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ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA AND OTHER RELATED VARIETIES.

JEAN-YVES CHARBONNEL AND MOUCHIRA ZAITER

Abstract. In this note, we discuss some varieties which are constructed analogously to the isospectral commuting varieties. These varieties are subvarieties of varieties having very simple desingularizations. For instance, this is the case of the nullcone of any cartesian power of a reductive Lie algebra and we prove that it is normal. Moreover, as a byproduct of these investigations and a Ginzburg's result, we get that the normalizations of the isospectral commuting variety and the commuting variety have rational singularities.

Contents

1. Introduction

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and B the normalizer of $\mathfrak{b}$ in G.

1.1. **Main results.** By definition, $\mathcal{B}^{(k)}$ is the subset of elements $(x_1, \dots, x_k)$ of $\mathfrak{g}^k$ such that $x_1, \dots, x_k$ are in a same Borel subalgebra of $\mathfrak{g}$. This subset of $\mathfrak{g}^k$ is closed and contains two interesting subsets: the generalized commuting variety of $\mathfrak{g}$, denoted by $\mathcal{C}^{(k)}$ and the nullcone of $\mathfrak{g}^k$ denoted by $\mathcal{N}^{(k)}$. According to [Mu65, Ch.2, §1, Theorem], for $(x_1, \dots, x_k)$ in $\mathcal{B}^{(k)}$, $(x_1, \dots, x_k)$ is in $\mathcal{N}^{(k)}$ if and only if $x_1, \dots, x_k$ are nilpotent. By definition, $\mathcal{C}^{(k)}$ is the closure in $\mathfrak{g}^k$ of the set of elements whose all components are in a same Cartan subalgebra. According to a Richardson Theorem [Ri79], $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$.

There is a natural projective morphism $G\times_B \mathfrak{b}^k \to \mathcal{B}^{(k)}$. For $k = 1$, this morphism is not birational but for $k \geq 2$, it is birational. Furthermore, denoting by $\mathcal{X}$ the subvariety of elements (x, y) of $\mathfrak{g} \times \mathfrak{h}$ such that y is in the closure of the orbit of x under G, the canonical morphism $G \times_B \mathfrak{b} \to \mathcal{X}$ is

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projective and birational and $\mathfrak{g}$ is the categorical quotient of $\mathcal{X}$ under the action of $W(\mathcal{R})$ on the factor $\mathfrak{h}$. For $k \geq 2$, the inverse image of $\mathcal{B}^{(k)}$ by the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$ is not irreducible but the canonical action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of its irreducible components. Denoting by $\mathcal{B}_{\mathcal{X}}^{(k)}$ one of these components, we have a commutative diagram

$$\begin{array}{ccc} G \times_B \mathfrak{b}^k & \longrightarrow & \mathcal{B}_{\mathcal{X}}^{(k)} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & \mathcal{B}^{(k)} & \end{array}$$

with η the restriction to $\mathcal{B}_{\mathcal{X}}^{(k)}$ of the canonical projection ϖ from $\mathcal{X}^k$ to $\mathfrak{g}^k$. The first main theorem of this note is the following theorem:

Theorem 1.1. (i) *The variety $\mathcal{B}_{\mathcal{X}}^{(k)}$ is normal. Moreover, for $k \geq 2$, $\mathcal{B}_{\mathcal{X}}^{(k)}$ is the normalization of $\mathcal{B}^{(k)}$ and η is the normalization morphism.*

(ii) *The variety $\mathcal{N}^{(k)}$ is normal. Moreover, for $k \geq 2$, it is the underlying variety to a non reduced well defined subscheme of $\mathfrak{g}^k$.*

(iii) *The algebra $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ is a free extension of $\mathrm{S}(\mathfrak{h}^k)$ and $\Bbbk[\mathcal{B}^{(k)}]$ is a free extension of $\mathrm{S}(\mathfrak{h}^k)^{W(\mathcal{R})}$.*

According to Ting Xue and K. Vilonen, in general $\mathcal{N}^{(k)}$ and $\mathcal{B}_{\mathcal{X}}^{(k)}$ have no rational singularities for $k \geq 2$. In the study of the generalized commuting variety, the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of G plays an important role. Denoting by X the closure in $\mathrm{Gr}_\ell(\mathfrak{b})$ of the orbit of $\mathfrak{h}$ under B, $G.X$ is the closure of the orbit of $G.\mathfrak{h}$ and we have the following result:

Theorem 1.2. *Let X' be the set of centralizers of regular elements of $\mathfrak{g}$ whose semisimple components is regular or subregular.*

(i) *All element of X is a commutative algebraic subalgebra of $\mathfrak{g}$.*

(ii) *For x in $\mathfrak{g}$, the set of elements of $G.X$ containing x has dimension at most $\dim \mathfrak{g}^x - \ell$.*

(iii) *The sets $X' \setminus B.\mathfrak{h}$ and $G.X' \setminus G.\mathfrak{h}$ are dense in $X \setminus B.\mathfrak{h}$ and $G.X \setminus G.\mathfrak{h}$ respectively.*

(iv) *The sets X' and $G.X'$ are smooth big open subsets of X and $G.X$ respectively.*

Let $\mathfrak{X}_{0,k}$ be the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$ and let Γ be a desingularization of X in the category of B-varieties. Let E be the tautological bundle over X and set:

$$E_{\mathrm{s}} := E \times_X \Gamma, \qquad E_{\mathrm{s}}^{(k)} := \underbrace{E_{\mathrm{s}} \times_\Gamma \cdots \times_\Gamma E_{\mathrm{s}}}_{k \text{ factors}}.$$

Then $E_{\mathrm{s}}^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$. Set: $\mathcal{C}_{\mathrm{n}}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The following theorem is the second main result of this note:

Theorem 1.3. (i) *The variety $\mathcal{C}_{\mathrm{n}}^{(k)}$ is irreducible and $G \times_B E_{\mathrm{s}}^{(k)}$ is a desingularization of $\mathcal{C}_{\mathrm{n}}^{(k)}$.*

(ii) *For $k = 2$, the normalizations of $\mathcal{C}_{\mathrm{n}}^{(k)}$ and $\mathcal{C}^{(k)}$ have rational singularities.*

The proof of Assertion (ii) is a consequence of the proof of Assertion (i), and the deep result of Ginzburg [Gi12] which asserts that the normalization of $\mathcal{C}_{\mathrm{n}}^{(2)}$ is Gorenstein.

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1.2. **Notations.** • An algebraic variety is a finite type reduced scheme over $\Bbbk$.

• For V a vector space, its dual is denoted by V^* and the augmentation ideal of its symmetric algebra $\mathrm{S}(V)$ is denoted by $\mathrm{S}_+(V)$. For A a graded algebra over $\mathbb{N}$, A_+ is the ideal generated by the homogeneous elements of positive degree.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $\Bbbk[X]$ is the algebra of regular functions on X and $\Bbbk(X)$ is the field of rational functions on X when X is irreducible. When X is smooth, the sheaf of regular differential forms of top degree on X is denoted by Ω_X.

• For X an algebraic variety and for $\mathcal{M}$ a sheaf on X, $\Gamma(V, \mathcal{M})$ is the space of local sections of $\mathcal{M}$ over the open subset V of X. For i a nonnegative integer, $\mathrm{H}^i(X, \mathcal{M})$ is the i-th group of cohomology of $\mathcal{M}$. For example, $\mathrm{H}^0(X, \mathcal{M}) = \Gamma(X, \mathcal{M})$.

Lemma 1.4. [EGAII, Corollaire 5.4.3] *Let X be an irreducible affine algebraic variety and let Y be a desingularization of X. Then $\mathrm{H}^0(Y, \mathcal{O}_Y)$ is the integral closure of $\Bbbk[X]$ in its fraction field.*

• For K a group and for E a set with a group action of K, E^K is the set of invariant elements of E under K. The following lemma is straightforward and will be used in the proof of Corollary 2.14.

Lemma 1.5. *Let A be an algebra generated by the subalgebras A_1 and A_2. Let K be a group with a group action of K on A_2. Suppose that the following conditions are verified:*

(1) *$A_1 \cap A_2$ is contained in A_2^K,*
(2) *A is a free A_2-module having a basis contained in A_1,*
(3) *A_1 is a free $A_1 \cap A_2$-module having the same basis.*

Then there exists a unique group action of K on the algebra A extending the action of K on A_2 and fixing all the elements of A_1. Moreover, if $A_1 \cap A_2 = A_2^K$ then $A^K = A_1$.

• For E a set and k a positive integer, E^k denotes its k-th cartesian power. If E is finite, its cardinality is denoted by $|E|$. If E is a vector space, for $x = (x_1, \dots, x_k)$ in E^k, P_x is the subspace of E generated by $x_1, \dots, x_k$. Moreover, there is a canonical action of $\mathrm{GL}_k(\Bbbk)$ in E^k given by:

$$(a_{i,j}, 1 \leq i, j \leq k).(x_1, \dots, x_k) := (\sum_{j=1}^{k} a_{i,j} x_j, i = 1, \dots, k)$$

In particular, the diagonal action of G in $\mathfrak{g}^k$ commutes with the action of $\mathrm{GL}_k(\Bbbk)$.

• For $\mathfrak{a}$ reductive Lie algebra, its rank is denoted by $\ell_{\mathfrak{a}}$ and the dimension of its Borel subalgebras is denoted by $\mathrm{b}_{\mathfrak{a}}$. In particular, $\dim \mathfrak{a} = 2\mathrm{b}_{\mathfrak{a}} - \ell_{\mathfrak{a}}$.

• If E is a subset of a vector space V, denote by $\mathrm{span}(E)$ the vector subspace of V generated by E. The grassmanian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$. By definition, a *cone* of V is a subset of V invariant under the natural action of $\Bbbk^* := \Bbbk \setminus \{0\}$ and a *multicone* of V^k is a subset of V^k invariant under the natural action of $(\Bbbk^*)^k$ on V^k.

Lemma 1.6. *Let X be an open cone of V and let S be a closed multicone of $X \times V^{k-1}$. Denoting by S_1 the image of S by the first projection, $S_1 \times \{0\} = S \cap (X \times \{0\})$. In particular, S_1 is closed in X.*

Proof. For x in X, x is in S_1 if and only if for some $(v_2, \ldots, v_k)$ in V^{k-1}, $(x, tv_2, \ldots, tv_k)$ is in S for all t in $\Bbbk$ since S is a closed multicone of $X \times V^{k-1}$, whence the lemma. □

• The dual of $\mathfrak{g}$ is denoted by $\mathfrak{g}^*$ and it identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle ., . \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

• Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Denote by $\mathcal{R}$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and by $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by Π. The neutral elements of G and $W(\mathcal{R})$ are denoted by $1_{\mathfrak{g}}$ and $1_{\mathfrak{h}}$ respectively. For α in $\mathcal{R}$, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator x_α of $\mathfrak{g}^\alpha$ is chosen so that $\langle x_\alpha, x_{-\alpha} \rangle = 1$ for all α in $\mathcal{R}$. As usual, the half sum of positive roots is denoted by ρ.

• The normalizers of $\mathfrak{b}$ and $\mathfrak{h}$ in G are denoted by B and $N_G(\mathfrak{h})$ respectively. For x in $\mathfrak{b}$, $\overline{x}$ is the element of $\mathfrak{h}$ such that $x - \overline{x}$ is in the nilpotent radical $\mathfrak{u}$ of $\mathfrak{b}$.

• For X an algebraic B-variety, denote by $G \times_B X$ the quotient of $G \times X$ under the right action of B given by $(g, x).b := (gb, b^{-1}.x)$. More generally, for k positive integer and for X an algebraic B^k-variety, denote by $G^k \times_{B^k} X$ the quotient of $G^k \times X$ under the right action of B^k given by $(g, x).b := (gb, b^{-1}.x)$ with g and b in G^k and B^k respectively.

Lemma 1.7. *Let P and Q be parabolic subgroups of G such that P is contained in Q. Let X be a Q-variety and let Y be a closed subset of X, invariant under P. Then $Q.Y$ is a closed subset of X. Moreover, the canonical map from $Q \times_P Y$ to $Q.Y$ is a projective morphism.*

Proof. Since P and Q are parabolic subgroups of G and since P is contained in Q, Q/P is a projective variety. Denote by $Q \times_P X$ and $Q \times_P Y$ the quotients of $Q \times X$ and $Q \times Y$ under the right action of P given by $(g, x).p := (gp, p^{-1}.x)$. Let $g \mapsto \overline{g}$ be the quotient map from Q to Q/P. Since X is a Q-variety, the map

$$Q \times X \longrightarrow Q/P \times X \qquad (g, x) \longmapsto (\overline{g}, g.x)$$

defines through the quotient an isomorphism from $Q \times_P X$ to $Q/P \times X$. Since Y is a P-invariant closed subset of X, $Q \times_P Y$ is a closed subset of $Q \times_P X$ and its image by the above isomorphism equals $Q/P \times Q.Y$. Hence $Q.Y$ is a closed subset of X since Q/P is a projective variety. From the commutative diagram:

$$\begin{array}{ccc} Q \times_P Y & \longrightarrow & Q/P \times Q.Y \\ & \searrow & \downarrow \\ & & Q.Y \end{array}$$

we deduce that the map $Q \times_P Y \to Q.Y$ is a projective morphism. □

• For $k \geq 1$ and for the diagonal action of B in $\mathfrak{b}^k$, $\mathfrak{b}^k$ is a B-variety. The canonical map from $G \times \mathfrak{b}^k$ to $G \times_B \mathfrak{b}^k$ is denoted by $(g, x_1, \ldots, x_k) \mapsto \overline{(g, x_1, \ldots, x_k)}$. Let $\mathcal{B}^{(k)}$ and $\mathcal{N}^{(k)}$ be the images of $G \times \mathfrak{b}^k$ and $G \times \mathfrak{u}^k$ respectively by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$ so that $\mathcal{B}^{(k)}$ and $\mathcal{N}^{(k)}$ are closed subsets of $\mathfrak{g}^k$ by Lemma 1.7. Let $\mathcal{B}^{(k)}_{\mathrm{n}}$ be the normalization of $\mathcal{B}^{(k)}$ and let η be the normalization morphism. We have the commutative diagram:

$$\begin{array}{ccc} G \times_B \mathfrak{b}^k & \xrightarrow{\gamma_{\mathrm{n}}} & \mathcal{B}^{(k)}_{\mathrm{n}} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & \mathcal{B}^{(k)} & \end{array}$$

Let $\mathcal{N}_{\mathrm{n}}^{(k)}$ be the normalization of $\mathcal{N}^{(k)}$ and let $\varkappa$ be the normalization morphism. We have the commutative diagram:

$$\begin{array}{ccc} G \times_B \mathfrak{u}^k & \xrightarrow{\upsilon_{\mathrm{n}}} & \mathcal{N}_{\mathrm{n}}^{(k)} \\ & {\scriptstyle\upsilon}\searrow \quad \swarrow{\scriptstyle\varkappa} & \\ & \mathcal{N}^{(k)} & \end{array}$$

with υ the restriction of γ to $G \times_B \mathfrak{u}^k$.

• Let i be the injection $(x_1, \ldots, x_k) \mapsto \overline{(1_{\mathfrak{g}}, x_1, \ldots, x_k)}$ from $\mathfrak{b}^k$ to $G \times_B \mathfrak{b}^k$. Then $\iota := \gamma \circ i$ and $\iota_{\mathrm{n}} := \gamma_{\mathrm{n}} \circ i$ are closed embeddings of $\mathfrak{b}^k$ into $\mathcal{B}^{(k)}$ and $\mathcal{B}_{\mathrm{n}}^{(k)}$ respectively. In particular, $\mathcal{B}^{(k)} = G.\iota(\mathfrak{b}^k)$ and $\mathcal{B}_{\mathrm{n}}^{(k)} = G.\iota_{\mathrm{n}}(\mathfrak{b}^k)$.

• Let e be the sum of the x_β's, β in Π, and let h be the element of $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ such that $\beta(h) = 2$ for all β in Π. Then there exists a unique f in $[\mathfrak{g}, \mathfrak{g}]$ such that (e, h, f) is a principal $\mathfrak{sl}_2$-triple. The one parameter subgroup of G generated by $\mathrm{ad}h$ is denoted by $t \mapsto h(t)$. The Borel subalgebra containing f is denoted by $\mathfrak{b}_-$ and its nilpotent radical is denoted by $\mathfrak{u}_-$. Let B_- be the normalizer of $\mathfrak{b}_-$ in G and let U and U_- be the unipotent radicals of B and B_- respectively.

Lemma 1.8. *Let $k \geq 2$ be an integer. Let X be an affine variety and set $Y := \mathfrak{b}^k \times X$. Let Z be a closed B-invariant subset of Y under the group action given by $g.(v_1, \ldots, v_k, x) = (g(v_1), \ldots, g(v_k), x)$ with $(g, v_1, \ldots, v_k)$ in $B \times \mathfrak{b}^k$ and x in X. Then $Z \cap \mathfrak{h}^k \times X$ is the image of Z by the projection $(v_1, \ldots, v_k, x) \mapsto (\overline{v_1}, \ldots, \overline{v_k}, x)$.*

Proof. For all v in $\mathfrak{b}$,

$$\overline{v} = \lim_{t \to 0} h(t)(v)$$

whence the lemma since Z is closed and B-invariant. □

• For $x \in \mathfrak{g}$, let x_{s} and x_{n} be the semisimple and nilpotent components of x in $\mathfrak{g}$. Denote by $\mathfrak{g}^x$ and G^x the centralizers of x in $\mathfrak{g}$ and G respectively. For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and for A a subgroup of G, set:

$$\mathfrak{a}^x := \mathfrak{a} \cap \mathfrak{g}^x \qquad A^x := A \cap G^x$$

The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

and denote by $\mathfrak{g}_{\mathrm{reg,ss}}$ the set of regular semisimple elements of $\mathfrak{g}$. Both $\mathfrak{g}_{\mathrm{reg}}$ and $\mathfrak{g}_{\mathrm{reg,ss}}$ are G-invariant dense open subsets of $\mathfrak{g}$. Setting $\mathfrak{h}_{\mathrm{reg}} := \mathfrak{h} \cap \mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{b}_{\mathrm{reg}} := \mathfrak{b} \cap \mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{u}_{\mathrm{reg}} := \mathfrak{u} \cap \mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{g}_{\mathrm{reg,ss}} = G(\mathfrak{h}_{\mathrm{reg}})$, $\mathfrak{g}_{\mathrm{reg}} = G(\mathfrak{b}_{\mathrm{reg}})$ and $G(\mathfrak{u}_{\mathrm{reg}})$ is the set of regular elements of the nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ of $\mathfrak{g}$.

Lemma 1.9. *Let $k \geq 2$ be an integer and let x be in $\mathfrak{g}^k$. For O open subset of $\mathfrak{g}_{\mathrm{reg}}$, $P_x \cap O$ is not empty if and only if for some g in $\mathrm{GL}_k(\Bbbk)$, the first component of $g.x$ is in O.*

Proof. Since the components of $g.x$ are in P_x for all g in $\mathrm{GL}_k(\Bbbk)$, the condition is sufficient. Suppose that $P_x \cap O$ is not empty and denote by $x_1, \ldots, x_k$ the components of x. For some $(a_1, \ldots, a_k)$ in $\Bbbk^k \setminus \{0\}$,

$$a_1 x_1 + \cdots + a_k x_k \in O$$

Let i be such that $a_i \neq 0$ and let τ be the transposition of $\mathfrak{S}_k$ such that $\tau(1) = i$. Denoting by g the element of $\mathrm{GL}_k(\Bbbk)$ such that $g_{1,j} = a_{\tau(j)}$ for $j = 1, \ldots, k$, $g_{j,j} = 1$ for $j = 2, \ldots, k$ and $g_{j,l} = 0$ for $j \geq 2$ and $j \neq l$, the first component of $g\tau.x$ is in O. □

• Denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra of $\mathfrak{g}$-invariant elements of $\mathrm{S}(\mathfrak{g})$. Let $p_1, \dots, p_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degree $d_1, \dots, d_\ell$ respectively. Choose the polynomials $p_1, \dots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \dots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, consider a shift of p_i in the direction y: $p_i(x + ty)$ with $t \in \Bbbk$. Expanding $p_i(x + ty)$ as a polynomial in t, we obtain

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y) t^m; \qquad \forall (t, x, y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{1}$$

where $y \mapsto (m!)p_i^{(m)}(x, y)$ is the derivative at x of p_i at the order m in the direction y. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Remark that $p_i^{(0)}(x, y) = p_i(x)$ while $p_i^{(d_i)}(x, y) = p_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.10. The family $\mathcal{P}_x := \{p_i^{(m)}(x, .);\ 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $\mathrm{S}(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. We say that the family $\mathcal{P}_x$ is constructed by the *argument shift method*.

• Let $i \in \{1, \dots, \ell\}$. For x in $\mathfrak{g}$, denote by $\varepsilon_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle \varepsilon_i(x), y \rangle = \frac{\mathrm{d}}{\mathrm{d}t} p_i(x + ty)\,|_{t=0}$$

for all y in $\mathfrak{g}$. Thereby, ε_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. According to [Ko63, Theorem 9], for x in $\mathfrak{g}$, x is in $\mathfrak{g}_{\mathrm{reg}}$ if and only if $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ are linearly independent. In this case, $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$.

Denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ defined by the equality:

$$\varepsilon_i(x + ty) = \sum_{m=0}^{d_i - 1} \varepsilon_i^{(m)}(x, y) t^m, \qquad \forall (t, x, y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{2}$$

and set:

$$V_{x,y} := \mathrm{span}(\{\varepsilon_i^{(0)}(x, y), \dots, \varepsilon_i^{(d_i - 1)}(x, y),\ i = 1, \dots, \ell\})$$

for (x, y) in $\mathfrak{g} \times \mathfrak{g}$. According to [Bol91, Corollary 2], $V_{x,y}$ has dimension $\mathrm{b}_{\mathfrak{g}}$ if and only if $P_{x,y}$ has dimension 2 and $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$.

