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Abstract

In this paper, we consider the probabilistic modeling of media exhibiting uncertainties on material symmetries. More specifically, we address both the construction of a stochastic model and the definition of a methodology allowing the numerical simulation (and consequently, the inverse experimental identification) of random elasticity tensors whose mean distance (in a sense to be defined) to a given class of material symmetry is specified. Following the eigensystem characterization of the material symmetries, the proposed approach relies on the probabilistic model derived in [25], allowing the variance of selected eigenvalues of the elasticity tensor to be partially prescribed. In this context, a new methodology (regarding in particular the parametrization of the model) is defined and illustrated in the case of transversely isotropic materials. The efficiency of the approach is demonstrated by computing the mean distance of the random elasticity tensor to a given material symmetry class, the distance and projection onto the space of transversely isotropic tensors being defined by considering the Riemmanian metric and the Euclidean projection, respectively. It is shown that the methodology allows the above distance to be (partially) reduced as the overall level of statistical fluctuations increases, no matter the initial distance of the mean model used in the simulations. A comparison between this approach and the initial nonparametric approach introduced in [37] is finally provided.

Key words: Elasticity tensor; Material Symmetry; Maximum Entropy Principle; Probabilistic Model; Uncertainty.

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1. Introduction

The increasing use and modeling of heterogeneous materials with complex microstructures, such as fiber- or nano-reinforced composites and live tissues, gives rise to many scientific challenges defined at various scales. Among these aspects, the macroscopic modeling of such materials, while trivial at first sight, is still a complicated task when the material symmetries of the medium are not supposed *a priori*. It is worth noticing that (**i**) such symmetry properties are usually assumed for the sake of convenience and/or simplicity (rather than experimentally identified), and that (**ii**) all materials are likely to present a slightly anisotropic behavior which may or may not be taken into account, depending on whether the anisotropic contributions are considered as negligible or not. It should further be pointed out that the question on how to properly define material symmetry constraints also arises in the mesoscale computational modeling and experimental identification of random elastic microstructures (see the remark at the end of this section).

This paper then addresses the stochastic modeling of uncertain elasticity tensors with unknown material symmetries. No matter the scale at which the representation is performed, the complex material can be conceptually "replaced" as an homogeneous medium whose linear behavior is modeled by a constitutive equation defined by an overall fourth-order elasticity tensor, denoted by $\mathbb{C}$, exhibiting uncertainties on the symmetry class to which it belongs, as well as possible intrinsic randomness (a mesoscopic modeling resulting in the definition of the so-called *apparent*, stochastic properties; see [14] [15] [31] [29] for a detailed discussion and the definition of inequalities between the apparent and *effective* elasticity tensors). Consequently, the matrix representation $[\mathbf{C}] \in \mathcal{M}_6^+(\mathbb{R})$ (where $\mathcal{M}_6^+(\mathbb{R})$ is the set of all the 6×6 symmetric positive-definite real matrices) of the elasticity tensor $\mathbb{C}$ has to be modeled as a random matrix whose probabilistic model must be constructed. With reference to the fundamental points introduced above, it should be pointed out at this stage that such a derivation turns out to be useful (**i**) in the more general framework of computational stochastic mechanics, when the randomness arising from mesoscopic features (see [9] for an example of such an application, for instance) or macroscopic uncertainties on material symmetries may have to be taken into account at the coarse scale (that is, the scale at which engineering structural applications are carried out); (**ii**) in the context of experimental identification, when no *a priori* assumptions can be made about the material symmetries exhibited by the microstructure under consideration, or when such assumptions have to be relaxed.

Such a construction can be achieved within two general stochastic frameworks. A first approach, referred to as a *parametric* one, consists in modeling the uncertainties on all the non-zero components of the random matrix $[\mathbf{C}]$ and then requires a prior choice regarding the class of symmetry to which all the realizations will belong. Clearly, this material symmetry class should include the expected (or usually assumed) symmetry class and would thus correspond, in the most general case, to the class of fourth-order anisotropic elasticity tensors. The parametric approach would then require the construction of a probabilistic model for 21 statistically dependent real-valued random variables, so that the joint probability distribution should be constructed on $\mathbb{R}^{21}$. Such a construction is clearly intractable in practice, and it should be pointed out that the complexity of the construction still remains for higher material symmetries.

An alternative to this modeling relies on the direct construction of a nonparametric probabilistic model for the matrix-valued random variable corresponding to the matrix representation $[\mathbf{C}]$. Such an approach was first introduced for the anisotropic material class in [36] [37] (and experimentally identified in [13] for instance), in a slightly different context. In the sequel, we will refer to this probabilistic model as the *nonparametric probabilistic model for anisotropic media*. The construction is based on the use of the Maximum Entropy Principle (MEP; see Section 3.1) under the following set of constraints:

(**i**) The usual normalization condition for the probability density function;

(**ii**) The constraints induced by the symmetries and positiveness properties;

(**iii**) The mean function, which is assumed to be given;

(**iv**) The constraint related to the existence of the second-order moment of the inverse matrix norm (allowing, together with point ((**ii**)), the stochastic non-uniform ellipticity property to be preserved).

Following a similar methodology, Das and Ghanem [9] replaced constraint (**iv**) by an uniform ellipticity condition consisting in introducing a deterministic non-zero lower bound (for the elasticity tensor) which has to be defined and/or constructed. While these nonparametric probabilistic models benefit from both their mathematical background and simplicity (and especially regarding the experimental identification, because of a minimal parametrization), they basically induce anisotropic statistical fluctuations which cannot be preferably assigned to a set of components of the random

matrix (for instance, with respect to material symmetries). This fact was recently pointed out in [39], where the authors propose a new class with the nonparametric probabilistic approach by adding a new parameter allowing the distance of the anisotropy class to be measured with respect to the isotropic symmetry, and to be partly controlled. Such developments are in the class of the generalized probabilistic approach of uncertainties [38] corresponding to a coupling between the parametric probabilistic approach and the nonparametric one. Nevertheless, for lower material symmetries, such an approach requires the construction of a parametric probabilistic model on a space of high dimension (5 for transverse isotropy, 9 for orthotropy, etc.).

In this paper, we specifically address the construction of a probabilistic model and the definition of a methodology dedicated to the modeling of an anisotropic elasticity tensor, under the constraint that the mean distance of the random elasticity matrix $[\mathbf{C}]$ to a given material symmetry class is specified. The paper is organized as follows.

We present, in Section 2, the general framework associated with material symmetry classes. We introduce distances in the set of elasticity tensors (from which projections onto the set of elasticity tensors with given symmetries can be defined), as well as the eigensystem characterization of material symmetries. In particular, such definitions will be used in order to define a set of constraints on the stochastic eigenvalues of the random elasticity tensor, so that the (mean) distance to a material symmetry class can be specified.

The probabilistic model for symmetric positive-definite random matrices verifying a stochastic ellipticity condition (with constrained variances on selected eigenvalues), derived in [25] and used in this paper, is recalled in Section 3. A new parametrization is further introduced and the strategy for simulating independent realizations of the random elasticity tensor is briefly recalled.

The methodology is finally exemplified in Section 4, where we consider the case of a symmetry constraint defined with respect to transverse isotropy. Such an application typically corresponds to the case of unidirectional composites, for instance. Moreover, the results obtained by using the proposed approach are compared with the ones derived from the nonparametric probabilistic approach for anisotropic media.

Remark: naturally, the constraints on material symmetries considered in this paper typically hold when dealing with the mesoscale stochastic modeling

of heterogeneous materials, which have to be modeled as non-homogeneous random media. However, such a modeling would require the construction of a probabilistic model for a tensor-valued random field, which is outside the scope of this work and will be presented in a forthcoming paper.

