HAL open science

Existence of supersonic traveling waves for the Frenkel-Kontorova model

Samar Issa, Mustapha Jazar, Régis Monneau

To cite this version:

Samar Issa, Mustapha Jazar, Régis Monneau. Existence of supersonic traveling waves for the Frenkel-Kontorova model. Differential and integral equations, 2013, 26 (3-4), pp.321-353. hal-00684236

HAL Id: hal-00684236

https://hal.science/hal-00684236v1

Submitted on 31 Mar 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Existence of supersonic traveling waves for the Frenkel-Kontorova model

S. Issa [*], M. Jazar [†] & R. Monneau [‡]

March 31, 2012

Abstract

In this paper, we study the standard one-dimensional (non-overdamped) Frenkel-Kontorova (FK) model describing the motion of atoms in a lattice. For this model we show that for any supersonic velocity $c > 1$, there exist bounded traveling waves moving with velocity c. The profile of these traveling waves is a phase transition between limit states k_- in $-\infty$ and k_+ in $+\infty$. Those limit states are some integers which reflect the assumed 1-periodicity of the periodic potential inside the FK model. For every $c > 1$, we show that we can always find k_- and k_+ such that $k_+ - k_-$ is an odd integer. Furthermore for $c \geq \sqrt{\frac{25}{24}}$, we show that we can take $k_+ - k_- = 1$. These traveling waves are limits of minimizers of a certain energy functional defined on a bounded interval, when the length of the interval goes to infinity. Our method of proof uses a concentration compactness type argument which is based on a cleaning lemma for minimizers of this functional.

AMS Classification: 35A15, 35C07, 37K60.
Keywords: Frenkel-Kontorova model, Traveling wave, variational method, cleaning lemma, concentration-compactness.

1 Introduction

In this introduction, we first present in Subsection 1.1 the problem that we study in this paper. In Subsection 1.2 we introduce properties and definitions useful to state our main results in Subsection 1.3. Subsection 1.4 is devoted to a brief review of the literature and in Subsection 1.5 we give a sketch of the strategy for proving our main results. Finally Subsection 1.5 presents the organization of the paper.

1.1 Setting of the problem

We recall that the Frenkel-Kontorova (FK) model (introduced in [8]) is a model for a one-dimensional chain of atoms of position $q_n(t) \in \mathbb{R}$ depending on the time $t \in \mathbb{R}$ and solving the following system of ODEs for each $n \in \mathbb{Z}$:

$$\frac{d^2 q_n}{dt^2} = q_{n+1} + q_{n-1} - 2q_n + W'(q_n) \tag{1.1}$$

[*]LaMA-Liban, Université Libanaise. Tripoli, Liban. Email: samar_issa@live.com
[†]LaMA-Liban, Université Libanaise. Tripoli, Liban. Email: mjazar@ul.edu.lb
[‡]Université Paris-Est, CERMICS, Ecole des Ponts ParisTech, 6 et 8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2

where W is a periodic potential reflecting the periodicity of the lattice of atoms.
The FK model arises in the description of a broad range of physical phenomena, including crystal dislocation, plastic deformation (see for instance [3, 4] and the references therein).

In the present paper, we make the following assumption on the potential W:

$$\begin{cases} W \in C^2(\mathbb{R}) & \\ W(a+1) = W(a) & \text{for every } a \in \mathbb{R}, \\ W(a) > 0 = W(0) = W'(0) & \text{for every } a \in \mathbb{R}\backslash\mathbb{Z}, \\ W''(0) > 0. & \end{cases} \tag{1.2}$$

A traveling wave for equation (1.1), moving with velocity $c \in \mathbb{R}$, is by definition a particular solution of the form

$$q_n(t) = u(n - ct)$$

This means that u is solution of the following "advance-delay" differential equation

$$c^2 u''(x) = u(x+1) + u(x-1) - 2u(x) + W'(u(x)) \qquad \text{for every } x \in \mathbb{R}. \tag{1.3}$$

Notice that by assumption (1.2), every integer is a constant solution of (1.3). Our goal is to construct non trivial solutions of (1.3) which are phase transitions between two integer constant states for $x = -\infty$ and $x = +\infty$. To this end, we introduce the following condition at infinity:

$$\begin{cases} u(\pm\infty) \in \mathbb{Z}, \\ u(+\infty) - u(-\infty) \in 1 + 2\mathbb{Z}. \end{cases} \tag{1.4}$$

Indeed, in our proof we show that the non trivial phase transitions u that we construct also satisfy that $u(+\infty) - u(-\infty)$ is an odd integer.

1.2 First properties and definitions

In order to describe our main results in the next subsection, we first mention a decay property of any solution u of (1.3) and will introduce a few notations, like the functional whose (1.3) is the Euler-Lagrange equation.

Definition 1.1. (Exponential decay property)
We say that a function u has the exponential decay property at $+\infty$ (resp. at $-\infty$) if there exist constants $C > 0$ and $\lambda > 0$ such that

$$|u(x) - k_+| \le Ce^{-\lambda x} \qquad \textit{for } x \ge 0, \textit{ and for some constant } k_+ \in \mathbb{Z},$$

$$(\textit{resp.} \quad |u(x) - k_-| \le Ce^{\lambda x} \qquad \textit{for } x \le 0, \textit{ and for some constant } k_- \in \mathbb{Z}.)$$

We show the following result

Theorem 1.2. (Exponential decay)
Let $u \in L^\infty(\mathbb{R})$ be a solution of (1.3). If u has a limit $u(+\infty) \in \mathbb{Z}$ (resp. $u(-\infty) \in \mathbb{Z}$), then u has the exponential decay property at $+\infty$ (resp. at $-\infty$).

We now define mathematically the energy functional (which is physically the opposite of the action integral of the system) whose equation (1.3) is the Euler-Lagrange equation:

$$J(u) = \int_{\mathbb{R}} \left\{ \frac{c^2}{2}|u'(x)|^2 - \frac{1}{2}|(Du)(x)|^2 + W(u(x)) \right\} dx,$$

where

$$(Du)(x) := u(x + \tfrac{1}{2}) - u(x - \tfrac{1}{2}).$$

Notice that by the exponential decay (Theorem 1.2), the integral defining J is convergent for any bounded solution u of (1.3) with integer limits at infinity. More generally, the functional J is defined for all $u \in H$, with

$$H := \left\{ u \in H^1_{loc}(\mathbb{R});\ u' \in L^2(\mathbb{R}) \text{ and } W(u) \in L^1(\mathbb{R}) \right\}.$$

Note that each minimizer of J on H, if it exists, is solution of (1.3). For $k \in \mathbb{Z}\backslash\{0\}$ and $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ 1 & \text{if } x \geq 1, \end{cases}$$

and

$$1 - \varphi(-x) = \varphi(x), \tag{1.5}$$

set

$$E(k) = \inf_{\psi \in C_c^\infty(\mathbb{R})} J(k\varphi + \psi),$$

where we recall that $C_c^\infty(\mathbb{R})$ is the space of smooth functions with compact support. By direct calculation, we can show that E is independent of the choice of φ. Therefore $E(k)$ can be interpreted as the "minimal energy" of a phase transition between the constant states 0 and k.

