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# The Energy-Momentum tensor on low dimensional $\text{Spin}^c$ manifolds

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On a compact surface endowed with any  $\text{Spin}^c$  structure, we give a formula involving the Energy-Momentum tensor in terms of geometric quantities. A new proof of a Bär-type inequality for the eigenvalues of the Dirac operator is given. The round sphere  $\mathbb{S}^2$  with its canonical  $\text{Spin}^c$  structure satisfies the limiting case. Finally, we give a spinorial characterization of immersed surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  by solutions of the generalized Killing spinor equation associated with the induced  $\text{Spin}^c$  structure on  $\mathbb{S}^2 \times \mathbb{R}$ .

*Keywords:*  $\text{Spin}^c$  structures, Dirac operator, eigenvalues, Energy-Momentum tensor, compact surfaces, isometric immersions.

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## 1 Introduction

On a compact Spin surface, Th. Friedrich and E.C. Kim proved that any eigenvalue  $\lambda$  of the Dirac operator satisfies the equality [9, Thm. 4.5]:

$$\lambda^2 = \frac{\pi\chi(M)}{\text{Area}(M)} + \frac{1}{\text{Area}(M)} \int_M |T^\psi|^2 v_g, \quad (1.1)$$

where  $\chi(M)$  is the Euler-Poincaré characteristic of  $M$  and  $T^\psi$  is the field of quadratic forms called the Energy-Momentum tensor. It is given on the complement set of zeroes of the eigenspinor  $\psi$  by

$$T^\psi(X, Y) = g(\ell^\psi(X), Y) = \frac{1}{2} \text{Re} \left( X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right),$$

for every  $X, Y \in \Gamma(TM)$ . Here  $\ell^\psi$  is the field of symmetric endomorphisms associated with the field of quadratic forms  $T^\psi$ . We should point out that since

$\psi$  is an eigenspinor, the zero set is discret [3]. The proof of Equality (1.1) relies mainly on a local expression of the covariant derivative of  $\psi$  and the use of the Schrödinger-Lichnerowicz formula. This equality has many direct consequences. First, since the trace of  $\ell^\psi$  is equal to  $\lambda$ , we have by the Cauchy-Schwarz inequality that  $|\ell^\psi|^2 \geq \frac{(tr(\ell^\psi))^2}{n} = \frac{\lambda^2}{2}$ , where  $tr$  denotes the trace of  $\ell^\psi$ . Hence, Equality (1.1) implies the Bär inequality [2] given by

$$\lambda^2 \geq \lambda_1^2 := \frac{2\pi\chi(M)}{Area(M)}. \quad (1.2)$$

Moreover, from Equality (1.1), Th. Friedrich and E.C. Kim [9] deduced that  $\int_M \det(T^\psi)v_g = \pi\chi(M)$ , which gives an information on the Energy-Momentum tensor without knowing the eigenspinor nor the eigenvalue. Finally, for any closed surface  $M$  in  $\mathbb{R}^3$  of constant mean curvature  $H$ , the restriction to  $M$  of a parallel spinor on  $\mathbb{R}^3$  is a generalized Killing spinor of eigenvalue  $-H$  with Energy-Momentum tensor equal to the Weingarten tensor  $II$  (up to the factor  $-\frac{1}{2}$ ) [21] and we have by (1.1)

$$H^2 = \frac{\pi\chi(M)}{Area(M)} + \frac{1}{4Area(M)} \int_M |II|^2 v_g.$$

Indeed, given any surface  $M$  carrying such a spinor field, Th. Friedrich [8] showed that the Energy-Momentum tensor associated with this spinor satisfies the Gauss-Codazzi equations and hence  $M$  is locally immersed into  $\mathbb{R}^3$ .

Having a  $\text{Spin}^c$  structure on manifolds is a weaker condition than having a  $\text{Spin}$  structure because every  $\text{Spin}$  manifold has a trivial  $\text{Spin}^c$  structure. Additionally, any compact surface or any product of a compact surface with  $\mathbb{R}$  has a  $\text{Spin}^c$  structure carrying particular spinors. In the same spirit as in [14], when using a suitable conformal change, the second author [23] established a Bär-type inequality for the eigenvalues of the Dirac operator on a compact surface endowed with any  $\text{Spin}^c$  structure. In fact, any eigenvalue  $\lambda$  of the Dirac operator satisfies

$$\lambda^2 \geq \lambda_1^2 := \frac{2\pi\chi(M)}{Area(M)} - \frac{1}{Area(M)} \int_M |\Omega| v_g, \quad (1.3)$$

where  $i\Omega$  is the curvature form of the connection on the line bundle given by the  $\text{Spin}^c$  structure. Equality is achieved if and only if the eigenspinor  $\psi$  associated with the first eigenvalue  $\lambda_1$  is a Killing  $\text{Spin}^c$  spinor, i.e., for every  $X \in \Gamma(TM)$  the eigenspinor  $\psi$  satisfies

$$\begin{cases} \nabla_X \psi = -\frac{\lambda_1}{2} X \cdot \psi, \\ \Omega \cdot \psi = i|\Omega|\psi. \end{cases} \quad (1.4)$$

Here  $X \cdot \psi$  denotes the Clifford multiplication and  $\nabla$  the spinorial Levi-Civita connection [7].

Studying the Energy-Momentum tensor on a compact Riemannian  $\text{Spin}$  or  $\text{Spin}^c$  manifolds has been done by many authors, since it is related to several geometric situations. Indeed, on compact  $\text{Spin}$  manifolds, J.P. Bourguignon and P. Gauduchon [5] proved that the Energy-Momentum tensor appears naturally

in the study of the variations of the spectrum of the Dirac operator. Th. Friedrich and E.C. Kim [10] obtained the Einstein-Dirac equation as the Euler-Lagrange equation of a certain functional. The second author extended these last two results to  $\text{Spin}^c$  manifolds [24]. Even if it is not a computable geometric invariant, the Energy-Momentum tensor is, up to a constant, the second fundamental form of an isometric immersion into a  $\text{Spin}$  or  $\text{Spin}^c$  manifold carrying a parallel spinor [21, 24]. For a better understanding of the tensor  $q^\varphi$  associated with a spinor field  $\varphi$ , the first author [12] studied Riemannian flows and proved that, if the normal bundle carries a parallel spinor  $\psi$ , the tensor  $q^\varphi$  associated with  $\varphi$  (the restriction of  $\psi$  to the flow) is the O'Neill tensor of the flow.

