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# Maximum principle and bang-bang property of time optimal controls for Schrödinger type systems

Jérôme Lohéac\*      Marius Tucsnak\*<sup>†</sup>

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## Abstract

We consider the time optimal control problem, with a point target, for a class of infinite dimensional systems with a dynamics governed by an abstract Schrödinger type equation. The main results establish a Pontryagin type maximum principle and give sufficient conditions for the bang-bang property of optimal controls. The results are then applied to some systems governed by partial differential equations. The paper ends by a discussion of possible extensions and by stating some open problems.

**Key words.** Time optimal control, Pontryagin's maximum principle, Bang-Bang property, Schrödinger equation

## 1 Introduction

Time optimal control is a classical problem for linear finite dimensional systems. For these systems, among the interesting properties of time optimal controls, the facts that they satisfy Pontryagin's maximum principle and that they are *bang-bang*, are well-known (see Bellman, Glicksberg and Gross [3]). The first extensions of these results to infinite dimensional linear systems have been given in Fattorini's paper [5]. The progress made in this field has been successively reported in the books of Lions [15] and of Fattorini [7]. In particular, the bang-bang property of time optimal controls has been quite rapidly established for invertible input operators. In the case of systems governed by PDE's, this means, roughly speaking, that the control is active in the entire spatial domain where the PDE is considered. The situation in which the control is active only in a part of the considered domain (or of its boundary) has been considered only recently. The results in this direction are limited as far as we know, to systems governed by linear parabolic equations (see, for instance, Mizel and Seidman [17], Wang [21], Phung and Wang [18], Kunisch and Wang [13]). The methodology used in most of the above mentioned works, based on a special  $L^\infty$  null-controllability property of the considered system, does not seem applicable to systems which are time reversible, as those governed by wave, Schrödinger or Euler Bernoulli equations. The first aim of this paper is to show that for some of these systems, provided that they are exactly controllable in arbitrarily small time, Pontryagin's maximum principle holds in a standard form. Moreover, assuming that the system is approximatively controllable with controls active only for times  $t$  in a set of positive measure, we prove that time optimal controls

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\*E-mail: [jerome.loheac@iecn.u-nancy.fr](mailto:jerome.loheac@iecn.u-nancy.fr) and [marius.tucsnak@iecn.u-nancy.fr](mailto:marius.tucsnak@iecn.u-nancy.fr)

<sup>†</sup>Both authors are with Institut Élie Cartan UMR 7502, Nancy-Université, CNRS, INRIA, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France, and INRIA Nancy Grand Est, Projet CORIDA.

have the bang-bang property and they are unique. We provide several examples of applications of these results to systems governed by the Schrödinger or the Euler-Bernoulli plate equations.

Let us recall, in order to give the precise statement of our main result, some classical notation and definitions from control theory.

We first introduce some notation. Let  $X$  and  $Y$  be Hilbert spaces. If  $P \in \mathcal{L}(X; Y)$  then the *null-space* and the *range* of  $P$  are the subspaces of  $X$  and  $Y$  respectively defined by

$$\text{Ker } P = \{x \in X, Px = 0\}, \quad \text{Ran } P = \{Px, x \in X\}.$$

Throughout this paper,  $X$  and  $U$  are complex Hilbert spaces, identified with their duals. The inner product and the norm in  $X$  are denoted by  $\langle \cdot, \cdot \rangle_X$  and  $\|\cdot\|_X$ , respectively. We denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  a strongly continuous semigroup on  $X$  generated by an operator  $A : \mathcal{D}(A) \rightarrow X$ .

Let  $B \in \mathcal{L}(U, X)$  be a control operator, let  $z_0 \in X$  and let  $u \in L^2([0, \infty), U)$ . We consider the infinite dimensional system described by the equation

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0. \quad (1.1)$$

With the above notation, the solution  $z$  of (1.1) is defined by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \quad (t \geq 0), \quad (1.2)$$

where  $\Phi_t \in \mathcal{L}(L^2([0, t], U); X)$  is given by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma) d\sigma \quad (u \in L^2([0, \infty), U)). \quad (1.3)$$

The maps  $(\Phi_t)$  are called *input to state maps*.

Recall the following classical definitions (see, for instance, Tucsnak and Weiss [20, Sections 4.2 and 11.1]):

**Definition 1.1.**

- The pair  $(A, B)$  is said *approximatively controllable* in time  $\tau$  if  $\text{Ran } \Phi_\tau$  is dense in  $X$ .
- The pair  $(A, B)$  is said *exactly controllable* in time  $\tau$  if  $\text{Ran } \Phi_\tau = X$ .

We also need the following (less classical) definition.

**Definition 1.2.** Let  $e \subset [0, \tau]$  be a set of positive Lebesgue measure. The pair  $(A, B)$  is said *approximatively controllable in time  $\tau$  from  $e$*  if the range of the map  $\Phi_{\tau, e} \in \mathcal{L}(L^2([0, \tau], U), X)$  defined by

$$\Phi_{\tau, e} u = \int_e \mathbb{T}_{\tau-\sigma} Bu(\sigma) d\sigma \quad (u \in L^2([0, \tau], U))$$

is dense in  $X$ .

We are now in position to give a precise definition of time optimal controls.