2. On the varieties $\mathcal{B}^{(k)}$

Let $k \geq 2$ be an integer. According to the above notations, we have the commutative diagrams:

$$\begin{array}{ccc} G \times_B \mathfrak{b}^k & \xrightarrow{\gamma_{\mathfrak{n}}} & \mathcal{B}_{\mathfrak{n}}^{(k)} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & \mathcal{B}^{(k)} & \end{array} \qquad \begin{array}{ccc} G \times_B \mathfrak{u}^k & \xrightarrow{\upsilon_{\mathfrak{n}}} & \mathcal{N}_{\mathfrak{n}}^{(k)} \\ & {\scriptstyle\upsilon}\searrow \quad \swarrow{\scriptstyle\varkappa} & \\ & \mathcal{N}^{(k)} & \end{array}$$

Since the Borel subalgebras of $\mathfrak{g}$ are conjugate under G, $\mathcal{B}^{(k)}$ is the subset of elements of $\mathfrak{g}^k$ whose components are in a same Borel subalgebra and $\mathcal{N}^{(k)}$ are the elements of $\mathcal{B}^{(k)}$ whose all the components are nilpotent.

Lemma 2.1. (i) *The morphism γ from $G \times_B \mathfrak{b}^k$ to $\mathcal{B}^{(k)}$ is projective and birational. In particular, $G \times_B \mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(k)}$ has dimension $k\mathrm{b}_{\mathfrak{g}} + n$.*

(ii) *The morphism υ from $G \times_B \mathfrak{u}^k$ to $\mathcal{N}^{(k)}$ is projective and birational. In particular, $G \times_B \mathfrak{u}^k$ is a desingularization of $\mathcal{N}^{(k)}$ and $\mathcal{N}^{(k)}$ has dimension $(k + 1)n$.*

Proof. (i) Denote by $\Omega_{\mathfrak{g}}^{(2)}$ the subset of elements (x, y) of $\mathfrak{g}^2$ such that $P_{x,y}$ has dimension 2 and such that $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$. According to Lemma 1.7, γ is a projective morphism. For $1 \leq i < j \leq k$, let $\Omega_{i,j}^{(k)}$ be the inverse image of $\Omega_{\mathfrak{g}}^{(2)}$ by the projection

$$(x_1, \ldots, x_k) \longmapsto (x_i, x_j)$$

Then $\Omega_{i,j}^{(k)}$ is an open subset of $\mathfrak{g}^k$ whose intersection with $\mathcal{B}^{(k)}$ is not empty. Let $\Omega_{\mathfrak{g}}^{(k)}$ be the union of the $\Omega_{i,j}^{(k)}$. According to [Bol91, Corollary 2] and [Ko63, Theorem 9], for (x, y) in $\Omega_{\mathfrak{g}}^{(2)} \cap \mathcal{B}^{(2)}$, $V_{x,y}$ is the unique Borel subalgebra of $\mathfrak{g}$ containing x and y so that the restriction of γ to $\gamma^{-1}(\Omega_{\mathfrak{g}}^{(k)})$ is a bijection onto $\Omega_{\mathfrak{g}}^{(k)}$. Hence γ is birational. Moreover, $G \times_B \mathfrak{b}^k$ is a smooth variety as a vector bundle over the smooth variety G/B, whence the assertion since $G \times_B \mathfrak{b}^k$ has dimension $k\mathrm{b}_{\mathfrak{g}} + n$.

(ii) According to Lemma 1.7, υ is a projective morphism. Let $\mathcal{N}_{\mathrm{reg}}^{(k)}$ be the subset of elements of $\mathcal{N}^{(k)}$ whose at least one component is a regular element of $\mathfrak{g}$. Then $\mathcal{N}_{\mathrm{reg}}^{(k)}$ is an open subset of $\mathcal{N}^{(k)}$. Since a regular nilpotent element is contained in one and only one Borel subalgebra of $\mathfrak{g}$, the restriction of υ to $\upsilon^{-1}(\mathcal{N}_{\mathrm{reg}}^{(k)})$ is a bijection onto $\mathcal{N}_{\mathrm{reg}}^{(k)}$. Hence υ is birational. Moreover, $G \times_B \mathfrak{u}^k$ is a smooth variety as a vector bundle over the smooth variety G/B, whence the assertion since $G \times_B \mathfrak{u}^k$ has dimension $(k+1)n$. □

2.1. Denote by $\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}//G$ and $\pi_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}/W(\mathcal{R})$ the quotient maps, i.e the morphisms defined by the invariants. Recall $\mathfrak{g}//G = \mathfrak{h}/W(\mathcal{R})$, and let $\mathcal{X}$ be the following fiber product:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\overline{\gamma}} & \mathfrak{g} \\ \downarrow{\scriptstyle \overline{\rho}} & & \downarrow{\scriptstyle \pi_{\mathfrak{g}}} \\ \mathfrak{h} & \xrightarrow{\pi_{\mathfrak{h}}} & \mathfrak{h}/W(\mathcal{R}) \end{array}$$

where $\overline{\gamma}$ and $\overline{\rho}$ are the restriction maps. The actions of G and $W(\mathcal{R})$ on $\mathfrak{g}$ and $\mathfrak{h}$ respectively induce an action of $G \times W(\mathcal{R})$ on $\mathcal{X}$.

Lemma 2.2. (i) *There exists a well defined G-equivariant morphism γ_{n} from $G \times_B \mathfrak{b}$ to $\mathcal{X}$ such that γ is the composition of γ_{n} and $\overline{\gamma}$.*

(ii) *The morphism γ_{n} is projective and birational. Moreover, $\mathcal{X}$ is irreducible.*

(iii) *The subscheme $\mathcal{X}$ is normal. Moreover, every element of $\mathfrak{g}_{\mathrm{reg}} \times \mathfrak{h} \cap \mathcal{X}$ is a smooth point of $\mathcal{X}$.*

(iv) *The algebra $\Bbbk[\mathcal{X}]$ is the space of global sections of $\mathcal{O}_{G \times_B \mathfrak{b}}$ and $\Bbbk[\mathcal{X}]^G = \mathrm{S}(\mathfrak{h})$.*

Proof. (i) Since the map $(g, x) \mapsto (g(x), \overline{x})$ is constant on the B-orbits, there exists a uniquely defined morphism γ_{n} from $G \times_B \mathfrak{b}$ to $\mathfrak{g} \times \mathfrak{h}$ such that $(g(x), \overline{x})$ is the image by γ_{n} of the image of (g, x) in $G \times_B \mathfrak{b}$. The image of γ_{n} is contained in $\mathcal{X}$ since for all p in $\mathrm{S}(\mathfrak{g})^G$, $p(\overline{x}) = p(x) = p(g(x))$. Furthermore, γ_{n} verifies the condition of the assertion.

(ii) According to Lemma 1.7, γ_{n} is a projective morphism. Let (x, y) be in $\mathfrak{g} \times \mathfrak{h}$ such that $p(x) = p(y)$ for all p in $\mathrm{S}(\mathfrak{g})^G$. For some g in G, $g(x)$ is in $\mathfrak{b}$ and its semisimple component is y so that (x, y) is in the image of γ_{n}. As a result, $\mathcal{X}$ is irreducible as the image of the irreducible variety $G \times_B \mathfrak{b}$. Since for all (x, y) in $\mathcal{X} \cap \mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}_{\mathrm{reg}}$, there exists a unique w in $W(\mathcal{R})$ such that $y = w(x)$, the fiber of γ_{n} at any element $\mathcal{X} \cap G.(\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}_{\mathrm{reg}})$ has one element. Hence γ_{n} is birational, whence the assertion.

(iii) The morphism $\pi_{\mathfrak{h}}$ is finite, and so is $\overline{\gamma}$. Moreover $\pi_{\mathfrak{h}}$ is smooth over $\mathfrak{h}_{\mathrm{reg}}$, $\overline{\gamma}$ is smooth over $\mathfrak{g}_{\mathrm{reg}}$. Finally, $\pi_{\mathfrak{g}}$ is flat and all fibers are normal and Cohen-Macaulay. Thus the same holds for the

morphism $\overline{\rho}$. Since $\mathfrak{h}$ is smooth this implies that $\mathcal{X}$ is normal and Cohen-Macaulay by [MA86, Ch. 8, §23].

(iv) According to (ii), (iii) and Lemma 1.4, $\Bbbk[\mathcal{X}] = \mathrm{H}^0(G \times_B \mathfrak{b}, \mathcal{O}_{G\times_B\mathfrak{b}})$. Under the action of G in $\mathfrak{g} \times \mathfrak{h}$, $\Bbbk[\mathfrak{g} \times \mathfrak{h}]^G = \mathrm{S}(\mathfrak{g})^G \otimes_\Bbbk \mathrm{S}(\mathfrak{h})$ and its image in $\Bbbk[\mathcal{X}]$ by the quotient morphism equals $\mathrm{S}(\mathfrak{h})$. Moreover, since G is reductive, $\Bbbk[\mathcal{X}]^G$ is the image of $\Bbbk[\mathfrak{g} \times \mathfrak{h}]^G$ by the quotient morphism, whence the assertion. □

Proposition 2.3. [He76, Theorem B and Corollary] (i) *For $i > 0$, $\mathrm{H}^i(G/B, \mathcal{L}_0(\mathrm{S}(\mathfrak{b}^*))$ equals 0.*

(ii) *The variety $\mathcal{X}$ has rational singularities.*

Corollary 2.4. (i) *Let x and x' be in $\mathfrak{b}_{\mathrm{reg}}$ such that $(x', \overline{x'})$ is in $G.(x, \overline{x})$. Then x' is in $B(x)$.*

(ii) *For all w in $W(\mathcal{R})$, the map*

$$U_- \times \mathfrak{b}_{\mathrm{reg}} \longrightarrow \mathcal{X}, \qquad (g, x) \longmapsto (g(x), w(\overline{x}))$$

is an isomorphism onto a smooth open subset of $\mathcal{X}$.

Proof. (i) The semisimple components of x and x' are conjugate under B since they are conjugate to $\overline{x}$ under B. Let b and b' be in B such that $\overline{x}$ is the semisimple component of $b(x)$ and $b'(x')$. Then the nilpotent components of $b(x)$ and $b'(x')$ are regular nilpotent elements of $\mathfrak{g}^{\overline{x}}$, belonging to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^{\overline{x}}$ of $\mathfrak{g}^{\overline{x}}$. Hence x' is in $B(x)$.

(ii) Since the action of G and $W(\mathcal{R})$ on $\mathcal{X}$ commute, it suffices to prove the corollary for $w = 1_{\mathfrak{h}}$. Denote by θ the map

$$U_- \times \mathfrak{b}_{\mathrm{reg}} \longrightarrow \mathcal{X}, \qquad (g, x) \longmapsto (g(x), \overline{x}).$$

Let (g, x) and (g', x') be in $U_- \times \mathfrak{b}_{\mathrm{reg}}$ such that $\theta(g, x) = \theta(g', x')$. By (i), $x' = b(x)$ for some b in B. Hence $g^{-1}g'b$ is in G^x. Since x is in $\mathfrak{b}_{\mathrm{reg}}$, G^x is contained in B and $g^{-1}g'$ is in $U_- \cap B$, whence $(g, x) = (g', x')$ since $U_- \cap B = \{1_{\mathfrak{g}}\}$. As a result, θ is a dominant injective map from $U_- \times \mathfrak{b}_{\mathrm{reg}}$ to the normal variety $\mathcal{X}$. Hence θ is an isomorphism onto a smooth open subset of $\mathcal{X}$, by Zariski Main Theorem [Mu88, §9]. □

2.2. According to Lemma 2.1,(i), $G \times_B \mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and we have the commutative diagram:

$$\begin{array}{ccc} G \times_B \mathfrak{b}^k & \xrightarrow{\gamma_{\mathfrak{n}}} & \mathcal{B}^{(k)}_{\mathfrak{n}} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & \mathcal{B}^{(k)} & \end{array}$$

Lemma 2.5. *Let ϖ be the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. Denote by ι_k the map*

$$\mathfrak{b}^k \longrightarrow \mathcal{X}^k, \qquad (x_1, \dots, x_k) \longmapsto (x_1, \dots, x_k, \overline{x_1}, \dots, \overline{x_k}).$$

(i) *The map ι_k is a closed embedding of $\mathfrak{b}^k$ into $\mathcal{X}^k$.*

(ii) *The subvariety $\iota_k(\mathfrak{b}^k)$ of $\mathcal{X}^k$ is an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$.*

(iii) *The subvariety $\varpi^{-1}(\mathfrak{b}^k)$ of $\mathcal{X}^k$ is invariant under the canonical action of $W(\mathcal{R})^k$ in $\mathcal{X}^k$ and this action induces a simply transitive action of $W(\mathcal{R})^k$ on the set of irreducible components of $\varpi^{-1}(\mathfrak{b}^k)$.*

Proof. (i) The map

$$\mathfrak{b}^k \longrightarrow G^k \times \mathfrak{b}^k, \qquad (x_1, \dots, x_k) \longmapsto (1_{\mathfrak{g}}, \dots, 1_{\mathfrak{g}}, x_1, \dots, x_k)$$

defines through the quotient a closed embedding of $\mathfrak{b}^k$ in $G^k \times_{B^k} \mathfrak{b}^k$. Denote it by ι'. Let $\gamma_{\mathrm{n}}^{(k)}$ be the map

$$G^k \times_{B^k} \mathfrak{b}^k \longrightarrow \mathcal{X}^k, \qquad (x_1, \dots, x_k) \longmapsto (\gamma_{\mathrm{n}}(x_1), \dots, \gamma_{\mathrm{n}}(x_k)).$$

Then $\iota_k = \gamma_{\mathrm{n}}^{(k)} \circ \iota'$. Since γ_{n} is a projective morphism, ι_k is a closed morphism. Moreover, it is injective since $\varpi \circ \iota_k$ is the identity of $\mathfrak{b}^k$.

(ii) Since $\mathrm{S}(\mathfrak{h})$ is a finite extension of $\mathrm{S}(\mathfrak{h})^{W(\mathcal{R})}$, ϖ is a finite morphism. So $\varpi^{-1}(\mathfrak{b}^k)$ and $\mathfrak{b}^k$ have the same dimension. According to (i), $\iota_k(\mathfrak{b}^k)$ is an irreducible subvariety of $\omega^{-1}(\mathfrak{b}^k)$ of the same dimension, whence the assertion.

(iii) Since all the fibers of ϖ are invariant under the action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$, $\varpi^{-1}(\mathfrak{b}^k)$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\varpi^{-1}(\mathfrak{b}^k)$. For w in $W(\mathcal{R})^k$, set $Z_w := w.\iota_k(\mathfrak{b}^k)$. Then Z_w is an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$ for all w in $W(\mathcal{R})^k$ by (ii). For w in $W(\mathcal{R})^k$ such that $Z_w = \iota_k(\mathfrak{b}^k)$, for all $(x_1, \dots, x_k)$ in $\mathfrak{h}^k_{\mathrm{reg}}$, $(x_1, \dots, x_k, w.(x_1, \dots, x_k))$ is in $\iota_k(\mathfrak{b}^k)$ so that $(x_1, \dots, x_k)$ is invariant under w and w is the identity.

Let Z be an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$ and let Z_0 be its image by the map

$$(x_1, \dots, x_k, y_1, \dots, y_k) \longmapsto (\overline{x_1}, \dots, \overline{x_k}, y_1, \dots, y_k).$$

Since ϖ is G^k-equivariant and $\mathfrak{b}^k$ is invariant under B^k, $\varpi^{-1}(\mathfrak{b}^k)$ and Z are invariant under B^k. Hence by Lemma 1.8, Z_0 is closed. Moreover, since the image of the map

$$Z_0 \times \mathfrak{u}^k \longrightarrow \mathcal{X}^k, \qquad ((x_1, \dots, x_k, y_1, \dots, y_k), (u_1, \dots, u_k)) \longmapsto (x_1 + u_1, \dots, x_k + u_k, y_1, \dots, y_k)$$

is an irreducible subset of $\varpi^{-1}(\mathfrak{b}^k)$ containing Z, Z is the image of this map. Since Z_0 is contained in $\mathcal{X}^k$, Z_0 is contained in the image of the map

$$\mathfrak{h}^k \times W(\mathcal{R})^k \longrightarrow \mathfrak{h}^k \times \mathfrak{h}^k, \qquad (x_1, \dots, x_k, w_1, \dots, w_k) \longmapsto (x_1, \dots, x_k, w_1(x_1), \dots, w_k(x_k)).$$

Then, since $W(\mathcal{R})$ is finite and Z_0 is irreducible, for some w in $W(\mathcal{R})^k$, Z_0 is the image of $\mathfrak{h}^k$ by the map

$$(x_1, \dots, x_k) \longmapsto (x_1, \dots, x_k, w.(x_1, \dots, x_k))$$

and $Z = Z_w$, whence the assertion. □

Set $\mathfrak{Y} := G^k \times_{B^k} \mathfrak{b}^k$. The map

$$G \times \mathfrak{b}^k \longrightarrow G^k \times \mathfrak{b}^k, \qquad (g, v_1, \dots, v_k) \longmapsto (g, \dots, g, v_1, \dots, v_k)$$

defines through the quotient a closed immersion from $G \times_B \mathfrak{b}^k$ to $\mathfrak{Y}$. Denote it by ν. Consider the diagonal action of G on $\mathcal{X}^k$ and identify $G \times_B \mathfrak{b}^k$ with $\nu(G \times_B \mathfrak{b}^k)$ by the closed immersion ν.

Corollary 2.6. *Set* $\mathcal{B}_{\mathcal{X}}^{(k)} := G.\iota_k(\mathfrak{b}^k)$.

(i) *The subset* $\mathcal{B}_{\mathcal{X}}^{(k)}$ *is the image of* $G \times_B \mathfrak{b}^k$ *by* $\gamma_{\mathrm{n}}^{(k)}$. *Moreover, the restriction of* $\gamma_{\mathrm{n}}^{(k)}$ *to* $G \times_B \mathfrak{b}^k$ *is a projective birational morphism from* $G \times_B \mathfrak{b}^k$ *onto* $\mathcal{B}_{\mathcal{X}}^{(k)}$.

(ii) *The subset* $\mathcal{B}_{\mathcal{X}}^{(k)}$ *of* $\mathcal{X}^k$ *is an irreducible component of* $\varpi^{-1}(\mathcal{B}^{(k)})$.

(iii) *The subvariety* $\varpi^{-1}(\mathcal{B}^{(k)})$ *of* $\mathcal{X}^k$ *is invariant under* $W(\mathcal{R})^k$ *and this action induces a simply transitive action of* $W(\mathcal{R})^k$ *on the set of irreducible components of* $\varpi^{-1}(\mathcal{B}^{(k)})$.

(iv) *The subalgebra* $\Bbbk[\mathcal{B}^{(k)}]$ *of* $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)}]$ *equals* $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$ *with respect to the action of* $W(\mathcal{R})^k$ *on* $\varpi^{-1}(\mathcal{B}^{(k)})$.

Proof. (i) The variety G/B identifies with the diagonal of $(G/B)^k$ so that $G \times_B \mathfrak{b}^k$ is a closed subvariety of $G^k \times_{B^k} \mathfrak{b}^k$. Let $\gamma_{\mathcal{X}}$ be the restriction of $\gamma_{\mathrm{n}}^{(k)}$ to $G \times_B \mathfrak{b}^k$. Since $\iota_k = \gamma_{\mathrm{n}}^{(k)} \circ \iota'$, $G \times_B \mathfrak{b}^k = G.\iota'(\mathfrak{b}^{(k)})$ and $\gamma_{\mathrm{n}}^{(k)}$ is G-equivariant, $\mathcal{B}_{\mathcal{X}}^{(k)} = \gamma_{\mathcal{X}}(G \times_B \mathfrak{b}^k)$. Hence $\mathcal{B}_{\mathcal{X}}^{(k)}$ is closed in $\mathcal{X}^k$ and $\gamma_{\mathcal{X}}$ is a projective

morphism from $G \times_B \mathfrak{b}^k$ to $\mathcal{B}_{\mathcal{X}}^{(k)}$ since $\gamma_{\mathrm{n}}^{(k)}$ is a projective morphism. According to Lemma 2.1,(i), $\varpi \circ \gamma_{\mathcal{X}}$ is a birational morphism onto $\mathcal{B}^{(k)}$. Then $\gamma_{\mathcal{X}}$ is birational since $\varpi(\mathcal{B}_{\mathcal{X}}^{(k)}) = \mathcal{B}^{(k)}$, whence the assertion.

(ii) Since ϖ is a finite morphism, $\varpi^{-1}(\mathcal{B}^{(k)})$, $\mathcal{B}_{\mathcal{X}}^{(k)}$ and $\mathcal{B}^{(k)}$ have the same dimension, whence the assertion since $\mathcal{B}_{\mathcal{X}}^{(k)}$ is irreducible as an image of an irreducible variety.

(iii) Since the fibers of ϖ are invariant under $W(\mathcal{R})^k$, $\varpi^{-1}(\mathcal{B}^{(k)})$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\varpi^{-1}(\mathcal{B}^{(k)})$. Let Z be an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$. Since ϖ is G^k-equivariant, $\varpi^{-1}(\mathcal{B}^{(k)})$ and Z are invariant under the diagonal action of G. Moreover, $Z = G.(Z \cap \varpi^{-1}(\mathfrak{b}^k))$ since $\mathcal{B}^{(k)} = G.\mathfrak{b}^k$. Hence for some irreducible component Z_0 of $Z \cap \varpi^{-1}(\mathfrak{b}^k)$, $Z = G.Z_0$. According to Lemma 2.5,(iii), Z_0 is contained in $w.\iota_k(\mathfrak{b}^k)$ for some w in $W(\mathcal{R})^k$. Hence $Z = w.\mathcal{B}_{\mathcal{X}}^{(k)}$ since the actions of G^k and $W(\mathcal{R})^k$ on $\mathcal{X}^k$ commute and Z is an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$.

Let $w = (w_1, \dots, w_k)$ be in $W(\mathcal{R})^k$ such that $w.\mathcal{B}_{\mathcal{X}}^{(k)} = \mathcal{B}_{\mathcal{X}}^{(k)}$. Let x be in $\mathfrak{b}_{\mathrm{reg}}$ and let $i = 1, \dots, k$. Set:

$$z := (x_1, \dots, x_k, \overline{x_1}, \dots, \overline{x_k}) \text{ with } x_j := \begin{cases} x & \text{if} \quad j = i \\ x_j = e & \text{otherwise} \end{cases}.$$

Then there exists $(y_1, \dots, y_k)$ in $\mathfrak{b}^k$ and g in G such that

$$w.z = (g(y_1), \dots, g(y_k), \overline{y_1}, \dots, \overline{y_k}).$$

For some b in B, $b(y_i) = \overline{y_i}$ since y_i is a regular semisimple element, belonging to $\mathfrak{b}$. As a result, $gb^{-1}(\overline{y_i}) = x$ and $w_i(x) = \overline{y_i}$. Hence gb^{-1} is an element of $N_G(\mathfrak{b})$ representing w_i^{-1}. Furthermore, since $gb^{-1}(b(y_j)) = e$ for $j \neq i$, $b(y_j)$ is a regular nilpotent element belonging to $\mathfrak{b}$. Then, since there is one and only one Borel subalgebra containing a regular nilpotent element, $gb^{-1}(\mathfrak{b}) = \mathfrak{b}$ and $w_i = 1_{\mathfrak{b}}$. As a result, w is the identity of $W(\mathcal{R})^k$, whence the assertion.