Remark 1.3. *Notice that the potential W appearing in equation (1.3), is the opposite of the usual potential for phase transitions. Therefore we will construct transitions between two physically unstable phases.*

We show that E satisfies the following

Proposition 1.4. ***(Properties of the energy E)***
If $c > 1$, then the energy E satisfies the following properties:

(i) $E(0) = 0$.

(ii) $E(k) = E(-k) > 0$ *for all* $k \in \mathbb{Z}\backslash\{0\}$.

(iii) $E(k) \leq E(k-p) + E(p)$ *for all* $k \in \mathbb{Z}\backslash\{0\}$ *and for all* $p \in \mathbb{Z}\backslash\{0, k\}$.

Now, let us state the following two definitions

Definition 1.5. ***(k-transition)***
We say a function u is a $k-$ transition if and only if it satisfies the following conditions

$$\begin{cases} u \in H \cap C^2(\mathbb{R}), \\ u(\pm\infty) \in \mathbb{Z}, \\ u(+\infty) - u(-\infty) = k, \\ J(u) = E(k), \\ u \text{ solves } (1.3). \end{cases}$$

Notice that in our definition, a k-transition u enjoys the property of minimality of its energy among the whole class of transitions between the phase 0 and the phase k.

Definition 1.6. (Stability)
We say that $k \in \mathbb{Z}\backslash\{0\}$ is stable if and only if $E(k) < E(k-p) + E(p)$ for every $p \in \mathbb{Z}\backslash\{0, k\}$.

Notice that our stability definition simply requires the strict inequality in the property (iii) of Proposition 1.4.

1.3 Main results

Our main results are:

Theorem 1.7. (Existence of a traveling wave solution)
Let $c > 1$, then there exists at least one solution $u \in C^2(\mathbb{R})$ of (1.3)-(1.4).

This result is a straightforward corollary of the following two theorems:

Theorem 1.8. (Stability implies existence of k-transitions)
For $c > 1$, if k is stable, then there exists a k-transition.

Theorem 1.9. (Existence of a stable odd integer)
For $c > 1$, there exists $k \in 1 + 2\mathbb{Z}$ which is stable.

Theorem 1.10. ($k = 1$ is stable)
For every $c \geq \sqrt{\frac{25}{24}}$, the integer 1 is stable and then there exists a 1-transition.

Our approach is quite general. It could be used to study existence of solutions to more general equations, like:

$$c^2 u''(x) = V'(u(x+1) - u(x)) - V'(u(x) - u(x-1)) + W'(u(x)) \tag{1.6}$$

for some convex potential V of interaction, or also for interactions not restricted to nearest neighbors.
Even if it is not covered by this paper, it would be interesting to study the limits $c \to +\infty$ and $c \to 1$, and also to study the uniqueness of the phase transitions. Notice that we do not know if there exists a 1-transition if $1 < c < \sqrt{\frac{25}{24}}$.

Remark 1.11. *(First integral for the solutions of equation (1.3))*
For all $c \in \mathbb{R}$, we can show that every solution u of equation (1.3) satisfies the following first integral (when the sum is convergent):

$$\sum_{i \in \mathbb{Z}} \left\{ \frac{c^2}{2} u'^2(x+i) - W\left(u(x+i)\right) + \frac{1}{2}\left(u(x+i+1) - u(x+i)\right)^2 \right\} = constant.$$

But we were not able to use this remarkable property.

1.4 Brief review of the literature

More general models of the type (1.6) have been studied in the literature. Fermi, Pasta and Ulam first studied the FPU lattice (case $W = 0$) with cubic and quadratic potentials V (see [7]). Several works followed dealing with this lattice. For instance for the potential

$$V(p) = ab^{-1}(e^{-bp} + bp - 1),$$

Toda [23] found explicit formula's for the traveling waves. We mention the paper of Friesecke and Wattis [9], that studied broader range of nonlinear interaction potentials, namely the superquadratic growth. The authors showed the existence of supersonic solitary waves, using a variational approach (the concentration-compactness principle) with prescribed average potential energy. Further results in this path were given by Smets and Willem [22] (with prescribed velocity using the mountain pass theorem), Pankov and Pflüger [20] and Iooss [11].

The case $W \not\equiv 0$ is more complicated. We recall that if W is 1-periodic, the model is called a Frenkel-Kontorova or disrete Sine-Gordon lattice. In the case of harmonic interaction $V(x) = \frac{1}{2}x^2$, Iooss and Kirchgässner [12] established the existence of small amplitude waves (see also [5] for a case with non nearest interactions). We mention [10] for the construction of periodic traveling waves with W concave quadratic. Let us also mention [21] where traveling waves, which are "linear plus periodic", are constructed for general equations like (1.6).
In [15], the authors showed rigorously the existence of subsonic heteroclinic wave solutions of a FK model, where the on-site potential is taken piecewise quadratic (see also [16] for formal results in this direction). Heteroclinic traveling waves are also consctructed in [14] for a cosine potential W.

Let us notice that our problem has some common features (like lack of maximum principle, possible oscillations of solutions) with the study of critical points of functionals like the following one

$$\int_{\mathbb{R}} \frac{1}{2} \left\{ \left[(u'')^2 + \beta u'^2 \right] + \frac{1}{4}(u^2-1)^2 \right\} \, dx$$

We refer for instance to [1] for a nice review of results about this problem. For $\beta > 0$, the Euler-Lagrange equation is called the Extended Fisher-Kolmogorov equation, while for $\beta < 0$, it called the Swift-Hohenberg equation. Heteroclinic (oscillating) solutions are constructed in [1] (see also [2]) using the clipping method introduced in [13]. The clipping procedure reduces the size of the interval of some oscillating candidate, with a new candidate with "lower energy" (see for instance Lemma 9 in [1]). This interesting tool is used for removing spurious oscillations from minimizing sequences, and even if it is rather different, can be compared to our cleaning lemma (Lemma 5.2).

1.5 Strategy of the proofs

In our paper we prove the existence of phase transitions using a new approach. As usual, we first avoid a direct study of equation (1.3), but instead try to find a solution of a variational problem. This consists in looking for minimizers of the functional J on H. For a fixed $R > 0$, we replace H by H_R defined for $\ell \in \mathbb{Z}$ by

$$H_R := \left\{ u \in H^1_{loc}(\mathbb{R}) \text{ such that } u(x+2R) - u(x) = \ell \right\}, \tag{1.7}$$

and J by J_R defined by

$$J_R(u) = \int_{-R}^{R} \left\{ \frac{c^2}{2}(u')^2 - \frac{1}{2}(Du)^2 + W(u) \right\} \tag{1.8}$$

and we look for minimizers of the problem

$$\inf_{v \in H_R} J_R(v). \tag{1.9}$$

<u>Existence of a ℓ-transition.</u>
We first build "ℓ-transitions" u_R on "intervals of length $2R$", and then take the limit as R goes to infinity. The limit function u of u_R is not necessarily a ℓ-transition. In the simplest case, we may have the situation sketched on Figure 1, where the ℓ-transition is splitted in two smaller transitions ℓ_1 and ℓ_2.
We show that this situation can only occur if

$$E(\ell_1) + E(\ell_2) \leq E(\ell). \tag{1.10}$$

In particular, if ℓ is stable, then it is impossible. Indeed, (1.10) can be shown using an argument similar to the concentration-compactness argument of Lions (see [17]).