2. Definition of a distance of an elasticity tensor with arbitrary material symmetries to the class of elasticity tensors exhibiting given symmetry properties

2.1. Representation of the elasticity tensor

In the following, we will consider the matrix representation $[C] \in \mathcal{M}_6^+(\mathbb{R})$ of the fourth-order tensor $\mathbb{C}$ (with components $\mathbb{C}_{ijk\ell}$) belonging to the set of elasticity tensors (verifying the usual symmetry and positiveness properties), also known as the Kelvin formulation and defined as follows [23]:

$$[C] = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \sqrt{2}\mathbb{C}_{1123} & \sqrt{2}\mathbb{C}_{1113} & \sqrt{2}\mathbb{C}_{1112} \\ & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \sqrt{2}\mathbb{C}_{2223} & \sqrt{2}\mathbb{C}_{2213} & \sqrt{2}\mathbb{C}_{2212} \\ & & \mathbb{C}_{3333} & \sqrt{2}\mathbb{C}_{3323} & \sqrt{2}\mathbb{C}_{3313} & \sqrt{2}\mathbb{C}_{3312} \\ & & & 2\mathbb{C}_{2323} & 2\mathbb{C}_{2313} & 2\mathbb{C}_{2312} \\ & Sym. & & & 2\mathbb{C}_{1313} & 2\mathbb{C}_{1312} \\ & & & & & 2\mathbb{C}_{1212} \end{bmatrix}. \tag{1}$$

Otherwise stated, the summation on repeated indices is assumed throughout the paper.

2.2. Definition of distances in the set of elasticity tensors

The question of defining the distance between elasticity tensors (or equivalently, of finding the closest approximation of an elasticity tensor with arbitrary symmetry to an elasticity tensor with given symmetries) has been largely investigated, especially within the context of geophysical applications (see [5] for instance). The calculation of such a distance typically arises in inverse and/or experimental identification, when one wants to reduce the complexity of a model by best fitting anisotropic measurements to a model with given higher symmetries (see [11] [28] and the references therein).

Several metrics have been introduced in the literature to quantify the distance between two elasticity tensors. The most widely used metrics are the Euclidean, Log-Euclidean [2] and Riemannian metrics [26], denoted by d_E, d_{LE} and d_R respectively, and defined for any elasticity tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ by:

$$d_E(\mathbb{C}_1, \mathbb{C}_2) = \|\mathbb{C}_2 - \mathbb{C}_1\|, \tag{2}$$

$$d_{LE}(\mathbb{C}_1, \mathbb{C}_2) = \|\log(\mathbb{C}_2) - \log(\mathbb{C}_1)\|, \tag{3}$$

$$d_R(\mathbb{C}_1, \mathbb{C}_2) = \|\log\left(\mathbb{C}_1^{-1/2}\mathbb{C}_2\mathbb{C}_1^{-1/2}\right)\|. \tag{4}$$

In addition, we introduce the inner product of two fourth-order elasticity tensors $\mathbb{C}$ and $\mathbb{D}$, and its associated norm, such that:

$$< \mathbb{C}, \mathbb{D} >= \mathbb{C}_{ijk\ell}\mathbb{D}_{ijk\ell}, \quad \|\mathbb{C}\| =< \mathbb{C}, \mathbb{C} >^{1/2} . \tag{5}$$

The convention stated in Eq. (1) ensures the preservation of the norm, no matter the representation of the elasticity tensor:

$$\|\mathbb{C}\| = \|[C]\|_{\mathrm{F}},$$

where $\|[C]\|_{\mathrm{F}} = \sqrt{\mathrm{Tr}\left([C]^2\right)}$ is the Frobenius norm (or Hilbert-Schmidt norm) of the real symmetric matrix $[C]$. Furthermore, the Log-Euclidean and Riemmanian metrics have the additional property of invariance by inversion,

$$d_{LE}(\mathbb{C}_1, \mathbb{C}_2) = d_{LE}({\mathbb{C}_1}^{-1}, {\mathbb{C}_2}^{-1}),$$
$$d_R(\mathbb{C}_1, \mathbb{C}_2) = d_R({\mathbb{C}_1}^{-1}, {\mathbb{C}_2}^{-1}),$$

which makes them more attractive than the Euclidean metric when dealing with elasticity tensors. This property for the Log-Euclidean and Riemmanian metrics is particularly important when one is interested in defining the closest approximation (together with its reference frame) of a tensor belonging to a given symmetry class, since the result should then be independent on whether the stiffness or compliance tensor is considered.

2.3. Projection onto the set of elasticity tensors with given material symmetries

Let $\mathcal{C}^{\mathrm{Sym}}$ be a class of elasticity tensors with given symmetries (isotropy, transverse isotropy, orthotropy, etc.). Let $\mathbb{C}$ be a fourth-order elasticity tensor having an arbitrary symmetry, with components $\mathbb{C}_{ijk\ell}$ with respect to a given frame $\mathfrak{R} = (0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. We then denote by $\mathbb{C}^{\mathrm{Sym}} = \mathcal{P}^{\mathrm{Sym}}(\mathbb{C})$ the projection of $\mathbb{C}$ onto $\mathcal{C}^{\mathrm{Sym}}$, calculated by using one of the distance d introduced in the previous section, such that:

$$\mathbb{C}^{\mathrm{Sym}} = \underset{\widetilde{\mathbb{C}} \in \mathcal{C}^{\mathrm{Sym}}}{\mathrm{Arg\,min}}\ d(\mathbb{C}, \widetilde{\mathbb{C}}). \tag{6}$$

As an example, let $\mathbb{C}^{\mathrm{TI}}$ be the projection of $\mathbb{C}$ onto the set of all the elasticity tensors exhibiting transverse isotropy with respect to $\mathbf{e}_3$. Using the

Euclidean distance d_E, it can then be shown that the matrix $[C^{\mathrm{TI}}]$ is given by [27]:

$$[C^{\mathrm{TI}}] = \begin{bmatrix} \mathbb{C}^{\mathrm{TI}}_{1111} & \mathbb{C}^{\mathrm{TI}}_{1122} & \mathbb{C}^{\mathrm{TI}}_{1133} & 0 & 0 & 0 \\ & \mathbb{C}^{\mathrm{TI}}_{1111} & \mathbb{C}^{\mathrm{TI}}_{1133} & 0 & 0 & 0 \\ & & \mathbb{C}^{\mathrm{TI}}_{3333} & 0 & 0 & 0 \\ & & & 2\mathbb{C}^{\mathrm{TI}}_{2323} & 0 & 0 \\ & Sym. & & & 2\mathbb{C}^{\mathrm{TI}}_{2323} & 0 \\ & & & & & \mathbb{C}^{\mathrm{TI}}_{1111} - \mathbb{C}^{\mathrm{TI}}_{1122} \end{bmatrix}, \quad (7)$$

where

$$\mathbb{C}^{\mathrm{TI}}_{1111} = \frac{1}{8}\left(3\mathbb{C}_{1111} + 3\mathbb{C}_{2222} + 2\mathbb{C}_{1122} + 4\mathbb{C}_{1212}\right), \quad (8)$$

$$\mathbb{C}^{\mathrm{TI}}_{1122} = \frac{1}{8}\left(\mathbb{C}_{1111} + \mathbb{C}_{2222} + 6\mathbb{C}_{1122} - 4\mathbb{C}_{1212}\right), \quad (9)$$

$$\mathbb{C}^{\mathrm{TI}}_{1133} = \frac{1}{2}\left(\mathbb{C}_{1133} + \mathbb{C}_{2233}\right), \quad (10)$$

$$\mathbb{C}^{\mathrm{TI}}_{3333} = \mathbb{C}_{3333}, \quad (11)$$

$$\mathbb{C}^{\mathrm{TI}}_{2323} = \frac{1}{2}\left(\mathbb{C}_{2323} + \mathbb{C}_{1313}\right). \quad (12)$$

It should be pointed out that such a projection could also be derived by using either the Log-Euclidean or the Riemannian metric. However, such projections would require the numerical solving of the corresponding optimization problem and consequently, would largely increase the computational time associated in the context of the probabilistic analysis. For this reason, the Euclidean projection will be used in this work.