(iv) Since the fibers of ϖ are invariant under $W(\mathcal{R})^k$, $\Bbbk[\mathcal{B}^{(k)}]$ is contained in $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$. Let p be in $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$. Since $W(\mathcal{R})$ is a finite group, p is the restriction to $\varpi^{-1}(\mathcal{B}^{(k)})$ of an element q of $\Bbbk[\mathcal{X}]^{\otimes k}$, invariant under $W(\mathcal{R})^k$. Since $\Bbbk[\mathcal{X}]^{W(\mathcal{R})} = \mathrm{S}(\mathfrak{g})$, q is in $\mathrm{S}(\mathfrak{g})^{\otimes k}$ by Lemma 2.1,(iv), and p is in $\Bbbk[\mathcal{B}^{(k)}]$, whence the assertion. □

2.3. For α a positive root, denote by $\mathfrak{h}_\alpha$ the kernel of α and by S_α the closure of $U(\mathfrak{h}_\alpha)$ in $\mathfrak{b}$. For β in Π, set:

$$\mathfrak{u}_\beta := \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} \mathfrak{g}^\beta, \qquad \mathfrak{b}_\beta := \mathfrak{h}_\beta \oplus \mathfrak{u}_\beta.$$

Lemma 2.7. *For α in $\mathcal{R}_+$, let $\mathfrak{h}'_\alpha$ be the set of subregular elements belonging to $\mathfrak{h}_\alpha$.*

(i) *For α in $\mathcal{R}_+$, S_α is a subvariety of codimension 2 of $\mathfrak{b}$. Moreover, it is contained in $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$.*

(ii) *For β in Π, $S_\beta = \mathfrak{b}_\beta$.*

(iii) *The S_α's, $\alpha \in \mathcal{R}_+$, are the irreducible components of $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$.*

Proof. (i) For x in $\mathfrak{h}'_\alpha$, $\mathfrak{b}^x = \mathfrak{h} + \Bbbk x_\alpha$. Hence $U(\mathfrak{h}'_\alpha)$ has dimension $n - 1 + \ell - 1$, whence the assertion since $U(\mathfrak{h}'_\alpha)$ is dense in S_α and $\mathfrak{h}'_\alpha$ is contained in $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$.

(ii) For β in Π, $U(\mathfrak{h}'_\beta)$ is contained in $\mathfrak{b}_\beta$ since $\mathfrak{b}_\beta$ is an ideal of $\mathfrak{b}$, whence the assertion by (i).

(iii) According to (i), it suffices to prove that $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$ is the union of the S_α's. Let x be in $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$. According to [V72], for some g in G and for some β in Π, x is in $g(\mathfrak{b}_\beta)$. Since $\mathfrak{b}_\beta$ is an ideal of $\mathfrak{b}$, by Bruhat's decomposition of G, for some b in B and for some w in $W(\mathcal{R})$, $b^{-1}(x)$ is in $w(\mathfrak{b}_\beta) \cap \mathfrak{b}$.

By definition,

$$w(\mathfrak{b}_\beta) = w(\mathfrak{h}_\beta) \oplus w(\mathfrak{u}_\beta) = \mathfrak{h}_{w(\beta)} \oplus \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} \mathfrak{g}^{w(\alpha)}.$$

So,

$$w(\mathfrak{b}_\beta) \cap \mathfrak{b} = \mathfrak{h}_{w(\beta)} \oplus \mathfrak{u}_0 \text{ with } \mathfrak{u}_0 := \bigoplus_{\substack{\alpha \in \mathcal{R}_+ \setminus \{\beta\} \\ w(\alpha) \in \mathcal{R}_+}} \mathfrak{g}^{w(\alpha)}.$$

The subspace $\mathfrak{u}_0$ of $\mathfrak{u}$ is a subalgebra, not containing $\mathfrak{g}^{w(\beta)}$. Then, denoting by U_0 the closed subgroup of U whose Lie algebra is $\operatorname{ad}\mathfrak{u}_0$,

$$\overline{U_0(\mathfrak{h}_{w(\beta)})} = w(\mathfrak{b}_\beta) \cap \mathfrak{b}$$

since the left hand side is contained in the right hand side and has the same dimension. As a result, x is in $S_{w(\beta)}$ since $S_{w(\beta)}$ is B-invariant, whence the assertion. □

Recall that θ is the map

$$U_- \times \mathfrak{b}_{\mathrm{reg}} \longrightarrow \mathcal{X}, \qquad (g, x) \longmapsto (g(x), \overline{x})$$

and denote by W'_k the inverse image of $\theta(U_- \times \mathfrak{b}_{\mathrm{reg}})$ by the projection

$$\mathcal{B}_{\mathcal{X}}^{(k)} \longrightarrow \mathcal{X}, \qquad (x_1, \dots, x_k, y_1, \dots, y_k) \longmapsto (x_1, y_1).$$

Lemma 2.8. *Let W_k be the subset of elements (x, y) of $\mathcal{B}_{\mathcal{X}}^{(k)}$ ($x \in \mathfrak{g}^k, y \in \mathfrak{b}^k$) such that $P_x \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty.*

(i) *The subset W'_k of $\mathcal{B}_{\mathcal{X}}^{(k)}$ is a smooth open subset. Moreover, the map*

$$U_- \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1} \longrightarrow W'_k, \qquad (g, x_1, \dots, x_k) \longmapsto (g(x_1), \dots, g(x_k), \overline{x_1}, \dots, \overline{x_k}).$$

is an isomorphism of varieties.

(ii) *The subset $\mathcal{B}_{\mathcal{X}}^{(k)}$ of $\mathfrak{g}^k \times \mathfrak{b}^k$ is invariant under the canonical action of $\mathrm{GL}_k(\Bbbk)$.*

(iii) *The subset W_k of $\mathcal{B}_{\mathcal{X}}^{(k)}$ is a smooth open subset. Moreover, W_k is the $G \times \mathrm{GL}_k(\Bbbk)$-invariant set generated by W'_k.*

(iv) *The subvariety $\mathcal{B}_{\mathcal{X}}^{(k)} \setminus W_k$ of $\mathcal{B}_{\mathcal{X}}^{(k)}$ has codimension at least $2k$.*

Proof. (i) According to Corollary 2.4,(ii), θ is an isomorphism onto a smooth open subset of $\mathcal{X}$. As a result, W'_k is an open subset of $\mathcal{B}_{\mathcal{X}}^{(k)}$ and the map

$$U_- \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1} \longrightarrow W'_k, \qquad (g, x_1, \dots, x_k) \longmapsto (g(x_1), \dots, g(x_k), \overline{x_1}, \dots, \overline{x_k})$$

is an isomorphism whose inverse is given by

$$\begin{array}{rl} W'_k \longrightarrow & U_- \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1} \\ (x_1, \dots, x_k) \longmapsto & (\theta^{-1}(x_1, \overline{x_1})_1, \theta^{-1}(x_1, \overline{x_1})_1(x_1), \dots, \theta^{-1}(x_1, \overline{x_1})_1(x_k)) \end{array}$$

with θ^{-1} the inverse of θ and $\theta^{-1}(x_1, \overline{x_1})_1$ the component of $\theta^{-1}(x_1, \overline{x_1})$ on U_-, whence the assertion since $U_- \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1}$ is smooth.

(ii) For $(x_1, \dots, x_k)$ in $\mathfrak{b}^k$ and for $(a_{i,j}, 1 \le i, j \le k)$ in $\mathrm{GL}_k(\Bbbk)$,

$$\overline{\sum_{j=1}^{k} a_{i,j} x_j} = \sum_{j=1}^{k} a_{i,j} \overline{x_j}$$

so that $\iota_k(\mathfrak{b}^k)$ is invariant under the action of $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{g}^k \times \mathfrak{h}^k$ defined by

$$(a_{i,j}, 1 \leq i, j \leq k).(x_1, \ldots, x_k, y_1, \ldots, y_k) := (\sum_{j=1}^{k} a_{i,j}x_j,\ j = 1, \ldots, k, \sum_{j=1}^{k} a_{i,j}y_j,\ j = 1, \ldots, k),$$

whence the assertion since $\mathcal{B}_{\mathcal{X}}^{(k)} = G.\iota_k(\mathfrak{b}^k)$ and the actions of G and $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{g}^k \times \mathfrak{h}^k$ commute.

(iii) According to (i), $G.W_k'$ is a smooth open subset of $\mathcal{B}_{\mathcal{X}}^{(k)}$. Moreover, $G.W_k'$ is the subset of elements (x, y) such that the first component of x is regular. So, by (ii) and Lemma 1.9, $W_k = \mathrm{GL}_k(\Bbbk).(G.W_k')$, whence the assertion.

(iv) According to Corollary 2.6,(i), $\mathcal{B}_{\mathcal{X}}^{(k)}$ is the image of $G \times_B \mathfrak{b}^k$ by the restriction $\gamma_{\mathcal{X}}$ of $\gamma_{\mathfrak{n}}^{(k)}$ to $G \times_B \mathfrak{b}^k$. Then $\mathcal{B}_{\mathcal{X}}^{(k)} \setminus W_k$ is contained in the image of $G \times_B (\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}})^k$ by $\gamma_{\mathcal{X}}$. As a result, by Lemma 2.7,

$$\dim \mathcal{B}_{\mathcal{X}}^k \setminus W_k \leq n + k(\mathrm{b}_{\mathfrak{g}} - 2),$$

whence the assertion. □

2.4. For E a B-module, denote by $\mathcal{L}_0(E)$ the sheaf of local sections of the vector bundle $G \times_B E$ over G/B. Let Δ be the diagonal of $(G/B)^k$ and let $\mathcal{J}_\Delta$ be its ideal of definition in $\mathcal{O}_{(G/B)^k}$. The variety G/B identifies with Δ so that $\mathcal{O}_{(G/B)^k}/\mathcal{J}_\Delta$ is isomorphic to $\mathcal{O}_{G/B}$. For E a B^k-module, denote by $\mathcal{L}(E)$ the sheaf of local sections of the vector bundle $G^k \times_{B^k} E$ over $(G/B)^k$.

Lemma 2.9. *Let E be a B^k-module and let A be a trivial B^k-module. Denote by $\overline{E}$ the B-module defined by the diagonal action of B on E.*

(i) *The short sequence of $\mathcal{O}_{(G/B)^k}$-modules*

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{G^k \times_{B^k} \mathfrak{b}^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}_0(\overline{E}) \longrightarrow 0$$

is exact.

(ii) *The space $\mathrm{H}^0((G/B)^k, \mathcal{L}(E \otimes_{\Bbbk} A))$ is equal to $\mathrm{H}^0((G/B)^k, \mathcal{L}(E)) \otimes_{\Bbbk} A$.*

Proof. (i) Since $\mathcal{L}(E)$ is a locally free $\mathcal{O}_{(G/B)^k}$-module, the short sequence of $\mathcal{O}_{(G/B)^k}$-modules

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{O}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow 0$$

is exact, whence the assertion since $\mathcal{O}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E)$ is isomorphic to $\mathcal{L}_0(\overline{E})$.

(ii) Since A is a trivial B^k-module, the vector bundle $G^k \times_{B^k} (E \otimes_{\Bbbk} A)$ is isomorphic to $(G^k \times_{B^k} E) \otimes_{\Bbbk} A$, whence

$$\mathcal{L}(E \otimes_{\Bbbk} A) = \mathcal{L}(E) \otimes_{\Bbbk} A \quad \text{and} \quad \mathrm{H}^0((G/B)^k, \mathcal{L}(E \otimes_{\Bbbk} A)) = \mathrm{H}^0((G/B)^k, \mathcal{L}(E)) \otimes_{\Bbbk} A.$$

□

From Lemma 2.9 results a canonical morphism

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(E)) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{E}))$$

for all B^k-module E. According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle ., . \rangle$, the duals of $\mathfrak{b}$ and $\mathfrak{u}$ identify with $\mathfrak{b}_-$ and $\mathfrak{u}_-$ respectively so that $\mathfrak{b}_-$ and $\mathfrak{u}_-$ are B-modules.

Lemma 2.10. (i) *The algebra $\Bbbk[\mathcal{B}_{\mathfrak{n}}^{(k)}]$ is equal to $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k)}))$.*

(ii) *The algebra $\Bbbk[\mathcal{N}_{\mathfrak{n}}^{(k)}]$ is equal to $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k)}))$.*

(iii) *The algebra* $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ *is the image of the morphism*

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k))) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k)})).$$

Proof. (i) Since $G \times_B \mathfrak{b}^k$ is a desingularization of the normal variety $\mathcal{B}_{\mathrm{n}}^{(k)}$, $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is the space of global sections of $\mathcal{O}_{G\times_B\mathfrak{b}^k}$ by Lemma 1.4. Let π be the bundle projection of the fiber bundle $G\times_B\mathfrak{b}^k$. Since $\mathrm{S}(\mathfrak{b}_-^k)$ is the space of polynomial functions on $\mathfrak{b}^k$,

$$\pi_*(\mathcal{O}_{G\times_B\mathfrak{b}^k}) = \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k)}),$$

whence the assertion.

(ii) By Lemma 2.1,(ii), $G\times_B\mathfrak{u}^k$ is a desingularization of $\mathcal{N}_{\mathrm{n}}^{(k)}$ so that $\Bbbk[\mathcal{N}_{\mathrm{n}}^{(k)}]$ is the space of global sections of $\mathcal{O}_{G\times_B\mathfrak{u}^k}$ by Lemma 1.4. Denoting by π_0 the bundle projection of $G\times_B\mathfrak{u}^k$,

$$\pi_{0*}(\mathcal{O}_{G\times_B\mathfrak{u}^k}) = \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k)}),$$

whence the assertion.

(iii) Since $G^k\times_{B^k}\mathfrak{b}^k$ is isomorphic to $(G\times_B\mathfrak{b})^k$,

$$\mathrm{H}^0((G/B)^k, \mathcal{O}_{G^k\times_{B^k}\mathfrak{b}^k}) = \mathrm{H}^0(G/B, \mathcal{O}_{G\times_B\mathfrak{b}})^{\otimes k}.$$

By (i),

$$H^0(G/B, \mathcal{O}_{G\times_B\mathfrak{b}}) = \mathrm{H}^0(G/B, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-)) = \Bbbk[\mathcal{X}]$$

since $G\times_B\mathfrak{b}$ is a desingularization of $\mathcal{X}$ by Lemma 2.1,(i) and (ii), whence

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k)) = \Bbbk[\mathcal{X}^k].$$

By definition, $\mathcal{B}_{\mathcal{X}}^{(k)}$ is a closed subvariety of $\mathcal{X}^k$. According to Corollary 2.6, $\Bbbk[\mathcal{B}^{(k)}]$ is a subalgebra of $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ having the same fraction field. Hence $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ is a subalgebra of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ having the same fraction field. Then by (i), the image of the morphism

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathfrak{b}_-^k)) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k)}))$$

is equal to $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$. □

Proposition 2.11. *Let A be the image of the morphism*

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{u}_-^k))) \longrightarrow \mathrm{H}^0((G/B)^k, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k)})).$$

(i) *The algebra* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ *is equal to* $\Bbbk[\mathcal{N}_{\mathrm{n}}^{(k)}]\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h}^k)$.

(ii) *The algebra* $\Bbbk[\mathcal{B}_{\mathcal{X}}^{(k)}]$ *is equal to* $A\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h}^k)$.

Proof. Since $\mathfrak{b}_-^k$ is the direct sum of $\mathfrak{u}_-^k$ and $\mathfrak{h}^k$,

$$\mathrm{S}(\mathfrak{b}_-^k) = \mathrm{S}(\mathfrak{u}_-^k)\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h}^k).$$

In the B-module $\mathfrak{b}_-$, $\mathfrak{h}$ is a subspace of invariant elements. Then $\mathrm{S}(\mathfrak{h}^k)$ is a trivial B^k-module. So, by Lemma 2.10,(i) and (ii), and Lemma 2.9,(ii),

$$\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] = \Bbbk[\mathcal{N}_{\mathrm{n}}^{(k)}]\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h}^k).$$

Moreover,

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k))) = \mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{u}_-^k)))\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h}^k),$$

whence the proposition by Lemma 2.10,(iii). □

Remark 2.12. According to Proposition 2.11, $\mathrm{S}(\mathfrak{h}^k)$ is embedded in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ and by Lemma 2.10,(iii), the embedding is given by the map

$$\mathrm{S}(\mathfrak{h}^k) \longrightarrow \Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}], \qquad p \longmapsto ((x_1, \dots, x_k, y_1, \dots, y_k) \mapsto p(y_1, \dots, y_k).$$

Denote by Φ this map.

Theorem 2.13. (i) *The variety $\mathcal{B}_{\mathfrak{X}}^{(k)}$ is normal. In particular, $\mathcal{B}_{\mathrm{n}}^{(k)} = \mathcal{B}_{\mathfrak{X}}^{(k)}$, $\gamma_{\mathrm{n}} = \gamma_{\mathfrak{X}}$ and the normalization morphism of $\mathcal{B}^{(k)}$ is the restriction of ϖ to $\mathcal{B}_{\mathfrak{X}}^{(k)}$.*

(ii) *The image of Φ is equal to $\Bbbk[\mathcal{B}_{\mathfrak{X}}]^G$. Moreover, $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\Bbbk[\mathcal{B}_{\mathfrak{X}}]^G$.*

(iii) *The subalgebras $\Bbbk[\mathcal{B}^{(k)}]^G$ and $\Phi(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]^G$ are equal.*

Proof. (i) Since $\mathrm{S}(\mathfrak{h}^k)$ is a polynomial algebra of dimension $k\ell$, the depth of localizations of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ at every maximal ideal is at least $k\ell$ by Proposition 2.11,(ii). In particular, it is at least 2 since $k \geq 2$. According to Lemma 2.8,(iii) and (iv), $\mathcal{B}_{\mathfrak{X}}^{(k)}$ is smooth in codimension 1. Then $\mathcal{B}_{\mathfrak{X}}^{(k)}$ is normal by Serre's normality criterion [Bou98, §1, n°10, Théorème 4], whence the assertion by Corollary 2.6,(i).

(ii) Since $\mathcal{B}_{\mathfrak{X}}^{(k)}$ is a closed subvariety of $\mathfrak{X}^k$ and $\Bbbk[\mathfrak{X}]$ is generated by $\mathrm{S}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{h})$, $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ is generated by $\mathrm{S}(\mathfrak{h}^k)$ and the image of $\mathrm{S}(\mathfrak{g}^k)$ in $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ which is equal to $\Bbbk[\mathcal{B}^{(k)}]$. For p in $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$, denote by $\overline{p}$ the element of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ such that

$$\overline{p}(x_1, \dots, x_k) := p(x_1, \dots, x_k, x_1, \dots, x_k).$$

Then the restriction of $p - \Phi(\overline{p})$ to $\iota_k(\mathfrak{h}^k)$ equals 0. Moreover, if p is in $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]^G$, $p - \Phi(\overline{p})$ is G-invariant so that $p = \Phi(\overline{p})$, whence the assertion.

(iii) For $(x_1, \dots, x_k)$ in $\mathfrak{h}^k$, for w in $W(\mathcal{R})$ and for g_w a representative of w in $N_G(\mathfrak{h})$, we have

$$(w(x_1), \dots, w(x_k), w(x_1), \dots, w(x_k)) = g_w.(x_1, \dots, x_k, w(x_1), \dots, w(x_k))$$

so that the subalgebra $\Bbbk[\mathcal{B}^{(k)}]^G$ of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]^G$ is contained in $\Phi(\mathrm{S}(\mathfrak{h}^k))^{W(\mathcal{R})}$ by (ii). Moreover, since G is reductive, $\Bbbk[\mathcal{B}^{(k)}]^G$ is the image of $(\mathrm{S}(\mathfrak{g})^{\otimes k})^G$ by the restriction morphism. According to [J07, Theorem 2.9 and some remark], the restriction morphism $(\mathrm{S}(\mathfrak{g})^{\otimes k})^G \to (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is surjective, whence the equality $\Bbbk[\mathcal{B}^{(k)}]^G = \Phi(\mathrm{S}(\mathfrak{h}^k))^{W(\mathcal{R})}$. □

Corollary 2.14. *Let M be a graded complement to $\Bbbk[\mathcal{B}^{(k)}]^G_+\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

(i) *The space M contains a basis of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{\otimes k}]$ over $\mathrm{S}(\mathfrak{h})^{\otimes k}$.*

(ii) *The intersection of M and $\mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ is different from 0.*

Proof. (i) Since M is a graded complement to $\Bbbk[\mathcal{B}^{(k)}]^G_+\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$, by induction on l,

$$\Bbbk[\mathcal{B}^{(k)}] = M\Bbbk[\mathcal{B}^{(k)}]^G + (\Bbbk[\mathcal{B}^{(k)}]^G_+)^l\Bbbk[\mathcal{B}^{(k)}].$$

Hence $\Bbbk[\mathcal{B}^{(k)}] = M\Bbbk[\mathcal{B}^{(k)}]^G$ since $\Bbbk[\mathcal{B}^{(k)}]$ is graded. Then, by Theorem 2.13,(ii), $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}] = M\mathrm{S}(\mathfrak{h})^{\otimes k}$. In particular,

$$\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}] = M + \mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}].$$

Then M contains a graded complement M' to $\mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ in $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$, whence the assertion by Proposition 2.11,(i).