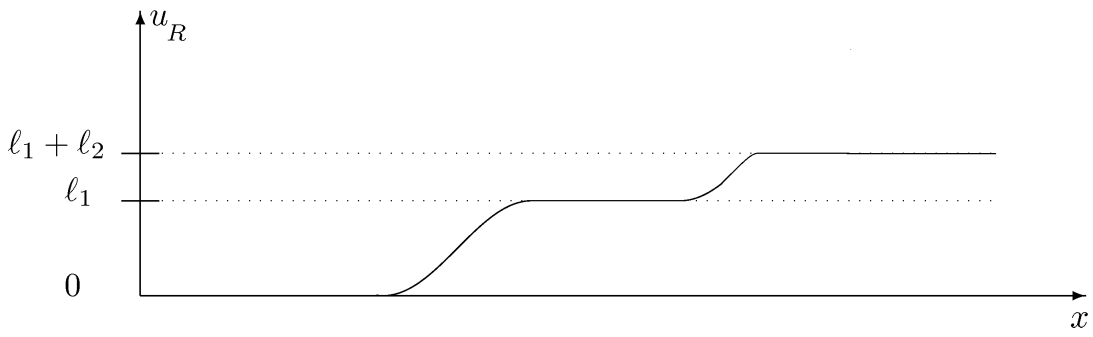

Figure 1: Splitting of an ℓ-transition u_R in $\ell_1 + \ell_2$

Existence of a stable odd integer ℓ.
Firstly, we choose $\ell \in 1 + 2\mathbb{Z}$ such that

$$E(\ell) = \inf_{k \in 1+2\mathbb{Z}} E(k).$$

We assume by contradiction that we have a splitting of ℓ in a finite sequence of integers, i.e.

$$\ell = \ell_1 + \ldots + \ell_N \qquad \text{with } \ell_i \in \mathbb{Z} \text{ for all } i = 1, \ldots, N.$$

Then there is at least one integer $\ell_{i_0} \in 1 + 2\mathbb{Z}$ such that

$$E(\ell_{i_0}) + E(\ell - \ell_{i_0}) \leq E(\ell).$$

From the properties of the energy E (Proposition 1.4) and the definition of the stability of ℓ (Definition 1.6), we deduce that

$$E(\ell_{i_0}) = E(\ell) \text{ and } E(\ell - \ell_{i_0}) = 0,$$

and then $\ell_{i_0} = \ell$.

Justification of the previous simple scenario.
First, we have a BV estimate for a velocity $c > 1$, which claims that

$$\int_{[-R,R]} \left| \frac{d}{dx} \beta(u_R) \right| \, dx \leq C \quad \text{with} \quad \beta(v) = \int_0^v \sqrt{W(w)} \, dw, \tag{1.11}$$

where C is independent of R (but depends on $c > 1$).
Because of the bound (1.11), the solution has to be close to constants on large intervals and, in order to minimize the energy, these constants have to be integers. We get a control on the length of the transition between two integers k_1 and k_2 using the cleaning lemma, (see Lemma 5.2). Indeed the cleaning lemma states that if u_R is close to an integer k on two intervals I_1 and I_2, contained in $(-R, R)$ and each of length larger or equal to 2, then u_R is also close to k on the convex hull of $I_1 \cup I_2$. Using these arguments, we can pass to the limit $R \to +\infty$, and show the existence of a ℓ-transition.

1.6 Organization of the article

In Section 2, we prove the exponential decay property (Theorem 1.2).
In Section 3, we construct a minimizer u_R for problem (1.9) on an interval of finite length $2R$ (see Proposition 3.4).
In Section 4, we show basic bounds on the sequence $(u_R)_R$ both in energy and on its total variation.
In Section 5, we prove a cleaning lemma for u_R (see Lemma 5.2).
Then in Section 6, we study the distance from u_R to $\mathbb{Z}$ and prove some bounds on the "jumps" of u_R, uniformly with respect to R (see Proposition 6.1). In particular we show that u_R stays close to some integers on long enough intervals.
In Section 7, we give the proof of Proposition 1.4 and show the convergence of the sequence $(u_R)_R$ to a function u which is a solution of equation (1.3) (see Proposition 7.1). We also show that the limit u enjoys some additional minimal energy properties (see Proposition 7.2).
Finally in Section 8, we prove Theorem 1.8 as a consequence of Propositions 7.1 and 7.2. We also prove Theorems 1.9 and 1.10.

2 An exponential decay property and proof of Theorem 1.2

The main goal of this section is to show the following exponential decay property whose Theorem 1.2 is a straightforward corollary.

Proposition 2.1. ***(Exponential decay)***
There exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, then there exist two constants λ, $\kappa > 0$ such that for any solution $u \in L^\infty(\mathbb{R})$ of (1.3), we have the following properties:

(i) *If $|u(x)| \leq \delta$ for every $x \geq 0$, then*

$$|u(x)| \leq \kappa\delta e^{-\lambda x} \qquad \text{for } x \geq 0.$$

(ii) *If $|u(x)| \leq \delta$ for every $x \leq 0$, then*

$$|u(x)| \leq \kappa\delta e^{-\lambda |x|} \qquad \text{for } x \leq 0.$$

(iii) *If $|u(x)| \leq \delta$ for every $x \in I$, where I is a bounded interval in $\mathbb{R}$, then*

$$|u(x)| \leq \kappa\delta e^{-\lambda dist(x, \partial I)} \qquad \text{for } x \in I,$$

where ∂I is the boundary of I.

In order to prove Proposition 2.1, we introduce for $r \in \mathbb{R}$, the quantity

$$M_r(u) := \sup_{x \geq r} |u(x)|$$

and prove first the following result:

Lemma 2.2. ***(Basic estimate)***
There exist $\delta_0 > 0$, $\mu \in (0, 1)$ and $L > 0$ such that for all $u \in L^\infty(\mathbb{R})$ solution of equation (1.3), we have

$$\text{If } M_0(u) \leq \delta_0, \qquad \text{then} \qquad M_{r+L}(u) \leq \mu M_r(u) \qquad \text{for all } r \geq 0.$$

Proof of Lemma 2.2.
Suppose that the result of this lemma is false. Then for every sequences

$$\begin{cases} L_n \to +\infty, \\ \delta_n \to 0, \\ \mu_n \to 1, \end{cases}$$

there exists a solution $u_n \in L^\infty(\mathbb{R})$ of equation (1.3) such that

$$M_0(u_n) \leq \delta_n \qquad \text{and} \qquad M_{L_n+r_n}(u_n) > \mu_n M_{r_n}(u_n) \qquad \text{for some } r_n \geq 0.$$

Case 1: the suppremum $M_{L_n+r_n}(u_n)$ is not reached at infinity
In this case there exists $x_n \in [L_n + r_n, +\infty)$ such that

$$M_{L_n+r_n}(u_n) = |u_n(x_n)| =: \varepsilon_n.$$

Then we have

$$\delta_n \geq M_0(u_n) \geq M_{L_n+r_n}(u_n) = \varepsilon_n \to 0. \tag{2.12}$$

Set

$$v_n(x) := \varepsilon_n^{-1} u_n(x + x_n),$$

then v_n is solution of the following equation

$$c^2 v_n''(x) = v_n(x+1) + v_n(x-1) - 2v_n(x) + \varepsilon_n^{-1} W'(v_n \varepsilon_n). \tag{2.13}$$