Remark: the closest approximation $\widehat{\mathbb{C}^{\mathrm{Sym}}}$ of $\mathbb{C}$, as well as its reference frame $\widehat{\mathfrak{R}^{\mathrm{Sym}}}$, could be defined as:

$$\widehat{\mathbb{C}^{\mathrm{Sym}}} = \mathcal{P}^{\mathrm{Sym}}\left(\widetilde{\mathbb{C}}\left([Q^{\mathrm{opt}}]\right)\right), \quad (13)$$

in which $[Q^{\mathrm{opt}}]$ would be computed by solving the following optimization problem:

$$[Q^{\mathrm{opt}}] = \underset{[Q]\in \mathrm{SO}(3)}{\mathrm{Arg\,min}}\ d\left(\widetilde{\widetilde{\mathbb{C}}}\left([Q]\right), \mathcal{P}^{\mathrm{Sym}}\left(\widetilde{\widetilde{\mathbb{C}}}\left([Q]\right)\right)\right), \quad (14)$$

where d stands either for d_{LE} or d_R, SO(3) is the group of all 3-by-3 real orthogonal matrices (with determinant equal +1) and $\widetilde{\widetilde{\mathbb{C}}}\left([Q]\right)$ would be defined by:

$$\widetilde{\widetilde{\mathbb{C}}}\left([Q]\right)_{ijk\ell} = [Q]_{ip}[Q]_{jq}[Q]_{kr}[Q]_{\ell s}\mathbb{C}_{pqrs}. \quad (15)$$

Alternatively, Eq. (14) could be formalized by using the matrix representation of the elasticity tensor (given by Eq. (1)) and by considering a parametrization of the orthogonal group SO(6) (see [22]). Such a characterization was studied in [6], in which the use of some quantities, invariant with respect to SO(3) (such as the trace of the second-order symmetric Voigt and dilatation tensors), is proposed so as to make the minimization procedure easier. Numerical procedures for estimating such approximations and their reference frames (or "effective orientations") have been proposed and successfully applied in [21] and [20] (for transversely isotropic and orthotropic tensors, respectively); see also [19]. Note finally that the question of defining such approximations in the presence of measurement errors was recently addressed in [3].

However, due to the presence of uncertainties which are taken into account through the probabilistic model inducing statistical fluctuations, there would not be additional benefit to optimize the distance with respect to $[Q]$. Consequently, we will limit the development in considering the minimization with respect to the distance (see Eq. (6)).

2.4. Eigensystem characterization of the material symmetry classes

The classification of material symmetries has been investigated by many researchers and was historically based on crystallographic considerations. Quite recently, other approaches, in which the elasticity tensors were classified either by considering the set of admitted (minor) symmetry planes (see [7] [8]) or with respect to symmetry groups (see [12] [16]), were proposed. Both approaches basically result in the definition of eight symmetry classes, as shown in [7].

In this work, we follow the second approach and more precisely, we consider the eigensystem-based coordinate-free characterization of the symmetry classes, as defined in [4] (see the references therein for previous applications of this approach, such as the pioneering work [30] in the case of transversely isotropic media defined by a known axis). This characterization allows one to define coordinate-free conditions which are necessary and sufficient for the identification of the material symmetry class of a given elasticity tensor. These conditions are formulated through the definition of the multiplicities of the eigenvalues and through a set of constraints for the corresponding eigenspaces.

In this framework, it should be noted that (**i**) the use of the classical random ensembles from the Random Matrix Theory [24] (such that GOE, GUE and so on) generally implies all the random eigenvalues of the random matrix representation to be of multiplicity one (no matter the eigenvalue

multiplicity orders of the random matrix mean value), and that (**ii**) the corresponding random eigenspaces cannot explicitly be described and constrained. It can then be deduced that the distance of each realization of the random elasticity tensor to a given (and closest) symmetry class can be controlled (albeit within a limited extent) by enforcing the closeness of a few selected random eigenvalues. Furthermore, while the above classical ensembles of the random matrices do not allow the mean value of the random eigenvalues to be specified, they allow the closeness of several eigenvalues to be partially controlled by constraining their respective variances. The present stochastic modeling will thus rely on the probabilistic model for symmetric positive-definite random matrices with prescribed variances on several eigenvalues, derived in [25] and recalled in the next section.

3. Probabilistic model of random matrix [C]

3.1. Model derivation

The construction of the model relies on the use of the MEP, introduced in [17] [18] [33] for random vectors. We recall that such an approach allows one to explicitly construct probability distributions using the *available information* only, so that no additional bias is introduced by the probabilistic modeling. The MEP consists in maximizing the measure of entropy S, defined as:

$$S = -\int_{\mathcal{M}_6^+(\mathbb{R})} p_{[\mathbf{C}]}\left([C]\right) \ln\left(p_{[\mathbf{C}]}\left([C]\right)\right) \mathrm{d}C, \tag{16}$$

with respect to the probability density function $p_{[\mathbf{C}]}$, where $[C] \mapsto p_{[\mathbf{C}]}\left([C]\right)$ is the probability density function from $\mathcal{M}_6^+\left(\mathbb{R}\right)$ into $\mathbb{R}^+$ defining the probability distribution $P_{[\mathbf{C}]} = p_{[\mathbf{C}]}\left([C]\right) \mathrm{d}C$ of random matrix $[\mathbf{C}]$ with values in $\mathcal{M}_6^+\left(\mathbb{R}\right)$. The volume measure $\mathrm{d}C$ on the set $\mathcal{M}_6^S\left(\mathbb{R}\right)$ of all the symmetric 6×6 real matrices is written as $\mathrm{d}C = 2^{15/2} \prod_{1 \leq i \leq j \leq 6} \mathrm{d}[C]_{ij}$ (see [34]). The optimization problem (16) is solved under the following set of constraints:

$$\int_{\mathcal{M}_6^+(\mathbb{R})} p_{[\mathbf{C}]}\left([C]\right) \mathrm{d}C = 1, \tag{17}$$

$$\mathrm{E}\left\{[\mathbf{C}]\right\} = \int_{\mathcal{M}_6^+(\mathbb{R})} [C]\, p_{[\mathbf{C}]}\left([C]\right) \mathrm{d}C = [\underline{C}], \tag{18}$$

$$\int_{\mathcal{M}_6^+(\mathbb{R})} \ln\left(\det\left([C]\right)\right) p_{[\mathbf{C}]}\left([C]\right) \mathrm{d}C = \beta, \quad |\beta| < +\infty, \tag{19}$$

$$\mathrm{E}\left\{\left(\underline{\boldsymbol{\varphi}}^{i\mathrm{T}}\,[\mathbf{C}]\,\underline{\boldsymbol{\varphi}}^{i}\right)^2\right\} = s_i^2\underline{\lambda}_i^2, \quad i \in \mathcal{I} \subseteq [1,6], \tag{20}$$

where $\det([C])$ and $[C]^{\mathrm{T}}$ are the determinant and the transpose of $[C]$, $\mathrm{E}\{\cdot\}$ denotes the mathematical expectation and $\left\{\left(\underline{\lambda}_i, \underline{\boldsymbol{\varphi}}^i\right)\right\}_i$ are the eigenvalues and eigenvectors of the mean matrix $[\underline{C}]$. The family $\{s_i\}_{i\in\mathcal{I}}$ is a set of m parameters which are supposed to be either assumed or computed from an experimental inverse identification.

The set of constraints defined by Eqs. (17-19) basically corresponds to the one previously used and studied in [34] [35]. Eq. (17) is the classical normalization condition for the probability density function, while Eq. (18) means that the mean matrix is supposed to be known *a priori*. Eq. (19) implies the existence of the second-order moment of the inverse random matrix norm (see [34] [35]). Finally, the set of constraints defined by Eq. (20) allows one to partially prescribe the variances of m ($m \leq 6$) selected random eigenvalues $\{\lambda_i\}_{i=1}^m$ of $[\mathbf{C}]$ (see [25] for a discussion).