(ii) Suppose that $M' = M$. We expect a contradiction. According to (i), the canonical maps

$$M \otimes_{\Bbbk} \mathrm{S}(\mathfrak{h})^{\otimes k} \longrightarrow \Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}], \qquad M \otimes_{\Bbbk} \Bbbk[\mathcal{B}^{(k)}]^G \longrightarrow \Bbbk[\mathcal{B}^{(k)}]$$

are isomorphisms. Then, according to Lemma 1.5, there exists a group action of $W(\mathcal{R})$ on $\Bbbk[\mathcal{B}_{\mathfrak{n}}^{(k)}]$ extending the diagonal action of $W(\mathcal{R})$ in $\mathrm{S}(\mathfrak{h})^{\otimes k}$ and such that $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]^{W(\mathcal{R})} = \Bbbk[\mathcal{B}^{(k)}]$ since $\Bbbk[\mathcal{B}^{(k)}] \cap \mathrm{S}(\mathfrak{h})^{\otimes k} = (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ by Theorem 2.13,(iii). Moreover, since $W(\mathcal{R})$ is finite, the subfield of invariant elements of the fraction field of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ is the fraction field of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]^{W(\mathcal{R})}$. Hence the action of $W(\mathcal{R})$ in $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ is trivial since $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ and $\Bbbk[\mathcal{B}^{(k)}]$ have the same fraction field, whence the contradiction since $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is strictly contained in $\mathrm{S}(\mathfrak{h})^{\otimes k}$. □

Corollary 2.15. *The restriction of $\gamma_{\mathfrak{X}}$ to $\gamma_{\mathfrak{X}}^{-1}(W_k)$ is an isomorphism onto W_k.*

Proof. Since the subset of Borel subalgebras containing a regular element is finite, the fibers of $\gamma_{\mathfrak{X}}$ over the elements of W_k are finite. Indeed, according to Zariski Main Theorem [Mu88, §9], they have only one element since $\mathcal{B}_{\mathfrak{X}}^{(k)}$ is normal by Theorem 2.13 and since $\gamma_{\mathfrak{X}}$ is projective and birational. So, the restriction of $\gamma_{\mathfrak{X}}$ to $\gamma_{\mathfrak{X}}^{-1}(W_k)$ is a bijection onto the open subset W_k, whence the corollary by Zariski Main Theorem [Mu88, §9] since W_k is smooth by Lemma 2.8. □

3. On the nullcone

Let $k \geq 2$ be an integer. Let I be the ideal of $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ generated by $\mathrm{S}_+(\mathfrak{h}^k)$ and let N be the subscheme of $\mathcal{B}_{\mathfrak{X}}^{(k)}$ defined by I. Recall that according to Theorem 2.13,(i), η is the canonical projection from $\mathcal{B}_{\mathfrak{X}}^{(k)}$ onto $\mathcal{B}^{(k)}$.

Lemma 3.1. *Set $\mathcal{N}_{\mathfrak{X}}^{(k)} := \eta^{-1}(\mathcal{N}^{(k)})$.*

(i) *The variety $\mathcal{N}_{\mathfrak{X}}^{(k)}$ equals $\gamma_{\mathfrak{X}}(G \times_B \mathfrak{u}^k)$.*

(ii) *The nullvariety of I in $\mathcal{B}_{\mathfrak{X}}^{(k)}$ equals $\mathcal{N}_{\mathfrak{X}}^{(k)}$.*

Proof. (i) By definition, $\gamma^{-1}(\mathcal{N}^{(k)}) = G \times_B \mathfrak{u}^k$. Then, since $\gamma = \eta \circ \gamma_{\mathfrak{X}}$, $\mathcal{N}_{\mathfrak{X}}^{(k)} = \gamma_{\mathfrak{X}}(G \times_B \mathfrak{u}^k)$.

(ii) Let $\mathcal{V}_I$ be the nullvariety of I in $\mathcal{B}_{\mathfrak{X}}^{(k)}$. According to Remark 2.12, for $(g, x_1, \dots, x_k)$ in $G \times \mathfrak{b}^k$, $\gamma_{\mathfrak{X}}(\overline{(g, x_1, \dots, x_k)})$ is a zero of I if and only if $x_1, \dots, x_k$ are nilpotent, whence the assertion. □

Theorem 3.2. *Let I_0 be the ideal of $\Bbbk[\mathcal{B}^{(k)}]$ generated by $\Bbbk[\mathcal{B}^{(k)}]_+^G$.*

(i) *The variety $\mathcal{N}_{\mathfrak{X}}^{(k)}$ is a normal variety and I is its ideal of definition in $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$. In particular, I is prime.*

(ii) *The variety $\mathcal{N}^{(k)}$ is normal.*

(iii) *The ideal I_0 is strictly contained in the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

Proof. (i) According to Theorem 2.13,(i) and Proposition 2.11,(i),

$$\Bbbk[\mathcal{B}_{\mathfrak{X}}^k] = \Bbbk[\mathcal{N}_{\mathfrak{n}}^k] \otimes_{\Bbbk} \mathrm{S}(\mathfrak{h}^k).$$

Hence $\Bbbk[N] = \Bbbk[\mathcal{N}_{\mathfrak{n}}^{(k)}]$. As a result, N is an irreducible normal scheme and I is a prime ideal. So, by Lemma 3.1,(ii), $\mathcal{N}_{\mathfrak{X}}^{(k)}$ is a normal variety.

(ii) According to Theorem 2.13,(ii), $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. So, the restriction to $\Bbbk[\mathcal{B}^{(k)}]$ of the quotient map from $\Bbbk[\mathcal{B}_{\mathfrak{X}}^{(k)}]$ to $\Bbbk[\mathcal{N}_{\mathfrak{X}}^{(k)}]$ is surjective. Furthermore, the image of $\Bbbk[\mathcal{B}^{(k)}]$ by this morphism equals $\Bbbk[\mathcal{N}^{(k)}]$ since $\mathcal{N}_{\mathfrak{X}}^{(k)} = \eta^{-1}(\mathcal{N}^{(k)})$, whence $\Bbbk[\mathcal{N}^{(k)}] = \Bbbk[\mathcal{N}_{\mathfrak{X}}^{(k)}]$. As a result, $\mathcal{N}^{(k)}$ is normal by (i).

(iii) Since $\Bbbk[\mathcal{B}^{(k)}]_+^G$ is contained in $\mathrm{S}_+(\mathfrak{h}^k)$, I_0 is contained in $I \cap \Bbbk[\mathcal{B}^{(k)}]$. According to Lemma 3.1,(ii) and (i), $I \cap \Bbbk[\mathcal{B}^{(k)}]$ is the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$. Let M be a graded complement of $\Bbbk[\mathcal{B}^{(k)}]_+^G \Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$. According to Corollary 2.14,(ii), $I \cap M$ is different from 0. Hence I_0 is strictly contained in $I \cap \Bbbk[\mathcal{B}^{(k)}]$, whence the assertion. □

4. Main varieties

Denote by X the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B. According to Lemma 1.7, $G.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G.

4.1. For α in $\mathcal{R}$, denote by $\mathfrak{h}_\alpha$ the kernel of α. Set $V_\alpha := \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha$ and denote by X_α the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of V_α under B.

Lemma 4.1. *Let α be in $\mathcal{R}_+$. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$ and let P be its normalizer in G.*

(i) *The subset $P.X$ of $\mathrm{Gr}_\ell(\mathfrak{g})$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under P.*

(ii) *The closed set X_α of $\mathrm{Gr}_\ell(\mathfrak{g})$ is an irreducible component of $X \setminus B.\mathfrak{h}$.*

(iii) *The set $P.X_\alpha$ is an irreducible component of $P.X \setminus P.\mathfrak{h}$.*

(iv) *The varieties $X \setminus B.\mathfrak{h}$ and $P.X \setminus P.\mathfrak{h}$ are equidimenional of codimension 1 in X and $P.X$ respectively.*

Proof. (i) Since X is a B-invariant closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$, $P.X$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ by Lemma 1.7. Hence $\overline{P.\mathfrak{h}}$ is contained in $P.X$ since $\mathfrak{h}$ is in X, whence the assertion since $\overline{P.\mathfrak{h}}$ is a P-invariant subset containing X.

(ii) Denoting by H_α the coroot of α,

$$\lim_{t\to\infty} \exp(t\,\mathrm{ad}\, x_\alpha)(\frac{-1}{2t}H_\alpha) = x_\alpha.$$

So V_α is in the closure of the orbit of $\mathfrak{h}$ under the one parameter subgroup of G generated by $\mathrm{ad}\, x_\alpha$. As a result, X_α is a closed subset of $X \setminus B.\mathfrak{h}$ since V_α is not a Cartan subalgebra. Moreover, X_α has dimension $n-1$ since the normalizer of V_α in $\mathfrak{g}$ is $\mathfrak{h} + \mathfrak{g}^\alpha$. Hence X_α is an irreducible component of $X \setminus B.\mathfrak{h}$ since X has dimension n.

(iii) Since X_α is a B-invariant closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$, $P.X_\alpha$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ by Lemma 1.7. According to (ii), $P.X_\alpha$ is contained in $P.X \setminus P.\mathfrak{h}$ and it has dimension $\dim \mathfrak{p} - \ell - 1$, whence the assertion since $P.X$ has dimension $\dim \mathfrak{p} - \ell$.

(iv) Let P_u be the unipotent radical of P and let L be the reductive factor of P whose Lie algebra contains $\mathrm{ad}\,\mathfrak{h}$. Denote by $N_L(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in L. Since $B.\mathfrak{h}$ and $P.\mathfrak{h}$ are isomorphic to U and $L/N_L(\mathfrak{h}) \times P_\mathrm{u}$ respectively, they are affine open subsets of X and $P.X$ respectively, whence the assertion by [EGAIV, Corollaire 21.12.7]. □

For x in V, set:

$$V_x := \mathrm{span}(\{\varepsilon_1(x), \dots, \varepsilon_\ell(x)\}).$$

Lemma 4.2. *Let Δ be the set of elements (x, V) of $\mathfrak{g} \times G.X$ such that x is in V.*

(i) *For (x, V) in $\mathfrak{b} \times X$, (x, V) is in the closure of $B.(\mathfrak{h}_{\mathrm{reg}} \times \{\mathfrak{h}\})$ in $\mathfrak{b} \times \mathrm{Gr}_\ell(\mathfrak{b})$ if and only if x is in V.*

(ii) *The set Δ is the closure in $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$ of $G.(\mathfrak{h}_{\mathrm{reg}} \times \{\mathfrak{h}\})$.*

(iii) *For (x, V) in Δ, V_x is contained in V.*

Proof. (i) Let Δ' be the subset of elements (x, V) of $\mathfrak{b} \times X$ such that x is in V and let Δ'_0 be the closure of $B.(\mathfrak{h}_{\mathrm{reg}} \times \{\mathfrak{h}\})$ in $\mathfrak{b} \times \mathrm{Gr}_\ell(\mathfrak{b})$. Then Δ' is a closed subset of $\mathfrak{b} \times \mathrm{Gr}_\ell(\mathfrak{b})$ containing Δ'_0. Let (x, V) be in Δ'. Let E be a complement to V in $\mathfrak{b}$ and let Ω_E be the set of complements to E in $\mathfrak{g}$. Then Ω_E is an open neighborhood of V in $\mathrm{Gr}_\ell(\mathfrak{b})$. Moreover, the map

$$\mathrm{Hom}_{\Bbbk}(V, E) \xrightarrow{\kappa} \Omega_E, \qquad \varphi \longmapsto \kappa(\varphi) := \mathrm{span}(\{v + \varphi(v) \mid v \in V\}).$$

is an isomorphism of varieties. Let Ω_E^c be the inverse image of the set of Cartan subalgebras. Then 0 is in the closure of Ω_E^c in $\mathrm{Hom}_{\Bbbk}(V, E)$ since V is in X. For all φ in Ω_E^c, $(x + \varphi(x), \kappa(\varphi))$ is in Δ_0'. Hence (x, V) is in Δ_0'.

(ii) Let (x, V) be in Δ. For some g in G, $g(V)$ is in X. So by (i), $(g(x), g(V))$ is in Δ_0' and (x, V) is in the closure of $G.(\mathfrak{h}_{\mathrm{reg}} \times \{\mathfrak{h}\})$ in $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$, whence the assertion.

(iii) For $i = 1, \dots, \ell$, let Δ_i be the set of elements (x, V) of Δ such that $\varepsilon_i(x)$ is in V. Then Δ_i is a closed subset of $\mathfrak{g} \times G.X$, invariant under the action of G in $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$ since ε_i is a G-equivariant map. For all (g, x) in $G \times \mathfrak{h}_{\mathrm{reg}}$, $(g(x), g(\mathfrak{h}))$ is in Δ_i since $\varepsilon_i(g(x))$ centralizes $g(x)$. Hence $\Delta_i = \Delta$ since $G.(\mathfrak{h}_{\mathrm{reg}} \times \{\mathfrak{h}\})$ is dense in Δ by (ii). As a result, for all V in $G.X$ and for all x in V, $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ are in V. □

Corollary 4.3. *Let (x, V) be in Δ and let $\mathfrak{z}$ be the center of $\mathfrak{g}^{x_s}$.*

(i) *The subspace $\mathfrak{z}$ is contained in V_x and V.*

(ii) *The space V is an algebraic, commutative subalgebra of $\mathfrak{g}$.*

Proof. (i) If x is regular semisimple, V is a Cartan subalgebra of $\mathfrak{g}$. Suppose that x is not regular semisimple. Denote by $\mathfrak{z}$ the center of $\mathfrak{g}^{x_s}$. Let $\mathfrak{N}_{\mathfrak{g}^{x_s}}$ be the nilpotent cone of $\mathfrak{g}^{x_s}$ and let Ω_{reg} be the regular nilpotent orbit of $\mathfrak{g}^{x_s}$. For all y in Ω_{reg}, $x_s + y$ is in $\mathfrak{g}_{\mathrm{reg}}$ and $\varepsilon_1(x_s + y), \dots, \varepsilon_\ell(x_s + y)$ is a basis of $\mathfrak{g}^{x_s+y}$ by [Ko63, Theorem 9]. Then for all z in $\mathfrak{z}$, there exist regular functions on Ω_{reg}, $a_{1,z}, \dots, a_{\ell,z}$, such that

$$z = a_{1,z}(y)\varepsilon_1(x_s + y) + \cdots + a_{\ell,z}(y)\varepsilon_\ell(x_s + y)$$

for all y in Ω_{reg}. Furthermore, these functions are uniquely defined by this equality. Since $\mathfrak{N}_{\mathfrak{g}^{x_s}}$ is a normal variety and $\mathfrak{N}_{\mathfrak{g}^{x_s}} \setminus \Omega_{\mathrm{reg}}$ has codimension 2 in $\mathfrak{N}_{\mathfrak{g}^{x_s}}$, the functions $a_{1,z}, \dots, a_{\ell,z}$ have regular extensions to $\mathfrak{N}_{\mathfrak{g}^{x_s}}$. Denoting again by $a_{i,z}$ the regular extension of $a_{i,z}$ for $i = 1, \dots, \ell$,

$$z = a_{1,z}(y)\varepsilon_1(x_s + y) + \cdots + a_{\ell,z}(y)\varepsilon_\ell(x_s + y)$$

for all y in $\mathfrak{N}_{\mathfrak{g}^{x_s}}$. As a result, $\mathfrak{z}$ is contained in V_x. Hence $\mathfrak{z}$ is contained in V by Lemma 4.2,(iii).

(ii) Since the set of commutative subalgebras of dimension ℓ is closed in $\mathrm{Gr}_\ell(\mathfrak{g})$, V is a commutative subalgebra of $\mathfrak{g}$. According to (i), the semisimple and nilpotent components of the elements of V are contained in V. For x in $V \setminus \mathfrak{N}_{\mathfrak{g}}$, all the replica of x_s are contained in the center of $\mathfrak{g}^{x_s}$. Hence V is an algebraic subalgebra of $\mathfrak{g}$ by (i). □

4.2. For s in $\mathfrak{h}$, denote by X^s the subset of elements of X, contained in $\mathfrak{g}^s$.

Lemma 4.4. *Let s be in $\mathfrak{h}$ and let $\mathfrak{z}$ be the center of $\mathfrak{g}^s$.*

(i) *The set X^s is the closure in $\mathrm{Gr}_\ell(\mathfrak{g}^s)$ of the orbit of $\mathfrak{h}$ under B^s.*

(ii) *The set of elements of $G.X$ containing $\mathfrak{z}$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G^s.*

Proof. (i) Set $\mathfrak{p} := \mathfrak{g}^s + \mathfrak{b}$, let P be the normalizer of $\mathfrak{p}$ in G and let $\mathfrak{p}_u$ be the nilpotent radical of $\mathfrak{p}$. For g in P, denote by $\overline{g}$ its image by the canonical projection from P to G^s. Let Z be the closure in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathrm{Gr}_\ell(\mathfrak{g})$ of the image of the map

$$B \longrightarrow \mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b}), \qquad g \longmapsto (g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$$

and let Z' be the subset of elements (V, V') of $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b})$ such that

$$V' \subset \mathfrak{g}^s \cap \mathfrak{b} \quad \text{and} \quad V \subset V' \oplus \mathfrak{p}_u.$$

Then Z' is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b})$ and Z is contained in Z' since $(g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$ is in Z' for all g in B. Since $\mathrm{Gr}_\ell(\mathfrak{b})$ is a projective variety, the images of Z by the projections $(V, V') \mapsto V$

and $(V, V') \mapsto V'$ are closed in $\mathrm{Gr}_\ell(\mathfrak{b})$ and they equal X and $\overline{B^s.\mathfrak{h}}$ respectively. Furthermore, $\overline{B^s.\mathfrak{h}}$ is contained in X^s.

Let V be in X^s. For some V' in $\mathrm{Gr}_\ell(\mathfrak{b})$, (V, V') is in Z. Since

$$V \subset \mathfrak{g}^s,\ V' \subset \mathfrak{g}^s,\ V \subset V' \oplus \mathfrak{p}_\mathfrak{u},$$

$V = V'$ so that V is in $\overline{B^s.\mathfrak{h}}$, whence the assertion.

(ii) Since $\mathfrak{z}$ is contained in $\mathfrak{h}$, all element of $\overline{G^s.\mathfrak{h}}$ is an element of $G.X$ containing $\mathfrak{z}$. Let V be in $G.X$, containing $\mathfrak{z}$. Since V is a commutative subalgebra of $\mathfrak{g}^s$ and since $\mathfrak{g}^s \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}^s$, for some g in G^s, $g(V)$ is contained in $\mathfrak{b} \cap \mathfrak{g}^s$. So, one can suppose that V is contained in $\mathfrak{b}$. According to the Bruhat decomposition of G, since X is B-invariant, for some b in U and for some w in $W(\mathcal{R})$, V is in $bw.X$. Set:

$$\mathcal{R}_{+,w} := \{\alpha \in \mathcal{R}_+ \mid w(\alpha) \in \mathcal{R}_+\}, \qquad \mathcal{R}'_{+,w} := \{\alpha \in \mathcal{R}_+ \mid w(\alpha) \notin \mathcal{R}_+\},$$

$$\mathfrak{u}_1 := \bigoplus_{\alpha \in \mathcal{R}_{+,w}} \mathfrak{g}^{w(\alpha)}, \qquad \mathfrak{u}_2 := \bigoplus_{\alpha \in -\mathcal{R}'_{+,w}} \mathfrak{g}^{w(\alpha)}, \qquad \mathfrak{u}_3 := \bigoplus_{\alpha \in \mathcal{R}'_{+,w}} \mathfrak{g}^{w(\alpha)},$$

$$B^w := wBw^{-1}, \qquad \mathfrak{b}^w := \mathfrak{h} \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_3,$$

so that $\mathrm{ad}\,\mathfrak{b}^w$ is the Lie algebra of B^w and $w.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B^w. Moreover, $\mathfrak{u}$ is the direct sum of $\mathfrak{u}_1$ and $\mathfrak{u}_2$. For $i = 1, 2$, denote by U_i the closed subgroup of U whose Lie algebra is $\mathrm{ad}\,\mathfrak{u}_i$. Then $U = U_2U_1$ and $b = b_2b_1$ with b_i in U_i for $i = 1, 2$. Since $w^{-1}(\mathfrak{u}_1)$ is contained in $\mathfrak{u}$ and X is invariant under B, $b_2b_1w.X = b_2w.X$. Then $b_2^{-1}(V)$ is in $w.X$ and

$$b_2^{-1}(V) \subset \mathfrak{b} \cap \mathfrak{b}^w = \mathfrak{h} \oplus \mathfrak{u}_1$$

since V is contained in $\mathfrak{b}$. Set:

$$\mathfrak{u}_{2,1} := \mathfrak{u}_2 \cap \mathfrak{g}^s, \qquad \mathfrak{u}_{2,2} := \mathfrak{u}_2 \cap \mathfrak{p}_\mathfrak{u}$$

and for $i = 1, 2$, denote by $U_{2,i}$ the closed subgroup of U_2 whose Lie algebra is $\mathrm{ad}\,\mathfrak{u}_{2,i}$. Then $\mathfrak{u}_2$ is the direct sum of $\mathfrak{u}_{2,1}$ and $\mathfrak{u}_{2,2}$ and $U_2 = U_{2,1}U_{2,2}$ so that $b_2 = b_{2,1}b_{2,2}$ with $b_{2,i}$ in $U_{2,i}$ for $i = 1, 2$. As a result, $\mathfrak{z}$ is contained in $b_{2,1}^{-1}(V)$ and $b_{2,2}^{-1}(\mathfrak{z})$ is contained $\mathfrak{h} \oplus \mathfrak{u}_1$. Hence $b_{2,2}^{-1}(\mathfrak{z}) = \mathfrak{z}$ since $\mathfrak{u}_1 \cap \mathfrak{u}_{2,2} = \{0\}$.

Suppose $b_{2,2} \neq 1_\mathfrak{g}$. We expect a contradiction. For some x in $\mathfrak{u}_{2,2}$, $b_{2,2} = \exp(\mathrm{ad}\,x)$. The space $\mathfrak{u}_{2,2}$ is a direct sum of root spaces since so are $\mathfrak{u}_2$ and $\mathfrak{p}_\mathfrak{u}$. Let $\alpha_1, \ldots, \alpha_m$ be the positive roots such that the corresponding root spaces are contained in $\mathfrak{u}_{2,2}$. They are ordered so that for $i \leq j$, $\alpha_j - \alpha_i$ is a positive root if it is a root. For $i = 1, \ldots, m$, let c_i be the coordinate of x at x_{α_i} and let i_0 be the smallest integer such that $c_{i_0} \neq 0$. For all z in $\mathfrak{z}$,

$$b_{2,2}^{-1}(z) - z - c_{i_0}\alpha_{i_0}(z)x_{\alpha_{i_0}} \in \bigoplus_{j > i_0} \mathfrak{g}^{\alpha_j},$$

whence the contradiction since for some z in $\mathfrak{z}$, $\alpha_{i_0}(z) \neq 0$. As a result, $b_{2,1}^{-1}(V)$ is an element of $w.X = \overline{B^w.\mathfrak{h}}$, contained in $\mathfrak{g}^s$. So, by (i), $b_{2,1}^{-1}(V)$ and V are in $\overline{G^s.\mathfrak{h}}$, whence the assertion. □

4.3. Define a torus of $\mathfrak{g}$ as a commutative algebraic subalgebra of $\mathfrak{g}$ whose all elements are semisimple. For x in $\mathfrak{g}$, denote by Z_x the subset of elements of $G.X$ containing x and by $(G^x)_0$ the identity component of G^x.