**Definition 1.3.** Let  $z_0, z_1 \in X$  with  $z_0 \neq z_1$ . A function  $u^* \in L^\infty([0, \infty), U)$  is said a *time optimal control* for the pair  $(A, B)$ , associated to the initial state  $z_0$  and the final state  $z_1$ , if there exists  $\tau^* > 0$  such that

1.  $z_1 = \mathbb{T}_{\tau^*} z_0 + \Phi_{\tau^*} u^*$  and  $\|u^*\|_{L^\infty([0, \tau^*], U)} \leq 1$ ;

2. If  $\tau > 0$  is such that there exists  $u \in L^\infty([0, \tau], U)$  with

$$z_1 = \mathbb{T}_\tau z_0 + \Phi_\tau u, \quad \|u\|_{L^\infty([0, \tau], U)} \leq 1,$$

then  $\tau \geq \tau^*$ .

Our first main result provides a class of infinite dimensional system for which the maximum principle from the linear finite dimensional case can be extended in its classical form.

**Theorem 1.4.** *Suppose that  $(A, B)$  is exactly controllable in any time  $\tau > 0$ . Then, for every  $z_0, z_1 \in X$ ,  $z_0 \neq z_1$ , there exists a time optimal control  $u^*$  steering  $z_0$  to  $z_1$  in time  $\tau^* = \tau^*(z_0, z_1)$ . Moreover, there exists  $\eta \in X$ ,  $\eta \neq 0$ , such that*

$$\langle B^* \mathbb{T}_{\tau^* - t}^* \eta, u(t) \rangle_U = \max_{\substack{v \in U, \\ \|v\|_U \leq 1}} \langle B^* \mathbb{T}_{\tau^* - t}^* \eta, v \rangle_U \quad (t \in (0, \tau^*) \text{ a.e.}). \quad (1.4)$$

Our second main result shows that, under an extra assumption, the time optimal controls in the above theorem are bang-bang.

**Corollary 1.5.** *With the notation and the assumptions in Theorem 1.4, assume moreover that the pair  $(A, B)$  is approximatively controllable in time  $\tau^*$  from any  $e \subset [0, \tau^*]$  of positive measure. Then the time optimal control  $u^*$  is bang-bang, in the sense that*

$$\|u^*(t)\|_U = 1 \quad (t \in [0, \tau^*] \text{ a.e.}). \quad (1.5)$$

Moreover, the time optimal control is unique.

## 2 Some background on infinite dimensional systems

**Remark 2.1.** *Notice that if  $B$  is bounded and the pair  $(A, B)$  is exactly controllable in time  $\tau$  then  $\Phi_\tau(L^\infty([0, \tau], U)) = X$ . In other words, we can exactly control the system in time  $\tau$  by using inputs in  $L^\infty([0, \tau], U)$  (instead of taking controls in  $L^2([0, \tau], U)$  as in the Definition 1.1). Indeed, let*

$$R_\tau = \int_0^\tau \mathbb{T}_t B B^* \mathbb{T}_t^* dt \in \mathcal{L}(X),$$

be the controllability Gramian in time  $\tau$  for the pair  $(A, B)$ . Given  $z_0 \in X$ , the exact controllability in time  $\tau$  of the pair  $(A, B)$  implies that  $R_\tau$  is invertible and that the function  $u \in L^2([0, \tau], U)$  defined by

$$u = \Phi_\tau^* R_\tau^{-1} z_0 \quad (t \in [0, T]), \quad (2.1)$$

where  $\Phi_t^* \in \mathcal{L}(X, L^2([0, \tau], U))$  has the property  $\Phi_\tau u = z_0$ . On the other hand, it is well known (see, for instance [20, Section 4.4]) that, since  $B$  is bounded, we have

$$(\Phi_\tau^* w_0)(t) = B^* \mathbb{T}_{\tau - t}^* w_0 \quad (w_0 \in X, t \in [0, \tau]).$$

The last two formulas imply that the exact control defined in (2.1) writes

$$u(t) = B^* \mathbb{T}_{\tau - t}^* R_\tau^{-1} z_0 \quad (t \in [0, T]),$$

so that  $u \in C([0, \tau], X)$ .

**Definition 2.2.** The  $L^\infty$  reachable space at time  $t$  of the pair  $(A, B)$  is defined as

$$R_t^\infty = \Phi_t(L^\infty([0, t], U)). \quad (2.2)$$

By a slight variation of the argument in [7, Lemma 2.1.1] one can check that, for each  $t > 0$ , when endowed with the norm

$$\|x\|_{R_t^\infty} = \inf \{ \|u\|_{L^\infty([0, t], U)}, \Phi_t u = x \} \quad (x \in X), \quad (2.3)$$

$R_t^\infty$  is a Banach space.

**Proposition 2.3.** *Let  $0 \leq \sigma \leq t$ . Then we have the continuous inclusions*

$$R_\sigma^\infty \subset R_t^\infty \subset X.$$

*Proof.* It is clear that the above inclusions hold. To show that they are continuous, we take  $x \in R_t^\infty$ , so that there exists  $u \in L^\infty([0, t], U)$  with  $x = \Phi_t u$ . Since  $\Phi_t \in \mathcal{L}(L^2([0, t], U), X)$  it follows that there exists  $M > 0$  such that  $\|x\|_X \leq M \|u\|_{L^\infty([0, t], U)}$  for every  $u \in L^\infty([0, t], U)$  satisfying  $\Phi_t u = x$ . Consequently, we have

$$\|x\|_X \leq M \inf_{\substack{u \in L^\infty([0, t], U) \\ \Phi_t u = x}} \|u\|_{L^\infty([0, t], U)} = M \|x\|_{R_t^\infty},$$

so that the inclusion  $R_t^\infty \subset X$  is continuous.