By the definition of v_n, we have

$$|v_n(0)| = 1.$$

On the other hand, we have

$$\begin{aligned} M_{-L_n}(v_n) &= \varepsilon_n^{-1} \sup_{x \geq -L_n} |u_n(x + x_n)| \\ &\leq \varepsilon_n^{-1} \sup_{y \geq r_n} |u_n(y)| \\ &= \varepsilon_n^{-1} M_{r_n}(u_n) \\ &< \varepsilon_n^{-1} \frac{M_{L_n+r_n}(u_n)}{\mu_n} = \frac{1}{\mu_n}. \end{aligned}$$

Hence, we have

$$|v_n(x)| < \frac{1}{\mu_n} \qquad \text{for every } x \in [-L_n, +\infty). \tag{2.14}$$

Then (2.12) implies

$$\varepsilon_n^{-1} W'(v_n \varepsilon_n) = W''(0) \cdot v_n + o_{\varepsilon_n}(1) \qquad \text{on } [-L_n, +\infty), \tag{2.15}$$

since $W'(0) = 0$, with $o_{\varepsilon_n}(1) \to 0$ when ε_n goes to zero.
Relations (2.13), (2.14) and (2.15) imply the existence of a constant $C_1 > 0$ such that

$$|v_n''(x)| \leq C_1 \quad \text{for every} \quad x \in [-L_n + 1, +\infty).$$

Consequently there exist a subsequence, still denoted by $(v_n)_n$, and an element $v \in W^{2,\infty}(\mathbb{R})$ such that $v_n \to v$ uniformly on every compact set of $\mathbb{R}$.
Passing to the limit in (2.14), we get

$$\sup_{x \in \mathbb{R}} |v(x)| \leq 1 = |v(0)|.$$

Using (2.13), we also have that v is solution of:

$$c^2 v''(x) = v(x+1) + v(x-1) - 2v(x) + W''(0) \cdot v \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.16}$$

Applying Fourier transform to equation (2.16), we obtain:

$$\phi(\xi)\hat{v}(\xi) = 0, \tag{2.17}$$

with $\phi(\xi) = c^2\xi^2 + (e^{i\xi} + e^{-i\xi} - 2) + W''(0)$ where $\hat{v}$ is the Fourrier transform of v and $\xi \in \mathbb{R}$. Because $W''(0) > 0$, we show easily that $\phi > 0$. This implies that $v = 0$. This contradicts the fact that $|v(0)| = 1$.

Case 2: the suppremum $M_{L_n+r_n}(u_n)$ is "reached at infinity"
In this case, there exists $x_n \in [L_n + r_n, +\infty)$ such that

$$\varepsilon_n := |u_n(x_n)| \geq \frac{n}{n+1} M_{L_n+r_n}(u_n),$$

and we conclude similarly. □

Proof of Proposition 2.1
In the sequel, we will show only statement (i). Statements (ii) and (iii) can be proven in the same manner.
Let $\delta \in (0, \delta_0)$, such that $|u(x)| \leq \delta$ for every $x \geq 0$. Then by Lemma 2.2, there exists $\mu \in (0,1)$ and there exists $L > 0$ such that

$$M_{L+r} \leq \mu M_r \qquad \text{for all } r \geq 0.$$

where M_r stand for $M_r(u)$.
On the one hand, we have for $n \in \mathbb{N}$

$$M_{nL} \leq \mu M_{(n-1)L} \leq \mu^2 M_{(n-2)L} \leq \ldots \leq \mu^n M_0 \leq \mu^n \delta.$$

If $x \in [nL, (n+1)L]$ for some $n \geq 0$, we have

$$\begin{aligned} \mu^n &= \exp\left(n \ln(\mu)\right) \\ &\leq \exp\left(\frac{x \ln(\mu)}{L} - \ln(\mu)\right). \end{aligned}$$

This implies that $|u(x)| \leq \beta\delta e^{-\lambda x}$, where $\lambda = -\frac{\ln(\mu)}{L} > 0$ and $\beta = \frac{1}{\mu} > 0$. □

3 Minimizing on a bounded interval

In this section, we consider the minimization problem (1.9) and show the existence of a minimizer (see Proposition 3.4). To this end, we define the bounded interval

$$\Omega_R = (-R, R)$$

and prove first some preliminary lemmata with the notation J_R, H_R introduced in Subsection 1.5.

Lemma 3.1. ***(Control of the term Du)***
For all $u \in H_R$, we have the following inequality:

$$\int_{\Omega_R} \left(u\left(x + \frac{1}{2}\right) - u\left(x - \frac{1}{2}\right)\right)^2 dx \leq \int_{\Omega_R} (u'(x))^2 \, dx. \tag{3.18}$$

Proof of Lemma 3.1.
For every $u \in H_R$, we have:

$$\begin{aligned}\int_{\Omega_R}\left(u\left(x+\frac{1}{2}\right)-u\left(x-\frac{1}{2}\right)\right)^2\,dx &\leq \int_{\Omega_R}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} u'(x+t)\,dt\right)^2\,dx\\ &\leq \int_{\Omega_R}\int_{-\frac{1}{2}}^{\frac{1}{2}} |u'(x+t)|^2\,dt\,dx\\ &= \int_{\Omega_R}(u'(x))^2\,dx,\end{aligned}$$

where we have used Cauchy-Schwarz inequality in the second line, Fubini theorem and the $2R$-periodicity of u' in the third line. □

Lemma 3.2. ***(Poincaré-Wirtinger inequality in H_R)***
For every $R > 0$, there exists a constant $C_R > 0$, such that

$$\int_{\Omega_R}\left(u(x)-\frac{1}{|\Omega_R|}\int_{\Omega_R} u\,dx\right)^2\,dx \leq C_R\int_{\Omega_R}\left(u'(x)\right)^2\,dx \text{ for every } u \in H_R. \tag{3.19}$$

Proof of Lemma 3.2.
See [6, Theorem 1, page 275]. □

Lemma 3.3. (Coercivity of J_R)
For all $c > 1$, the functional J_R is coercive on H_R for the semi-norm $\|u\|_{H_R} = \|u'\|_{L^2(\Omega_R)}$ and satisfies: $J_R \geq 0$ on H_R.

Proof of Lemma 3.3.
For all $u \in H_R$, we have:

$$\begin{aligned}J_R(u) &:= \frac{c^2}{2}\int_{\Omega_R}(u'(x))^2\,dx - \frac{1}{2}\int_{\Omega_R}(Du(x))^2\,dx + \int_{\Omega_R} W(u(x))\,dx\\ &\geq \frac{1}{2}(c^2-1)\int_{\Omega_R}(u'(x))^2\,dx + \int_{\Omega_R} W(u(x))\,dx \qquad (3.20)\\ &\geq \frac{1}{2}(c^2-1)\int_{\Omega_R}(u'(x))^2\,dx,\end{aligned}$$

where we have used Lemma 3.1 in the second line and the fact that $W \geq 0$ in the third line. We deduce that:

$$J_R(u) \to \infty \qquad \text{as } \|u\|_{H_R} \to \infty.$$

□

Proposition 3.4. ***(Existence of a minimizer for problem (1.9))***
Let $c > 1$, then J_R has at least one minimizer u_R on H_R. Moreover u_R is solution of (1.3) and $u_R \in C^2(\mathbb{R})$.