In this paper, we give a formula corresponding to (1.1) for any eigenspinor  $\psi$  of the square of the Dirac operator on compact surfaces endowed with any  $\text{Spin}^c$  structure (see Theorem 3.1). It is motivated by the following two facts: First, when we consider eigenvalues of the square of the Dirac operator, another tensor field is of interest. It is the skew-symmetric tensor field  $Q^\psi$  given by

$$Q^\psi(X, Y) = g(q^\psi(X), Y) = \frac{1}{2} \text{Re} \left( X \cdot \nabla_Y \psi - Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right),$$

for all vector fields  $X, Y \in \Gamma(TM)$ . This tensor was studied by the first author in the context of Riemannian flows [12]. Second, we consider any compact surface  $M$  immersed in  $\mathbb{S}^2 \times \mathbb{R}$  where  $\mathbb{S}^2$  is the round sphere equipped with a metric of curvature one. The  $\text{Spin}^c$  structure on  $\mathbb{S}^2 \times \mathbb{R}$ , induced from the canonical one on  $\mathbb{S}^2$  and the  $\text{Spin}$  structure on  $\mathbb{R}$ , admits a parallel spinor [22]. The restriction to  $M$  of this  $\text{Spin}^c$  structure is still a  $\text{Spin}^c$  structure with a generalized Killing spinor [24].

In Section 2, we recall some basic facts on  $\text{Spin}^c$  structures and the restrictions of these structures to hypersurfaces. In Section 3 and after giving a formula corresponding to (1.1) for any eigenspinor  $\psi$  of the square of the Dirac operator, we deduce a formula for the integral of the determinant of  $T^\psi + Q^\psi$  and we establish a new proof of the Bär-type inequality (1.3). In Section 4, we consider the 3-dimensional case and treat examples of hypersurfaces in  $\mathbb{C}\mathbb{P}^2$ . In the last section, we come back to the question of a spinorial characterisation of surfaces in  $\mathbb{S}^2 \times \mathbb{R}$ . Here we use a different approach than the one in [25]. In fact, we prove that given any surface  $M$  carrying a generalized Killing spinor associated with a particular  $\text{Spin}^c$  structure, the Energy-Momentum tensor satisfies the four compatibility equations stated by B. Daniel [6]. Thus there exists a local immersion of  $M$  into  $\mathbb{S}^2 \times \mathbb{R}$ .

## 2 Preliminaries

In this section, we begin with some preliminaries concerning  $\text{Spin}^c$  structures and the Dirac operator. Details can be found in [18], [20], [7], [23] and [24].

**The Dirac operator on  $\text{Spin}^c$  manifolds:** Let  $(M^n, g)$  be a Riemannian manifold of dimension  $n \geq 2$  without boundary. We denote by  $\text{SOM}$

the  $\mathrm{SO}_n$ -principal bundle over  $M$  of positively oriented orthonormal frames. A  $\mathrm{Spin}^c$  structure of  $M$  is a  $\mathrm{Spin}_n^c$ -principal bundle  $(\mathrm{Spin}^c M, \pi, M)$  and an  $\mathbb{S}^1$ -principal bundle  $(\mathbb{S}^1 M, \pi, M)$  together with a double covering given by  $\theta : \mathrm{Spin}^c M \rightarrow \mathrm{SOM} \times_M \mathbb{S}^1 M$  such that  $\theta(ua) = \theta(u)\xi(a)$ , for every  $u \in \mathrm{Spin}^c M$  and  $a \in \mathrm{Spin}_n^c$ , where  $\xi$  is the 2-fold covering of  $\mathrm{Spin}_n^c$  over  $\mathrm{SO}_n \times \mathbb{S}^1$ . Let  $\Sigma M := \mathrm{Spin}^c M \times_{\rho_n} \Sigma_n$  be the associated spinor bundle where  $\Sigma_n = \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$  and  $\rho_n : \mathrm{Spin}_n^c \rightarrow \mathrm{End}(\Sigma_n)$  denotes the complex spinor representation. A section of  $\Sigma M$  will be called a spinor field. The spinor bundle  $\Sigma M$  is equipped with a natural Hermitian scalar product denoted by  $(\cdot, \cdot)$ . We define an  $L^2$ -scalar product  $\langle \psi, \varphi \rangle = \int_M (\psi, \varphi) v_g$ , for any spinors  $\psi$  and  $\varphi$ . Additionally, any connection 1-form  $A : T(\mathbb{S}^1 M) \rightarrow i\mathbb{R}$  on  $\mathbb{S}^1 M$  and the connection 1-form  $\omega^M$  on  $\mathrm{SOM}$ , induce a connection on the principal bundle  $\mathrm{SOM} \times_M \mathbb{S}^1 M$ , and hence a covariant derivative  $\nabla$  on  $\Gamma(\Sigma M)$  [7, 24]. The curvature of  $A$  is an imaginary valued 2-form denoted by  $F_A = dA$ , i.e.,  $F_A = i\Omega$ , where  $\Omega$  is a real valued 2-form on  $\mathbb{S}^1 M$ . We know that  $\Omega$  can be viewed as a real valued 2-form on  $M$  [7, 17]. In this case  $i\Omega$  is the curvature form of the associated line bundle  $L$ . It is the complex line bundle associated with the  $\mathbb{S}^1$ -principal bundle via the standard representation of the unit circle. For every spinor  $\psi$ , the Dirac operator is locally defined by

$$D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi,$$

where  $(e_1, \dots, e_n)$  is a local oriented orthonormal tangent frame and “ $\cdot$ ” denotes the Clifford multiplication. The Dirac operator is an elliptic, self-adjoint operator with respect to the  $L^2$ -scalar product and verifies, for any spinor field  $\psi$ , the Schrödinger-Lichnerowicz formula

$$D^2\psi = \nabla^* \nabla \psi + \frac{1}{4} S\psi + \frac{i}{2} \Omega \cdot \psi \quad (2.1)$$

where  $\Omega \cdot$  is the extension of the Clifford multiplication to differential forms given by  $(e_i^* \wedge e_j^*) \cdot \psi = e_i \cdot e_j \cdot \psi$ . For any spinor  $\psi \in \Gamma(\Sigma M)$ , we have [13]

$$(i\Omega \cdot \psi, \psi) \geq -\frac{c_n}{2} |\Omega|_g |\psi|^2, \quad (2.2)$$

where  $|\Omega|_g$  is the norm of  $\Omega$ , with respect to  $g$  given by  $|\Omega|_g^2 = \sum_{i < j} (\Omega_{ij})^2$  in any orthonormal local frame and  $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$ . Moreover, equality holds in (2.2) if and only if  $\Omega \cdot \psi = i\frac{c_n}{2} |\Omega|_g \psi$ .