To prove the continuity of the inclusion  $R_\sigma^\infty \subset R_t^\infty$ , it suffices to note that if  $x \in R_\sigma^\infty$  and  $u \in L^\infty([0, \sigma], U)$  are such that  $x = \Phi_\sigma u$  then  $\tilde{u} \in L^\infty([0, t], U)$  defined for almost every  $s \in [0, t]$  by  $\tilde{u}(s) = \chi_{(t-\sigma, t)}(s)u(s-t+\sigma)$ , where  $\chi_{(t-\sigma, t)}$  denotes the characteristic function of the set  $(t-\sigma, t)$ , has the property  $x = \Phi_t \tilde{u}$  and  $\|\tilde{u}\|_{L^\infty([0, t], U)} = \|u\|_{L^\infty([0, \sigma], U)}$ .  $\square$

From Remark 2.1 it follows that if  $(A, B)$  is exactly controllable in some time  $\tau > 0$  then  $R_\tau^\infty = X$ . This fact, combined with Proposition 2.3 and with the closed graph theorem clearly imply the result below.

**Corollary 2.4.** *Assume that  $(A, B)$  is exactly controllable in some time  $\tau > 0$ . Then the norms  $\|\cdot\|_{R_\tau^\infty}$  and  $\|\cdot\|_X$  are equivalent.*

**Remark 2.5.** *If  $R_t^\infty = X$ , the control cost  $C_t$  in time  $t$  is defined by*

$$C_t = \sup_{x \neq 0} \frac{\|x\|_{R_t^\infty}}{\|x\|_X} \quad (t > 0). \quad (2.4)$$

*It is easily seen that the map  $t \mapsto C_t$  is non-increasing.*

We also need some concepts of observability theory. Let  $Y$  be another Hilbert space and let  $C \in \mathcal{L}(X, Y)$  be an *observation operator* for the semigroup  $\mathbb{T}$ . For  $\tau > 0$  we consider the *initial state to output map*  $\Psi_\tau$  defined by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \quad (z_0 \in X, t \in [0, \tau]).$$

The operators  $\Psi_\tau$  are briefly called *output maps* corresponding to the pair  $(A, C)$ . If  $e \subset [0, \tau]$  is a set of positive measure, we consider the restriction of the above defined initial state to output map to a set of positive measure  $e \subset [0, \tau]$ , which is defined by

$$\Psi_{\tau, e} \in \mathcal{L}(X, L^2([0, \tau], U)), \quad \Psi_{\tau, e} = \chi_e \Psi_\tau,$$

where  $\chi_e$  is the characteristic function of  $e$ .

We denote by  $\Psi_\tau^d$  (respectively  $\Psi_{\tau,e}^d$ ) the output maps corresponding (respectively the restriction to a set of positive measure  $e \subset [0, \tau]$  of these output maps) corresponding to the pair  $(A^*, B^*)$ , i.e., we set

$$\begin{aligned} (\Psi_\tau^d z_0)(t) &= B^* \mathbb{T}_t^* z_0 \quad (z_0 \in X, t \in [0, \tau]), \\ \Psi_{\tau,e}^d &\in \mathcal{L}(X, L^2([0, \tau], U)), \quad \Psi_{\tau,e}^d = \chi_e \Psi_\tau^d. \end{aligned} \quad (2.5)$$

There are several generalizations of the concept of observability to infinite dimensional linear systems. In this paper we need only the following one.

**Definition 2.6.** Let  $\tau > 0$  and  $e \subset [0, \tau]$  be a set of positive measure. The pair  $(A, C)$  is said approximately observable from  $e$  if  $\text{Ker } \Psi_{\tau,e} = \{0\}$ .

We have the following duality result, which is a simple consequence of the corresponding result for  $e = [0, \tau]$  (see, for instance [16, Proposition 2.4]).

**Proposition 2.7.** Let  $\tau > 0$ ,  $e \subset [0, \tau]$  a set of positive measure and let

$$e' = \{\tau - t \mid t \in e\}.$$

Then

$$\Phi_{\tau,e} = (\Psi_{\tau,e'}^d)^* \mathbf{J}_\tau, \quad (2.6)$$

where  $(\Psi_{\tau,e'}^d)^* \in \mathcal{L}(L^\infty([0, \tau]; U), X)$  is the dual operator of  $\Psi_{\tau,e'}^d$  and  $\mathbf{J}_\tau$  is the reflection operator on  $L^2([0, \tau]; U)$ , defined by  $\mathbf{J}_\tau u(t) = u(\tau - t)$  (Notice that  $\mathbf{J}_\tau$  is self-adjoint and also unitary.).

In particular, the pair  $(A, B)$  is approximately controllable in time  $\tau$  from a set of positive measure  $e \subset [0, \tau]$  if and only if

$$\text{Ker } \Psi_{\tau,e'}^d = \{0\}.$$

**Remark 2.8.** If  $e = [0, \tau]$  then formula (2.6) simply writes

$$\Phi_\tau = (\Psi_\tau^d)^* \mathbf{J}_\tau. \quad (2.7)$$

### 3 Main results

In this section we extend the maximum principle, well-known for finite dimensional systems, to a class of infinite dimensional systems. To this aim, we continue to use all the notation in Section 2. In particular we denote by  $X$  the state space (assumed to be Hilbert),  $A$  stands for the semigroup generator and the control operator (assumed to be bounded) is still denoted by  $B$ . We also continue to use the notation  $R_t^\infty$ , introduced in (2.2), for the space which is reachable by  $L^\infty$  input functions. Moreover, for every  $t > 0$ , the closed unit ball in  $R_t^\infty$  is:

$$B_t^\infty(1) = \{\Phi_t u, u \in L^\infty([0, t], U), \|u\|_{L^\infty([0, t], U)} \leq 1\}.$$

Before deriving the maximum principle, we note the following existence result for time optimal controls. We skip the proof of this result since it can be obtained by simple adaptation of the proofs in [7, Lemma 3.1.1 and Theorem 3.1.2].

