} d\mu. \end{cases}$$

We can conclude the proof by letting Q tend to infinity. \square

4.2. Proof of Theorem 3.2. Cases (i) and (ii) are direct consequences of Theorem 3.1, applying inequality (10) to the function $q \geq c > 0$. In case (iii), we first prove that (M, g) has in fact polynomial volume growth of degree k less than $2 + \frac{4a}{1-4a}$, this follows from the assumption $c > 0$.

◇ *Case (iii), Preliminaries.* Assume that $B(x_0, Q)$ is a J -stable ball. Take $\xi(t) = (1 + \epsilon t)^{-\alpha} - (1 + \epsilon Q)^{-\alpha}$ for $\epsilon, \alpha > 0$. Then,

$$\begin{cases} (1 - 2a)(\xi')^2 - 2a\xi\xi'' = \alpha\epsilon^2[(1 - 4a)\alpha - 2a](1 + \epsilon t)^{-2\alpha-2} \\ + 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha}(1 + \epsilon t)^{-\alpha-2}. \end{cases}$$

Applying (9) to ξ we find,

$$(16) \quad \begin{cases} c \int_{B(x_0, Q)} \xi^2(r) d\mu \leq (2\pi a + \|P\|_1)\xi^2(0) \\ + \epsilon^2\alpha[(1 - 4a)\alpha - 2a] \int_{B(x_0, Q)} (1 + \epsilon r)^{-2\alpha-2} d\mu \\ + 2a\epsilon^2\alpha(\alpha + 1)(1 + \epsilon Q)^{-\alpha} \int_{B(x_0, Q)} (1 + \epsilon r)^{-\alpha-2} d\mu. \end{cases}$$

Call respectively A_2 and A_3 the last two terms in the right-hand side of the preceding inequality.

Assume that there exists a positive constant C_k such that $V(B(x_0, t)) \leq C_k(1 + t)^k$, for all $t > 0$. Then,

$$(17) \quad \begin{cases} \int_{B(x_0, Q)} (1 + \epsilon r)^{-\beta} d\mu \leq C_k(1 + \epsilon Q)^{-\beta}(1 + Q)^k \\ + \beta\epsilon C_k \int_0^Q (1 + \epsilon t)^{-\beta-1}(1 + t)^k dt. \end{cases}$$

Since (M, g) is complete non-compact, we can let Q tend to infinity in (16).

◇ *Case (iii) continued.* Define k_0 by

$$(18) \quad k_0 = \inf\{k \mid \exists C_k \text{ such that } V(B(x_0, t)) \leq C_k(1 + t)^k, \forall t > 0\}.$$

Claim: $k_0 < 2 + \frac{4a}{1-4a}$. Indeed if not, let k_1 be such that $k_0 < k_1 < k_0 + \frac{1}{2}$. Choose α such that $2\alpha + 2 = k_1 + \frac{1}{2}$, and $\epsilon = 1$. Using (17), one finds that the term A_2 in (16) is uniformly bounded when Q tends to infinity. Similarly, one sees that the term A_3 tends to zero as Q tends to infinity. It follows that for any $R > 0$, one has that

$$(19) \quad c \int_{B(x_0, R)} (1 + r)^{-2\alpha} d\mu \leq C(k_1),$$

which implies that $cV(B(x_0, R)) \leq C(k_1)(1 + R)^{2\alpha} \leq C(k_1)(1 + R)^{k_0-1}$. This contradicts the definition of k_0 .

Since $k_0 < 2 + \frac{4a}{1-4a}$, the assumption of Theorem 3.1 (iii) is satisfied and we can conclude. \square

4.3. Proof of Proposition 3.3.

◇ Assume that $\Delta + W$ has finite index on $C_0^1(M)$. Then there exists a compact $K \subset M$ such that $\Delta + W$ is non-negative on $C_0^1(M \setminus K)$. Take ϕ to be a smooth function with compact support, such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in a compact neighborhood of K . Given any $\psi \in C_0^1(M)$, write ψ as $\psi = \phi\psi + (1 - \phi)\psi$. An easy computation gives,

$$(20) \quad \begin{cases} \int_M |d\psi|^2 + W\psi^2 = \\ \int_M |d((1 - \phi)\psi)|^2 + W((1 - \phi)\psi)^2 \\ + \int_M W(\phi^2 + 2\phi(1 - \phi))\psi^2 \\ - \frac{1}{2} \int_M \psi^2 \Delta((1 - \phi)^2) - \int_M \psi^2 |d\phi|^2 \\ + 2 \int_M \phi(1 - \frac{1}{2}\phi) |d\psi|^2. \end{cases}$$

Because $\Delta + W$ is non-negative in $M \setminus K$, and because of our choice of ϕ , the first and fourth terms in the right-hand side of (20) are non-negative. The other terms can be written as $-\int_M P\psi^2$, where the function P is defined by

$$(21) \quad \begin{cases} P := |d\phi|^2 - \Delta(\phi(1 - \frac{1}{2}\phi)) \\ -W\phi^2 - 2\phi(1 - \phi)W. \end{cases}$$

Recall that W is locally integrable and that ϕ is smooth with compact support. It follows that P is locally integrable, with compact support. By (20), the operator $\Delta + W + P$ is non-negative on $C_0^1(M)$, as stated.

◇ Assume that there exists a function P , which is locally integrable with compact support, such that $\Delta + W + P$ is non-negative on $C_0^1(M)$. Let K be a compact neighborhood of the support of P . Then,

$$0 \leq \int_M |d\psi|^2 + W\psi^2 + P\psi^2 = \int_M |d\psi|^2 + W\psi^2,$$

for any $\psi \in C_0^1(M \setminus K)$, and this means that $\Delta + W$ is non-negative on $C_0^1(M \setminus K)$. By a result of B. Devyver [2], this implies that $\Delta + W$ has finite index on $C_0^1(M)$. \square

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Pierre Bérard
 Université Grenoble 1
 Institut Fourier (UJF-CNRS)
 B.P. 74
 38402 Saint Martin d'Hères Cedex
 France
 Pierre.Berard@ujf-grenoble.fr

Philippe Castillon
 Université Montpellier II
 Département des sciences mathématiques CC 51
 I3M (UMR 5149)
 34095 Montpellier Cedex 5
 France
 Philippe.Castillon@univ-montp2.fr