Every Spin manifold has a trivial  $\mathrm{Spin}^c$  structure [7, 19]. In fact, we choose the trivial line bundle with the trivial connection whose curvature  $i\Omega$  is zero. Also every Kähler manifold  $M$  of complex dimension  $m$  has a canonical  $\mathrm{Spin}^c$  structure. Let  $\times$  by the Kähler form defined by the complex structure  $J$ , i.e.  $\times(X, Y) = g(JX, Y)$  for all vector fields  $X, Y \in \Gamma(TM)$ . The complexified cotangent bundle

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0} M \oplus \Lambda^{0,1} M$$

decomposes into the  $\pm i$ -eigenbundles of the complex linear extension of the complex structure. Thus, the spinor bundle of the canonical  $\mathrm{Spin}^c$  structure is given by

$$\Sigma M = \Lambda^{0,*} M = \bigoplus_{r=0}^m \Lambda^{0,r} M,$$

where  $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$  is the bundle of  $r$ -forms of type  $(0,1)$ . The line bundle of this canonical  $\text{Spin}^c$  structure is given by  $L = (K_M)^{-1} = \Lambda^m(\Lambda^{0,1}M)$ , where  $K_M$  is the canonical bundle of  $M$  [7, 19]. This line bundle  $L$  has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by  $i\Omega = -i\rho$ , where  $\rho$  is the Ricci form given by  $\rho(X, Y) = \text{Ric}(JX, Y)$ . We point out that the canonical  $\text{Spin}^c$  structure on every Kähler manifold carries a parallel spinor [7, 22].

**Spin<sup>c</sup> hypersurfaces and the Gauss formula:** Let  $\mathcal{Z}$  be an oriented  $(n+1)$ -dimensional Riemannian  $\text{Spin}^c$  manifold and  $M \subset \mathcal{Z}$  be an oriented hypersurface. The manifold  $M$  inherits a  $\text{Spin}^c$  structure induced from the one on  $\mathcal{Z}$ , and we have [24]

$$\Sigma M \simeq \begin{cases} \Sigma \mathcal{Z}|_M & \text{if } n \text{ is even,} \\ \Sigma^+ \mathcal{Z}|_M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field  $X$ , tangent to  $M$ , is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)|_M, \quad (2.3)$$

where  $\psi \in \Gamma(\Sigma \mathcal{Z})$  (or  $\psi \in \Gamma(\Sigma^+ \mathcal{Z})$  if  $n$  is odd),  $\varphi$  is the restriction of  $\psi$  to  $M$ , “ $\cdot$ ” is the Clifford multiplication on  $\mathcal{Z}$ , “ $\bullet$ ” that on  $M$  and  $\nu$  is the unit normal vector. The connection 1-form defined on the restricted  $\mathbb{S}^1$ -principal bundle  $(P_{\mathbb{S}^1}M := P_{\mathbb{S}^1}\mathcal{Z}|_M, \pi, M)$ , is given by  $A = A^{\mathcal{Z}}|_M : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})|_M \rightarrow i\mathbb{R}$ . Then the curvature 2-form  $i\Omega$  on the  $\mathbb{S}^1$ -principal bundle  $P_{\mathbb{S}^1}M$  is given by  $i\Omega = i\Omega^{\mathcal{Z}}|_M$ , which can be viewed as an imaginary 2-form on  $M$  and hence as the curvature form of the line bundle  $L^M$ , the restriction of the line bundle  $L^{\mathcal{Z}}$  to  $M$ . For every  $\psi \in \Gamma(\Sigma \mathcal{Z})$  ( $\psi \in \Gamma(\Sigma^+ \mathcal{Z})$  if  $n$  is odd), the real 2-forms  $\Omega$  and  $\Omega^{\mathcal{Z}}$  are related by [24]

$$(\Omega^{\mathcal{Z}} \cdot \psi)|_M = \Omega \bullet \varphi - (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi. \quad (2.4)$$

We denote by  $\nabla^{\Sigma \mathcal{Z}}$  the spinorial Levi-Civita connection on  $\Sigma \mathcal{Z}$  and by  $\nabla$  that on  $\Sigma M$ . For all  $X \in \Gamma(TM)$ , we have the spinorial Gauss formula [24]:

$$(\nabla_X^{\Sigma \mathcal{Z}} \psi)|_M = \nabla_X \varphi + \frac{1}{2} II(X) \bullet \varphi, \quad (2.5)$$

where  $II$  denotes the Weingarten map of the hypersurface. Moreover, Let  $D^{\mathcal{Z}}$  and  $D^M$  be the Dirac operators on  $\mathcal{Z}$  and  $M$ , after denoting by the same symbol any spinor and its restriction to  $M$ , we have

$$\nu \cdot D^{\mathcal{Z}} \varphi = \tilde{D} \varphi + \frac{n}{2} H \varphi - \nabla_{\nu}^{\Sigma \mathcal{Z}} \varphi, \quad (2.6)$$

where  $H = \frac{1}{n} \text{tr}(II)$  denotes the mean curvature and  $\tilde{D} = D^M$  if  $n$  is even and  $\tilde{D} = D^M \oplus (-D^M)$  if  $n$  is odd.