**Proposition 3.1.** *With the above notation and assumptions, assume moreover that  $z_0, z_1 \in X$ ,  $z_0 \neq z_1$  are such that there exists  $t > 0$  with*

$$z_1 - \mathbb{T}_t z_0 \in B_t^\infty(1).$$

*Then there exists  $\tau^*(z_0, z_1) > 0$  such that*

$$\tau^*(z_0, z_1) = \min\{t > 0 \mid z_1 - \mathbb{T}_t z_0 \in B_t^\infty(1)\} > 0 \quad (3.1)$$

In other words  $\tau^*$  is the minimal time in which  $z_0$  can be steered to  $z_1$  by controls satisfying  $\|u(t)\| \leq 1$  for almost every  $t$ . Therefore, any  $u^* \in L^\infty([0, \tau^*], U)$  satisfying

$$z_1 - \mathbb{T}_{\tau^*} z_0 = \Phi_{\tau^*} u^*, \quad (3.2)$$

is called a *time optimal control* for the pair  $(A, B)$ .

Let now give the proof of Theorem 1.4.

*Proof of Theorem 1.4.* The existence of a time optimal control  $u^*$  is given by Proposition 3.1. To prove the maximum principle, we first show, in the spirit of [7], that  $z_1 - \mathbb{T}_{\tau^*} z_0 \in \partial B_{\tau^*}^\infty(1)$ , the boundary being taken in the sense of the topology of  $R_{\tau^*}^\infty$  and hence, according to Corollary 2.4, in the sense of the topology of  $X$ . Assume, by contradiction, that  $z_1 - \mathbb{T}_{\tau^*} z_0 \notin \partial B_{\tau^*}^\infty(1)$ . This implies the existence of  $r \in (0, 1)$  such that  $\frac{1}{r}(z_1 - \mathbb{T}_{\tau^*} z_0) \in B_{\tau^*}^\infty(1)$ , i.e., the existence of  $u_1 \in L^\infty([0, \tau^*], U)$ ,  $\|u_1\|_{L^\infty([0, \tau^*], U)} \leq r < 1$  such that  $z_1 - \mathbb{T}_{\tau^*} z_0 = \Phi_{\tau^*} u_1$ . Let  $t < \tau^*$ . Then

$$\begin{aligned} z_1 - \mathbb{T}_t z_0 &= \mathbb{T}_{\tau^*} z_0 - \mathbb{T}_t z_0 + \Phi_{\tau^*} u_1 = \\ &= \Phi_t u_1 + \mathbb{T}_{\tau^*} z_0 - \mathbb{T}_t z_0 + \int_0^t (\mathbb{T}_{\tau^* - \sigma} - \mathbb{T}_{t - \sigma}) B u_1(\sigma) d\sigma + \int_t^{\tau^*} \mathbb{T}_{\tau^* - \sigma} B u_1(\sigma) d\sigma. \end{aligned}$$

Let

$$\varphi(t, \tau^*) = \mathbb{T}_{\tau^*} z_0 - \mathbb{T}_t z_0 + \int_0^t (\mathbb{T}_{\tau^* - \sigma} - \mathbb{T}_{t - \sigma}) B u_1(\sigma) d\sigma + \int_t^{\tau^*} \mathbb{T}_{\tau^* - \sigma} B u_1(\sigma) d\sigma \quad (t \in (0, \tau^*)),$$

so that

$$z_1 - \mathbb{T}_t z_0 = \Phi_t u_1 + \varphi(t, \tau^*). \quad (3.3)$$

It is easily seen that  $\lim_{t \rightarrow \tau^*} \varphi(t, \tau^*) = 0$  in  $X$ . From Remark 2.5, we have that  $\|\varphi(t, \tau^*)\|_{R_t^\infty} \leq C_t \|\varphi(t, \tau^*)\|_X$ , with  $t \mapsto C_t$  a non-increasing function. This means that for  $t$  close enough to  $\tau^*$  there exists a control  $u_2 \in L^\infty([0, t], U)$  with

$$\|u_2\|_{L^\infty([0, t], U)} \leq 1 - r, \quad \Phi_t u_2 = \varphi(t, \tau^*).$$

The above formula and (3.3) imply, denoting  $u = u_1 + u_2$ , that

$$z_1 - \mathbb{T}_t z_0 = \Phi_t u, \quad \|u\|_{L^\infty([0, t], U)} \leq 1,$$

which contradicts the fact that  $\tau^*$  is minimal in the sense of (3.1). We have thus shown that indeed  $z_1 - \mathbb{T}_{\tau^*} z_0 \in \partial B_{\tau^*}^\infty(1)$ . This implies, using a consequence of the Hahn-Banach theorem (see, for instance, [1, Theorem 1.13]) and the fact that the interior of  $B_{\tau^*}^\infty(1)$  is non empty, that there exists  $\eta \in X$ ,  $\eta \neq 0$  such that

$$\langle \eta, z_1 - \mathbb{T}_{\tau^*} z_0 \rangle_X \geq \langle \eta, x \rangle_X \quad (x \in B_{\tau^*}^\infty(1)).$$

The above formula, combined to the definition (3.2) of the time optimal control implies that

$$\langle \eta, \Phi_{\tau^*} u^* \rangle_X \geq \langle \eta, \Phi_{\tau^*} v \rangle_X \quad (v \in L^\infty([0, \tau^*], U), \quad \|v\|_{L^\infty([0, \tau^*], U)} \leq 1). \quad (3.4)$$