Lemma 4.5. *Let x be in $\mathfrak{N}_\mathfrak{g}$ and let Z be an irreducible component of Z_x. Suppose that some element of Z is not contained in $\mathfrak{N}_\mathfrak{g}$.*

(i) *For some torus $\mathfrak{s}$ of $\mathfrak{g}^x$, all element of a dense open subset of Z contains a conjugate of $\mathfrak{s}$ under $(G^x)_0$.*

(ii) *For some s in $\mathfrak{s}$ and for some irreducible component Z_1 of Z_{s+x}, Z is the closure in* $\mathrm{Gr}_\ell(\mathfrak{g})$ *of* $(G^x)_0.Z_1$.

(iii) *If Z_1 has dimension smaller than* $\dim \mathfrak{g}^{s+x} - \ell$, *then Z has dimension smaller than* $\dim \mathfrak{g}^x - \ell$.

Proof. (i) After some conjugation by an element of G, we can suppose that $\mathfrak{g}^x \cap \mathfrak{b}$ and $\mathfrak{g}^x \cap \mathfrak{h}$ are a Borel subalgebra and a maximal torus of $\mathfrak{g}^x$ respectively. Let Z_0 be the subset of elements of Z contained in $\mathfrak{b}$ and let $(B^x)_0$ be the identity component of B^x. Since Z is an irreducible component of Z_x, Z is invariant under $(G^x)_0$ and $Z = (G^x)_0.Z_0$. Since $(G^x)_0/(B^x)_0$ is a projective variety, according to the proof of Lemma 1.7, $(G^x)_0.Z_*$ is a closed subset of Z for all closed subset Z_* of Z. Hence for some irreducible component Z_* of Z_0, $Z = (G^x)_0.Z_*$. According to Corollary 4.3,(ii), for all V in Z_*, there exists a torus $\mathfrak{s}$, contained in $\mathfrak{g}^x \cap \mathfrak{h}$ and verifying the following two conditions:

(1) V is contained in $\mathfrak{s} + (\mathfrak{g}^x \cap \mathfrak{u})$,

(2) V contains a conjugate of $\mathfrak{s}$ under $(B^x)_0$.

Let $\mathfrak{s}$ be a torus of maximal dimension verifying Conditions (1) and (2) for some V in Z_*. By hypothesis, $\mathfrak{s}$ has positive dimension. Let $Z_\mathfrak{s}$ be the subset of elements of Z_* verifying Conditions (1) and (2) with respect to $\mathfrak{s}$. By maximality of $\dim \mathfrak{s}$, for V in $Z_* \setminus Z_\mathfrak{s}$, $\dim V \cap \mathfrak{u} > \ell - \dim \mathfrak{s}$ or $\dim V \cap \mathfrak{u} = \ell - \dim \mathfrak{s}$ and V is contained in $\mathfrak{s}' + \mathfrak{u}$ for some torus of dimension $\dim \mathfrak{s}$, different from $\mathfrak{s}$. By rigidity of tori, $\mathfrak{s}$ is not in the closure in $\mathrm{Gr}_{\dim \mathfrak{s}}(\mathfrak{h})$ of the set of tori different from $\mathfrak{s}$. Hence $Z_* \setminus Z_\mathfrak{s}$ is a closed subset of Z_* since for all V in $\overline{Z_* \setminus Z_\mathfrak{s}}$, $\dim V \cap \mathfrak{u}$ has dimension at least $\ell - \dim \mathfrak{s}$. As a result, $(G^x)_0.Z_\mathfrak{s}$ contains a dense open subset whose all elements contain a conjugate of $\mathfrak{s}$ under $(G^x)_0$.

(ii) For some s in $\mathfrak{s}$, $\mathfrak{g}^s$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Let Z^s be the subset of elements of Z containing s. Then Z^s is contained in Z_{s+x} and according to Corollary 4.3,(i), Z^s is the subset of elements of Z, containing $\mathfrak{s}$. By (i), for some irreducible component Z_1' of Z^s, $(G^x)_0.Z_1'$ is dense in Z. Let Z_1 be an irreducible component of Z_{s+x}, containing Z_1'. According to Corollary 4.3,(ii), Z_1 is contained in Z_x since x is the nilpotent component of $s + x$. So $Z_1 = Z_1'$ and $(G^x)_0.Z_1$ is dense in Z.

(iii) Since Z_1 is an irreducible component of Z_{s+x}, Z_1 is invariant under the identity component of G^{s+x}. Moreover, G^{s+x} is contained in G^x since x is the nilpotent component of $s + x$. As a result, by (ii),

$$\dim Z \leq \dim \mathfrak{g}^x - \dim \mathfrak{g}^{s+x} + \dim Z_1,$$

whence the assertion. □

Denote by C_h the G-invariant closed cone generated by h.

Lemma 4.6. *Suppose $\mathfrak{g}$ semisimple. Let Γ be the closure in $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$ of the image of the map*

$$\Bbbk^* \times G \longrightarrow \mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g}) \qquad (t, g) \longmapsto (tg(h), g(\mathfrak{h}))$$

and let Γ_0 be the inverse image of the nilpotent cone by the first projection.

(i) *The subvariety Γ of $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$ has dimension $2n + 1$. Moreover, Γ is contained in Δ.*

(ii) *The varieties C_h and $G.X$ are the images of Γ by the first and second projections respectively.*

(iii) *The subvariety Γ_0 of Γ is equidimensional of codimension* 1.

(iv) *For x nilpotent in $\mathfrak{g}$, the subvariety of elements V of $G.X$, containing x and contained in $\overline{G(x)}$, has dimension at most* $\dim \mathfrak{g}^x - \ell$.

Proof. (i) Since the stabilizer of $(h, \mathfrak{h})$ in $\Bbbk^* \times G$ equals $\{1\} \times H$, Γ has dimension $2n + 1$. Since $tg(h)$ is in $g(\mathfrak{h})$ for all (t, g) in $\Bbbk^* \times G$ and Δ is a closed subset of $\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g})$, Γ is contained in Δ.

(ii) Since $\mathrm{Gr}_\ell(\mathfrak{g})$ is a projective variety, the image of Γ by the first projection is closed in $\mathfrak{g}$. So, it equals C_h since it is contained in C_h and it contains the cone generated by $G.h$. Let π be the canonical map from $\mathfrak{g} \setminus \{0\}$ to the projective space $\mathbb{P}(\mathfrak{g})$ of $\mathfrak{g}$ and let $\widetilde{\Gamma}$ be the image of $\Gamma \cap (\mathfrak{g} \setminus \{0\}) \times \mathrm{Gr}_\ell(\mathfrak{g})$ by the map $(x, V) \mapsto (\pi(x), V)$. Since C_h is a closed cone, $\widetilde{\Gamma}$ is a closed subset of $\mathbb{P}(\mathfrak{g}) \times \mathrm{Gr}_\ell(\mathfrak{g})$. Hence the image of $\tilde{\Gamma}$ by the second projection is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$. So, it equals $\overline{G.\mathfrak{h}}$ since it is contained in $\overline{G.\mathfrak{h}}$ and it contains $G.\mathfrak{h}$. As a result, the image of Γ by the second projection equals $\overline{G.\mathfrak{h}}$ since it is contained in $\overline{G.\mathfrak{h}}$ and it contains the image of $\widetilde{\Gamma}$ by the second projection.

(iii) The subvariety C_h of $\mathfrak{g}$ has dimension $2n + 1$ and the nullvariety of p_1 in C_h is contained in $\mathfrak{N}_\mathfrak{g}$ since it is the nullvariety in $\mathfrak{g}$ of the polynomials $p_1, \ldots, p_\ell$. Hence $\mathfrak{N}_\mathfrak{g}$ is the nullvariety of p_1 in C_h and Γ_0 is the nullvariety in Γ of the function $(x, V) \mapsto p_1(x)$. So Γ_0 is equidimensional of codimension 1 in Γ.

(iv) Let T be the subset of elements V of $G.X$, containing x and contained in $\overline{G(x)}$. Denote by Γ_T the inverse image of $\overline{G.T}$ by the projection from Γ to $G.X$. Then Γ_T is contained in Γ_0. Since x is in all element of T and since Γ_T is invariant under G, the image of Γ_T by the first projection equals $\overline{G(x)}$. Moreover, $\{x\} \times T = \{x\} \times G.X \cap \Gamma_T$. Hence

$$\dim \Gamma_T = \dim T + \dim \mathfrak{g} - \dim \mathfrak{g}^x.$$

By (i) and (iii),

$$\dim \Gamma_T \leq \dim \mathfrak{g} - \ell$$

since Γ_T is contained in Γ_0. Hence T has dimension at most $\dim \mathfrak{g}^x - \ell$. □

When $\mathfrak{g}$ is semisimple, denote by $(G.X)_\mathrm{u}$ the subset of elements of $G.X$ contained in $\mathfrak{N}_\mathfrak{g}$.

Corollary 4.7. *Suppose* $\mathfrak{g}$ *semisimple. Let* x *be in* $\mathfrak{N}_\mathfrak{g}$.
(i) *The variety* $(G.X)_\mathrm{u}$ *has dimension at most* $2n - \ell$.
(ii) *The variety* $Z_x \cap (G.X)_\mathrm{u}$ *has dimension at most* $\dim \mathfrak{g}^x - \ell$.

Proof. (i) Let T be an irreducible component of $(G.X)_\mathrm{u}$ and let Δ_T be its inverse image by the canonical projection from Δ to $G.X$. Then Δ_T is a vector bundle of rank ℓ over T. So it has dimension $\dim T + \ell$. Let Y be the projection of Δ_T onto $\mathfrak{g}$. Since T is an irreducible projective variety, Y is an irreducible closed subvariety of $\mathfrak{g}$ contained in $\mathfrak{N}_\mathfrak{g}$. The subvariety $(G.X)_\mathrm{u}$ of $G.X$ is invariant under G since so is $\mathfrak{N}_\mathfrak{g}$. Hence Δ_T and Y are G-invariant and for some y in $\mathfrak{N}_\mathfrak{g}$, $Y = \overline{G(y)}$. Denoting by F_y the fiber at y of the projection $\Delta_T \to Y$, V is contained in $\overline{G(y)}$ and contains y for all V in F_y. So, by Lemma 4.6,(iv),

$$\dim F_y \leq \dim \mathfrak{g}^y - \ell.$$

Since the projection is G-equivariant, this inequality holds for the fibers at the elements of $G(y)$. Hence,

$$\dim \Delta_T \leq \dim \mathfrak{g} - \ell \text{ and } \dim T \leq 2n - \ell.$$

(ii) Let Z be an irreducible component of $Z_x \cap (G.X)_\mathrm{u}$ and let T be an irreducible component of $(G.X)_\mathrm{u}$, containing Z. Let Δ_T and Y be as in (i). Then $G(x)$ is contained in Y and the inverse image of $\overline{G(x)}$ in Δ_T has dimension at least $\dim G(x) + \dim Z$. So, by (i),

$$\dim G(x) + \dim Z \leq \dim \mathfrak{g} - \ell,$$

whence the assertion. □

Theorem 4.8. *For x in $\mathfrak{g}$, the variety of elements of $G.X$, containing x, has dimension at most* $\dim \mathfrak{g}^x - \ell$.

Proof. Prove the theorem by induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ is commutative, $G.X = \{\mathfrak{g}\}$. If the derived Lie algebra of $\mathfrak{g}$ is simple of dimension 3, $G.X$ has dimension 2 and for x not in the center of $\mathfrak{g}$, $Z_x = \{\mathfrak{g}^x\}$. Suppose the theorem true for all reductive Lie algebra of dimension strictly smaller than $\dim \mathfrak{g}$. Let x be in $\mathfrak{g}$. Since $G.X$ has dimension $\dim \mathfrak{g} - \ell$, we can suppose that x is not in the center of $\mathfrak{g}$. If x is not nilpotent, $\mathfrak{g}^{x_s}$ has dimension strictly smaller than $\dim \mathfrak{g}$ and all element of $G.X$ containing x is contained in $\mathfrak{g}^{x_s}$ by Corollary 4.3,(i), whence the theorem in this case by induction hypothesis. As a result, by Lemma 4.5, for all x in $\mathfrak{g}$, all irreducible component of Z_x, containing an element not contained in $\mathfrak{N}_\mathfrak{g}$, has dimension at most $\dim \mathfrak{g}^x - \ell$.

Let $\mathfrak{z}_\mathfrak{g}$ be the center of $\mathfrak{g}$ and let x be a nilpotent element of $\mathfrak{g}$. Denoting by Z'_x the subset of elements of $\overline{G.(\mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}])}$ containing x, Z_x is the image of Z'_x by the map $V \mapsto V + \mathfrak{z}_\mathfrak{g}$, whence the theorem by Corollary 4.7. □

4.4. Let s be in $\mathfrak{b} \setminus \{0\}$. Set $\mathfrak{p} := \mathfrak{g}^s + \mathfrak{b}$ and denote by $\mathfrak{p}_\mathrm{u}$ the nilpotent radical of $\mathfrak{p}$. Let P be the normalizer of $\mathfrak{p}$ in G and let P_u be its unipotent radical. For a nilpotent orbit Ω of G^s in $\mathfrak{g}^s$, denote by $\Omega^\#$ the induced orbit by Ω from $\mathfrak{g}^s$ to $\mathfrak{g}$.

Lemma 4.9. *Let Y be a G-invariant irreducible closed subset of $\mathfrak{g}$ and let Y' be the union of G-orbits of maximal dimension in Y. Suppose that s is the semisimple component of an element x of Y'. Denote by Ω the orbit of x_n under G^s and set $Y_1 := \mathfrak{z} + \overline{\Omega} + \mathfrak{p}_\mathrm{u}$.*

(i) *The subset Y_1 of $\mathfrak{p}$ is closed and invariant under P.*

(ii) *The subset $G(Y_1)$ of $\mathfrak{g}$ is a closed subset of dimension* $\dim \mathfrak{z} + \dim G(x)$.

(iii) *For some nonempty open subset Y'' of Y', the conjugacy class of $\mathfrak{g}^{y_s}$ under G does not depend on the element y of Y''.*

(iv) *For a good choice of x in Y'', Y is contained in $G(Y_1)$.*

Proof. (i) By [Ko63, §3.2, Lemma 5], G^s is connected and $P = P_\mathrm{u} G^s$. For all y in $\mathfrak{p}$ and for all g in P_u, $g(y)$ is in $y + \mathfrak{p}_\mathrm{u}$. Hence Y_1 is invariant under P since it is invariant under G^s. Moreover, it is a closed subset of $\mathfrak{p}$ since $\mathfrak{z} + \overline{\Omega}$ is a closed subset of $\mathfrak{g}^s$.

(ii) According to (i) and Lemma 1.7, $G(Y_1)$ is a closed subset of $\mathfrak{g}$. According to [CMa93, Theorem 7.1.1], $\Omega^\# \cap (\Omega + \mathfrak{p}_\mathrm{u})$ is a P-orbit and the centralizers in $\mathfrak{g}$ of its elements are contained in $\mathfrak{p}$. For y in $\Omega^\# \cap (\Omega + \mathfrak{p}_\mathrm{u})$ and for g in G, if $g(y)$ is in Y_1 then it is in $\Omega + \mathfrak{p}_\mathrm{u}$ since it is nilpotent. So, for y in $\Omega^\# \cap (\Omega + \mathfrak{p}_\mathrm{u})$, the subset of elements g of G such that $g(y)$ is in Y_1 has dimension $\dim \mathfrak{p}$. As a result,

$$\dim G(Y_1) = \dim G \times_P Y_1 = \dim \mathfrak{p}_\mathrm{u} + \dim Y_1.$$

Since $\dim \mathfrak{g}^x = \dim \mathfrak{g}^s - \dim \Omega$,

$$\begin{aligned} \dim Y_1 &= \dim \mathfrak{z} + \dim \mathfrak{p}_\mathrm{u} + \dim \mathfrak{g}^s - \dim \mathfrak{g}^x \\ \dim G(Y_1) &= \dim \mathfrak{z} + 2\dim \mathfrak{p}_\mathrm{u} + \dim \mathfrak{g}^s - \dim \mathfrak{g}^x \\ &= \dim \mathfrak{z} + \dim G(x). \end{aligned}$$

(iii) Let τ be the canonical morphism from $\mathfrak{g}$ to its categorical quotient $\mathfrak{g}//G$ under G and let Z be the closure in $\mathfrak{g}//G$ of $\tau(Y)$. Since Y is irreducible, Z is irreducible and there exists an irreducible component $\widetilde{Z}$ of the preimage of Z in $\mathfrak{b}$ whose image in $\mathfrak{g}//G$ equals Z. Since the set of conjugacy classes under G of the centralizers of the elements of $\mathfrak{b}$ in $\mathfrak{g}$ is finite, for some nonempty open subset $Z^\#$ of $\widetilde{Z}$, the centralizers of its elements are conjugate under G. The image of $Z^\#$ in $\mathfrak{g}//G$

contains a dense open subset Z' of Z. Let Y'' be the inverse image of Z' by the restriction of τ to Y'. Then Y'' is a dense open subset of Y and the centralizers in $\mathfrak{g}$ of the semisimple components of its elements are conjugate under G.

(iv) Suppose that x is in Y''. Let Z_Y be the set of elements y of Y'' such that $\mathfrak{g}^{y_s} = \mathfrak{g}^s$. Then $G.Z_Y = Y''$. For all nilpotent orbit Ω of G^s in $\mathfrak{g}^s$, set:

$$Y_\Omega = \mathfrak{z} + \overline{\Omega} + \mathfrak{p}_u$$

Then Z_Y is contained in the union of the Y_Ω's. Hence Y'' is contained in the union of the $G(Y_\Omega)$'s. According to (ii), $G(Y_\Omega)$ is a closed subset of $\mathfrak{g}$. Hence Y is contained in the union of the $G(Y_\Omega)$'s since Y'' is dense in Y. Then Y is contained in $G(Y_\Omega)$ for some Ω since Y is irreducible and there are finitely many nilpotent orbits in $\mathfrak{g}^s$, whence the assertion. □

Theorem 4.10. *(i) The variety $G.X$ is the union of $G.\mathfrak{h}$ and the $G.X_\beta$'s, $\beta \in \Pi$.*

(ii) *The variety X is the union of $U.\mathfrak{h}$ and the X_α's, $\alpha \in \mathcal{R}_+$.*

Proof. Let $\mathfrak{z}_\mathfrak{g}$ be the center of $\mathfrak{g}$ and let μ be the map

$$\mathrm{Gr}_{\ell'}([\mathfrak{g},\mathfrak{g}]) \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}), \qquad V \longmapsto \mathfrak{z}_\mathfrak{g} + V$$

with ℓ' the rank of $[\mathfrak{g},\mathfrak{g}]$ and set:

$$X_d := \overline{B.(\mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}])}, \qquad X_{\alpha,d} := \overline{B.(V_\alpha \cap [\mathfrak{g},\mathfrak{g}])}$$

for α in $\mathcal{R}_+$. Then X, $G.X$, X_α, $G.X_\alpha$ are the images of X_d, $G.X_d$, $X_{\alpha,d}$, $G.X_{\alpha,d}$ by μ respectively. So we can suppose $\mathfrak{g}$ semisimple.

(i) For $\ell = 1$, $\mathfrak{g}$ is simple of dimension 3. In this case, $G.X$ is the union of $G.\mathfrak{h}$ and $G.\mathfrak{g}^e$. So, we can suppose $\ell \geq 2$. According to Lemma 4.1,(iii), for α in $\mathcal{R}_+$, $G.X_\alpha$ is an irreducible component of $G.X \setminus G.\mathfrak{h}$. Moreover, for all β in $\Pi \cap W(\mathcal{R})(\alpha)$, $G.X_\alpha = G.X_\beta$ since V_α and V_β are conjugate under $N_G(\mathfrak{h})$.

Let T be an irreducible component of $G.X \setminus G.\mathfrak{h}$. Set:

$$\Delta_T := \Delta \cap \mathfrak{g} \times T$$

and denote by Y the image of Δ_T by the first projection. Then Y is closed in $\mathfrak{g}$ since $\mathrm{Gr}_\ell(\mathfrak{g})$ is a projective variety. Since Δ_T is a vector bundle over T and since T is irreducible, Δ_T is irreducible and so is Y. Since T is an irreducible component of $G.X \setminus G.\mathfrak{h}$, T, Δ_T and Y are G-invariant. According to Lemma 4.1,(iii), T has codimension 1 in $G.X$. Hence, by Corollary 4.7,(i) Y is not contained in the nilpotent cone since $\ell \geq 2$. Let Y' be the set of elements x of Y such that $\mathfrak{g}^x$ has minimal dimension. According to Lemma 4.9,(ii) and (iv), for x in a G-invariant dense subset Y'' of Y',

$$\dim Y \leq \dim G(x) + \dim \mathfrak{z}$$

with $\mathfrak{z}$ the center of $\mathfrak{g}^{x_s}$ and according to Theorem 4.8,

$$\dim \Delta_T \leq \dim G(x) + \dim \mathfrak{z} + \dim \mathfrak{g}^x - \ell = \dim \mathfrak{g} + \dim \mathfrak{z} - \ell$$

Hence Δ_T has dimension at most $2n + \dim \mathfrak{z}$ and $\dim \mathfrak{z} = \ell - 1$ since T has codimension 1 in $G.X$. Let x be in Y'' such that x_s is in $\mathfrak{h}$. Then x_s is subregular and $\mathfrak{z}$ is the kernel of a positive root α. Denoting by $\mathfrak{s}_\alpha$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$, $\mathfrak{g}^{x_s}$ is the direct sum of $\mathfrak{h}_\alpha$ and $\mathfrak{s}_\alpha$. Since the maximal commutative subalgebras of $\mathfrak{s}_\alpha$ have dimension 1, a commutative subalgebra of dimension ℓ of $\mathfrak{g}^{x_s}$ is either a Cartan subalgebra of $\mathfrak{g}$ or conjugate to V_α under the adjoint group of $\mathfrak{g}^{x_s}$. As a result, V_α is in T and $T = \overline{G.V_\alpha} = G.X_\alpha$ since T is G-invariant, whence the assertion.