Proof of Proposition 3.4
Let $(u_n)_n$ be a minimizing sequence of J_R in H_R, i.e. $(u_n)_n \subset H_R$ and $J_R(u_n) \to \inf_{v\in H_R} J_R(v)$.

Step 1: Extraction of a subsequence
Since $J_R(u_n) \to \alpha$, then there exists a constant $M > 0$ such that:

$$|J_R(u_n)| \leq M \qquad \text{for every } n \in \mathbb{N}.$$

On the one hand, using (3.20) we have

$$\frac{1}{2}(c^2-1)\int_{\Omega_R}(u_n'(x))^2\,dx \quad \leq \quad J_R(u_n)\leq M,$$

Hence we obtain the following bound:

$$\int_{\Omega_R}\left(u_n'(x)\right)^2\,dx\leq\frac{2M}{c^2-1}:=C_1.$$

Moreover, by Poincaré-Wirtinger inequality (Lemma 3.2), we have:

$$\int_{\Omega_R}\left(u_n(x)-\frac{1}{|\Omega_R|}\int_{\Omega_R}u_n\right)^2\,dx\leq C_R\int_{\Omega_R}\left(u_n'(x)\right)^2\,dx\leq C_RC_1.$$

For every $n\in\mathbb{N}$, we define: $\tilde{u}_n:=u_n-\left\lfloor\frac{1}{|\Omega_R|}\int_{\Omega_R}u_n\right\rfloor\in H_R$ where $\lfloor a\rfloor$ denotes the floor integer part of a real a. Then

$$0\leq\frac{1}{|\Omega_R|}\int_{\Omega_R}\tilde{u}_n\leq 1,\qquad\text{and}\qquad J_R(\tilde{u}_n)=J_R(u_n).$$

Consequently the sequence $(\tilde{u}_n)_n$ is bounded in $H^1(\Omega_R)$, implying that there is a subsequence of $(\tilde{u}_n)_n$, still denoted by $(\tilde{u}_n)_n$, and an element $u\in H^1(\Omega_R)$, such that:

$$\tilde{u}_n\rightharpoonup u=:u_R\qquad\text{weakly in } H^1(\Omega_R). \tag{3.21}$$

Since $|\Omega_R|$ is bounded, then by Rellich-Kondrachov theorem, we have:

$$\tilde{u}_n\to u\qquad\text{strongly in } L^2(\Omega_R),\text{ up to a subsequence.} \tag{3.22}$$

Hence, by the Lebesgue inverse theorem, we have

$$\tilde{u}_n(x)\to u(x)\qquad\text{for a.e. } x\in\Omega_R,\text{ up to a subsequence,}$$

and in particular we get

$$\ell=\lim_{n\to\infty}\left[\tilde{u}_n(x+2R)-\tilde{u}_n(x)\right]=u(x+2R)-u(x),$$

which implies that $u\in H_R$.

Step 2: $\mathbf{J_R(u)=\inf_{v\in H_R}J_R(v)}$

From relation (3.21) and by lower semi-continuity, we get

$$\int_{\Omega_R}\left(u'(x)\right)^2\,dx\leq\liminf_{n\to\infty}\int_{\Omega_R}\left(u_n'(x)\right)^2\,dx. \tag{3.23}$$

Property (3.22) and the periodicity property of $D\tilde{u}_n$ imply:

$$\int_{\Omega_R}(Du(x))^2\,dx=\lim_{n\to\infty}\int_{\Omega_R}(D\tilde{u}_n(x))^2\,dx. \tag{3.24}$$

On the other hand, since W is a bounded and continuous function on $\mathbb{R}$, then Lebesgue dominated convergence theorem implies:

$$\lim_{n\to\infty}\int_{\Omega_R}W\left(\tilde{u}_n(x)\right)\,dx=\int_{\Omega_R}W\left(u(x)\right)\,dx. \tag{3.25}$$

Relations (3.23), (3.24) and (3.25) imply:

$$J_R(u) \leq \liminf_{n\to\infty} J_R(\tilde{u}_n) = \inf_{v\in H_R} J_R(v).$$

Since $u \in H_R$, then $J_R(u) = \inf_{v\in H_R} J_R(v)$.

Step 3: conclusion
By classical arguments, we get that $u = u_R$ solves the Euler-Lagrange equation associated to J_R:

$$-c^2u'' - D^2u + W'(u) = 0 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}). \tag{3.26}$$

From the fact that $\int_{\Omega_R} u'^2\, dx \leq C$, we deduce that $u \in L^\infty_{loc}(\mathbb{R})$ and then from (3.26), we get $u \in W^{2,\infty}_{loc}(\mathbb{R})$, which by bootstrap implies $u \in C^2(\mathbb{R})$. □

4 Basic uniform bounds on $(u_R)_R$

In this section, we denote by u_R the solution given in Proposition 3.4. We give some basic bounds on u_R, independent on R, that will be useful later in other sections.

Lemma 4.1. ***(Bound on the energy)***
Let u_R be the function constructed in Proposition 3.4, then there exists a constant $M > 0$, independent on $R \geq 1$, such that:

$$J_R(u_R) \leq M. \tag{4.27}$$

Proof of Lemma 4.1.
Define the function v_R as follows:

$$v_R(x) = \begin{cases} 0 & \text{if } -R \leq x \leq 0, \\ \ell x & \text{if } 0 < x \leq 1, \\ \ell & \text{if } 1 < x \leq R, \\ v_R(x+2R) - v_R(x) = \ell & \text{if } x \notin [-R, R]. \end{cases}$$

Since $v_R \in H_R$, and u_R is a minimizer of J_R on H_R, then

$$\begin{aligned} J_R(u_R) &\leq J_R(v_R) \\ &\leq \frac{c^2}{2}\int_{\Omega_R} (v_R')^2\, dx + \int_{\Omega_R} W(v_R)\, dx \\ &= \frac{c^2}{2}\int_0^1 (v_R')^2\, dx + \int_0^1 W(v_R)\, dx \\ &\leq \frac{c^2\ell^2}{2} + \|W\|_{L^\infty(\mathbb{R})} =: M. \end{aligned}$$

This ends the proof. □

Lemma 4.2. ***(Bound on the total variation of $\beta(u_R)$)***
Let us define $\beta(v) := \int_0^v \sqrt{W(x)}\, dx$. Then

$$TV(\beta(u_R), \Omega_R) := \int_{\Omega_R} \left| \frac{d}{dx}\beta(u_R(x)) \right| dx \leq \frac{M}{\sqrt{2(c^2-1)}} := M' , \tag{4.28}$$

where M is the constant given in (4.27).

Proof of Lemma 4.2.
We have

$$\begin{aligned} J_R(u_R) &\geq \frac{c^2-1}{2}\int_{\Omega_R}(u_R')^2\,dx + \int_{\Omega_R} W(u_R(x))\,dx \\ &\geq \sqrt{2(c^2-1)}\int_{\Omega_R}|u_R'|\sqrt{W(u_R)}\,dx \\ &= \sqrt{2(c^2-1)}\int_{\Omega_R}\left|\frac{d}{dx}\beta(u_R)\right|\,dx, \end{aligned} \tag{4.29}$$

where, in the first line we have used (3.20), and in the second line we have used Young's inequality. We conclude to (4.28) using (4.27). □

Remark 4.3. *Notice that the trick used in the second line of (4.29) is sometimes called the Modica-Mortola trick, see [19] (see also [18]).*

Lemma 4.4. ***(Bounds on W)***
There exist $\gamma_2 > \gamma_1 > 0$ such that

$$\gamma_1\left[dist\ (a,\mathbb{Z})\right]^2 \leq W(a) \leq \gamma_2\left[dist\ (a,\mathbb{Z})\right]^2. \tag{4.30}$$

The proof is left to the reader using (1.2).