### 3 The 2-dimensional case

In this section, we consider compact surfaces endowed with any  $\text{Spin}^c$  structure. We have

**Theorem 3.1** *Let  $(M^2, g)$  be a Riemannian manifold and  $\psi$  an eigenspinor of the square of the Dirac operator  $D^2$  with eigenvalue  $\lambda^2$  associated with any  $\text{Spin}^c$  structure. Then we have*

$$\lambda^2 = \frac{S}{4} + |T^\psi|^2 + |Q^\psi|^2 + \Delta f + |Y|^2 - 2Y(f) + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right),$$

where  $f$  is the real-valued function defined by  $f = \frac{1}{2}\ln|\psi|^2$  and  $Y$  is a vector field on  $TM$  given by  $g(Y, Z) = \frac{1}{|\psi|^2}\text{Re}(D\psi, Z \cdot \psi)$  for any  $Z \in \Gamma(TM)$ .

**Proof.** Let  $\{e_1, e_2\}$  be an orthonormal frame of  $TM$ . Since the spinor bundle  $\Sigma M$  is of real dimension 4, the set  $\{\frac{\psi}{|\psi|}, \frac{e_1 \cdot \psi}{|\psi|}, \frac{e_2 \cdot \psi}{|\psi|}, \frac{e_1 \cdot e_2 \cdot \psi}{|\psi|}\}$  is orthonormal with respect to the real product  $\text{Re}(\cdot, \cdot)$ . The covariant derivative of  $\psi$  can be expressed in this frame as

$$\nabla_X \psi = \delta(X)\psi + \alpha(X) \cdot \psi + \beta(X)e_1 \cdot e_2 \cdot \psi, \quad (3.1)$$

for all vector fields  $X$ , where  $\delta$  and  $\beta$  are 1-forms and  $\alpha$  is a  $(1, 1)$ -tensor field. Moreover  $\beta$ ,  $\delta$  and  $\alpha$  are uniquely determined by the spinor  $\psi$ . In fact, taking the scalar product of (3.1) respectively with  $\psi, e_1 \cdot \psi, e_2 \cdot \psi, e_1 \cdot e_2 \cdot \psi$ , we get  $\delta = \frac{d(|\psi|^2)}{2|\psi|^2}$  and

$$\alpha(X) = -\ell^\psi(X) + q^\psi(X) \quad \text{and} \quad \beta(X) = \frac{1}{|\psi|^2}\text{Re}(\nabla_X \psi, e_1 \cdot e_2 \cdot \psi).$$

Using (2.1), it follows that

$$\lambda^2 = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |\alpha|^2 + |\beta|^2 + |\delta|^2 + \frac{1}{4}S + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).$$

Now it remains to compute the term  $|\beta|^2$ . We have

$$\begin{aligned} |\beta|^2 &= \frac{1}{|\psi|^4}\text{Re}(\nabla_{e_1} \psi, e_1 \cdot e_2 \cdot \psi)^2 + \frac{1}{|\psi|^4}\text{Re}(\nabla_{e_2} \psi, e_1 \cdot e_2 \cdot \psi)^2 \\ &= \frac{1}{|\psi|^4}\text{Re}(D\psi - e_2 \cdot \nabla_{e_2} \psi, e_2 \cdot \psi)^2 + \frac{1}{|\psi|^4}\text{Re}(D\psi - e_1 \cdot \nabla_{e_1} \psi, e_1 \cdot \psi)^2 \\ &= g(Y, e_1)^2 + g(Y, e_2)^2 + \frac{|d(|\psi|^2)|^2}{4|\psi|^4} - g(Y, \frac{d(|\psi|^2)}{|\psi|^2}) \\ &= |Y|^2 - 2Y(f) + \frac{|d(|\psi|^2)|^2}{4|\psi|^4}, \end{aligned}$$

which gives the result by using the fact that  $\Delta f = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + \frac{|d(|\psi|^2)|^2}{2|\psi|^4}$ .  $\square$

**Remark 3.2** *Under the same conditions as Theorem 3.1, if  $\psi$  is an eigenspinor of  $D$  with eigenvalue  $\lambda$ , we get*

$$\lambda^2 = \frac{S}{4} + |T^\psi|^2 + \Delta f + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).$$

In fact, in this case  $Y = 0$  and

$$\begin{aligned} 0 &= \text{Re}(D\psi, e_1 \cdot e_2 \cdot \psi) = \text{Re}(e_1 \cdot \nabla_{e_1} \psi + e_2 \cdot \nabla_{e_2} \psi, e_1 \cdot e_2 \cdot \psi) \\ &= \text{Re}(-e_2 \cdot \nabla_{e_1} \psi + e_1 \cdot \nabla_{e_2} \psi, \psi) = 2Q^\psi(e_1, e_2)|\psi|^2. \end{aligned} \quad (3.2)$$

This was proven by Friedrich and Kim in [9] for a Spin structure on  $M$ .

In the following, we will give an estimate for the integral  $\int_M \det(T^\psi + Q^\psi)v_g$  in terms of geometric quantities, which has the advantage that it does not depend on the eigenvalue  $\lambda$  nor on the eigenspinor  $\psi$ . This is a generalization of the result of Friedrich and Kim in [9] for Spin structures.

**Theorem 3.3** *Let  $M$  be a compact surface and  $\psi$  any eigenspinor of  $D^2$  associated with eigenvalue  $\lambda^2$ . Then we have*

$$\int_M \det(T^\psi + Q^\psi)v_g \geq \frac{\pi\chi(M)}{2} - \frac{1}{4} \int_M |\Omega|v_g. \quad (3.3)$$

Equality in (3.3) holds if and only if either  $\Omega$  is zero or has constant sign.