Using next (2.5) and (2.7) it follows that

$$\int_0^{\tau^*} \langle B^* \mathbb{T}_{\tau^*-t}^* \eta, u^*(t) \rangle_U dt \geq \int_0^{\tau^*} \langle B^* \mathbb{T}_{\tau^*-t}^* \eta, v(t) \rangle_U dt \quad (\|v\|_{L^\infty([0, \tau^*], U)} \leq 1). \quad (3.5)$$

Using Lemma 2.2.1 of [7] and the fact that  $t \mapsto B^* \mathbb{T}_{\tau^*-t}^* \eta$  is  $U$ -weakly measurable, we obtain that

$$\sup_{\substack{v \in L^\infty([0, \tau^*], U), \\ \|v\|_{L^\infty([0, \tau^*], U)} \leq 1}} \int_0^{\tau^*} \langle B^* \mathbb{T}_{\tau^*-t}^* \eta, v(t) \rangle_U dt = \int_0^{\tau^*} \|B^* \mathbb{T}_{\tau^*-t}^* \eta\|_U dt,$$

and hence we have

$$\int_0^{\tau^*} \langle B^* \mathbb{T}_{\tau^*-t}^* \eta, u^*(t) \rangle_U dt = \int_0^{\tau^*} \|B^* \mathbb{T}_{\tau^*-t}^* \eta\|_U dt.$$

On the other hand, we clearly have

$$\langle B^* \mathbb{T}_{\tau^*-t}^* \eta, u^*(t) \rangle_U \leq \|B^* \mathbb{T}_{\tau^*-t}^* \eta\|_U \quad (t \in [0, \tau^*] \quad \text{a.e.}).$$

The last estimate and (3.5) imply that

$$\langle B^* \mathbb{T}_{\tau^*-t}^* \eta, u^*(t) \rangle_U = \|B^* \mathbb{T}_{\tau^*-t}^* \eta\|_U \quad (t \in [0, \tau^*] \quad \text{a.e.}),$$

which implies (1.4). □

We conclude this Section by giving the proof of Corollary 1.5.

*Proof of Corollary 1.5.* We know from Theorem 1.4 that there exists  $\eta \in X$ ,  $\eta \neq 0$ , such that we have (1.4). On the other hand, the fact that the pair  $(A, B)$  is approximatively controllable in time  $\tau^*$  from any  $e \in [0, \tau^*]$  of positive measure, together with Proposition 2.7, ensures that  $B^* \mathbb{T}_{\tau^*-t}^* \eta \neq 0$  for almost every  $t \in [0, \tau^*]$ . Therefore, (1.4) implies that the time optimal control is given by

$$u^*(t) = \frac{1}{\|B^* \mathbb{T}_{\tau^*-t}^* \eta\|_U} B^* \mathbb{T}_{\tau^*-t}^* \eta \quad (t \in [0, \tau^*] \quad \text{a.e.}),$$

which clearly implies (1.5).

The uniqueness of  $u^*$  is can be proved in a standard way, using the strict convexity of  $U$  and the bang-bang property of every time optimal control. Indeed, assume by contradiction that there exist two time optimal controls  $u_1^* \neq u_2^*$ . Then for  $u^* = \frac{1}{2}(u_1^* + u_2^*) \in L^\infty([0, \tau^*], U)$ , we have  $\|u^*\|_{L^\infty([0, \tau^*], U)} \leq 1$ ,  $\|u^*(t)\|_U < 1$  for  $t$  in some set of positive measure and  $\Phi_{\tau^*} u^* = z_1 - \mathbb{T}_{\tau^*} z_0$ . Thus  $u^*$  is a time optimal control which does not satisfy the bang-bang property, so that we have obtained a contradiction. □



## 4 Application to the Schrödinger equation

We first give a preliminary lemma for which, although the result seems quite simple, we did not find the exact statement in the literature. The idea of this lemma, using a unique continuation result of Privalov, is borrowed from Reifler and Vogt [19] (see also Hante, Sigalotti and Tucsnak [8])

**Lemma 4.1.** *Let  $I \subset \mathbb{Z}$ ,  $(\lambda_n)_{n \in I}$  be a sequence of real numbers bounded from below (or from above), let  $e \subset \mathbb{R}$  a bounded set of positive measure and let  $(a_n)_{n \in I} \in \ell^1(I, \mathbb{C})$ . Assume that*

$$\sum_{n \in I} a_n e^{i\lambda_n t} = 0 \quad (t \in e).$$

Then

$$\sum_{n \in I} a_n e^{i\lambda_n t} = 0 \quad (t \in \mathbb{R}).$$

*Proof.* Let  $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \text{Im}s > 0\}$ . We define the map

$$\begin{aligned} f: \mathbb{C}_+ &\rightarrow \mathbb{C} \\ s &\mapsto \sum_{n \in I} a_n e^{i\lambda_n s} \end{aligned}$$

Since  $(a_n)$  is in  $\ell^1$  and  $(\lambda_n)$  is bounded from below, we clearly have that  $f$  is complex analytic on  $\mathbb{C}_+$  and continuous up to the boundary (i.e., up to the real axis). Using the fact that  $f$  vanishes on  $e \subset \mathbb{R}$  of positive measure and Privalov's uniqueness Theorem, [22, Vol. II, ch. XIV, Theorem 1.9], we obtain that  $f$  vanishes on  $\overline{\mathbb{C}_+}$ , thus on  $\mathbb{R}$ .  $\square$