Let $\mu_0 \in \mathbb{R}$, $(\alpha - 1) \in \mathbb{R}$, $[M_1] \in \mathcal{M}_6^S(\mathbb{R})$ and $\{\tau_i' \in \mathbb{R}\}_{i=1}^6$ be the Lagrange multipliers associated with the constraints (17-20). It can then be shown that the probability density function $[C] \mapsto p_{[\mathbf{C}]}([C])$ takes the form:

$$p_{[\mathbf{C}]}([C]) = k_1 \left(\det([C])\right)^{\alpha-1} \exp\left(-\mathrm{tr}\left([M_1]^{\mathrm{T}}[C]\right) - \sum_{i\in\mathcal{I}} \tau_i' \left(\underline{\boldsymbol{\varphi}}^{i\mathrm{T}}\,[C]\,\underline{\boldsymbol{\varphi}}^{i}\right)^2\right),$$

where $\mathrm{tr}([C])$ is the trace of $[C]$ and k_1 is a normalization constant. Let $[\underline{\Phi}]$ be the matrix whose columns are the eigenvectors of the mean matrix $[\underline{C}]$ corresponding to the constrained eigenvalues gathered in the diagonal matrix $[\underline{\Lambda}]$. It can easily be proven that $[\underline{C}] - [\underline{C}][\underline{\Phi}][\underline{\Lambda}]^{-1}[\underline{\Phi}]^{\mathrm{T}}[\underline{C}]$ is positive (positive-definite if $m = 0$) and consequently, there is a rectangular $n \times (n - m)$ real matrix $[D]$ such that:

$$[D][D]^{\mathrm{T}} = [\underline{C}] - [\underline{C}][\underline{\Phi}][\underline{\Lambda}]^{-1}[\underline{\Phi}]^{\mathrm{T}}[\underline{C}]. \tag{21}$$

Matrix $[D]$ can be obtained from a singular value decomposition. The random matrix $[\mathbf{C}]$ is next written as:

$$[\mathbf{C}] = [\underline{L}]\,[\mathbf{G}]\,[\underline{L}]^{\mathrm{T}}, \tag{22}$$

where $[\underline{L}] = \left[[\underline{C}][\Phi][\Lambda]^{-1/2} \quad [D]\right]$ and $[\mathbf{G}]$ is a random matrix whose probability density function can be readily obtained as:

$$p_{[\mathbf{G}]}([G]) = k_2 \left(\det([G])\right)^{\alpha-1} \exp\left(-\mathrm{tr}\left([M]^{\mathrm{T}}[G]\right) - \sum_{i=1}^{m} \tau_i [G]_{ii}{}^2\right), \tag{23}$$

where k_2 is a new normalization constant and the following changes of variables were performed: $[M] = [\underline{L}]^{\mathrm{T}} [M_1] [\underline{L}]$, $\tau_i = \tau_i' \underline{\lambda}_i^2$. Finally, the random matrix $[\mathbf{G}]$ is expressed as:

$$[\mathbf{G}] = [\mathbf{H}] [\mathbf{H}]^{\mathrm{T}}, \tag{24}$$

in which $[\mathbf{H}]$ is a lower triangular random matrix with probability density function:

$$\begin{aligned} p_{[\mathbf{H}]}([H]) = \quad & k_3 \left(\prod_{\ell=1}^{6} [H]_{\ell\ell}{}^{5-\ell+2\alpha} \right) \\ & \times \exp\left(-\mathrm{tr}\left([H]^{\mathrm{T}} [M]^{\mathrm{T}} [H] \right) - \sum_{i=1}^{m} \tau_i \left(\sum_{\ell=1}^{i} [H]_{i\ell}{}^2 \right)^2 \right), \end{aligned} \tag{25}$$

where k_3 is an appropriate normalization constant. It can be shown that $[\mathbf{G}]$ is such that

$$\mathrm{E}\left\{ [\mathbf{G}] \right\} = [I_6], \tag{26}$$

and

$$\mathrm{E}\left\{ [\mathbf{G}]_{ii}{}^2 \right\} = s_i^2, \tag{27}$$

where $[I_6]$ is the 6×6 identity matrix. From Eq. (26), it can be deduced that $[M]$ is a diagonal matrix, so that the corresponding Lagrange multipliers can be put into the vectorial form $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_6) = ([M]_{11}, \ldots, [M]_{66})$ and Eq. (25) can be written as:

$$\begin{aligned} p_{[\mathbf{H}]}([H]) = \quad & \prod_{i=1}^{m} \widehat{k}_i [H]_{ii}{}^{5-i+2\alpha} \exp\left(-\mu_i \left(\sum_{\ell=1}^{i} [H]_{i\ell}{}^2 \right) - \tau_i \left(\sum_{\ell=1}^{i} [H]_{i\ell}{}^2 \right)^2 \right) \\ & \times \prod_{i=m+1}^{n} \widehat{k}_i [H]_{ii}{}^{5-i+2\alpha} \exp\left[-\mu_i [H]_{ii}{}^2 \right] \\ & \times \prod_{i=m+1}^{n} \prod_{\ell=1}^{i-1} \widehat{k}_{i\ell} \exp\left[-\mu_i [H]_{i\ell}{}^2 \right], \end{aligned} \tag{28}$$

where $\left\{ \widehat{k}_i \right\}_i$ and $\left\{ \widehat{k}_{i\ell} \right\}_{i\ell}$ are normalization constants. From Eq. (28), it is seen that:

- the terms $[\mathbf{H}]_{i\ell}$ are all independent from each other and from the other elements for $i > m$ and $i > \ell$, and are normally distributed with mean 0 and standard deviation $1/\sqrt{2\mu_i}$. Moreover, it can be proven that $\mu_i = 5/2 + \alpha$ for $i > m$.

- the terms $[\mathbf{H}]_{ii}$ are all independent from each other and from the other elements for $i > m$ and are distributed according to:

$$p_{[\mathbf{H}]_{ii}}\left([H]_{ii}\right) = \widehat{k}_i {[H]_{ii}}^{5-i+2\alpha} \exp\left[-\mu_i {[H]_{ii}}^2\right], \ [H]_{ii} \geq 0,$$

with $\mu_i = 5/2 + \alpha$.

- for each i fixed in $1, \ldots, m$, the random variables $[\mathbf{H}]_{i\ell}$, for $\ell = 1, ..., i$ are statistically dependent, while the families of random variables $\{[\mathbf{H}]_{11}\}$, $\{[\mathbf{H}]_{21}, [\mathbf{H}]_{22}\}$, ..., $\{[\mathbf{H}]_{m\ell}, \ell = 1, \ldots, m\}$ are independent. For each i fixed in $1, \ldots, m$, the joint probability density function of the random variables $[\mathbf{H}]_{i1}, \ldots, [\mathbf{H}]_{ii}$ is written as:

$$p_{[\mathbf{H}]_{i1},\ldots,[\mathbf{H}]_{ii}}\left([H]_{i1}, \ldots, [H]_{ii}\right) = \widehat{k}_i {[H]_{ii}}^{5-i+2\alpha} \exp\left(-\mu_i \left(\sum_{\ell=1}^{i} {[H]_{i\ell}}^2\right) - \tau_i \left(\sum_{\ell=1}^{i} {[H]_{i\ell}}^2\right)^2\right).$$

Making use of a change of variables, it can also be proven that the marginal distribution of component $[\mathbf{G}]_{ii}$ is:

$$p_{[\mathbf{G}]_{ii}}\left([G]_{ii}\right) = a_i {[G]_{ii}}^{3/2+\alpha} \exp\left(-\mu_i [G]_{ii} - \tau_i {[G]_{ii}}^2\right), \tag{29}$$

where $[G]_{ii} > 0$ (resp. $[G]_{ii} \geq 0$) when $3/2 + \alpha$ is a real positive number (resp. a positive integer) and a_i is a normalization constant.

Let δ be the parameter allowing the level of statistical fluctuations of random matrix $[\mathbf{G}]$ to be characterized:

$$\delta^2 = \frac{1}{6} \mathrm{E}\left\{ \| [\mathbf{G}] - [I_6] \|_{\mathrm{F}}^2 \right\}. \tag{30}$$

Making use of Eq. (26) and taking advantage of both the algebraic and probabilistic properties of components $[\mathbf{H}]_{i\ell}$, it can further be shown that:

$$\delta^2 = \frac{1}{6} \sum_{i=1}^{m} s_i^2 - \frac{7 - (m/6)(7 + 2\alpha)}{5 + 2\alpha}. \tag{31}$$

Finally, we introduce the following measure of statistical fluctuations on the random matrix $[\mathbf{C}]$:

$$\delta_{\mathbf{C}}^2 = \frac{\mathrm{E}\left\{ \| [\mathbf{C}] - [\underline{C}] \|_{\mathrm{F}}^2 \right\}}{\| [\underline{C}] \|_{\mathrm{F}}^2}. \tag{32}$$

3.2. Parametrization of the probabilistic model

From the previous sections, it is seen that the probabilistic model is initially parametrized by a set of $2m+1$ Lagrange multipliers, namely α, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m)$. However, enforcing Eq. (26) yields a set of m uncoupled equations in terms of (α, μ_i, τ_i) $(1 \leq i \leq m)$ which can be numerically solved for, say, parameter μ_i, and used to reduce the parametrization to $m+1$ parameters. Let $q = 3/2 + \alpha$. In the following, we will assume that q is a positive integer. The system of equations then reads:

$$\int_{\mathbb{R}^+} p_{[\mathbf{G}]_{ii}}\left([G]_{ii}\right) d[G]_{ii} = 1, \tag{33}$$

$$\int_{\mathbb{R}^+} [G]_{ii} p_{[\mathbf{G}]_{ii}}\left([G]_{ii}\right) d[G]_{ii} = 1, \tag{34}$$

for $i \in \mathcal{I}$ and where $[G]_{ii} \mapsto p_{[\mathbf{G}]_{ii}}\left([G]_{ii}\right)$ is given by Eq. (29).