(ii) According to Lemma 4.1,(ii), for α in $\mathcal{R}_+$, X_α is an irreducible component of $X \setminus B.\mathfrak{h}$. Let $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ be its simple factors. For $j = 1, \dots, m$, denote by X_j the closure in $\mathrm{Gr}_{\ell_{\mathfrak{g}_j}}(\mathfrak{g}_j)$ of the orbit of $\mathfrak{h} \cap \mathfrak{g}_j$. Then $X = X_1 \times \cdots \times X_m$ and the complement to $B.\mathfrak{h}$ in X is the union of the

$$X_1 \times \cdots \times X_{j-1} \times (X_j \setminus B.(\mathfrak{h} \cap \mathfrak{g}_j)) \times X_{j+1} \times \cdots \times X_m$$

So, we can suppose $\mathfrak{g}$ simple. Consider

$$\mathfrak{b} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_\ell = \mathfrak{g}$$

an increasing sequence of parabolic subalgebras verifying the following condition: for $i = 0, \dots, \ell - 1$, there is no parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that

$$\mathfrak{p}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_{i+1}.$$

For $i = 0, \dots, \ell$, let P_i be the normalizer of $\mathfrak{p}_i$ in G, let $\mathfrak{p}_{i,\mathrm{u}}$ be the nilpotent radical of $\mathfrak{p}_i$ and let $P_{i,\mathrm{u}}$ be the unipotent radical of P_i. For $i = 0, \dots, \ell$ and for α in $\mathcal{R}_+$, set $X_i := P_i.X$ and $X_{i,\alpha} := P_i.X_\alpha$. Prove by induction on $\ell - i$ that for all sequence of parabolic subalgebras verifying the above condition, the $X_{i,\alpha}$'s, $\alpha \in \mathcal{R}_+$, are the irreducible components of $X_i \setminus P_i.\mathfrak{h}$.

For $i = \ell$, it results from (i). Suppose that it is true for $i + 1$. According to Lemma 4.1,(iii), the $X_{i,\alpha}$'s are irreducible components of $X_i \setminus P_i.\mathfrak{h}$.

Claim 4.11. Let T be an irreducible component of $X_i \setminus P_i.\mathfrak{h}$ such that P_i is its stabilizer in P_{i+1}. Then $T = X_{i,\alpha}$ for some α in $\mathcal{R}_+$.

Proof. According to the induction hypothesis, T is contained in $X_{i+1,\alpha}$ for some α in $\mathcal{R}_+$. According to Lemma 4.1,(iv), T has codimension 1 in X_i so that $P_{i+1}.T$ and $X_{i+1,\alpha}$ have the same dimension. Then they are equal and T contains $\mathfrak{g}^x$ for some x in $\mathfrak{b}_{\mathrm{reg}}$ such that x_{s} is a subregular element belonging to $\mathfrak{h}$. Denoting by α' the positive root such that $\alpha'(x_{\mathrm{s}}) = 0$, $\mathfrak{g}^x = V_{\alpha'}$ since $V_{\alpha'}$ is the commutative subalgebra contained in $\mathfrak{b}$ and containing $\mathfrak{h}_{\alpha'}$, which is not Cartan, so that $T = X_{i,\alpha'}$. □

Suppose that $X_i \setminus P_i.\mathfrak{h}$ is not the union of the $X_{i,\alpha}$'s, $\alpha \in \mathcal{R}_+$. We expect a contradiction. Let T be an irreducible component of $X_i \setminus P_i.\mathfrak{h}$, different from $X_{i,\alpha}$ for all α. According to Claim 4.11 and according to the condition verified by the sequence, T is invariant under P_{i+1}. Moreover, according to Claim 4.11, it is so for all sequence $\mathfrak{p}'_0, \dots, \mathfrak{p}'_\ell$ of parabolic subalgebras verifying the above condition and such that $\mathfrak{p}'_j = \mathfrak{p}_j$ for $j = 0, \dots, i$. As a result, for all simple root β such that $\mathfrak{g}^{-\beta}$ is not in $\mathfrak{p}_i$, T is invariant under the one parameter subgroup of G generated by $\mathrm{ad}\,\mathfrak{g}^{-\beta}$. Hence T is invariant under G. It is impossible since for x in $\mathfrak{g} \setminus \{0\}$, the orbit $G(x)$ is not contained in $\mathfrak{p}_i$ since $\mathfrak{g}$ is simple, whence the assertion. □

4.5. Let X' be the subset of $\mathfrak{g}^x$ with x in $\mathfrak{b}_{\mathrm{reg}}$ such that x_{s} is regular or subregular. For α in $\mathcal{R}_+$, denote by θ_α the map

$$\Bbbk \longrightarrow X, \qquad t \longmapsto \exp(t \mathrm{ad}\, x_\alpha).\mathfrak{h}.$$

According to [Sh94, Ch. VI, Theorem 1], θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$, also denoted by θ_α. Set $Z_\alpha := \theta_\alpha(\mathbb{P}^1(\Bbbk))$ and $X'_\alpha := B.Z_\alpha$ so that $X'_\alpha = U.\mathfrak{h} \cup B.V_\alpha$.

Lemma 4.12. *Let α be in $\mathcal{R}_+$ and let V be in X. Denote by $\overline{V}$ the image of V by the projection $x \mapsto \overline{x}$.*

(i) *For x in $\mathfrak{h}$, x is subregular if and only if $V_x = \mathfrak{h}_\gamma$ for some positive root γ.*

(ii) *If $\overline{V}$ has dimension $\ell - 1$, then $\overline{V} = V_{\overline{x}} = \mathfrak{h}_\gamma$ for some x in V and for some γ in $\mathcal{R}_+$.*

(iii) *If $\overline{V} = \mathfrak{h}_\alpha$, then V is conjugate to V_α under B.*

Proof. (i) First of all, since $\varepsilon_1, \ldots, \varepsilon_\ell$ are G-equivariant maps, V_x is contained in the center of $\mathfrak{g}^x$ for all x in $\mathfrak{g}$. Then for x in $\mathfrak{h}$, V_x is the center of $\mathfrak{g}^x$ by Corollary 4.3,(i), whence the assertion.

(ii) For γ in $\mathcal{R}_+$, set:

$$V_\gamma := V \cap (\mathfrak{h}_\gamma + \mathfrak{u}).$$

Suppose $\overline{V} = \ell - 1$. Then V does not contain regular semisimple element. Hence V is the union of $V_\gamma, \gamma \in \mathcal{R}_+$. Since V is irreducible and $\mathcal{R}_+$ is finite, $V = V_\gamma$ for some γ in $\mathcal{R}_+$ so that $\overline{V} = \mathfrak{h}_\gamma$ since $\overline{V}$ has dimension $\ell - 1$. Let x be in V such that $\overline{x}$ is subregular. Then $V_{\overline{x}} = \mathfrak{h}_\gamma$ by (i).

(iii) Suppose $\overline{V} = \mathfrak{h}_\alpha$. By (ii), $V_{\overline{x}} = \mathfrak{h}_\alpha$ for some x in V. Let b be in B such that $b(x_s) = \overline{x}$. Then $b(V)$ centralizes $\mathfrak{h}_\alpha$ by Corollary 4.3,(i). Moreover, $b(V)$ is not a Cartan subalgebra since $\overline{V}$ does not contain regular semsimple element. The centralizer of $\mathfrak{h}_\alpha$ in $\mathfrak{b}$ equals $\mathfrak{h} + \mathfrak{g}^\alpha$ and V_α is the commutative algebra of dimension ℓ contained in $\mathfrak{h} + \mathfrak{g}^\alpha$ which is not a Cartan subalgebra, whence the assertion. □

Corollary 4.13. *Let α be a positive root.*

(i) *The sets X'_α and $G.X'_\alpha$ are open subsets of X and $G.X$ respectively.*

(ii) *The sets X' and $G.X'$ are big open subsets of X and $G.X$ respectively.*

Proof. (i) Prove that X'_α is a neighborhood of V_α in X. Denote by H_α the coroot of α and set:

$$E' := \bigoplus_{\gamma \in \mathcal{R}_+ \setminus \{\alpha\}} \mathfrak{g}^\gamma, \qquad E := \Bbbk H_\alpha \oplus E'.$$

Let Ω_E be the set of subspaces V of $\mathfrak{b}$ such that E is a complement to V in $\mathfrak{b}$ and let Ω'_E be the complement in $X \cap \Omega_E$ to the union of $X_\gamma, \gamma \in \mathcal{R}_+ \setminus \{\alpha\}$. Then Ω'_E is an open neighborhood of V_α in X. Let V be in Ω'_E such that V is not a Cartan subalgebra and denote by $\overline{V}$ its image by the projection $x \mapsto \overline{x}$. Then V is contained in $\overline{V} + \mathfrak{u}$ so that $\mathfrak{h} = \Bbbk H_\alpha + \overline{V}$. Since V is not a Cartan subalgebra, $\overline{V}$ has dimension $\ell - 1$. Hence $\overline{V} = \mathfrak{h}_\gamma$ for some positive root γ by Lemma 4.12,(ii). According to Lemma 4.12,(iii), V is conjugate to V_γ under B. Then $\alpha = \gamma$ and V is in X'_α since V is not in X_δ for all positive root δ different from α. Hence Ω'_E is contained in X'_α so that X'_α is a neighborhood of V_α in X. As a result, X'_α is an open subset of X since it is the union of $B.V_\alpha$ and the open subset $U.\mathfrak{h}$ of X. Hence $G.(X \setminus X'_\alpha)$ is a closed subset of $G.X$ by Lemma 1.7, whence the assertion.

(ii) By definition, X' is the union of the X'_α's, $\alpha \in \mathcal{R}_+$. Hence X' is an open subset of X by (i). Moreover, by Theorem 4.10,(ii), $X \setminus X'$ is the union of the $X_\alpha \setminus X'$'s, $\alpha \in \mathcal{R}_+$. Then X' is a big open subset of X since, for all α, $X_\alpha \setminus X'$ is strictly contained in the irreducible subvariety X_α of X.

Since $G.X'$ is the union of the $G.X'_\alpha$'s, $\alpha \in \mathcal{R}_+$, $G.X'$ is an open subset of $G.X$ by (i). Moreover, by Theorem 4.10,(i), $G.X \setminus G.X'$ is the union of the $G.X_\beta \setminus G.X'$'s, $\beta \in \Pi$. Hence $G.X'$ is a big open subset of $G.X$ since, for all β, $G.X_\beta \setminus G.X'$ is strictly contained in the irreducible subvariety $G.X_\beta$ of $G.X$. □

Proposition 4.14. *The sets X' and $G.X'$ are smooth big open subsets of X and $G.X$ respectively.*

Proof. According to Corollary 4.13,(ii), it remains to prove that X' and $G.X'$ are smooth open subsets of X and $G.X$ respectively. Denote by π the canonical projection from Δ onto $G.X$ and set $\Delta_0 := \pi^{-1}(X)$. Let μ be the map

$$\mathfrak{g}_{\mathrm{reg}} \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}) \qquad x \longmapsto \mathfrak{g}^x$$

and let μ_0 be its restriction to $\mathfrak{b}_{\mathrm{reg}}$. Then μ is a regular map. Denoting by Γ_μ and Γ_{μ_0} the graphs of μ and μ_0 respectively, Γ_μ and Γ_{μ_0} are smooth varieties contained in Δ and Δ_0 respectively since for x in $\mathfrak{g}_{\mathrm{reg,ss}}$, $\mathfrak{g}^x$ is a Cartan subalgebra, contained in $\mathfrak{b}$ when x is in $\mathfrak{b}$. Set:

$$\Gamma'_\mu := \Gamma_\mu \cap \pi^{-1}(G.X') = \Delta \cap \mathfrak{g}_{\mathrm{reg}} \times G.X' \quad \text{and} \quad \Gamma'_{\mu_0} := \Gamma_{\mu_0} \cap \pi^{-1}(X') = \Delta \cap \mathfrak{b}_{\mathrm{reg}} \times X'$$

Then Γ'_μ is a smooth variety as an open susbet of Γ_μ and Γ'_μ is an open subset of $\pi^{-1}(G.X')$ such that $\pi(\Gamma'_\mu) = G.X'$ since all element of $G.X'$ contains regular elements. In the same way, Γ'_{μ_0} is a smooth open subset of $\pi^{-1}(X')$ such that $\pi(\Gamma'_{\mu_0}) = X'$. As a result, Γ'_μ and Γ'_{μ_0} are smooth fiber bundles over $G.X'$ and X' respectively since Δ and Δ_0 are vector bundles over $G.X$ and X respectively. Hence $G.X'$ and X' are smooth varieties by [MA86, Ch. 8, Theorem 23.7]. □

5. On the generalized isospectral commuting variety

Let $k \geq 2$ be an integer. According to Section 2, we have the commutative diagram

$$\begin{array}{ccc} G \times_B \mathfrak{b}^k & \xrightarrow{\gamma_{\mathrm{n}}} & \mathcal{B}_{\mathrm{n}}^{(k)} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & \mathcal{B}^{(k)} & \end{array}$$

with $\mathcal{B}_{\mathrm{n}}^{(k)}$ the normalization of $\mathcal{B}^{(k)}$ and η the normalization morphism. By Theorem 2.13,(i) and Lemma 2.5,(i), ι_k is a closed embedding of $\mathfrak{b}^k$ into $\mathcal{B}_{\mathrm{n}}^{(k)}$, $\mathcal{B}_{\mathrm{n}}^{(k)}$ is the closure of $G.\iota_k(\mathfrak{b}^k)$ in $\mathcal{X}^k$ and η is the restriction to $\mathcal{B}_{\mathrm{n}}^{(k)}$ of the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. Denote by $\mathcal{C}^{(k)}$ the closure of $G.\mathfrak{b}^k$ in $\mathfrak{g}^k$ with respect to the diagonal action of G in $\mathfrak{g}^k$ and set $\mathcal{C}_{\mathrm{n}}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The varieties $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{n}}^{(k)}$ are called *generalized commuting variety* and *generalized isospectral commuting variety* respectively. For $k = 2$, $\mathcal{C}_{\mathrm{n}}^{(k)}$ is the isospectral commuting variety considered by M. Haiman in [Ha99, §8] and [Ha02, §7.2].

5.1. Set:

$$E^{(k)} := \{(u, x_1, \ldots, x_k) \in X \times \mathfrak{b}^k \mid u \ni x_1, \ldots, u \ni x_k\}.$$

Lemma 5.1. *Denote by $E^{(k,*)}$ the intersection of $E^{(k)}$ and $U.\mathfrak{b} \times (\mathfrak{g}_{\mathrm{reg,ss}} \cap \mathfrak{b})^k$ and for w in $W(\mathcal{R})$, denote by θ_w the map*

$$E^{(k)} \longrightarrow \mathfrak{b}^k \times \mathfrak{h}^k, \qquad (u, x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k, w(\overline{x_1}), \ldots, w(\overline{x_k})).$$

(i) *Denoting by $\mathfrak{X}_{0,k}$ the image of $E^{(k)}$ by the projection $(u, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$, $\mathfrak{X}_{0,k}$ is the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$ and $\mathcal{C}^{(k)}$ is the image of $G \times \mathfrak{X}_{0,k}$ by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$.*
(ii) *For all w in $W(\mathcal{R})$, $\theta_w(E^{(k,*)})$ is dense in $\theta_w(E^{(k)})$.*

Proof. (i) Since X is a projective variety, $\mathfrak{X}_{0,k}$ is a closed subset of $\mathfrak{b}^k$. The variety $E^{(k)}$ is irreducible of dimension $n + k\ell$ as a vector bundle of rank $k\ell$ over the irreducible variety X. So, $B.(\{\mathfrak{h}\} \times \mathfrak{h}^k)$ is dense in $E^{(k)}$ and $\mathfrak{X}_{0,k}$ is the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$, whence the assertion by Lemma 1.7.

(ii) Since $U.\mathfrak{b} \times (\mathfrak{g}_{\mathrm{reg,ss}} \cap \mathfrak{b})^k$ is an open susbet of $X \times \mathfrak{b}^k$, $E^{(k,*)}$ is an open subset of $E^{(k)}$. Moreover, it is a dense open subset since $E^{(k)}$ is irreducible, whence the assertion since θ_w is a morphism of algebraic varieties. □

5.2. Let s be in $\mathfrak{h}$ and let G^s be the centralizer of s in G. According to [Ko63, §3.2, Lemma 5], G^s is connected. Denote by $\mathcal{R}_s$ the set of roots whose kernel contains s and denote by $W(\mathcal{R}_s)$ the Weyl group of $\mathcal{R}_s$. Let $\mathfrak{z}_s$ be the center of $\mathfrak{g}^s$.

Lemma 5.2. *Let $x = (x_1, \dots, x_k)$ be in $\mathcal{C}^{(k)}$ verifying the following conditions:*

(1) *s is the semisimple component of x_1,*
(2) *for z in P_x, the centralizer in $\mathfrak{g}$ of the semisimple component of z has dimension at least $\dim \mathfrak{g}^s$.*

Then for $i = 1, \dots, k$, the semisimple component of x_i is contained in $\mathfrak{z}_s$.

Proof. Since x is in $\mathcal{C}^{(k)}$, $[x_i, x_j] = 0$ for all (i, j). Suppose that for some i, the semisimple component $x_{i,\mathrm{s}}$ of x_i is not in $\mathfrak{z}_s$. A contradiction is expected. Since $[x_1, x_i] = 0$, for all t in $\Bbbk$, $s + tx_{i,\mathrm{s}}$ is the semisimple component of $x_1 + tx_i$. Moreover, after conjugation by an element of G^s, we can suppose that $x_{i,\mathrm{s}}$ is in $\mathfrak{h}$. Since $\mathcal{R}$ is finite, there exists t in $\Bbbk^*$ such that the subset of roots whose kernel contains $s + tx_{i,\mathrm{s}}$ is contained in $\mathcal{R}_s$. Since $x_{i,\mathrm{s}}$ is not in $\mathfrak{z}_s$, for some α in $\mathcal{R}_s$, $\alpha(s + tx_{i,\mathrm{s}}) \neq 0$ that is $\mathfrak{g}^{s+tx_{i,\mathrm{s}}}$ is strictly contained in $\mathfrak{g}^s$, whence the contradiction. □

For w in $W(\mathcal{R})$, set:

$$C_w := G^s w B/B, \qquad B^w := wBw^{-1}.$$

Lemma 5.3. [Hu95, §6.17, Lemma] *Let $\mathfrak{B}$ be the set of Borel subalgebras of $\mathfrak{g}$ and let $\mathfrak{B}_s$ be the set of Borel subalgebras of $\mathfrak{g}$ containing s.*

(i) *For all w in $W(\mathcal{R})$, C_w is a connected component of $\mathfrak{B}_s$.*
(ii) *For (w, w') in $W(\mathcal{R}) \times W(\mathcal{R})$, $C_w = C_{w'}$ if and only if $w'w^{-1}$ is in $W(\mathcal{R}_s)$.*
(iii) *The variety C_w is isomorphic to $G^s/(G^s \cap B^w)$.*

For x in $\mathcal{B}^{(k)}$, denote by $\mathfrak{B}_x$ the subset of Borel subalgebras containing P_x.