**Proof.** As in the previous theorem, the spinor  $D\psi$  can be expressed in the orthonormal frame of the spinor bundle. Thus the norm of  $D\psi$  is equal to

$$\begin{aligned} |D\psi|^2 &= \frac{1}{|\psi|^2} \operatorname{Re} (D\psi, \psi)^2 + \frac{1}{|\psi|^2} \sum_{i=1}^2 \operatorname{Re} (D\psi, e_i \cdot \psi)^2 + \frac{1}{|\psi|^2} \operatorname{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2 \\ &= (\operatorname{tr} T^\psi)^2 |\psi|^2 + |Y|^2 |\psi|^2 + \frac{1}{|\psi|^2} \operatorname{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2, \end{aligned} \quad (3.4)$$

where we recall that the trace of  $T^\psi$  is equal to  $-\frac{1}{|\psi|^2} \operatorname{Re} (D\psi, \psi)$ . On the other hand, by (3.2) we have that  $\frac{1}{|\psi|^2} \operatorname{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2 = 2|Q^\psi|^2 |\psi|^2$ . Thus Equation (3.4) reduces to

$$\frac{|D\psi|^2}{|\psi|^2} = (\operatorname{tr} T^\psi)^2 + |Y|^2 + 2|Q^\psi|^2.$$

Now with the use of the equality  $\operatorname{Re} (D^2\psi, \psi) = |D\psi|^2 - \operatorname{div}\xi$ , where  $\xi$  is the vector field given by  $\xi = |\psi|^2 Y$ , we get

$$\lambda^2 + \frac{1}{|\psi|^2} \operatorname{div}\xi = (\operatorname{tr} T^\psi)^2 + |Y|^2 + 2|Q^\psi|^2. \quad (3.5)$$

An easy computation leads to  $\frac{1}{|\psi|^2} \operatorname{div}\xi = \operatorname{div}Y + 2Y(f)$  where we recall that  $f = \frac{1}{2} \ln(|\psi|^2)$ . Hence substituting this formula into (3.5) and using Theorem 3.1 yields

$$\frac{S}{4} + \left( \frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2} \right) + \Delta f + \operatorname{div}Y = (\operatorname{tr} T^\psi)^2 + |Q^\psi|^2 - |T^\psi|^2 = 2\det(T^\psi + Q^\psi).$$

Finally integrating over  $M$  and using the Gauss-Bonnet formula, we deduce the required result with the help of Equation (2.2). Equality holds if and only if  $\Omega \cdot \psi = i|\Omega|\psi$ . In the orthonormal frame  $\{e_1, e_2\}$ , the 2-form  $\Omega$  can be written  $\Omega = \Omega_{12} e_1 \wedge e_2$ , where  $\Omega_{12}$  is a function defined on  $M$ . Using the decomposition of  $\psi$  into positive and negative spinors  $\psi^+$  and  $\psi^-$ , we find that the equality is attained if and only if

$$\Omega_{12} e_1 \cdot e_2 \cdot \psi^+ + \Omega_{12} e_1 \cdot e_2 \cdot \psi^- = i|\Omega_{12}|\psi^+ + i|\Omega_{12}|\psi^-,$$

which is equivalent to say that,

$$\Omega_{12}\psi^+ = -|\Omega_{12}|\psi^+ \quad \text{and} \quad \Omega_{12}\psi^- = |\Omega_{12}|\psi^-.$$



Now if  $\psi^+ \neq 0$  and  $\psi^- \neq 0$ , we get  $\Omega = 0$ . Otherwise, it has constant sign. In the last case, we get that  $\int_M |\Omega| v_g = 2\pi\chi(M)$ , which means that the l.h.s. of this equality is a topological invariant.  $\square$

Next, we will give another proof of the Bär-type inequality (1.3) for the eigenvalues of any  $\text{Spin}^c$  Dirac operator. The following theorem was proved by the second author in [23] using conformal deformation of the spinorial Levi-Civita connection.

**Theorem 3.4** *Let  $M$  be a compact surface. For any  $\text{Spin}^c$  structure on  $M$ , any eigenvalue  $\lambda$  of the Dirac operator  $D$  to which is attached an eigenspinor  $\psi$  satisfies*

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\text{Area}(M)} - \frac{1}{\text{Area}(M)} \int_M |\Omega| v_g. \quad (3.6)$$

*Equality holds if and only if the eigenspinor  $\psi$  is a  $\text{Spin}^c$  Killing spinor, i.e., it satisfies  $\Omega \cdot \psi = i|\Omega|\psi$  and  $\nabla_X \psi = -\frac{\lambda}{2}X \cdot \psi$  for any  $X \in \Gamma(TM)$ .*

**Proof.** With the help of Remark (3.2), we have that

$$\lambda^2 = \frac{S}{4} + |T^\psi|^2 + \Delta f + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right). \quad (3.7)$$

Substituting the Cauchy-Schwarz inequality, i.e.  $|T^\psi|^2 \geq \frac{\lambda^2}{2}$  and the estimate (2.2) into Equality (3.7), we easily deduce the result after integrating over  $M$ . Now the equality in (3.6) holds if and only if the eigenspinor  $\psi$  satisfies  $\Omega \cdot \psi = i|\Omega|\psi$  and  $|T^\psi|^2 = \frac{\lambda^2}{2}$ . Thus, the second equality is equivalent to say that  $\ell^\psi(X) = \frac{\lambda}{2}X$  for all  $X \in \Gamma(TM)$ . Finally, a straightforward computation of the spinorial curvature of the spinor field  $\psi$  gives in a local frame  $\{e_1, e_2\}$  after using the fact  $\beta = -(*\delta)$  that

$$\begin{aligned} \frac{1}{2}R_{1212} e_1 \cdot e_2 \cdot \psi &= \left(\frac{\lambda^2}{2} + e_1(\delta(e_1)) + e_2(\delta(e_2))\right) e_2 \cdot e_1 \cdot \psi - \lambda\delta(e_2)e_1 \cdot \psi \\ &\quad + \lambda\delta(e_1)e_2 \cdot \psi + \left(e_1(\delta(e_2)) - e_2(\delta(e_1))\right)\psi. \end{aligned}$$

Thus the scalar product with  $e_1 \cdot \psi$  and  $e_2 \cdot \psi$  implies that  $\delta = 0$ . Finally,  $\beta = 0$  and the eigenspinor  $\psi$  is a  $\text{Spin}^c$  Killing spinor.  $\square$

Now, we will give some examples where equality holds in (3.6) or in (3.3). Some applications of Theorem 3.1 are also given.

### Examples:

1. Let  $\mathbb{S}^2$  be the round sphere equipped with the standard metric of curvature one. As a Kähler manifold, we endow the sphere with the canonical  $\text{Spin}^c$  structure of curvature form equal to  $i\Omega = -i\kappa$ , where  $\kappa$  is the Kähler 2-form. Hence, we have  $|\Omega| = |\kappa| = 1$ . Furthermore, we mentioned that for the canonical  $\text{Spin}^c$  structure, the sphere carries parallel spinors, i.e., an eigenspinor associated with the eigenvalue 0 of the Dirac operator  $D$ . Thus equality holds in (3.6). On the other hand, the equality in (3.3) also holds, since the sign of the curvature form  $\Omega$  is constant.