For small and moderate values of both α and τ_i (typically, $\alpha \leq 80$ and $\tau_i \leq 1000$), a closed-form algebraic equation for parameter μ_i can be readily obtained and used as follows. Proceeding to the change of variable $r = \sqrt{2\tau_i}[G]_{ii}$ in Eq. (33), the normalization constant a_i is first derived as:

$$a_i = \frac{(2\tau_i)^{(q+1)/2}}{\Gamma(q+1)\mathrm{U}\left(q+\frac{1}{2}, \frac{\mu_i}{\sqrt{2\tau_i}}\right)\exp\left(\frac{\mu_i^2}{8\tau_i}\right)}, \tag{35}$$

where $\mathrm{U}(a, x)$ is the parabolic cylinder function of parameter a and argument x (see [40] [1]). Next, substituting Eq. (35) in Eq. (34) yields the following equation:

$$\frac{\mathrm{U}\left(q+3/2, \mu_i/\sqrt{2\tau_i}\right)}{\mathrm{U}\left(q+1/2, \mu_i/\sqrt{2\tau_i}\right)} = \frac{\sqrt{2\tau_i}}{q+1}, \tag{36}$$

which has to be solved for parameter μ_i. It should be noted that the resolution of Eq. (36) does not require the normalization constant to be calculated.

For large values of parameters α and τ_i, Eq. (36) may be tricky to solve in practice because of numerical infinite values. In this case, the integral equations (33) and (34) have to be solved numerically, requiring the normalization constant to be calculated for each trial of parameter μ_i. This computation can be performed by writting Eq. (33) as:

$$\int_{\mathbb{R}^+} \exp\left(a_i' + q\log\left([G]_{ii}\right) - \mu_i[G]_{ii} - \tau_i{[G]_{ii}}^2\right) d[G]_{ii} = 1, \tag{37}$$

where log is the Neperian logarithm, $a_i' = \log\left(a_i\right)$ is a new normalization constant which is used to rescale the integrand and which may be computed using any optimization algorithm.

3.3. Strategy for simulating realizations of random matrix $[\mathbf{C}]$

The realizations of random matrix $[\mathbf{C}]$ can be readily obtained from the ones of matrix $[\mathbf{H}]$, using Eqs. (24) and (22). In the sequel, $Z \propto \mathcal{N}(a,b)$ (resp. $Z \propto \mathcal{G}(a,b)$) means that the random variable Z is normally (resp. Gamma) distributed with mean value a and standard deviation b (resp. with parameters a and b).

The terms $[\mathbf{H}]_{i\ell}$, with $i > m$ and $i > \ell$, are easily generated as Gaussian random variables, with mean 0 and standard deviation $1/\sqrt{2\mu_i}$ (with $\mu_i = 5/2 + \alpha$).

The terms $[\mathbf{H}]_{ii}$, $i > m$, can be simulated as:

$$[\mathbf{H}]_{ii} = \sqrt{\frac{Y_{ii}}{\mu_i}}, \tag{38}$$

where $Y_{ii} \propto \mathcal{G}(3 + \alpha - i/2, 1)$ and $\mu_i = 5/2 + \alpha$.

An efficient simulation algorithm of the terms $[\mathbf{H}]_{i\ell}$, $i \leq m$, has been proposed in [25] and is recalled below:

1. Generate the random matrix $[\mathbf{H}']$ whose components are defined by:

$$[\mathbf{H}']_{ii} = \sqrt{\frac{Y'_{ii}}{\mu_i}},\ Y'_{ii} \propto \mathcal{G}(3 + \alpha - i/2, 1), \tag{39}$$

   for $i = 1, \ldots, m$ and by:

$$[\mathbf{H}']_{i\ell} \propto \mathcal{N}\left(0, 1/\sqrt{2\mu_i}\right), \tag{40}$$

   for $i = 1, \ldots, m$ and $\ell = 1, \ldots, i-1$.
2. Compute the diagonal random matrix $[\mathbf{G}']$ with components:

$$[\mathbf{G}']_{ii} = \sum_{\ell=1}^{i} {[\mathbf{H}']_{i\ell}}^2, \quad i = 1, \ldots, m. \tag{41}$$

3. Generate the terms $[\mathbf{G}]_{ii}$ $(i = 1, \ldots, m)$ according to Eq. (29) by using an algorithm by rejection (see [10]) for instance.
4. Compute the terms $[\mathbf{H}]_{i\ell}$ as:

$$[\mathbf{H}]_{i\ell} = [\mathbf{H}']_{i\ell} \sqrt{\frac{[\mathbf{G}]_{ii}}{[\mathbf{G}']_{ii}}}. \tag{42}$$

4. Application

In this application, we consider three mean (or nominal) deterministic models, whose elasticity tensors belong to the anisotropic class and are more or less close to the transverse isotropy symmetry. Their respective distances to transverse isotropy will be first calculated. Then, for one of these mean models, we use the presented probabilistic model and methodology allowing the uncertainties to be taken into account. These probabilistic models depend on parameters α and $\boldsymbol{\tau}$. We then analyze the dependence of the global statistical fluctuations and of the mean distance to transverse isotropy with respect to these parameters. Finally, we compare the results given by this theory with the ones from the nonparametric probabilistic approach with anisotropic statistical fluctuations.

4.1. Definition of the mean models and quantification of their distances to the transverse isotropy class

For this application, we will consider three mean models $[\underline{C}_S]$, $[\underline{C}_M]$ and $[\underline{C}_H]$, respectively corresponding to a small, medium and large distance to transverse isotropy. They are obtained using a perturbation of the typical elasticity matrix of a carbon-epoxy unidirectional composite (with fibers aligned along axis $\mathbf{e}_3$) and are defined as follows (unit is GPa):

$$[\underline{C}_S] = \begin{bmatrix} 10.1036 & 0.5391 & 2.9625 & -0.0040 & 0.0071 & -0.0165 \\ 0.5391 & 10.1061 & 2.9782 & -0.0041 & -0.0070 & -0.0036 \\ 2.9625 & 2.9782 & 182.690 & 0.0197 & 0.0016 & 0.0145 \\ -0.0040 & -0.0041 & 0.0197 & 14.0339 & 0.0068 & 0.0008 \\ 0.0071 & -0.0070 & 0.0016 & 0.0068 & 14.0121 & -0.0103 \\ -0.0165 & -0.0036 & 0.0145 & 0.0008 & -0.0103 & 9.5552 \end{bmatrix},$$

$$[\underline{C}_M] = \begin{bmatrix} 10.3534 & 0.6895 & 2.9143 & -0.0283 & -0.0293 & 0.0004 \\ 0.6895 & 10.5845 & 3.0054 & 0.1794 & 0.2049 & 0.1017 \\ 2.9143 & 3.0054 & 183.1239 & 0.2110 & 0.0533 & 0.2894 \\ -0.0283 & 0.1794 & 0.2110 & 14.3502 & 0.3882 & 0.1864 \\ -0.0293 & 0.2049 & 0.0533 & 0.3882 & 14.5328 & 0.1060 \\ 0.0004 & 0.1017 & 0.2894 & 0.1864 & 0.1060 & 10.2945 \end{bmatrix},$$

$$[\underline{C}_H] = \begin{bmatrix} 10.4270 & 0.9722 & 3.4443 & 0.7987 & -0.0773 & 0.3999 \\ 0.9722 & 11.9611 & 2.5000 & 1.2461 & -0.5386 & -0.1726 \\ 3.4443 & 2.5000 & 186.1899 & -0.1625 & -0.1436 & 1.4450 \\ 0.7987 & 1.2461 & -0.1625 & 16.4521 & -0.4674 & 0.3480 \\ -0.0773 & -0.5386 & -0.1436 & -0.4674 & 15.9919 & 0.6151 \\ 0.3999 & -0.1726 & 1.4450 & 0.3480 & 0.6151 & 11.0544 \end{bmatrix}.$$