From the above Lemma, we can derive the following result:

**Lemma 4.2.** *Let  $X$  and  $Y$  be Hilbert spaces, let  $A_0$  be a self-adjoint and diagonalisable operator on  $X$  with domain  $\mathcal{D}(A_0)$  and let  $C \in \mathcal{L}(X, Y)$ . Moreover, assume that the spectrum  $\sigma(A_0)$  of  $A_0$  satisfies  $\sigma(A_0) \subset [m, \infty)$  for some  $m \in \mathbb{R}$ . Let  $z_0 \in X$  and  $z \in C^0(\mathbb{R}, X)$ ,  $y \in C^0(\mathbb{R}, Y)$  satisfying*

$$\dot{z}(t) = iA_0 z(t) \quad (t \in \mathbb{R}), \quad z(0) = z_0, \quad (4.1)$$

$$y(t) = Cz(t) \quad (t \in \mathbb{R}). \quad (4.2)$$

If

$$y(t) = 0 \quad (t \in e),$$

for some set  $e \subset \mathbb{R}$  of positive measure, then

$$y(t) = 0 \quad (t \in \mathbb{R}).$$

*Proof.* Since  $A_0$  is assumed to be self-adjoint and diagonalisable, there exists an orthonormal basis of eigenvectors  $(\varphi_n)_{n \in I}$  of  $A_0$ , with  $I \subset \mathbb{Z}$  and a corresponding sequence of eigenvalues  $(\lambda_n)_{n \in I}$ , where  $(\lambda_n)_{n \in I}$  is a sequence with values in  $[m, \infty)$ . The solution  $z$  of (4.1) writes

$$z(t) = \sum_{n \in I} a_n e^{i\lambda_n t} \varphi_n \quad (t \in \mathbb{R}),$$

where  $a_n = \langle z_0, \varphi_n \rangle_X$  for every  $n \in I$ . Hence  $y$  in (4.2) writes

$$y(t) = \sum_{n \in I} a_n e^{i\lambda_n t} C \varphi_n \quad (t \in \mathbb{R}).$$

For any  $v \in Y$ , we have

$$\langle y(t), v \rangle_Y = \sum_{n \in I} a_n e^{i\lambda_n t} \langle C \varphi_n, v \rangle_Y = \sum_{n \in I} a_n e^{i\lambda_n t} \langle C^* v, \varphi_n \rangle_X \quad (t \in \mathbb{R}).$$

Since  $(a_n)_n \in \ell^2(I, \mathbb{C})$  and  $(\langle C^* v, \varphi_n \rangle_X)_n \in \ell^2(I, \mathbb{C})$ , it follows that  $(a_n \langle C^* v, \varphi_n \rangle_X)_n \in \ell^1(I, \mathbb{C})$ . Moreover, we know that  $\langle y(t), v \rangle_Y = 0$  for every  $t \in e$  and that the sequence  $(\lambda_n)_{n \in I}$  is bounded from below, hence, using Lemma 4.1, we deduce that  $\langle y(t), v \rangle_Y = 0$  for every  $t \in \mathbb{R}$ . Since this property is true for every  $v \in Y$ , we conclude that  $y(t) = 0$  for every  $t \in \mathbb{R}$ .  $\square$

We next study a Schrödinger equation with distributed internal control. More precisely, we consider the system

$$\dot{z}(x, t) = -i\Delta z(x, t) + ia(x)z(x, t) + u(x, t)\chi_{\mathcal{O}}(x) \quad (x \in \Omega, \quad t \geq 0), \quad (4.3)$$

$$z(x, t) = 0 \quad (x \in \partial\Omega, \quad t \geq 0), \quad (4.4)$$

where  $\Omega \subset \mathbb{R}^n$  is an open set,  $\mathcal{O}$  is an open subset of  $\Omega$ ,  $\chi_{\mathcal{O}}$  is the characteristic function of  $\mathcal{O}$  and the control  $u$  is such that  $\|u(\cdot, t)\|_{L^2(\mathcal{O})} \leq 1$  for almost every  $t \geq 0$ .

**Proposition 4.3.** *Assume that one of the assumptions*

1. *The open set  $\Omega$  is bounded,  $\partial\Omega$  is of class  $C^2$ ,  $a \in L^\infty(\Omega, \mathbb{R})$  and  $\mathcal{O}$  satisfies the geometric optics condition, as described in Bardos, Lebeau and Rauch [2];*
2. *The open set  $\Omega$  is a rectangular domain,  $a$  is a constant and  $\mathcal{O}$  is an arbitrary nonempty open subset of  $\Omega$ ,*

*holds. Then for every  $z_0, z_1 \in L^2(\Omega)$ , with  $z_0 \neq z_1$ , there exists a unique time optimal control  $u^*$  steering the solution of (4.3)-(4.4) from  $z_0$  (at  $t = 0$ ) to  $z_1$  (at  $t = \tau^* = \tau^*(z_0, z_1)$ ). Moreover, there exists  $\eta \in L^2(\Omega)$ ,  $\eta \neq 0$ , such that*

$$\int_{\mathcal{O}} w(x, t) \overline{u^*}(x, t) dx = \max_{\substack{v \in L^2(\mathcal{O}), \\ \|v\|_{L^2(\mathcal{O})} \leq 1}} \int_{\mathcal{O}} w(x, t) \overline{v}(x) dx \quad (t \in [0, \tau^*] \quad \text{a.e.}), \quad (4.5)$$

where  $w$  is the solution of the adjoint problem

$$\dot{w}(x, t) = -i\Delta w(x, t) + i\overline{a}(x)w(x, t) \quad (x \in \Omega, \quad t \geq 0), \quad (4.6)$$

$$w(x, t) = 0 \quad (x \in \partial\Omega, \quad t \geq 0), \quad (4.7)$$

$$w(x, \tau^*) = \eta(x) \quad (x \in \Omega). \quad (4.8)$$

Finally,  $u^*$  has the bang-bang property

$$\|u^*(\cdot, t)\| = 1 \quad (t \in [0, \tau^*] \quad \text{a.e.}). \quad (4.9)$$