2. Let  $f : M \rightarrow \mathbb{S}^3$  be an isometric immersion of a surface  $M^2$  into the sphere equipped with its unique Spin structure and assume that the mean curvature  $H$  is constant. The restriction of a Killing spinor on  $\mathbb{S}^3$  to the surface  $M$  defines a spinor field  $\varphi$  solution of the following equation [11]

$$\nabla_X \varphi = -\frac{1}{2}II(X) \bullet \varphi + \frac{1}{2}J(X) \bullet \varphi, \quad (3.8)$$

where  $II$  denotes the second fundamental form of the surface and  $J$  is the complex structure of  $M$  given by the rotation of angle  $\frac{\pi}{2}$  on  $TM$ . It is easy to check that  $\varphi$  is an eigenspinor for  $D^2$  associated with the eigenvalue  $H^2 + 1$ . Moreover  $D\varphi = H\varphi + e_1 \cdot e_2 \cdot \varphi$ , so that  $Y = 0$ . Moreover the tensor  $T^\varphi = \frac{1}{2}II$  and  $Q^\varphi = \frac{1}{2}J$ . Hence by Theorem 3.1, and since the norm of  $\varphi$  is constant, we obtain

$$H^2 + \frac{1}{2} = \frac{1}{4}S + \frac{1}{4}|II|^2.$$

3. On two-dimensional manifolds, we can define another Dirac operator associated with the complex structure  $J$  given by  $\tilde{D} = Je_1 \cdot \nabla_{e_1} + Je_2 \cdot \nabla_{e_2} = e_2 \cdot \nabla_{e_1} - e_1 \cdot \nabla_{e_2}$ . Since  $\tilde{D}$  satisfies  $D^2 = (\tilde{D})^2$ , all the above results are also true for the eigenvalues of  $\tilde{D}$ .
4. Let  $M^2$  be a surface immersed in  $\mathbb{S}^2 \times \mathbb{R}$ . The product of the canonical Spin<sup>c</sup> structure on  $\mathbb{S}^2$  and the unique Spin structure on  $\mathbb{R}$  define a Spin<sup>c</sup> structure on  $\mathbb{S}^2 \times \mathbb{R}$  carrying parallel spinors [22]. Moreover, by the Schrödinger-Lichnerowicz formula, any parallel spinor  $\psi$  satisfies  $\Omega^{\mathbb{S}^2 \times \mathbb{R}} \cdot \psi = i\psi$ , where  $\Omega^{\mathbb{S}^2 \times \mathbb{R}}$  is the curvature form of the auxiliary line bundle. Let  $\nu$  be a unit normal vector field of the surface. We then write  $\partial t = T + f\nu$  where  $T$  is a vector field on  $TM$  with  $\|T\|^2 + f^2 = 1$ . On the other hand, the vector field  $T$  splits into  $T = \nu_1 + h\partial t$  where  $\nu_1$  is a vector field on the sphere. The scalar product of the first equation by  $T$  and the second one by  $\partial t$  gives  $\|T\|^2 = h$  which means that  $h = 1 - f^2$ . Hence the normal vector field  $\nu$  can be written as  $\nu = f\partial t - \frac{1}{f}\nu_1$ . As we mentioned before, the Spin<sup>c</sup> structure on  $\mathbb{S}^2 \times \mathbb{R}$  induces a Spin<sup>c</sup> structure on  $M$  with induced auxiliary line bundle. Next, we want to prove that the curvature form of the auxiliary line bundle of  $M$  is equal to  $i\Omega(e_1, e_2) = -if$ , where  $\{e_1, e_2\}$  denotes a local orthonormal frame on  $TM$ . Since the spinor  $\psi$  is parallel, we have by [22] that for all  $X \in T(\mathbb{S}^2 \times \mathbb{R})$  the equality  $\text{Ric}^{\mathbb{S}^2 \times \mathbb{R}} X \cdot \psi = i(X \lrcorner \Omega^{\mathbb{S}^2 \times \mathbb{R}}) \cdot \psi$ . Therefore, we compute

$$\begin{aligned} (\nu \lrcorner \Omega^{\mathbb{S}^2 \times \mathbb{R}}) \bullet \varphi &= (\nu \lrcorner \Omega^{\mathbb{S}^2 \times \mathbb{R}}) \cdot \nu \cdot \psi|_M = i\nu \cdot \text{Ric}^{\mathbb{S}^2 \times \mathbb{R}} \nu \cdot \psi|_M \\ &= -\frac{1}{f}i\nu \cdot \nu_1 \cdot \psi|_M = i\nu \cdot (\nu - f\partial t) \cdot \psi|_M \\ &= (-i\psi - if\nu \cdot \partial t \cdot \psi)|_M. \end{aligned}$$

Hence by Equation (2.4), we get that  $\Omega \bullet \varphi = -i(f\nu \cdot \partial t \cdot \psi)|_M$ . The scalar product of the last equality with  $e_1 \cdot e_2 \cdot \psi$  gives

$$\Omega(e_1, e_2)|\varphi|^2 = -f\text{Re}(i\nu \cdot \partial t \cdot \psi, e_1 \cdot e_2 \cdot \psi)|_M = -f\text{Re}(i\partial t \cdot \psi, \psi)|_M.$$

We now compute the term  $i\partial t \cdot \psi$ . For this, let  $\{e'_1, J e'_1\}$  be a local orthonormal frame of the sphere  $\mathbb{S}^2$ . The complex volume form acts as the identity on the spinor bundle of  $\mathbb{S}^2 \times \mathbb{R}$ , hence  $\partial t \cdot \psi = e'_1 \cdot J e'_1 \cdot \psi$ . But we have

$$\Omega^{\mathbb{S}^2 \times \mathbb{R}} \cdot \psi = -\rho \cdot \psi = -\times \cdot \psi = -e'_1 \cdot J e'_1 \cdot \psi.$$

Therefore,  $i\partial t \cdot \psi = \psi$ . Thus we get  $\Omega(e_1, e_2) = -f$ . Finally,

$$(i\Omega \bullet \varphi, \varphi) = f \operatorname{Re}(\nu \cdot \partial t \cdot \psi, \psi)|_M = -fg(\nu, \partial t)|\varphi|^2 = -f^2|\varphi|^2.$$

Hence Equality in Theorem 3.1 is just

$$H^2 = \frac{S}{4} + \frac{1}{4}|II|^2 - \frac{1}{2}f^2.$$

## 4 The 3-dimensional case

In this section, we will treat the 3-dimensional case.