Lemma 4.5. ***(Bounds on β)***
There exist two positive constants C_1 and C_2 such that

$$C_1\min(v^2,1) \leq |\beta(v)| \leq C_2 v^2 \qquad \text{for all } v\in\mathbb{R}. \tag{4.31}$$

The proof is also left to the reader using (4.30).

5 Cleaning lemma

Lemma 5.1. ***(Cleaning lemma for β)***
Let u_R be the function given in Proposition 3.4, $\delta\in(0,1]$ and a, b real numbers such that: $-R\leq a-1\leq a+1\leq b-1\leq b+1\leq R$. If

$$\sup_{x\in[a-1,a+1]\cup[b-1,b+1]}|u_R(x)| \leq \delta, \tag{5.32}$$

then there exists a positive constant C, independent of δ and R, such that

$$\int_I\left|\frac{d}{dx}\beta(u_R(x))\right|\,dx \leq C\delta^2 \qquad \text{where } I=[a,b]\subset\Omega_R, \tag{5.33}$$

with β defined in Lemma 4.2 above.

Proof of Lemma 5.1.
Set for every $u\in H_R$

$$E_I(u) := \frac{c^2}{2}\int_I(u')^2\,dx - \frac{1}{2}\int_I(Du)^2\,dx + \int_I W(u)\,dx.$$

Step 1: Bound on $E_I(u_R)$
We will prove

$$E_I(u_R) \geq \int_I \left[\frac{c^2-1}{2}(u_R')^2 + W(u_R) \right] \, dx - O(\delta^2). \tag{5.34}$$

Let us define a function $\tilde{u}$ as follows (which coincides with u_R on $[a,b]$)

$$\tilde{u}(x) := \begin{cases} 0 & \text{if } -\infty \leq x \leq a-1. \\ u_R(a)(x-(a-1)) & \text{if } a-1 \leq x \leq a, \\ u_R(x) & \text{if } a \leq x \leq b, \\ -u_R(b)(x-(b+1)) & \text{if } b \leq x \leq b+1, \\ 0 & \text{if } b+1 \leq x \leq +\infty. \end{cases}$$

Then we have:

$$\begin{aligned} J(\tilde{u}) &\geq \int_{\mathbb{R}} \left[\frac{c^2-1}{2} \left(\tilde{u}'\right)^2 + W\left(\tilde{u}\right) \right] \\ &\geq \int_I \left[\frac{c^2-1}{2} \left(u_R'\right)^2 + W\left(u_R\right) \right]. \end{aligned} \tag{5.35}$$

Notice that

$$\begin{aligned} \frac{1}{2}\int_{[a,b]} \left[(D\tilde{u})^2 - (Du_R)^2 \right] \, dx &= \frac{1}{2}\int_{[a,a+\frac{1}{2}]\bigcup[b-\frac{1}{2},b]} \left[(D\tilde{u})^2 - (Du_R)^2 \right] \, dx \\ &\geq -\frac{1}{2}\int_{[a,a+\frac{1}{2}]\bigcup[b-\frac{1}{2},b]} (Du_R)^2 \, dx \\ &\geq -2\delta^2, \end{aligned} \tag{5.36}$$

where we have used the bound (5.32). On the other hand, we have

$$\begin{aligned} J(\tilde{u}) - E_I(u_R) &= \int_{(-\infty,a)\bigcup(b,+\infty)} \left[\frac{c^2}{2}\tilde{u}'^2 - \frac{1}{2}(D\tilde{u})^2 + W(\tilde{u}) \right] \, dx - \frac{1}{2}\int_{[a,b]} \left[(D\tilde{u})^2 - (Du_R)^2 \right] \, dx \\ &\leq \int_{(a-1,a)\bigcup(b,b+1)} \left[\frac{c^2}{2}\tilde{u}'^2 + W(\tilde{u}) \right] \, dx - \frac{1}{2}\int_{[a,b]} \left[(D\tilde{u})^2 - (Du_R)^2 \right] \, dx \\ &\leq c^2\delta^2 + 2\gamma_2\delta^2 + 2\delta^2 \\ &= O(\delta^2), \end{aligned}$$

where we have used assumption (5.32), the fact that $W(a) \leq \gamma_2 a^2$ and (5.36). This computation, joint to (5.35), shows (5.34).

Step 2: Upper bound on $\int_I \left| \frac{d}{dx}\beta(u_R(x)) \right| \, dx$
In this Step, we will show that

$$\int_I \left| \frac{d}{dx}\beta(u_R(x)) \right| \, dx \leq \frac{1}{\sqrt{2(c^2-1)}} \left(O(\delta^2) + E_I(u_R) \right). \tag{5.37}$$

From (5.34), we have

$$\begin{aligned} E_I(u_R) + O(\delta^2) &\geq \int_I \left(\frac{c^2-1}{2} u_R'^2 + W(u_R) \right) \\ &\geq \sqrt{2(c^2-1)} \int_I \left| \frac{d}{dx}\beta(u_R(x)) \right| \, dx, \end{aligned}$$

where we have used (4.29). This show (5.37).

Step 3: Bounds on $E_I(u_R)$ in terms of δ^2
We will define a new candidate $\bar{u}$ (which coincides with u_R outside $[a,b]$), as follows:

$$\bar{u}(x) := \begin{cases} u_R(x) & \text{if } x \in \Omega_R \backslash [a,b], \\ -u_R(a)(x-(a+1)) & \text{if } x \in (a, a+1), \\ 0 & \text{if } x \in (a+1, b-1), \\ u_R(b)(x-(b-1)) & \text{if } x \in (b-1, b). \end{cases}$$

Since u_R minimizes J_R, then

$$J_R(u_R) \leq J_R(\bar{u}). \tag{5.38}$$

We want to use this inequality to estimate E_I. We have

$$J_R(u_R) = E_I(u_R) + E_{\Omega_R \backslash I}(u_R) \tag{5.39}$$

and

$$J_R(\bar{u}) = E_I(\bar{u}) + E_{\Omega_R \backslash I}(\bar{u}) \tag{5.40}$$

By relations (5.38), (5.39) and (5.40), we obtain the following inequality

$$\begin{aligned} E_I(u_R) &\leq J_R(\bar{u}) - E_{\Omega_R \backslash I}(u_R) \\ &\leq E_I(\bar{u}) + E_{\Omega_R \backslash I}(\bar{u}) - E_{\Omega_R \backslash I}(u_R) \\ &= E_I(\bar{u}) - \frac{1}{2} \int_{\Omega_R \backslash I} \left[(D\bar{u})^2 - (Du_R)^2 \right] \\ &= E_I(\bar{u}) - \frac{1}{2} \int_{[a-\frac{1}{2}, a] \bigcup [b, b+\frac{1}{2}]} \left[(D\bar{u})^2 - (Du_R)^2 \right] \\ &\leq E_I(\bar{u}) + \frac{1}{2} \int_{[a-\frac{1}{2}, a] \bigcup [b, b+\frac{1}{2}]} (Du_R)^2 \\ &\leq O(\delta^2). \end{aligned}$$

Therefore $E_I(u_R) \leq O(\delta^2)$.