Corollary 5.4. *Let $x = (x_1, \dots, x_k)$ be in $\mathcal{C}^{(k)}$. Suppose that x verifies Conditions* (1) *and* (2) *of Lemma 5.2. Then $\{C_w \cap \mathfrak{B}_x \mid w \in W(\mathcal{R})\}$ is the set of connected components of $\mathfrak{B}_x$.*

Proof. Since a Borel subalgebra contains the semisimple component of its elements and since s is the semisimple component of x_1, $\mathfrak{B}_x$ is contained in $\mathfrak{B}_s$. As a result, according to Lemma 5.3,(i), every connected component of $\mathfrak{B}_x$ is contained in C_w for some w in $W(\mathcal{R})$. Set $x_{\mathrm{n}} := (x_{1,\mathrm{n}}, \dots, x_{k,\mathrm{n}})$. Since $[x_i, x_j] = 0$ for all (i, j), P_x is contained in $\mathfrak{g}^s$. Let $\mathfrak{B}^s$ be the set of Borel subalgebras of $\mathfrak{g}^s$ and for y in $(\mathfrak{g}^s)^k$, let $\mathfrak{B}^s_y$ be the set of Borel subalgebras of $\mathfrak{g}^s$ containing P_y. According to [Hu95, Theorem 6.5], $\mathfrak{B}^s_{x_{\mathrm{n}}}$ is connected. Moreover, according to Lemma 5.2, the semisimple components of $x_1, \dots, x_k$ are in $\mathfrak{z}_s$ so that $\mathfrak{B}^s_{x_{\mathrm{n}}} = \mathfrak{B}^s_x$. Let w be in $W(\mathcal{R})$. According to Lemma 5.3,(iii), there is an isomorphism from $\mathfrak{B}^s$ to C_w. Moreover, the image of $\mathfrak{B}^s_x$ by this isomorphism equals $C_w \cap \mathfrak{B}_x$, whence the corollary. □

Corollary 5.5. *Let $x = (x_1, \dots, x_k)$ be in $\mathcal{C}^{(k)}$ verifying Conditions* (1) *and* (2) *of Lemma 5.2. Then $\eta^{-1}(x)$ is contained in the set of the $(x_1, \dots, x_k, w(x_{1,\mathrm{s}}), \dots, w(x_{k,\mathrm{s}}))$'s with w in $W(\mathcal{R})$.*

Proof. Since $\gamma = \eta \circ \gamma_{\mathrm{n}}$, $\eta^{-1}(x)$ is the image of $\gamma^{-1}(x)$ by γ_{n}. Furthermore, γ_{n} is constant on the connected components of $\gamma^{-1}(x)$ since $\eta^{-1}(x)$ is finite. Let C be a connected component of $\gamma^{-1}(x)$. Identifying $G \times_B \mathfrak{b}^k$ with the subvariety of elements (u, x) of $\mathfrak{B} \times \mathfrak{g}^k$ such that P_x is contained in u, C identifies with $(C_w \cap \mathfrak{B}_x) \times \{x\}$ for some w in $W(\mathcal{R})$ by Corollary 5.4. Then for some g in G^s and for some representative g_w of w in $N_G(\mathfrak{h})$, $gg_w(\mathfrak{b})$ contains P_x so that

$$\gamma_{\mathrm{n}}(C) = \{(x_1, \dots, x_k, \overline{(gg_w)^{-1}(x_1)}, \dots, \overline{(gg_w)^{-1}(x_k)})\}.$$

By Lemma 5.2, $x_{1,\mathrm{s}},\dots,x_{k,\mathrm{s}}$ are in $\mathfrak{z}_s$ so that $w^{-1}(x_{i,\mathrm{s}})$ is the semisimple component of $(gg_w)^{-1}(x_i)$ for $i=1,\dots,k$. Hence

$$\gamma_{\mathrm{n}}(C)=\{(x_1,\dots,x_k,w^{-1}(x_{1,\mathrm{s}}),\dots,w^{-1}(x_{k,\mathrm{s}}))\},$$

whence the corollary. □

Proposition 5.6. *The variety $\mathcal{C}_{\mathrm{n}}^{(k)}$ is irreducible and equal to the closure of $G.\iota_k(\mathfrak{b}^k)$ in $\mathcal{B}_{\mathrm{n}}^{(k)}$.*

Proof. Denote by $\overline{G.\iota_k(\mathfrak{b}^k)}$ the closure of $G.\iota_k(\mathfrak{b}^k)$ in $\mathcal{B}_{\mathrm{n}}^{(k)}$. Then $\overline{G.\iota_k(\mathfrak{b}^k)}$ is irreducible as the closure of an irreducible set. Since η is G-equivariant, $\eta(G.\iota_k(\mathfrak{b}^k))=G.\mathfrak{b}^k$. Hence $\eta(\overline{G.\iota_k(\mathfrak{b}^k)})=\mathcal{C}^{(k)}$ since η is a finite morphism and $\mathcal{C}^{(k)}$ is the closure of $G.\mathfrak{b}^k$ in $\mathfrak{g}^k$ by definition. So, it remains to prove that for all x in $\mathcal{C}^{(k)}$, $\eta^{-1}(x)$ is contained in $\overline{G.\iota_k(\mathfrak{b}^k)}$. There is a canonical action of $\mathrm{GL}_k(\Bbbk)$ on $\mathfrak{g}^k$ and $\mathcal{X}^k$. Since this action commutes with the action of G in $\mathcal{X}^k$, $\mathcal{B}_{\mathrm{n}}^{(k)}$ is invariant under $\mathrm{GL}_k(\Bbbk)$ and η is $\mathrm{GL}_k(\Bbbk)$-equivariant. As a result, since $\mathcal{C}^{(k)}$ and $G.\iota_k(\mathfrak{b}^k)$ are invariant under $\mathrm{GL}_k(\Bbbk)$, for x in $\mathcal{C}^{(k)}$, $\eta^{-1}(x')$ is contained in $\overline{G.\iota_k(\mathfrak{b}^k)}$ for all x' in P_x^k such that $P_{x'}=P_x$ if $\eta^{-1}(x)$ is contained in $\overline{G.\iota_k(\mathfrak{b}^k)}$. Then, according to Lemma 5.2, since η is G-equivariant, it suffices to prove that $\eta^{-1}(x)$ is contained in $\overline{G.\iota_k(\mathfrak{b}^k)}$ for x in $\mathcal{C}^{(k)}\cap\mathfrak{b}^k$ verifying Conditions (1) and (2) of Lemma 5.2 for some s in $\mathfrak{b}$.

According to Corollary 5.5,

$$\eta^{-1}(x)\subset\{(x_1,\dots,x_k,w(x_{1,\mathrm{s}}),\dots,w(x_{k,\mathrm{s}}))\mid w\in W(\mathcal{R})\}\text{ with }x=(x_1,\dots,x_k).$$

For s regular, P_x is contained in $\mathfrak{b}$ and $x_i=x_{i,\mathrm{s}}$ for $i=1,\dots,k$. By definition,

$$(w(x_1),\dots,w(x_k),w(x_1),\dots,w(x_k))\in\iota_k(\mathfrak{b}^k)$$

and for g_w a representative of w in $N_G(\mathfrak{h})$,

$$g_w^{-1}.(w(x_1),\dots,w(x_k),w(x_1),\dots,w(x_k))=(x_1,\dots,x_k,w(x_1),\dots,w(x_k)).$$

Hence $\eta^{-1}(x)$ is contained in $G.\iota_k(\mathfrak{b}^k)$. As a result, according to the notations of Lemma 5.1, for all w in $W(\mathcal{R})$, $\theta_w(E^{(k,*)})$ is contained in $G.\iota_k(\mathfrak{b}^k)$. Hence, by Lemma 5.1,(ii), $\theta_w(E^{(k)})$ is contained in $\overline{G.\iota_k(\mathfrak{b}^k)}$, whence the proposition. □

5.3. Let ϖ be the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. By Corollary 2.4,(ii), $\mathcal{B}_{\mathrm{n}}^{(k)}$ is an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$ and the action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of irreducible components of $\varpi^{-1}(\mathcal{B}^{(k)})$. According to Remark 2.12, there is an embedding Φ of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ into $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ given by

$$p\longmapsto((x_1,\dots,x_k,y_1,\dots,y_k)\mapsto p(y_1,\dots,y_k)).$$

This embedding identifies $\mathrm{S}(\mathfrak{h})^{\otimes k}$ with a subalgebra of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$.

Lemma 5.7. *Let Ψ be the restriction to $\mathrm{S}(\mathfrak{h})^{\otimes k}$ of the canonical map from $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ to $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$.*

(i) *The subvariety $\mathcal{C}_{\mathrm{n}}^{(k)}$ of $\mathcal{X}^k$ is invariant under the diagonal action of $W(\mathcal{R})$ in $\mathcal{X}^k$.*

(ii) *The map Ψ is an embedding of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ into $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$. Moreover, $\Psi(\mathrm{S}(\mathfrak{h})^{\otimes k})$ equals $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]^G$.*

(iii) *The image of $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ by Ψ equals $\Bbbk[\mathcal{C}^{(k)}]^G$.*

Proof. (i) For x in $\mathcal{B}_{\mathrm{n}}^{(k)}$ and w in $W(\mathcal{R})$, $\eta(x)=\eta(w.x)$, whence the assertion by Proposition 5.6.

(ii) According to Theorem 2.13,(ii), $\mathrm{S}(\mathfrak{h})^{\otimes k}$ equals $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$. Moreover, for P in $\mathrm{S}(\mathfrak{h})^{\otimes k}$, $P=0$ if $P(x)=0$ for all x in $\iota_k(\mathfrak{b}^k)$. Hence Ψ is injective. Since G is reductive, $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]^G$ is the image of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ by the quotient morphism, whence the assertion.

(iii) Since G is reductive, $\Bbbk[\mathcal{C}^{(k)}]^G$ is the image of $\Bbbk[\mathcal{B}^{(k)}]^G$ by the quotient morphism, whence the assertion since $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ equals $\Bbbk[\mathcal{B}^{(k)}]^G$ by Theorem 2.13,(iii). □

Identify $\mathrm{S}(\mathfrak{h})^{\otimes k}$ with a subalgebra of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ by Ψ.

Proposition 5.8. *Let $\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}$ and $\widetilde{\mathcal{C}^{(k)}}$ be the normalizations of $\mathcal{C}_{\mathrm{n}}^{(k)}$ and $\mathcal{C}^{(k)}$.*

(i) *The variety $\mathcal{C}^{(k)}$ is the categorical quotient of $\mathcal{C}_{\mathrm{n}}^{(k)}$ under the action of $W(\mathcal{R})$.*

(ii) *The variety $\widetilde{\mathcal{C}^{(k)}}$ is the categorical quotient of $\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}$ under the action of $W(\mathcal{R})$.*

Proof. (i) According to Theorem 2.13,(ii), $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Since $\mathcal{C}_{\mathrm{n}}^{(k)} = \eta^{-1}(\mathcal{C}^{(k)})$ by Proposition 5.6, the image of $\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ by the restriction morphism equals $\Bbbk[\mathcal{C}^{(k)}]$. Hence $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ is generated by $\Bbbk[\mathcal{C}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Then, by Lemma 5.7,(iii), $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]^{W(\mathcal{R})} = \Bbbk[\mathcal{C}^{(k)}]$.

(ii) Let K be the fraction field of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$. Since $\mathcal{C}_{\mathrm{n}}^{(k)}$ is a $W(\mathcal{R})$-variety, there is an action of $W(\mathcal{R})$ in K and $K^{W(\mathcal{R})}$ is the fraction field of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]^{W(\mathcal{R})}$ since $W(\mathcal{R})$ is finite. As a result, the integral closure $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]$ of $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$ in K is invariant under $W(\mathcal{R})$ and $\Bbbk[\widetilde{\mathcal{C}^{(k)}}]$ is contained in $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]^{W(\mathcal{R})}$ by (i). Let a be in $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]^{W(\mathcal{R})}$. Then a verifies a dependence integral equation over $\Bbbk[\mathcal{C}_{\mathrm{n}}^{(k)}]$,

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0$$

whence

$$a^m + (\frac{1}{|W(\mathcal{R})|}\sum_{w\in W(\mathcal{R})} w.a_{m-1})a^{m-1} + \cdots + \frac{1}{|W(\mathcal{R})|}\sum_{w\in W(\mathcal{R})} w.a_0 = 0$$

since a is invariant under $W(\mathcal{R})$ so that a is in $\Bbbk[\widetilde{\mathcal{C}^{(k)}}]$ by (i), whence the assertion. □

6. Desingularization

Let $k \geq 2$ be an integer. Let X, X' be as in Subsection 4.5. Denote by X_{n} the normalization of X and by θ_0 the normalization morphism. According to Proposition 4.14, X' identifies with a smooth big open subset of X_{n} and according to [Hir64], there exists a desingularization $(\Gamma, \pi_{\mathrm{n}})$ of X_{n} in the category of B-varieties such that the restriction of π_{n} to $\pi_{\mathrm{n}}^{-1}(X')$ is an isomorphism onto X'. Set $\pi = \theta_0 \circ \pi_{\mathrm{n}}$ so that (Γ, π) is a desingularization of X in the category of B-varieties. Recall that $\mathfrak{X}_{0,k}$ is the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$ and set $\mathfrak{X}_k := G \times_B \mathfrak{X}_{0,k}$. Then $\mathfrak{X}_k$ is a closed subvariety of $G \times_B \mathfrak{b}^k$.

Lemma 6.1. *Let E be the restriction to X of the tautological vector bundle of rank ℓ over $\mathrm{Gr}_\ell(\mathfrak{b})$ and let τ' be the canonical morphism from E to $\mathfrak{b}$.*

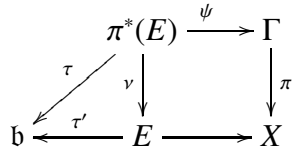
(i) *The morphism τ' is projective and birational.*

(ii) *Let ν be the canonical map from $\pi^*(E)$ to E. Then ν and $\tau := \tau' \circ \nu$ are B-equivariant birational projective morphisms from $\pi^*(E)$ to E and $\mathfrak{b}$ respectively. In particular, $\pi^*(E)$ is a desingularization of E and $\mathfrak{b}$.*

Proof. (i) By definition, E is the subvariety of elements (u, x) of $X \times \mathfrak{b}$ such that x is in u so that τ' is the projection from E to $\mathfrak{b}$. Since X is a projective variety, τ' is a projective morphism and $\tau'(E)$ is closed in $\mathfrak{b}$. Moreover, $\tau'(E)$ is B-invariant since τ' is a B-equivariant morphism and it contains $\mathfrak{h}$ since $\mathfrak{h}$ is in X. For x in $\mathfrak{h}_{\mathrm{reg}}$, $(\tau')^{-1}(x) = \{(\mathfrak{h}, x)\}$ since $\mathfrak{g}^x = \mathfrak{h}$. Hence τ' is a birational morphism and $\tau'(E) = \mathfrak{b}$ since $B(\mathfrak{h}_{\mathrm{reg}})$ is an open subset of $\mathfrak{b}$.

(ii) Since E is a vector bundle over X and since π is a projective birational morphism, ν is a projective birational morphism. Then τ is a projective birational morphism from $\pi^*(E)$ to $\mathfrak{b}$ by (i). It is B-equivariant since so are ν and τ'. Moreover, $\pi^*(E)$ is a desingularization of E and $\mathfrak{b}$ since $\pi^*(E)$ is smooth as a vector bundle over a smooth variety. □

Denote by ψ the canonical projection from $\pi^*(E)$ to Γ. Then, according to the above notations, we have the commutative diagram:



Lemma 6.2. *Let $E_{\mathrm{s}}^{(k)}$ be the fiber product $\pi^*(E)\times_\psi\cdots\times_\psi\pi^*(E)$ and let τ_k be the canonical morphism from $E_{\mathrm{s}}^{(k)}$ to $\mathfrak{b}^k$.*

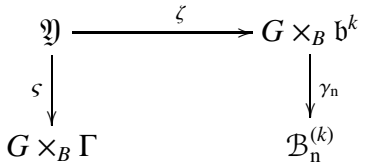
(i) *The vector bundle $E_{\mathrm{s}}^{(k)}$ over Γ is a vector subbundle of the trivial bundle $\Gamma\times\mathfrak{b}^k$. Moreover, $E_{\mathrm{s}}^{(k)}$ has dimension $k\ell+n$.*

(ii) *The morphism τ_k is a projective birational morphism from $E_{\mathrm{s}}^{(k)}$ onto $\mathfrak{X}_{0,k}$. Moreover, $E_{\mathrm{s}}^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$ in the category of B-varieties.*

Proof. (i) By definition, $E_{\mathrm{s}}^{(k)}$ is the subvariety of elements $(u,x_1,\dots,x_k)$ of $\Gamma\times\mathfrak{b}^k$ such that $x_1,\dots,x_k$ are in $\pi(u)$. Since X and Γ have dimension n, $E_{\mathrm{s}}^{(k)}$ has dimension $k\ell+n$ as a vector bundle of rank $k\ell$ over Γ.

(ii) Since Γ is a projective variety, τ_k is a projective morphism and $\tau_k(E_{\mathrm{s}}^{(k)})=\mathfrak{X}_{0,k}$ by Lemma 5.1,(i). For $(x_1,\dots,x_k)$ in $\mathfrak{b}^k_{\mathrm{reg,ss}}$, $\tau_k^{-1}(x_1,\dots,x_k)=\{(\mathfrak{g}^{x_1},(x_1,\dots,x_k))\}$ since $\mathfrak{g}^{x_1}$ is a Cartan subalgebra. Hence τ_k is a birational morphism, whence the assertion since $E_{\mathrm{s}}^{(k)}$ is a smooth variety as a vector bundle over the smooth variety Γ. □

Set $\mathfrak{Y}:=G\times_B(\Gamma\times\mathfrak{b}^k)$. The canonical projections from $G\times\Gamma\times\mathfrak{b}^k$ to $G\times\Gamma$ and $G\times\mathfrak{b}^k$ define through the quotients morphisms from $\mathfrak{Y}$ to $G\times_B\Gamma$ and $G\times_B\mathfrak{b}^k$. Denote by ς and ζ these morphisms. Then we have the following diagram:



The map $(g,x)\mapsto(g,\tau_k(x))$ from $G\times E_{\mathrm{s}}^{(k)}$ to $G\times\mathfrak{b}^k$ defines through the quotient a morphism $\overline{\tau_k}$ from $G\times_B E_{\mathrm{s}}^{(k)}$ to $\mathfrak{X}_k$.

Proposition 6.3. *Set $\xi:=\gamma_{\mathrm{n}}\circ\overline{\tau_k}$.*

(i) *The variety $G\times_B E_{\mathrm{s}}^{(k)}$ is a closed subvariety of $\mathfrak{Y}$.*

(ii) *The variety $G\times_B E_{\mathrm{s}}^{(k)}$ is a vector bundle of rank $k\ell$ over $G\times_B\Gamma$. Moreover, $G\times_B\Gamma$ and $G\times_B E_{\mathrm{s}}^{(k)}$ are smooth varieties.*

(iii) *The morphism ξ is a projective birational morphism from $G\times_B E_{\mathrm{s}}^{(k)}$ onto $\mathcal{C}_{\mathrm{n}}^{(k)}$.*

Proof. (i) According to Lemma 6.2,(i), $E_{\mathrm{s}}^{(k)}$ is a closed subvariety of $\Gamma\times\mathfrak{b}^k$, invariant under the diagonal action of B. Hence $G\times E_{\mathrm{s}}^{(k)}$ is a closed subvariety of $G\times\Gamma\times\mathfrak{b}^k$, invariant under the action of B, whence the assertion.

(ii) Since $E_{\mathrm{s}}^{(k)}$ is a B-equivariant vector bundle over Γ, $G\times_B E_{\mathrm{s}}^{(k)}$ is a G-equivariant vector bundle over $G\times_B\Gamma$. Since $G\times_B\Gamma$ is a fiber bundle over the smooth variety G/B with smooth fibers, $G\times_B\Gamma$ is a smooth variety. As a result, $G\times_B E_{\mathrm{s}}^{(k)}$ is a smooth variety.

(iii) According to Lemma 6.2,(ii), $\overline{\tau_k}$ is a projective birational morphism from $G \times_B E_{\mathrm{s}}^{(k)}$ to $\mathfrak{X}_k$. Since $\mathfrak{X}_{0,k}$ is a B-invariant closed subvariety of $\mathfrak{b}^k$, $\mathfrak{X}_k$ is closed in $G \times_B \mathfrak{b}^k$. According to Lemma 5.1,(i), $\gamma(\mathfrak{X}_k) = \mathcal{C}^{(k)}$. Moreover, $\gamma_{\mathrm{n}}(\mathfrak{X}_k)$ is an irreducible closed subvariety of $\mathcal{B}_{\mathrm{n}}^{(k)}$ since γ_{n} is a projective morphism by Lemma 1.7. Hence $\gamma_{\mathrm{n}}(\mathfrak{X}_k) = \mathcal{C}_{\mathrm{n}}^{(k)}$ by Proposition 5.6. For all z in $G.\iota_k(\mathfrak{h}_{\mathrm{reg}}^k)$, $|\gamma_{\mathrm{n}}^{-1}(z)| = 1$. Hence the restriction of γ_{n} to $\mathfrak{X}_k$ is a birational morphism onto $\mathcal{C}_{\mathrm{n}}^{(k)}$ since $G.\iota_k(\mathfrak{h}_{\mathrm{reg}}^k)$ is dense in $\mathcal{C}_{\mathrm{n}}^{(k)}$. Moreover, this morphism is projective since γ_{n} is projective. As a result, ξ is a projective birational morphism from $G \times_B E_{\mathrm{s}}^{(k)}$ onto $\mathcal{C}_{\mathrm{n}}^{(k)}$. □

The following corollary results from Lemma 6.2,(ii), Proposition 6.3,(ii) and (iii), and Lemma 1.4.

Corollary 6.4. *Let $\widetilde{\mathfrak{X}_{0,k}}$ and $\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}$ be the normalizations of $\mathfrak{X}_{0,k}$ and $\mathcal{C}_{\mathrm{n}}^{(k)}$ respectively. Then $\Bbbk[\widetilde{\mathfrak{X}_{0,k}}]$ and $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}]$ are the spaces of global sections of $\mathcal{O}_{E_{\mathrm{s}}^{(k)}}$ and $\mathcal{O}_{G\times_B E_{\mathrm{s}}^{(k)}}$ respectively.*

7. Rational singularities

Let $k \geq 2$ be an integer. Let X, X', X_{n}, θ_0, Γ, π_{n}, π, E, $E_{\mathrm{s}}^{(k)}$, ψ, ν, τ, τ_k be as in Section 6. We have the commutative diagram:

$$\begin{array}{ccccc} E_{\mathrm{s}}^{(k)} & \xrightarrow{\tau_k} & \mathfrak{X}_{0,k} & & \\ \downarrow{\scriptstyle \psi_k} & & & & \\ \Gamma & \xrightarrow{\pi_{\mathrm{n}}} & X_{\mathrm{n}} & \xrightarrow{\theta_0} & X \\ & \searrow^{\pi} & & & \end{array}$$

with ψ_k the canonical projection from $E_{\mathrm{s}}^{(k)}$ onto Γ.

7.1. Let $\mathfrak{g}'_{\mathrm{reg}}$ be the set of regular elements x such that x_{s} is regular or subregular and set $\mathfrak{b}'_{\mathrm{reg}} := \mathfrak{g}'_{\mathrm{reg}} \cap \mathfrak{b}$.

Lemma 7.1. (i) *The subset $\mathfrak{b}'_{\mathrm{reg}}$ of $\mathfrak{b}$ is a big open subset of $\mathfrak{b}$.*
(ii) *The subset $\mathfrak{g}'_{\mathrm{reg}}$ of $\mathfrak{g}$ is a big open subset of $\mathfrak{g}$.*

Proof. Let x be in $\mathfrak{g}'_{\mathrm{reg}} \setminus \mathfrak{g}_{\mathrm{reg,ss}}$. Let W be the set of elements y of $\mathfrak{g}^{x_{\mathrm{s}}}$ such that the restriction of $\mathrm{ad}\, y$ to $[x_{\mathrm{s}}, \mathfrak{g}]$ is injective. Then W is an open subset of $\mathfrak{g}^{x_{\mathrm{s}}}$, containing x, and the map

$$G \times W \longrightarrow \mathfrak{g}, \qquad (g, y) \longmapsto g(y)$$

is a submersion. Let $\mathfrak{z}$ be the center of $\mathfrak{g}^{x_{\mathrm{s}}}$ and set $\mathfrak{z}' := W \cap \mathfrak{z}$. For some open subset W' of W, containing x, for all y in W', the component of y on $\mathfrak{z}$ is in $\mathfrak{z}'$. Since $[\mathfrak{g}^{x_{\mathrm{s}}}, \mathfrak{g}^{x_{\mathrm{s}}}]$ is a simple algebra of dimension 3, $W' \cap \mathfrak{g}_{\mathrm{reg}}$ is contained in $\mathfrak{g}'_{\mathrm{reg}}$ and $G(W' \cap \mathfrak{g}_{\mathrm{reg}})$ is an open set, contained in $\mathfrak{g}'_{\mathrm{reg}}$ and containing x. As a result, $\mathfrak{g}'_{\mathrm{reg}}$ is an open subset of $\mathfrak{g}$ and $\mathfrak{b}'_{\mathrm{reg}}$ is an open subset of $\mathfrak{b}$.