**Theorem 4.1** *Let  $(M^3, g)$  be an oriented Riemannian manifold. For any  $\operatorname{Spin}^c$  structure on  $M$ , any eigenvalue  $\lambda$  of the Dirac operator to which is attached an eigenspinor  $\psi$  satisfies*

$$\lambda^2 \leq \frac{1}{\operatorname{vol}(M, g)} \int_M (|T^\psi|^2 + \frac{S}{4} + \frac{|\Omega|}{2}) v_g.$$

Equality holds if and only if the norm of  $\psi$  is constant and  $\Omega \cdot \psi = i|\Omega|\psi$ .

**Proof.** As in the proof of Theorem 3.1, the set  $\{\frac{\psi}{|\psi|}, \frac{e_1 \cdot \psi}{|\psi|}, \frac{e_2 \cdot \psi}{|\psi|}, \frac{e_3 \cdot \psi}{|\psi|}\}$  is orthonormal with respect to the real product  $\operatorname{Re}(\cdot, \cdot)$ . The covariant derivative of  $\psi$  can be expressed in this frame as

$$\nabla_X \psi = \eta(X)\psi + \ell(X) \cdot \psi, \quad (4.1)$$

for all vector fields  $X$ , where  $\eta$  is a 1-form and  $\ell$  is a  $(1, 1)$ -tensor field. Moreover  $\eta = \frac{d(|\psi|^2)}{2|\psi|^2}$  and  $\ell(X) = -\ell^\psi(X)$ . Using (2.1), it follows that

$$\begin{aligned} \lambda^2 &= \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |T^\psi|^2 + \frac{|d(|\psi|^2)|^2}{4|\psi|^4} + \frac{1}{4}S + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right) \\ &= \Delta f - \frac{|d(|\psi|^2)|^2}{2|\psi|^4} + |T^\psi|^2 + \frac{1}{4}S + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right). \end{aligned}$$

By the Cauchy-Schwarz inequality, we have  $\frac{1}{2}(i\Omega \cdot \psi, \frac{\psi}{|\psi|^2}) \leq \frac{1}{2}|\Omega|$ . Integrating over  $M$  and using the fact that  $|d(|\psi|^2)|^2 \geq 0$ , we get the result.  $\square$

**Example 4.2** *Let  $M^3$  be a 3-dimensional Riemannian manifold immersed in  $\mathbb{C}\mathbb{P}^2$  with constant mean curvature  $H$ . Since  $\mathbb{C}\mathbb{P}^2$  is a Kähler manifold, we endow it with the canonical  $\operatorname{Spin}^c$  structure whose line bundle has curvature equal to*

$-3i\kappa$ . Moreover, by the Schrödinger-Lichnerowicz formula we have that any parallel spinor  $\psi$  satisfies  $\Omega^{\mathbb{CP}^2} \cdot \psi = 6i\psi$ . As in the previous example, we compute

$$(\nu \lrcorner \Omega^{\mathbb{CP}^2}) \bullet \varphi = i(\nu \cdot \text{Ric}^{\mathbb{CP}^2}(\nu) \cdot \psi)|_M = -3i\varphi.$$

Finally,  $\Omega \bullet \varphi = 3i\varphi$ . Using Equation (2.6), we have that  $-\frac{3}{2}H$  is an eigenvalue of  $D$ . Since the norm of  $\varphi$  is constant, equality holds in Theorem 4.1 and hence

$$\frac{9}{4}H^2 + \frac{3}{2} = \frac{S}{4} + \frac{1}{4}|II|^2.$$

## 5 Characterization of surfaces in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we characterize the surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  by solutions of the generalized Killing spinors equation which are restrictions of parallel spinors of the canonical  $\text{Spin}^c$ -structure on  $\mathbb{S}^2 \times \mathbb{R}$  (see also [25] for a different proof). First recall the compatibility equations for characterization of surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  established by B. Daniel [6, Thm 3.3]:

**Theorem 5.1** *Let  $(M, g)$  be a simply connected Riemannian manifold of dimension 2,  $A : TM \rightarrow TM$  a field of symmetric operator and  $T$  a vector field on  $TM$ . We denote by  $f$  a real valued function such that  $f^2 + \|T\|^2 = 1$ . Assume that  $M$  satisfies the Gauss-Codazzi equations, i.e.  $G = \det A + f^2$  and*

$$d^\nabla A(X, Y) := (\nabla_X A)Y - (\nabla_Y A)X = f(g(Y, T)X - g(X, T)Y),$$

where  $G$  is the gaussian curvature, and the following equations

$$\nabla_X T = fA(X), \quad X(f) = -g(AX, T).$$

Then there exists an isometric immersion of  $M$  into  $\mathbb{S}^2 \times \mathbb{R}$  such that the Weingarten operator is  $A$  and  $\partial t = T + f\nu$ , where  $\nu$  is the normal vector field to the surface  $M$ .