*Proof.* Note first that, in the case in which the first assumption holds, it suffices to consider the case  $a \geq 0$ . Similarly, when the second assumption holds, it suffices to consider the case  $a = 0$ . Indeed, the case of an arbitrary  $a \in L^\infty(\Omega, \mathbb{R})$  (respectively an arbitrary  $a \in \mathbb{C}$ ) can be reduced to the case  $a(x) \geq 0$  (respectively  $a = 0$ ) by an obvious change of variables.

The result is a consequence of Theorem 1.4 and Corollary 1.5, with a particular choice of spaces and operators. More precisely let  $X = L^2(\Omega)$ ,  $U = L^2(\mathcal{O})$  and let the operator  $A$  be defined by

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad (4.10)$$

$$A\varphi = -i\Delta\varphi + ia(x)\varphi \quad (\varphi \in \mathcal{D}(A)). \quad (4.11)$$

The operator  $A$  is clearly skew-adjoint and diagonalisable.

The control operator  $B \in \mathcal{L}(U, X)$  is defined by

$$Bu = \chi_{\mathcal{O}}u \quad (u \in L^2(\mathcal{O})).$$

We note that, with the above defined spaces of operators, the system (4.3), (4.4) writes in the form  $\dot{z} = Az + Bu$ . Moreover, in order to apply Theorem 1.4, it suffices to check that the pair  $(A, B)$  is exactly controllable in any time  $\tau > 0$ . Under the first assumption of the present proposition, this fact is classical (it suffices, for instance, to combine Remark 7.4.4 and Theorem 6.7.2 from [20]). Under the second assumption in the present proposition, the exact controllability in any time of  $(A, B)$  has been proved in Jaffard [9] and Komornik [10]. Consequently, we can apply Theorem 1.4, so that we obtain (4.5).

In order to apply Corollary 1.5, we have to check that  $(A^*, B^*)$  is approximatively observable from any set of positive measure  $e \subset [0, \tau^*]$ . In PDE terms this means that we have to show that if a solution of (4.6) is such that

$$w(x, t) = 0 \quad (x \in \mathcal{O}, t \in e),$$

then

$$w(x, t) = 0 \quad (x \in \Omega, t \in \mathbb{R}).$$

To accomplish this goal, we note that, due to the fact that  $a \in L^\infty(\Omega)$ , the operator  $A$  in (4.10), (4.11) writes  $A = iA_0$  with  $A_0$  self-adjoint, diagonalisable and with a spectrum bounded from below. Hence, we can apply Lemma 4.2 to obtain that  $w(x, t) = 0$  for  $x \in \mathcal{O}$  and  $t \in \mathbb{R}$ . Consequently, using again the exact controllability in any time of the pair  $(A, B)$  (thus the exact observability in any time of  $(A^*, B^*)$ ), it follows that  $w(x, t) = 0$  for  $x \in \Omega$  and  $t \in \mathbb{R}$ . Finally, we can apply Corollary 1.5 to obtain (4.9).  $\square$

## 5 Concluding remarks and open questions

Our results in the previous section are, as far as we know, the first establishing the maximum principle and the bang-bang property for the time optimal controls in the case of a class of time reversible systems (of Schrödinger type) with controls localized in a strict subset of the spatial domain. If we take controls active in the whole spatial domain, one can easily adapt our methods to the wave or Euler-Bernoulli plate equation. More generally, the results in the previous can be easily adapted to systems of the form

$$\ddot{w} + A_0w = u, \quad (5.12)$$

where  $A_0$  is a positive operator in a Hilbert space  $H$  and  $u$  is the input function. We can easily check that the above equation can be written as a first order system satisfying the assumptions in Theorem 1.4 and Corollary 1.5, so that we have the maximum principle and the bang-bang property for the associated time optimal control problem. Since this question has been tackled directly, by quite similar methods, in Fattorini [6] and Krabs [11] we do not give the detailed argument here. Note that, taking  $H = L^2(\Omega)$ , where  $\Omega \subset \mathbb{R}^n$  is bounded and with a smooth boundary, and  $A_0$  to be the Dirichlet Laplacian (respectively the square of the Dirichlet Laplacian) in  $\Omega$ , (5.12) becomes the wave (respectively the Euler-Bernoulli plate) equation, with control active in all of  $\Omega$ . We also refer to Kunisch and Wachsmuth [12] for a theoretical and numerical study of the time optimal control problem for the wave equation.

An interesting and partially open question consists in obtaining the maximum principle and the bang-bang property of time optimal controls for systems governed by the Euler-Bernoulli plate equation with localized distributed control. The question is challenging, in particular, since most of the control theoretic results known for the Schrödinger equation have a natural counterpart for the Euler-Bernoulli plate equation (at least for hinged boundary conditions). It turns out that the situation seems unclear in the case of time optimal control problems, at least as far as the bang-bang property is concerned. More precisely, consider the one dimensional case (Euler-Bernoulli beam equation) with locally distributed control. The main technical obstacle (within our method) in obtaining the bang-bang property of time optimal controls consists in the fact that, in order to apply Corollary 1.5, one would need a counterpart of the uniqueness result in Lemma 4.1, with the sequence  $(\pm n^2)_{n \geq 1}$  (which is bounded neither from below nor from above) instead of  $(\lambda_n)$ . Therefore, even in one space dimension, establishing the uniqueness of time optimal controls and their bang-bang property are open questions.