We recall that any elasticity matrix exhibiting transverse isotropy has two eigenvalues of multiplicity two and two eigenvalues of multiplicity one. Making use of this property, we can obtain a rough characterization of the distance to transverse isotropy by computing the eigenvalues of these nominal models, which are:

- Mean model $[\underline{C}_S]$: $\underline{\lambda}_1 = 9.5498$, $\underline{\lambda}_2 = 9.5709$, $\underline{\lambda}_3 = 10.5417$, $\underline{\lambda}_4 = 14.0102$, $\underline{\lambda}_5 = 14.0359$, $\underline{\lambda}_6 = 182.7925$.
- Mean model $[\underline{C}_M]$: $\underline{\lambda}_1 = 9.7546$, $\underline{\lambda}_2 = 10.2847$, $\underline{\lambda}_3 = 11.0632$, $\underline{\lambda}_4 = 14.0443$, $\underline{\lambda}_5 = 14.8660$, $\underline{\lambda}_6 = 183.2265$.
- Mean model $[\underline{C}_H]$: $\underline{\lambda}_1 = 9.7907$, $\underline{\lambda}_2 = 11.0559$, $\underline{\lambda}_3 = 11.8578$, $\underline{\lambda}_4 = 15.8393$, $\underline{\lambda}_5 = 17.2267$, $\underline{\lambda}_6 = 186.3060$.

According to the eigensystem characterization of the symmetry class, the increase of the distance between the eigenvalues $\underline{\lambda}_1$ and $\underline{\lambda}_2$ as well as between $\underline{\lambda}_4$ and $\underline{\lambda}_5$ means that the distance to transverse isotropy is higher for $[\underline{C}_H]$ than for $[\underline{C}_S]$. The distances of the mean models to transverse isotropy, computed with respect to both the Euclidean and Riemannian metrics, are given in Tab. 1.

	$[\underline{C}_S]$	$[\underline{C}_M]$	$[\underline{C}_H]$
d_E	0.0537	1.0061	3.6851
d_R	0.0036	0.0702	0.2250

Table 1: Euclidean and Riemannian distances to transverse isotropy for the three mean models $[\underline{C}_S]$, $[\underline{C}_M]$ and $[\underline{C}_H]$.

These results allow the distance value to transverse isotropy to be correlated with the closeness of the eigenvalues.

4.2. Dependence of the distance to transverse isotropy and of the level of statistical fluctuations with respect to parameters α and $\boldsymbol{\tau}$

In this section, we investigate the capability of the proposed probabilistic model to describe uncertainties for elasticity tensors exhibiting material symmetries. For that, we carry out a parametric analysis of the probabilistic model to evaluate its capacity to generate realizations almost verifying material symmetries. Below, the application is limited to the class of transversely isotropic materials.

Following the previous section, the mean distance to transverse isotropy may be reduced by enforcing a small variance on the stochastic eigenvalues λ_1, λ_2, λ_4 and λ_5 (with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_4 \leq \lambda_5$). Consequently, the vector $\boldsymbol{\tau}$ is written as

$$\boldsymbol{\tau} = (\tau, \tau, 0, \tau, \tau, 0)\,, \tag{43}$$

where τ is a real positive parameter. In order to be consistent with the philosophy of the MEP, it should be pointed out here that enforcing the value of parameter τ_i, $i = 1, \ldots, m$, basically consists in setting, implicitly, a value of constraint parameter s_i (see Section 3.1). For each mean model, the matrices $[\underline{\Phi}]$ and $[\underline{\Lambda}]$, introduced in Section 3.1, are then respectively defined as:

$$[\underline{\Phi}] = \left[\underline{\boldsymbol{\varphi}}^1, \underline{\boldsymbol{\varphi}}^2, \underline{\boldsymbol{\varphi}}^4, \underline{\boldsymbol{\varphi}}^5\right], \quad [\underline{\Lambda}] = \begin{bmatrix} \underline{\lambda}_1 & 0 & 0 & 0 \\ 0 & \underline{\lambda}_2 & 0 & 0 \\ 0 & 0 & \underline{\lambda}_4 & 0 \\ 0 & 0 & 0 & \underline{\lambda}_5 \end{bmatrix}. \tag{44}$$

The mean and coefficient of variation of the six eigenvalues of the random elasticity matrix are estimated on a set of 2500 realizations by using mathematical statistics [32]. Note that this number of samples ensures the convergence of the statistical estimates. The mean and coefficient of variation are plotted (in semi-log scale) on Figs. 1 and 2, for $\alpha = 60$, mean model $[\underline{C}_S]$ and different values of τ (ranging from 0.1 to 10^4).

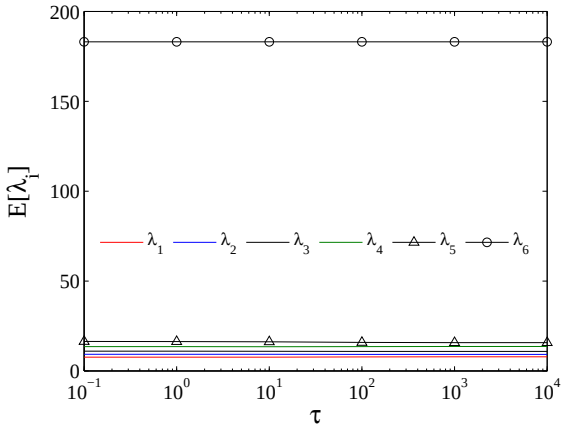

Figure 1: Plot of function $\tau \mapsto \mathrm{E}\{\lambda_i(\tau)\}$, for $\alpha = 60$ and mean model $[\underline{C}_S]$.

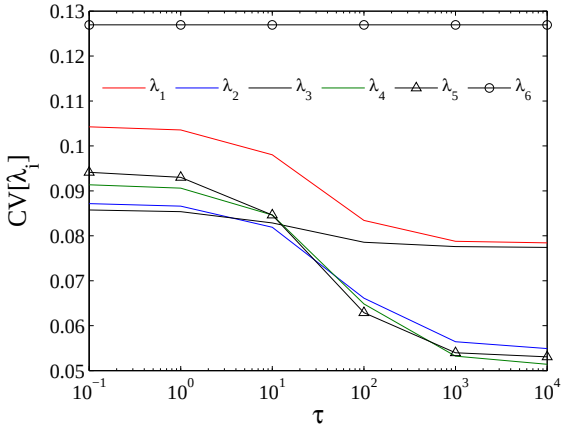

Figure 2: Plot of function $\tau \mapsto \mathrm{CV}\,\{\lambda_i(\tau)\}$, for $\alpha = 60$ and mean model $[\underline{C}_S]$.

First of all, it is seen that τ has a negligible effect on the mean values of the random eigenvalues, while large values of this parameter (typically, $\tau \geq 10^4$) imply a decrease of 25% (resp. 37%, 44% and 44%) of the coefficient of variation of λ_1 (resp. λ_2, λ_4 and λ_5). Furthermore, it is seen that all the coefficients of variation take similar values when τ tends to 0, which is consistent with the nonparametric probabilistic approach for the anisotropic class which may then be recovered. With such a probabilistic model, since all the eigenvalues still remains stochastic when $\delta_{\mathbf{C}} \neq 0$, the coefficients of variation corresponding to the constrained eigenvalues tend to a constant value (which is different from 0) when τ goes to infinity, which implies that the mean distance to transverse isotropy can only be specified within a limited range. As expected, the coefficients of variation of the unconstrained eigenvalues, namely λ_3 and λ_6, do not depend on τ (although the coefficient of variation of λ_3 presents a very small variation which may be due to its closeness of the constrained eigenvalues). All these comments can also be visualized on Figs. 3 and 4, where the probability density functions (estimated by using the kernel density estimation method) of the stochastic eigenvalues are plotted for $\tau = 1$ and $\tau = 10^4$ (using the mean model $[\underline{C}_S]$).