Step 4: Conclusion
Finally, by using steps 2 and 3, we obtain

$$\int_I \left| \frac{d}{dx} \beta(u_R) \right| \, dx \leq C\delta^2.$$

□

Lemma 5.2. ***(Cleaning lemma for u_R)***
There exists $\delta_0 \in (0,1]$ such that the following holds. Let u_R be the function given in Proposition 3.4, $\delta \in (0, \delta_0)$ and a, b real numbers such that: $-R \leq a-1 \leq a+1 \leq b-1 \leq b+1 \leq R$. If

$$\sup_{x \in [a-1, a+1] \cup [b-1, b+1]} |u_R(x)| \leq \delta,$$

then there exists a constant $\nu \geq 1$, independent of δ and R, such that

$$|u_R(x)| \leq \nu\delta \qquad \text{on} \qquad I = [a,b]. \tag{5.41}$$

Proof of Lemma 5.2.
By the cleaning lemma 5.1, we have

$$\int_I \left| \frac{d}{dx}\beta(u_R) \right| \, dx \leq C\delta^2.$$

Moreover, for every $x \in [a, b]$ we have

$$\beta(u_R(x)) = \beta(u_R(a)) + \int_a^x \frac{d}{dx}\left(\beta(u_R(x))\right) \, dy.$$

Hence using (4.31), we get

$$|\beta(u_R(x))| \leq C_2\delta^2 + C\delta^2 =: C_3\delta^2.$$

Using (4.31) again, we obtain

$$C_1 \min(u_R^2(x), 1) \leq |\beta(u_R(x))| \leq C_3\delta^2.$$

Consequently $\min(u_R^2(x), 1) \leq \frac{C_3}{C_1}\delta^2$. Hence for $\delta < \delta_0 := \min(1, \sqrt{C_1/C_3})$, we have

$$|u_R(x)| \leq \nu\delta \qquad \text{for all } x \in I,$$

with $\nu = \sqrt{C_3/C_1} \geq 1$. □

6 Uniform bounds on the "jumps" of u_R

In this section, we improve the bounds on u_R given in Section 4, using the cleaning lemma 5.2. The main result of this section is the following decomposition property of the solution u_R:

Proposition 6.1. ***(Uniform bounds on the "jumps" of u_R for finite R)***
There exist $\bar{N} \in \mathbb{N}$ and $\delta_1 > 0$, such that for all $\delta \in (0, \delta_1)$, there exists $L_\delta > 0$, such that for all R large enough, there exists u_R a solution of (1.3) constructed in Proposition 3.4, such that there exists an integer $K \in \{1, ..., \bar{N}\}$ such that the following holds.
There exists a finite sequence of intervals $(\tilde{I}_i)_{i=1,...,K}$, such that $\tilde{I}_i \subset \overline{\Omega}_R$ for each $i = 1, ..., K$ and

$$\begin{cases} |u_R(x) - k_i| \leq C\delta \quad \text{on} \quad \tilde{I}_i \quad \text{for some } k_i \in \mathbb{Z}, \quad \text{for each } i = 1, ..., K, \\ \sup \widetilde{I}_i \leq \inf \widetilde{I}_{i+1} \quad \text{for each} \quad i = 1, \cdots, K-1, \\ \left| \bigcup_{i=1}^{K} \tilde{I}_i \right| \geq 2R - L_\delta. \end{cases} \tag{6.42}$$

Moreover, we have the following bounds

$$\begin{cases} \sup_{\Omega_R} u_R - \inf_{\Omega_R} u_R \leq \bar{N}, \\ K \leq \bar{N}, \\ \sum_{i=1,...,K} |k_{i+1} - k_i| \leq \bar{N} \quad \text{with} \quad k_{K+1} := k_1 + \ell. \end{cases} \tag{6.43}$$

Proof of Proposition 6.1.
We do the proof in several steps. We start defining the following sets

$$E_\delta := \left\{ x \in \Omega_R \ ; \ dist \ (u_R(x), \mathbb{Z}) \leq \frac{\delta}{2} \right\},$$

and

$$F_\delta := \Omega_R \backslash E_\delta = \left\{ x \in \Omega_R \ ; \ dist \ (u_R(x), \mathbb{Z}) > \frac{\delta}{2} \right\}.$$

Step 1: $|E_\delta| \geq 2R - C_\delta$, with C_δ is independent on R
Let m_δ be a positive constant such that $W \geq m_\delta > 0$ on $\mathbb{R} \backslash \left(\mathbb{Z} + B_{\frac{\delta}{2}}(0) \right)$, then we have:

$$\begin{aligned} m_\delta |F_\delta| \leq \int_{F_\delta} W(u_R) \, dx \quad &\leq \int_{\Omega_R} W(u_R) \, dx \\ &\leq J_R(u_R) \leq M. \end{aligned}$$

where M independent of R, is the constant of Lemma 4.1. Notice that in the second line, we used (3.20) and $c > 1$. Hence

$$|F_\delta| \leq \frac{M}{m_\delta} =: C_\delta,$$

and

$$|E_\delta| \geq 2R - C_\delta.$$

Step 2: Bound on the oscillation of u_R
By Lemma 4.2, we have

$$TV(\beta(u_R), \Omega_R) \leq M', \tag{6.44}$$

where M' is the positive constant independent on R, given in inequality (4.28). Then

$$\max_{\Omega_R} \beta(u_R) - \min_{\Omega_R} \beta(u_R) \leq M'.$$

Since β is nondecreasing then we have

$$\max_{\Omega_R} \beta(u_R) - \min_{\Omega_R} \beta(u_R) = \beta \left(\max_{\Omega_R}(u_R) \right) - \beta \left(\min_{\Omega_R}(u_R) \right).$$

Using the 1-periodicity of W and the definition of β, we get

$$\left\lfloor \max_{\Omega_R} u_R - \min_{\Omega_R} u_R \right\rfloor \int_0^1 \sqrt{W(v)} \, dv \leq \int_{\min_{\Omega_R} u_R}^{\max_{\Omega_R} u_R} \sqrt{W(v)} \, dv \leq M'.$$

Hence

$$\max_{\Omega_R} u_R - \min_{\Omega_R} u_R \leq 1 + \frac{M'}{\int_0^1 \sqrt{W(v)} \, dv} = 1 + \frac{M'}{\beta(1)}. \tag{6.45}$$

Step 3: Definition of the sets E^k
Let k be an integer, and

$$E^k := \left\{ x \in \Omega_R \ ; \ |u_R(x) - k| \leq \frac{\delta}{2} \right\}.$$

By Step 2, there is a finite number of integers k, such that $E^k \neq \emptyset$. Let $\mathbb{K} = \left\{ k \in \mathbb{Z}, \ \ E^k \neq \emptyset \right\}$ and set $K :=$ card $(\mathbb{K})$. We write

$$\mathbb{K} = \left\{ \bar{k}_1, ..., \bar{k}_K \right\}$$

where the order of the $\bar{k}_j$'s is the same as the order of $(\inf E^{\bar{k}_j})$'s. In particular we have (with $|e_j| \leq \frac{\delta}{2} < \frac{1}{4}$)

$$\begin{aligned} M' \geq TV\left(\beta(u_R), \Omega_R\right) &\geq \inf_{(e_j)_j} \sum_{j=1}^{K-1} \left|\beta(\bar{k}_{j+1} + e_{j+1}) - \beta(\bar{k}_j + e_j)\right| \\ &\geq (K-1)\left[\beta(1-\frac{\delta}{2}) - \beta(\frac{\delta}{2})\right], \end{aligned}$$