However, we have a partial result, asserting that the maximum principle holds for the time optimal control problem for a system governed by the Euler-Bernoulli plate equation with locally distributed control. More precisely, consider the system

$$\ddot{z}(x, t) = -\Delta^2 z(x, t) + \chi_{\mathcal{O}}(x)u(x, t) \quad (x \in \Omega, t \geq 0), \quad (5.13)$$

$$z(x, t) = \Delta z(x, t) = 0 \quad (x \in \partial\Omega, t \geq 0), \quad (5.14)$$

where  $\Omega \subset \mathbb{R}^n$  is an open set,  $\mathcal{O}$  is an open subset of  $\Omega$ ,  $\chi_{\mathcal{O}}$  is the characteristic function of  $\mathcal{O}$  and the control  $u$  is such that  $\|u(\cdot, t)\|_{L^2(\mathcal{O})} \leq 1$  for almost every  $t \geq 0$ .

**Proposition 5.1.** *Assume that one of the assumptions*

1. *The open set  $\Omega$  is bounded,  $\partial\Omega$  is of class  $C^2$  and  $\mathcal{O}$  satisfies the geometric optics condition;*
2. *The open set  $\Omega$  is a rectangular domain and  $\mathcal{O}$  is an arbitrary nonempty open subset of  $\Omega$ ,*

*holds. Then for every  $f_0, f_1 \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $g_0, g_1 \in L^2(\Omega)$ , with  $(f_0, g_0) \neq (f_1, g_1)$ , there exists a time optimal control  $u^*$  steering the solution of (5.13)-(5.14) from  $(f_0, g_0)$  (at  $t = 0$ ) to  $(f_1, g_1)$  (at  $t = \tau^*$ ). Moreover, there exists  $(\eta_0, \eta_1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$ , with  $(\eta_0, \eta_1) \neq (0, 0)$  such that*

$$\int_{\mathcal{O}} \dot{w}(x, t) \overline{u^*}(x, t) dx = \max_{\substack{v \in L^2(\mathcal{O}), \\ \|v\|_{L^2(\mathcal{O})} \leq 1}} \int_{\mathcal{O}} \dot{w}(x, t) \overline{v}(x) dx, \quad (5.15)$$

where  $w$  is the solution of the adjoint problem,

$$\ddot{w}(x, t) = -\Delta^2 w(x, t) \quad (x \in \Omega, t \in \mathbb{R}), \quad (5.16)$$

$$w(x, t) = \Delta w(x, t) = 0 \quad (x \in \partial\Omega, t \in \mathbb{R}), \quad (5.17)$$

$$w(x, \tau^*) = \eta_0(x), \quad \dot{w}(x, \tau^*) = \eta_1(x) \quad (x \in \Omega). \quad (5.18)$$

*Proof.* The proof follows closely the reasoning used in the first part of the proof of Proposition 4.3. More precisely, it suffices to apply Theorem 1.4, with a particular choice of spaces and operators. Take  $X = [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$ ,  $U = L^2(\mathcal{O})$  and consider the operator  $A$  defined by

$$\mathcal{D}(A) = \{f \in H^4(\Omega) \cap H_0^1(\Omega), \Delta f = 0 \text{ on } \partial\Omega\} \times (H^2(\Omega) \cap H_0^1(\Omega)),$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Delta^2 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -\Delta^2 f \end{bmatrix} \quad \left( \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A) \right).$$

The control operator  $B \in \mathcal{L}(U, X)$  is defined by

$$Bu = \begin{bmatrix} 0 \\ \chi \mathcal{O}u \end{bmatrix} \quad (u \in L^2(\mathcal{O})).$$

We note that, with the above defined spaces of operators, the system (5.13), (5.14) writes in the form  $\dot{Z} = AZ + Bu$ , where  $Z = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}$ .

In order to apply Theorem 1.4, it suffices to check that the pair  $(A, B)$  is exactly controllable in any time  $\tau > 0$ . This fact is classical and follows from known (already used above) exact controllability results for the Schrödinger equation (see, for instance, [20, Proposition 7.5.1] or [14, Section 5]). Consequently, we can apply Theorem 1.4, so that we obtain (5.15).  $\square$

Another possible extension of the results in this work consists in studying the case of unbounded control operators. It is not difficult to see that our result in Theorem 1.4 can be extended to admissible (possibly unbounded) control operators, provided that we replace the exact controllability assumption with the condition  $R_t^\infty = X$  for every  $t > 0$ . Note that, for unbounded control operators, this condition is no longer a consequence of the exact controllability in arbitrarily small time. However, we did not include this generalization in our work for two reasons. Firstly, this would require more preliminaries and notation. Secondly, and more importantly, we found no relevant examples coming from PDE's with boundary control satisfying the condition  $R_t^\infty = X$  for every  $t > 0$ . Checking this condition is an open question, even for very simple examples such as the Schrödinger equation in one space dimension with Dirichlet boundary control.

Besides the theoretical open questions raised above, the numerical analysis of the time optimal control problem for infinite dimensional systems is an essentially open field, at least for point targets. An interesting track to tackle these issues could consist in adapting some of the results on the numerical approximation of norm optimal controls (see, for instance, Ervedoza and Zuazua [4] and references therein).

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