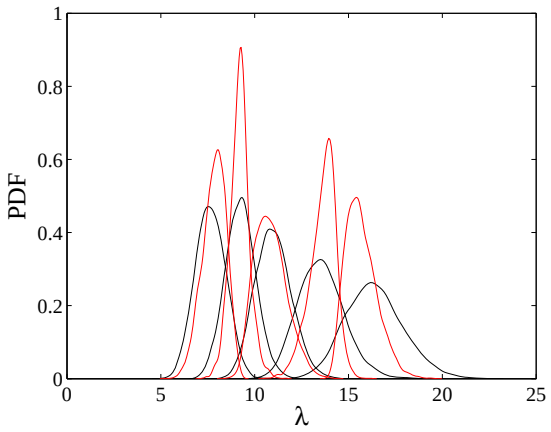

Figure 3: Plot of the PDFs of the random eigenvalues λ_i, $i = 1, ..., 5$, for $\tau = 1$ (black solid line) and $\tau = 10^4$ (red solid line). Mean model: $[\underline{C}_S]$.

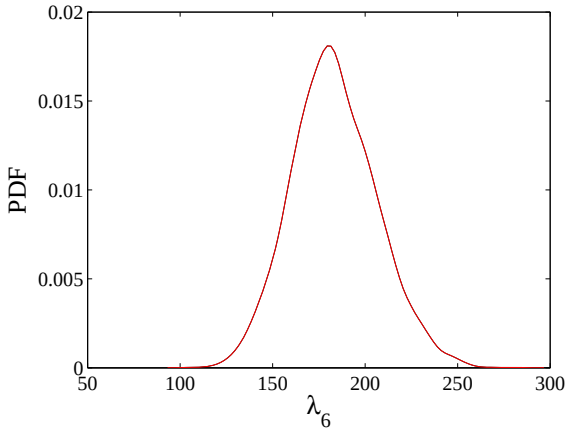

Figure 4: Plot of the PDF of the random eigenvalue λ_6, for $\tau = 1$ (black solid line) and $\tau = 10^4$ (red solid line). Mean model: $[\underline{C}_S]$.

The graph of $\tau \mapsto \delta_{\mathbf{C}}(\tau)$ is plotted on Figs. 5 and 6, for different values of parameters α and τ.

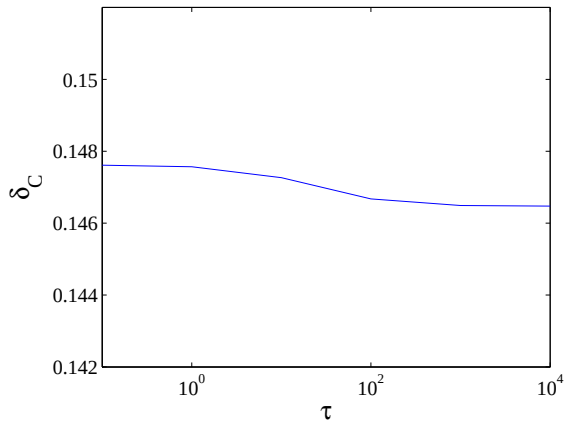

Figure 5: Plot of function $\tau \mapsto \delta_{\mathbf{C}}(\tau)$ for $\alpha = 60$ (mean model $[\underline{C}_S]$).

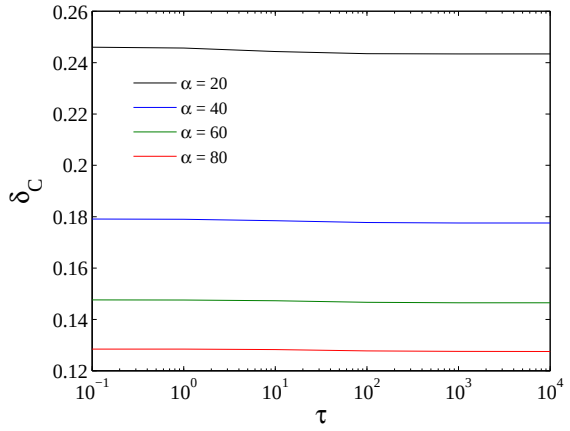

Figure 6: Plot of function $\tau \mapsto \delta_{\mathbf{C}}(\tau)$ for several values of parameter α (mean model $[\underline{C}_S]$).

It is seen that the level of statistical fluctuations is almost independent from parameter τ, with a decrease of $\delta_{\mathbf{C}}$ of 1% (resp. 0.8%, 0.8% and 0.7%) for $\alpha = 20$ (resp. $\alpha = 40$, 60 and 80) when τ runs over $[0.1, 10^4]$. Furthermore,

this level strongly depends on α: the larger the parameter α is, the smaller the overall level of fluctuation $\delta_{\mathbf{C}}$. Consequently, parameter α has to be used to calibrate the overall level of statistical fluctuations $\delta_{\mathbf{C}}$. The plots of $\tau \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}],[\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ are shown on Figs. 7 and 8, for different values of α (corresponding then to several levels of fluctuations).

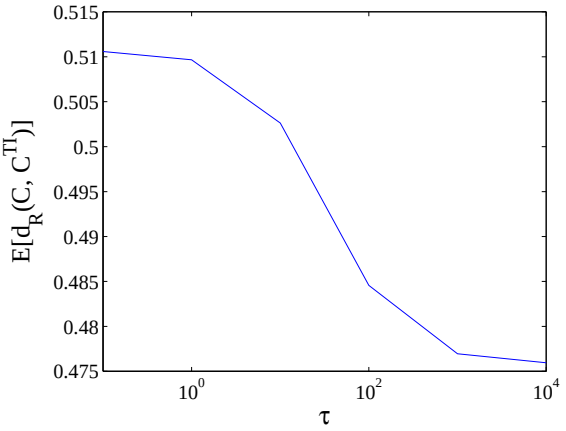

Figure 7: Plot of function $\tau \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}],[\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ for $\alpha = 60$ (mean model: $[\underline{C}_S]$).

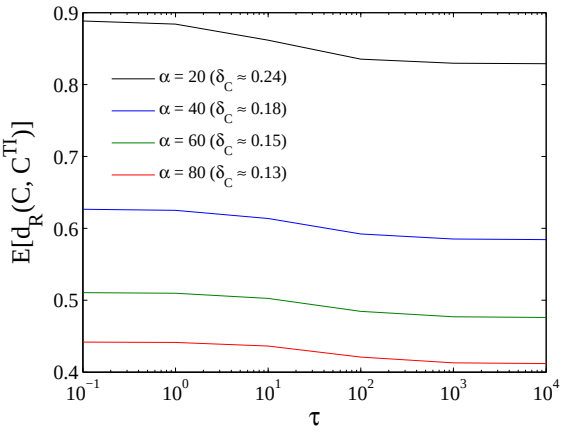

Figure 8: Plot of functions $\tau \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}],[\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ for several values of parameter α (mean model: $[\underline{C}_S]$).

It is seen that no matter the value of α, the mean distance to transverse isotropy can be reduced by increasing the value of parameter τ. For instance, in the case $\alpha = 60$ (corresponding to $\delta_{\mathbf{C}} = 0.15$), setting $\tau = 10^4$ yields a reduction of about 7% of the mean distance. It is also observed that this reduction is all the more important as the value of parameter α is small, which correspond to large levels of statistical fluctuations (and accordingly, to large variances of the eigenvalues).

We can now summarize the results above concerning the use of the mean distance for characterizing the material symmetries for a stochastic model of the elasticity tensor. For the mean model $[\underline{C}_S]$ (close to transverse isotropy), corresponding to the most severe case, the capability of the proposed probabilistic model to describe the symmetry classes from anisotropy to transverse isotropy is high when $\delta_{\mathbf{C}}$ is small. In counterpart, as $\delta_{\mathbf{C}}$ increases, this capability decreases due to the repulsion phenomena of the random eigenvalues, physical phenomena which cannot be avoided. Nevertheless, the proposed probabilistic model allows a significant transverse isotropy to be obtained for the random tensor when its global statistical fluctuations increase.