(i) Suppose that $\mathfrak{b} \setminus \mathfrak{b}'_{\mathrm{reg}}$ has an irreducible component Σ of codimension 1 in $\mathfrak{b}$. A contradiction is expected. Since Σ is invariant under B, $\Sigma \cap \mathfrak{h}$ is the image of Σ by the projection $x \mapsto \overline{x}$ by Lemma 1.8. Since Σ has codimension 1 in $\mathfrak{b}$, $\Sigma \cap \mathfrak{h} = \mathfrak{h}$ or $\Sigma = \Sigma \cap \mathfrak{h} + \mathfrak{u}$. Since Σ does not contain regular semisimple element, $\Sigma \cap \mathfrak{h}$ is an irreducible subset of codimension 1 of $\mathfrak{h}$, not containing regular semisimple elements. Hence $\Sigma \cap \mathfrak{h} = \mathfrak{h}_\alpha$ for some positive root α and $\Sigma \cap (\mathfrak{h}'_\alpha + \mathfrak{g}^\alpha) \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty, whence the contradiction.

(ii) Since $\mathfrak{b} \setminus \mathfrak{b}'_{\mathrm{reg}}$ is invariant under B, $\mathfrak{g} \setminus \mathfrak{g}'_{\mathrm{reg}} = G(\mathfrak{b} \setminus \mathfrak{b}'_{\mathrm{reg}})$ and

$$\dim \mathfrak{g} \setminus \mathfrak{g}'_{\mathrm{reg}} \leq n + \dim \mathfrak{b} \setminus \mathfrak{b}'_{\mathrm{reg}},$$

whence the assertion by (i). □

Setting $\mathfrak{b}_{\mathrm{reg},0} := \mathfrak{b}_{\mathrm{reg}}$ and $\mathfrak{b}_{\mathrm{reg},1} := \mathfrak{b}'_{\mathrm{reg}}$, let $V_{k,j}$ be the subset of elements x of $\mathfrak{X}_{0,k}$ such that $P_x \cap \mathfrak{b}_{\mathrm{reg},j}$ is not empty for $j = 0, 1$. By definition,

$$E^{(k)} := \{(u, x_1, \ldots, x_k) \in X \times \mathfrak{b}^k \mid u \ni x_1, \ldots, u \ni x_k\}$$

so that $E^{(k)}$ is a vector bundle over X. Denote by ρ_k the map

$$E^{(k)} \longrightarrow \mathfrak{X}_{0,k}, \qquad (u, x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k).$$

Proposition 7.2. *For $j = 0, 1$, let $V'_{k,j}$ be the subset of elements $x = (x_1, \ldots, x_k)$ of $\mathfrak{X}_{0,k}$ such that x_1 is in $\mathfrak{b}_{\mathrm{reg},j}$.*

(i) *For $j = 0, 1$, $V'_{k,j}$ is a smooth open subset of $\mathfrak{X}_{0,k}$.*

(ii) *For $j = 0, 1$, $V_{k,j}$ is a smooth open subset of $\mathfrak{X}_{0,k}$.*

(iii) *For $j = 0, 1$, $\rho_k^{-1}(V_{k,j})$ is a big open subset of $E^{(k)}$.*

Proof. (i) By definition, $V'_{k,j}$ is the intersection of $\mathfrak{X}_{0,k}$ and the open subset $\mathfrak{b}_{\mathrm{reg},j} \times \mathfrak{b}^{k-1}$ of $\mathfrak{b}^k$. Hence $V'_{k,j}$ is an open subset of $\mathfrak{X}_{0,k}$. For x_1 in $\mathfrak{b}_{\mathrm{reg},0}$, $(x_1, \ldots, x_k)$ is in $V'_{k,0}$ if and only if $x_2, \ldots, x_k$ are in $\mathfrak{g}^{x_1}$ by Corollary 4.3,(ii) and Lemma 6.2,(ii) since $\mathfrak{g}^{x_1}$ is in X. According to [Ko63, Theorem 9], for x in $\mathfrak{b}_{\mathrm{reg}}$, $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$. Hence the map

$$\begin{gathered}\mathfrak{b}_{\mathrm{reg}} \times \mathrm{M}_{k-1,\ell}(\Bbbk) \xrightarrow{\theta} V'_{k,0}, \\ (x, (a_{i,j}, 1 \leq i \leq k-1, 1 \leq j \leq \ell)) \longmapsto (x, \textstyle\sum_{j=1}^{\ell} a_{1,j}\varepsilon_j(x), \ldots, \sum_{j=1}^{\ell} a_{k-1,j}\varepsilon_j(x))\end{gathered}$$

is a bijective morphism. The open subset $\mathfrak{b}_{\mathrm{reg}}$ has a cover by open subsets V such that for some $e_1, \ldots, e_n$ in $\mathfrak{b}$, $\varepsilon_1(x), \ldots, \varepsilon_\ell(x), e_1, \ldots, e_n$ is a basis of $\mathfrak{b}$ for all x in V. Then there exist regular functions $\varphi_1, \ldots, \varphi_\ell$ on $V \times \mathfrak{b}$ such that

$$v - \sum_{j=1}^{\ell} \varphi_j(x, v)\varepsilon_j(x) \in \mathrm{span}(e_1, \ldots, e_n)$$

for all (x, v) in $V \times \mathfrak{b}$, so that the restriction of θ to $V \times \mathrm{M}_{k-1,\ell}(\Bbbk)$ is an isomorphism onto $\mathfrak{X}_{0,k} \cap V \times \mathfrak{b}^{k-1}$ whose inverse is

$$(x_1, \ldots, x_k) \longmapsto (x_1, ((\varphi_1(x_1, x_i), \ldots, \varphi_\ell(x_1, x_i)), i = 2, \ldots, k))$$

As a result, θ is an isomorphism and $V'_{k,0}$ is a smooth variety, whence the assertion since $V'_{k,1}$ is an open subset of $V'_{k,0}$.

(ii) The subvariety $\mathfrak{X}_{0,k}$ of $\mathfrak{b}^k$ is invariant under the natural action of $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{b}^k$ and $V_{k,j} = \mathrm{GL}_k(\Bbbk).V'_{k,j}$ by Lemma 1.9, whence the assertion by (i).

(iii) Since $V_{k,1}$ is contained in $V_{k,0}$, it suffices to prove the assertion for $j = 1$. Suppose that $E^{(k)} \setminus \rho_k^{-1}(V_{k,1})$ has an irreducible component Σ of codimension 1. A contradiction is expected. Denoting by π_0 the canonical projection $E^{(k)} \to X$, $\overline{\pi_0(\Sigma)}$ has codimension at most 1 in X. For u in $U.\mathfrak{h}$, u is a Cartan subalgebra and $\mathfrak{u}^k \setminus V_{k,1}$ has dimension $k(l-1)$. Hence $\overline{\pi_0(\Sigma)}$ has codimension 1 in X so that $\pi_0(\Sigma) \cap X'$ is not empty since X' is a big open subset of X by Corollary 4.13,(ii). For u in X', $u \cap \mathfrak{b}_{\mathrm{reg},1}$ is not empty. Then $\{u\} \times u^k$ is not contained in Σ for all u in a dense subset of $\pi_0(\Sigma)$, whence a contradiction since Σ has codimension 1. □

Corollary 7.3. *Let $j = 0, 1$.*

(i) *The restriction of ρ_k to $\rho_k^{-1}(V_{k,j})$ is an isomorphism onto $V_{k,j}$.*

(ii) *The morphism ρ_k is projective and birational and $V_{k,j}$ is a big open subset of $\mathfrak{X}_{0,k}$.*

Proof. (i) For $x = (x_1, \dots, x_k)$ in $V_{k,j}$ and u in X such that (u, x) is in $E^{(k)}$, $u = \mathfrak{g}^y$ for all y in $P_x \cap \mathfrak{b}_{\mathrm{reg}}$. Hence the restriction of ρ_k to $\rho_k^{-1}(V_{k,j})$ is injective. Then, by Zariski Main Theorem [Mu88, §9], the restriction of ρ_k to $\rho_k^{-1}(V_{k,j})$ is an isomorphism onto $V_{k,j}$ since $V_{k,j}$ is smooth.

(ii) Since X is a projective, ρ_k is a projective morphism. It is birational by (i). Then, by Proposition 7.2,(iii), $V_{k,j}$ is a big open subset of $\mathfrak{X}_{0,k}$. □

7.2. By definition, the restriction of π_{n} to $\pi_{\mathrm{n}}^{-1}(X')$ is an isomorphism onto X'. Identify $\pi_{\mathrm{n}}^{-1}(X')$ and X' by π_{n}. Set $E_{\mathrm{n}} := \theta_0^*(E)$ and denote by E'_{n} the inverse image of X' by the canonical map from E_{n} to X_{n}. Then E'_{n} is a smooth variety as a vector bundle above the smooth variety X'.

Lemma 7.4. *Denote by ν_{n} the canonical morphism from E_{n} to E and by ι_{n} the embedding $E'_{\mathrm{n}} \to E_{\mathrm{n}}$.*

(i) *There exists a well defined projective birational morphism σ_{n} from $\pi^*(E)$ to E_{n} such that $\nu = \nu_{\mathrm{n}} \circ \sigma_{\mathrm{n}}$. Moreover, E_{n} is normal.*

(ii) *The $\mathcal{O}_{E_{\mathrm{n}}}$-module $\iota_{\mathrm{n}*}(\Omega_{E'_{\mathrm{n}}})$ is free of rank* 1.

(iii) *The variety E_{n} is Gorenstein and has rational singularities. Moreover, the canonical module of E_{n} is free of rank* 1.

Proof. (i) Since E_{n} is a vectore bundle over X_{n}, E_{n} is a normal variety. Moreover, it is the normalization of E and ν_{n} is the normalization morphism, whence the assertion by Lemma 6.1,(ii).

(ii) Set $\tau_{\mathrm{n}} := \tau' \circ \nu_{\mathrm{n}}$ and denote by τ'_{n} the restriction of τ_{n} to E'_{n}. Let ω be a volume form on $\mathfrak{b}$. Then ${\tau'_{\mathrm{n}}}^*(\omega)$ is a global section of $\Omega_{E'_{\mathrm{n}}}$, without zero. For φ a local section of $\iota_{\mathrm{n}*}(\Omega_{E'_{\mathrm{n}}})$ above the open subset U of E_{n}, for some regular function ψ on $U \cap E'_{\mathrm{n}}$, $\psi {\tau'_{\mathrm{n}}}^*(\omega)$ is the restriction of φ to $U \cap E'_{\mathrm{n}}$. According to Proposition 4.14, X' is a big open subset of X_{n}. Hence E'_{n} is a big open subset of E_{n} and $U \cap E'_{\mathrm{n}}$ is a big open subset of U so that ψ has a regular extension to U. As a result, there exists a well defined morphism from $\iota_{\mathrm{n}*}(\Omega_{E'_{\mathrm{n}}})$ to $\mathcal{O}_{E_{\mathrm{n}}}$ whose inverse is the morphism

$$\mathcal{O}_{E_{\mathrm{n}}} \longrightarrow \iota_{\mathrm{n}*}(\Omega_{E'_{\mathrm{n}}}) , \qquad \psi \longmapsto \psi {\tau'_{\mathrm{n}}}^*(\omega),$$

whence the assertion.

(iii) Denote by μ the morphism

$$\mathcal{O}_{\pi^*(E)} \xrightarrow{\mu} \Omega_{\pi^*(E)} , \qquad \psi \longmapsto \psi \tau^*(\omega).$$

By Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}(\sigma_{\mathrm{n}})_*(\Omega_{\pi^*(E)})$ is isomorphic to $\iota_{\mathrm{n}*}(\Omega_{E'_{\mathrm{n}}})$ in the derived category of the coherent modules on E_{n}. So, by (ii), $\mathrm{R}^0\sigma_{\mathrm{n}*}(\mu)$ is an isomorphism from $\mathcal{O}_{E_{\mathrm{n}}}$ onto $\mathrm{R}^0(\sigma_{\mathrm{n}})_*(\Omega_{\pi^*(E)})$ since E_{n} is normal. Then, by [Hi91, §5], E_{n} is Gorenstein and has rational singularities. Again by (ii), the canonical module of E_{n} is free of rank 1 since it is equal to $\iota_{\mathrm{n}*}(\Omega_{E'_{\mathrm{n}}})$ by definition. □

Let $E_{\mathrm{n}}^{(k)}$ be the following fiber product:

$$\begin{array}{ccc} E_{\mathrm{n}}^{(k)} & \longrightarrow & E^{(k)} \\ \downarrow{\scriptstyle \rho_{\mathrm{n},k}} & & \downarrow{\scriptstyle \rho_k} \\ X_{\mathrm{n}} & \longrightarrow & X \end{array}$$

Then $E_{\mathrm{n}}^{(k)}$ is the normalization of $E^{(k)}$ since $E^{(k)}$ is a vector bundle over X.

Proposition 7.5. (i) *The variety $E_{\mathrm{s}}^{(k)}$ is a B-equivariant desingularization of $E_{\mathrm{n}}^{(k)}$.*

(ii) *The variety $E_{\mathrm{n}}^{(k)}$ is Gorenstein and has rational singularities.*

Proof. (i) The variety $E_{\mathrm{n}}^{(k)}$ is the normalization of $E^{(k)}$, whence a commutative diagram

$$\begin{array}{ccc} E_{\mathrm{s}}^{(k)} & \longrightarrow & E_{\mathrm{n}}^{(k)} \\ & \searrow & \downarrow \\ & & E^{(k)} \end{array}$$

According to Lemma 6.2,(ii), the diagonal arrow is a B-equivariant birational projective morphism. Hence the horizontal arrow is B-equivariant, birational and projective.

(ii) The variety $E_{\mathrm{n}}^{(k)}$ is a vector bundle over E_{n}. So, by Lemma 7.4,(iii), $E_{\mathrm{n}}^{(k)}$ is Gorenstein and has rational singularities. □

7.3. According to Lemma 2.2,(i) and Theorem 2.13,(i), ι_k is an embedding of $\mathfrak{b}^k$ into $\mathcal{B}_{\mathrm{n}}^{(k)}$. Moreover, $\iota_k(\mathfrak{X}_{0,k})$ is contained in $\mathcal{C}_{\mathrm{n}}^{(k)}$. Since $E_{\mathrm{n},k}$ and $\mathfrak{X}_{0,k}$ are B-varieties, we have a commutative diagram

$$\begin{array}{ccccc} G \times_B E_{\mathrm{n}}^{(k)} & \longrightarrow & G \times_B \mathfrak{X}_{0,k} & \longrightarrow & G \times_B \mathfrak{b}^k \\ & \searrow^{\kappa_{\mathrm{n},k}} & \downarrow & & \downarrow{\scriptstyle \gamma_{\mathrm{n}}} \\ & & \mathcal{C}_{\mathrm{n}}^{(k)} & \longrightarrow & \mathcal{B}_{\mathrm{n}}^{(k)} \end{array}$$

According to Proposition 7.2 and Corollary 7.3, $V_{k,1}$ identifies with a smooth big open subset of $E_{\mathrm{n}}^{(k)}$.

Lemma 7.6. (i) *The set $G \times_B V_{k,1}$ is a smooth big open subset of $G \times_B E_{\mathrm{n}}^{(k)}$.*

(ii) *The set $G.\iota_k(V_{k,1})$ is a smooth big open subset of $\mathcal{C}_{\mathrm{n}}^{(k)}$ and the restriction of $\kappa_{\mathrm{n},k}$ to $G \times_B V_{k,1}$ is an isomorphism onto $G.\iota_k(V_{k,1})$.*

(iii) *A global section of $\Omega_{G.\iota_k(V_{k,1})}$ has a regular extension to the smooth locus of $G \times_B E_{\mathrm{n}}^{(k)}$.*

Proof. (i) According to Proposition 7.2,(iii), $V_{k,1}$ is a smooth big open subset of $E_{\mathrm{n}}^{(k)}$, invariant under B. Then $G \times_B V_{k,1}$ is a smooth big open subset of $G \times_B E_{\mathrm{n}}^{(k)}$ since G/B is smooth.

(ii) Since $\gamma_{\mathrm{n}}^{-1}(G.\iota_k(V_{k,1}))$ equals $G \times_B V_{k,1}$ and since γ_{n} is projective and birational, $G.\iota_k(V_{k,1})$ is an open subset of $\mathcal{C}_{\mathrm{n}}^{(k)}$. Moreover, $G \times_B V_{k,1}$ is contained in the open subset $\gamma_{\mathrm{n}}^{-1}(W_k)$ of $G \times_B \mathfrak{b}^k$ and the restriction of γ_{n} to $\gamma_{\mathrm{n}}^{-1}(W_k)$ is an isomorphism onto W_k by Corollary 2.15, so that the restriction of γ_{n} to $G \times_B V_{k,1}$ is an isomorphism onto $G.\iota_k(V_{k,1})$, whence the assertion by (i).

(iii) The assertion results from (i), (ii) and Lemma A.1,(v). □

Denote by $\widetilde{\mathcal{C}_{\mathrm{n}}^{(k)}}$ and $\widetilde{\mathcal{C}^{(k)}}$ the normalizations of $\mathcal{C}_{\mathrm{n}}^{(k)}$ and $\mathcal{C}^{(k)}$ respectively.

Theorem 7.7. *The varieties $\widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}}$ and $\widetilde{\mathcal{C}^{(2)}}$ have rational singularities.*

Proof. According to Lemma 7.6,(ii), $\kappa_{\mathrm{n},k}$ is a birational morphism. It is projective since so are ρ_k and γ_{n} and G/B is projective. As a result, we have a commutative diagram

$$\begin{array}{ccc} G \times_B E_{\mathrm{n}}^{(2)} & \xrightarrow{\widetilde{\kappa_{\mathrm{n},2}}} & \widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}} \\ & {\scriptstyle \kappa_{\mathrm{n},2}}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & \mathcal{C}_{\mathrm{n}}^{(2)} & \end{array}$$

with μ the normalization morphism. Moreover, $\widetilde{\kappa_{\mathrm{n},2}}$ is a projective and birational morphism. By Lemma 7.6,(ii), $\mu^{-1}(G.\iota_2(V_{2,1}))$ is a smooth big open subset of $\widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}}$ and the restriction of μ to $\mu^{-1}(G.\iota_2(V_{2,1}))$ is an isomorphism onto $G.\iota_2(V_{2,1})$. So, by Lemma 7.6,(iii), all global section of $\Omega_{\mu^{-1}(G.\iota_2(V_{2,1}))}$ has a regular extension to the smooth locus of $G\times_B E_{\mathrm{n}}^{(2)}$, denoted by Y.

According to Proposition 6.3,(ii), $G\times_B E_{\mathrm{s}}^{(2)}$ is a desingularization of $\mathcal{C}_{\mathrm{n}}^{(2)}$ and by Proposition 7.5,(i), $E_{\mathrm{s}}^{(2)}$ is a desingularization of $E_{\mathrm{n}}^{(2)}$ with a B-equivariant desingularization morphism. Hence $G\times_B E_{\mathrm{s}}^{(2)}$ is a desingularization of $\widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}}$ and $G\times_B E_{\mathrm{n}}^{(2)}$. By Proposition 7.5,(ii), $E_{\mathrm{n}}^{(2)}$ has rational singularities. Hence $G\times_B E_{\mathrm{n}}^{(2)}$ has rational singularities as fiber bundle over a smooth variety with fibers having rational singularities. So, all global section of Ω_Y has a regular extension to $G\times_B E_{\mathrm{s}}^{(2)}$ by [KK73, p.50]. As a result, all global section of $\Omega_{\mu^{-1}(G.\iota_2(V_{2,1}))}$ has a regular extension to $G\times_B E_{\mathrm{s}}^{(2)}$. According to Proposition 5.6, $\widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}}$ is the normalization of the isospectral commuting variety and according to [Gi12, Theorem 1.3.4], $\widetilde{C_{\mathrm{n}}^{(2)}}$ is Gorenstein. Hence by [KK73, p.50], $\widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}}$ has rational singularities. By Proposition 5.8,(ii), $\widetilde{\mathcal{C}^{(2)}}$ is the categorical quotient of $\widetilde{\mathcal{C}_{\mathrm{n}}^{(2)}}$ under the action of $W(\mathcal{R})$. So, by [El81, Lemme 1], $\widetilde{\mathcal{C}^{(2)}}$ has rational singularities. □

Appendix A. Rational Singularities

Let X and Y be irreducible varieties. Denote by Y' the smooth locus of Y.

Lemma A.1. *Suppose that π is a projective birational morphism from Y to X verifying the following conditions for some smooth big open subset X' of X:*

(1) *the open subset $\pi^{-1}(X')$ of Y is big,*
(2) *the restriction of π to $\pi^{-1}(X')$ is an isomorphism onto X'.*

Then all regular form of top degree on X' has a unique regular extension to Y'.

Proof. According to Condition (2), $\pi^{-1}(X')$ is a dense open subset of Y'. Moreover, $\pi^{-1}(X')$ identifies with X'. Let ω be a differential form of top degree on X'. Since $\Omega_{Y'}$ is a locally free module of rank one, there is an affine open cover $O_1,\dots,O_k$ of Y' such that restriction of $\Omega_{Y'}$ to O_i is a free $\mathcal{O}_{O_i}$-module generated by some section ω_i. For $i=1,\dots,k$, set $O_i' := O_i\cap X'$. Let ω be a regular form of top degree on X'. For $i=1,\dots,k$, for some regular function a_i on O_i', $a_i\omega_i$ is the restriction of ω to O_i'. According to Condition (1), O_i' is a big open subset of O_i. Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for $1\le i,j\le k$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O_i'\cap O_j'$ and $O_i\cap O_j$ since $\Omega_{Y'}$ is torsion free as a locally free module. Let ω' be the global section of $\Omega_{Y'}$ extending the $a_i\omega_i$'s. Then ω' is a regular extension of ω to Y' and this extension is unique since X' is dense in Y' and $\Omega_{Y'}$ is torsion free. □

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