Now using this characterization theorem, we state our result:

**Theorem 5.2** *Let  $M$  be an oriented simply connected Riemannian manifold of dimension 2. Let  $T$  be a vector field and denote by  $f$  a real valued function such that  $f^2 + \|T\|^2 = 1$ . Denote by  $A$  a symmetric endomorphism field of  $TM$ . The following statements are equivalent:*

1. *There exists an isometric immersion of  $M$  into  $\mathbb{S}^2 \times \mathbb{R}$  of Weingarten operator  $A$  such that  $\partial t = T + f\nu$ , where  $\nu$  is the unit normal vector field of the surface.*
2. *There exists a  $\text{Spin}^c$  structure on  $M$  whose line bundle has a connection of curvature given by  $i\Omega = -if\kappa$ , such that it carries a non-trivial solution  $\varphi$  of the generalized Killing spinor equation  $\nabla_X \varphi = -\frac{1}{2}AX \bullet \varphi$ , with  $T \bullet \varphi = -f\varphi + \bar{\varphi}$ .*

**Proof.** We begin with  $1 \Rightarrow 2$ . The existence of such a  $\text{Spin}^c$  structure is assured by the restriction of the canonical one on  $\mathbb{S}^2 \times \mathbb{R}$ . Moreover, using the spinorial Gauss formula (2.5), any parallel spinor  $\psi$  on  $\mathbb{S}^2 \times \mathbb{R}$  induces a generalized Killing spinor  $\varphi = \psi|_M$  with  $A$  the Weingarten map of the surface  $M$ . Hence it remains to show the relation  $T \bullet \varphi = -f\varphi + \bar{\varphi}$ . In fact, using that  $\Omega^{\mathbb{S}^2 \times \mathbb{R}} \cdot \psi = i\psi$ , we write in the frame  $\{e_1, e_2, \nu\}$

$$\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, e_2)e_1 \cdot e_2 \cdot \psi + \Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \nu)e_1 \cdot \nu \cdot \psi + \Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_2, \nu)e_2 \cdot \nu \cdot \psi = i\psi. \quad (5.1)$$

By the previous example in Section 3, we know that  $\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, e_2) = -f$ . For the other terms, we compute

$$\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \nu) = \Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \frac{1}{f}\partial t - \frac{1}{f}T) = -\frac{1}{f}g(T, e_2)\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, e_2) = g(T, e_2),$$

where the term  $\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \partial t)$  vanishes since we can split  $e_1$  into a sum of vectors on the sphere and on  $\mathbb{R}$ . Similarly, we find that  $\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_2, \nu) = -g(T, e_1)$ . By substituting these values into (5.1) and taking Clifford multiplication with  $e_1 \cdot e_2$ , we get the desired property. For  $2 \Rightarrow 1$ , a straightforward computation for the spinorial curvature of the generalized Killing spinor  $\varphi$  yields on a local frame  $\{e_1, e_2\}$  of  $TM$  that

$$(-G + \det A)e_1 \bullet e_2 \bullet \varphi = -(d^\nabla A)(e_1, e_2) \bullet \varphi + if\varphi. \quad (5.2)$$

In the following, we will prove that the spinor field  $\theta := i\varphi - if\bar{\varphi} + JT \bullet \varphi$  is zero. For this, it is sufficient to prove that its norm vanishes. Indeed, we compute

$$|\theta|^2 = |\varphi|^2 + f^2|\varphi|^2 + \|T\|^2|\varphi|^2 - 2\text{Re}(i\varphi, if\bar{\varphi}) + 2\text{Re}(i\varphi, JT \bullet \varphi) \quad (5.3)$$

From the relation  $T \bullet \varphi = -f\varphi + \bar{\varphi}$  we deduce that  $\text{Re}(\varphi, \bar{\varphi}) = f|\varphi|^2$  and the equalities

$$g(T, e_1)|\varphi|^2 = \text{Re}(ie_2 \bullet \varphi, \varphi) \quad \text{and} \quad g(T, e_2)|\varphi|^2 = -\text{Re}(ie_1 \bullet \varphi, \varphi).$$

Therefore, Equation (5.3) becomes

$$\begin{aligned} |\theta|^2 &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2\text{Re}(i\varphi, JT \bullet \varphi) \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, e_1)\text{Re}(i\varphi, e_1 \bullet \varphi) + 2g(JT, e_2)\text{Re}(i\varphi, e_2 \bullet \varphi) \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, e_1)g(T, e_2)|\varphi|^2 - 2g(JT, e_2)g(T, e_1)|\varphi|^2 \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2g(JT, e_1)^2|\varphi|^2 - 2g(T, e_1)^2|\varphi|^2 \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2\|T\|^2|\varphi|^2 = 0. \end{aligned}$$

Thus, we deduce  $if\varphi = -f^2e_1 \cdot e_2 \cdot \varphi - fJT \cdot \varphi$ , where we use the fact that  $\bar{\varphi} = ie_1 \bullet e_2 \bullet \varphi$ . In this case, Equation (5.2) can be written as

$$(-G + \det A + f^2)e_1 \bullet e_2 \bullet \varphi = -((d^\nabla A)(e_1, e_2) + fJT) \bullet \varphi.$$

This is equivalent to say that both terms  $R_{1212} + \det A + f^2$  and  $(d^\nabla A)(e_1, e_2) + fJT$  are equal to zero. In fact, these are the Gauss-Codazzi equations in Theorem 5.1. In order to obtain the two other equations, we simply compute the

derivative of  $T \cdot \varphi = -f\varphi + \bar{\varphi}$  in the direction of  $X$  to get

$$\begin{aligned}
\nabla_X T \bullet \varphi + T \bullet \nabla_X \varphi &= \nabla_X T \bullet \varphi - \frac{1}{2} T \bullet A(X) \bullet \varphi \\
&= -X(f)\varphi - f\nabla_X \varphi + \nabla_X \bar{\varphi} \\
&= -X(f)\varphi + \frac{1}{2} f AX \bullet \varphi + \frac{1}{2} AX \bullet \bar{\varphi} \\
&= -X(f)\varphi + \frac{1}{2} f AX \bullet \varphi + \frac{1}{2} AX \bullet (T \bullet \varphi + f\varphi).
\end{aligned}$$

This reduces to  $\nabla_X T \bullet \varphi + g(T, A(X))\varphi = -X(f)\varphi + fA(X) \bullet \varphi$ . Hence we obtain  $X(f) = -g(A(X), T)$  and  $\nabla_X T = fA(X)$  which finishes the proof.  $\square$

**Remark 5.3** *The second condition in Theorem 5.2 is equivalent to the existence of a  $\text{Spin}^c$  structure whose line bundle  $L$  verifies  $c_1(L) = [\frac{i}{2\pi} f \times]$  and  $f \times$  is a closed 2-form. This  $\text{Spin}^c$  structure carries a non-trivial solution  $\varphi$  of the generalized Killing spinor equation  $\nabla_X \varphi = -\frac{1}{2} AX \bullet \varphi$ , with  $T \bullet \varphi = -f\varphi + \bar{\varphi}$ .*

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