4.3. Comparison between the proposed stochastic model and the nonparametric probabilistic approach for anisotropic materials

In this section, we compare the proposed stochastic model and the nonparametric probabilistic approach for anisotropic materials. In order to study the two probabilistic models, both of them are calibrated with respect to the level of statistical fluctuations $\delta_{\mathbf{C}}$. The parameter τ involved in the proposed analysis is set to 10^4 (thus yielding a reduction of the mean distance to transverse isotropy). Let $\hat{\lambda}_1 = \lambda_2 - \lambda_1$ and $\hat{\lambda}_4 = \lambda_5 - \lambda_4$. Following Sections 2.4 and 4.1, the mean distance to transverse isotropy may be reduced by imposing small values of the second-order moments $\mathrm{E}\left\{\hat{\lambda}_1^2\right\}$ and $\mathrm{E}\left\{\hat{\lambda}_4^2\right\}$. The plots of $\delta_{\mathbf{C}} \mapsto \mathrm{E}\left\{\hat{\lambda}_1^2\right\}$ and $\delta_{\mathbf{C}} \mapsto \mathrm{E}\left\{\hat{\lambda}_4^2\right\}$, obtained by considering the two probabilistic models, are shown on Fig. 9, for the three mean models ($[\underline{C}_S]$, $[\underline{C}_M]$ and $[\underline{C}_H]$) and for reasonable levels of statistical fluctuations ($\delta_{\mathbf{C}} \leq 0.12$). While the second-order moments obviously increase together with $\delta_{\mathbf{C}}$ for both the proposed model and the nonparametric approach with anisotropic fluctuations (no matter the mean model used in the simulations), it is seen that the proposed approach allows these moments to be significantly reduced, as shown in Tab. 2. The plot of $\delta_{\mathbf{C}} \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}], [\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ is shown on Fig. 10, for different nominal distances to transverse isotropy and for the two probabilistic approaches.

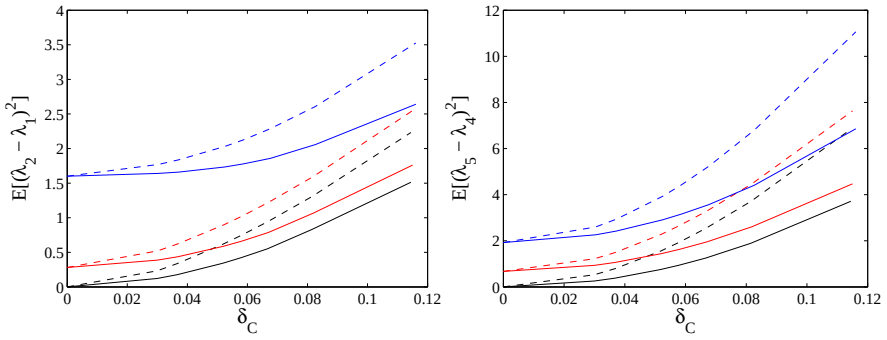

Figure 9: Plots of $\delta_{\mathbf{C}} \mapsto \mathrm{E}\left\{(\lambda_2 - \lambda_1)^2\right\}$ (left) and $\delta_{\mathbf{C}} \mapsto \mathrm{E}\left\{(\lambda_5 - \lambda_4)^2\right\}$ (right), for different initial distances to transverse isotropy ($[\underline{C}_S]$: black, $[\underline{C}_M]$: red, $[\underline{C}_H]$: blue). The nonparametric approach with anisotropic fluctuations appears in dashed line, while the proposed approach appears in solid line.

	$[\underline{C}_S]$	$[\underline{C}_M]$	$[\underline{C}_H]$
Reduction on $\mathrm{E}\left\{\hat{\lambda}_1^2\right\}$ (%)	32.2	30.9	25.1
Reduction on $\mathrm{E}\left\{\hat{\lambda}_4^2\right\}$ (%)	45.7	41.4	38

Table 2: Reduction of $\mathrm{E}\left\{\hat{\lambda}_1^2\right\}$ and $\mathrm{E}\left\{\hat{\lambda}_4^2\right\}$ obtained by using the proposed approach in comparison with the nonparametric probabilistic approach with anisotropic fluctuations.

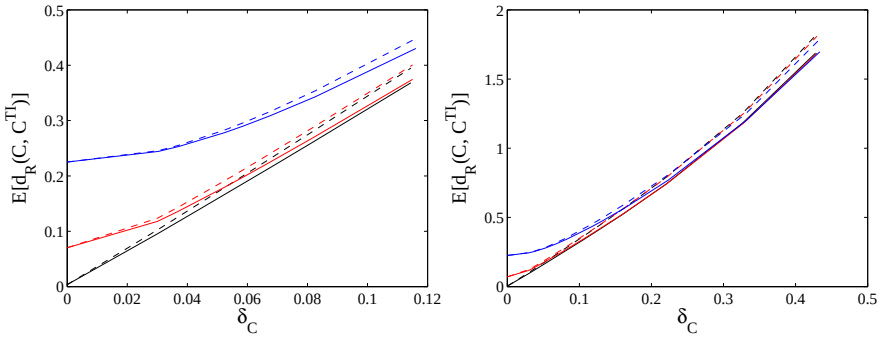

Figure 10: Plots of $\delta_{\mathbf{C}} \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}], [\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ for a small (left) and medium (right) range of fluctuation parameter $\delta_{\mathbf{C}}$ and for different initial distances to transverse isotropy ($[\underline{C}_S]$: black, $[\underline{C}_M]$: red, $[\underline{C}_H]$: blue). The nonparametric approach with anisotropic fluctuations appears in dashed line, while the proposed approach appears in solid line.

First of all, it is seen that the mean distance increases together with the overall level of statistical fluctuations, no matter the nominal model or the

considered probabilistic model. Then, for the three mean models used in the computations, the proposed approach allows the mean distance to be decreased (in comparison with the nonparametric model for anisotropic materials), the obtained reduction depending on $\delta_{\mathbf{C}}$. For mean model $[\underline{C}_M]$, the mean distance to transverse isotropy can thus be reduced by 7% for $\delta_{\mathbf{C}} \approx 0.4$ for instance. It is worth noticing that such a reduction strongly depends on the choice of the symmetry class with respect to which the mean distance is constrained, and that transverse isotropy clearly appears as one of the most severe case. Finally, it is shown that the mean model has a small effect on the mean distance to transverse isotropy for non negligible levels of statistical fluctuations (typically, for $\delta_{\mathbf{C}} \geq 0.3$).

5. Conclusion

In this paper, we considered the stochastic modeling for elasticity tensors with uncertain material symmetries. The proposed approach is based on the eigensystem characterization of the symmetry classes and allows the mean distance of the elasticity tensor to a given symmetry class to be partially controlled.

Making use of a probabilistic model for positive matrices proposed recently, we derived and exemplified a methodology, considering the case of a prescribed distance to transverse isotropy. In particular, it was shown that the mean distance can be significantly reduced in comparison with the nonparametric probabilistic model for anisotropic media. The application presented has been limited to transverse isotropy (that is the most severe case for the theory developed) and can easily be applied to any other class of symmetry.

In addition to its capability to represent different classes of symmetries, the probabilistic model proposed exhibits more parameters than the previous stochastic models developed in the literature (within the general framework of nonparametric probabilistic approaches for anisotropic media) and thus, it turns out to be especially suitable for the fundamental issue of inverse experimental identification under material symmetry uncertainties. It can also be used as a prior stochastic model for the development of computational approaches, where the underlying randomness arising from fine scale features may have to be taken into account at a coarse scale, for instance.

A natural extension of this work will deal with the derivation of nonparametric stochastic models for tensor-valued random fields under material symmetry constraints. Such developments are of particular interest for the

mesoscale modeling of heterogeneous random media and are currently under investigation.

Acknowledgments

This work was supported by the French Research Agency (ANR) in the context of TYCHE project.

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