Then

$$K \leq 1 + \frac{M'}{\beta(1-\frac{\delta}{2}) - \beta(\frac{\delta}{2})} \leq 1 + \frac{M'}{\beta(\frac{3}{4}) - \beta(\frac{1}{4})}. \tag{6.46}$$

Furthermore, we find that

$$E_\delta = \bigcup_{k \in \mathbb{K}} E^k.$$

Then by Step 1

$$|E_\delta| = \sum_{k \in \mathbb{K}} |E^k| \geq 2R - C_\delta.$$

Step 4: For $k \in \mathbb{K}$ fixed, covering of E^k by intervals I_i^k, each of length 2
We define the finite sequence $(x_i^k)_{i=0,\ldots,N_\delta^k}$, for $0 \leq i \leq N_\delta^k$, as follows:

1. $x_0^k = \inf\left\{x \in E^k,\ x \geq -R\right\}$,

2. $x_{i+1}^k := \inf\left\{x \in E^k,\ \ x_i^k + 2 \leq x \leq R\right\}$.

Let us define the following set $I_i^k = [x_i^k, x_i^k + 2]$, for every $0 \leq i \leq N_\delta^k$. We have also

$$E^k \subset \bigcup_{i=0,\ldots,N_\delta^k} I_i^k.$$

Notice that $I_i^k \subset \overline{\Omega}_R$ for $i = 0, \ldots, N_\delta^k - 1$, and that $I_{N_\delta^k}^k$ may not be included in Ω_R.

Step 5: Dichotomy on the sets I_i^k
For every interval I_i^k, there are two possibilities:

a. There is an element $y_i^k \in I_i^k$ such that $|u_R(y_i^k) - k| > \delta$.
Hence if $u_R(y_i^k) \geq u_R(x_i^k)$, then

$$\begin{aligned} \beta(u_R(y_i^k)) - \beta(u_R(x_i^k)) &\geq \int_{u_R(x_i^k)}^{u_R(y_i^k)} \sqrt{W(v)}\, dv \\ &\geq \int_{k+|u_R(x_i^k)-k|}^{k+(u_R(y_i^k)-k)} \sqrt{W(v)}\, dv \\ &\geq \int_{k+\frac{\delta}{2}}^{k+\delta} \sqrt{W(v)}\, dv \\ &\geq \sqrt{\gamma_1}\left[\frac{v^2}{2}\right]_{\frac{\delta}{2}}^{\delta} \\ &\geq C\delta^2, \end{aligned}$$

with $C = \frac{3}{8}\sqrt{\gamma_1}$, where we have used Lemma 4.4 with $\delta < \frac{1}{2}$.
The case $u_R(y_i^k) < u_R(x_i^k) \leq k + \frac{\delta}{2}$ is similar. Therefore

$$TV(\beta(u_R), I_i^k) \geq \left|\beta(u_R(y_i^k)) - \beta(u_R(x_i^k))\right| \geq C\delta^2.$$

b. Otherwise $|u_R(x) - k| \leq \delta$ for every $x \in I_i^k$.

Then we define two subsets of indices following the dichotomy:

$$M_a^k := \left\{i \in \{0, \dots, N_\delta^k\},\ \exists y_i^k \in I_i^k \text{ such that } |u_R(y_i^k) - k| > \delta\right\},$$

$$M_b^k := \left\{i \in \{0, \dots, N_\delta^k\} \text{ such that } |u_R(x) - k| \leq \delta,\ \ \forall x \in I_i^k\right\},$$

and the associated sets

$$E_\alpha^k = \bigcup_{i \in M_\alpha^k} I_i^k \qquad \text{where } \alpha = a, b.$$

We have: $E^k \subset E_a^k \cup E_b^k$.

Step 6: Total variation and $|E_a^k| \leq C'_\delta$
Since $I_{N_\delta^k}^k$ may not be included in Ω_R, we deduce the following bound from below

$$M' \geq TV(\beta(u_R), \Omega_R) \geq \sum_{i \in M_a^k \setminus \{N_\delta^k\}} TV(\beta(u_R), I_i^k) \geq C\delta^2 \left(\text{ card } (M_a^k) - 1\right).$$

This shows that $\text{card}(M_a^k) \leq 1 + \frac{M'}{C\delta^2}$. Consequently we have:

$$\begin{aligned} |E_a^k| &\leq \sum_{i \in M_a^k} |I_i^k| \\ &= 2 \text{ card}(M_a^k) \\ &\leq 2\left(1 + \frac{M'}{C\delta^2}\right) =: C'_\delta. \end{aligned}$$

Therefore the set $E_\alpha = \bigcup_{k \in \mathbb{K}} E_\alpha^k$ satisfies for $\alpha = a$

$$|E_a| \leq \text{card}(\mathbb{K}) C'_\delta = K C'_\delta.$$

Step 7: Replacement of E_b^k by a large interval
Let us consider the convex hull:

$$\hat{E}_b^k = \begin{cases} \text{conv}\left(\bigcup_{i \in M_b^k \setminus \{N_\delta^k\}} I_i^k\right) & \text{if } N_\delta^k \in M_b^k \text{ and } R \in \text{Int}(I_{N_\delta^k}^k), \\ \text{conv}\left(\bigcup_{i \in M_b^k} I_i^k\right) & \text{otherwise.} \end{cases}$$

where we recall that $\text{Int}(A)$ is the interior of a set A. Notice that by definition, $\hat{E}_b^k \subset \Omega_R$. Let us call

$$E_c^k = \begin{cases} I_{N_\delta^k}^k & \text{if } N_\delta^k \in M_b^k \text{ and } R \in \text{Int}(I_{N_\delta^k}^k), \\ \emptyset & \text{otherwise.} \end{cases}$$

Notice that $E_b^k \subset \hat{E}_b^k \bigcup E_c^k$ and $|E_c^k| \leq 2$ with $E_c^k \neq \emptyset$ for at most one $k \in \mathbb{K}$. This shows that

$$E_b \subset E_c \cup \left(\bigcup_{k \in \mathbb{K}} \hat{E}_b^k \right) \quad \text{with} \quad E_c = \bigcup_{k \in \mathbb{K}} E_c^k \quad \text{and} \quad |E_c| \leq 2 \tag{6.47}$$

If the cardinal of the set of indices defining $\hat{E}_b^k$ is bigger or equal 2, just take the first and last interval, and then apply the cleaning lemma 5.2 to obtain:

$$|u(x) - k| \leq \nu\delta \qquad \text{on } \widehat{E}_b^k, \tag{6.48}$$

with $\nu \geq 1$. If the cardinal is 1 then (6.48) is still true.

Step 8: Definition of L_δ
Recall that $E_\delta = \bigcup_{k \in \mathbb{K}} E^k \subset E_a \cup E_b$, with $|E_\delta| \geq 2R - C_\delta$ and $|E_a| \leq KC'_\delta$.
Therefore

$$|E_b| \geq 2R - C_\delta - KC'_\delta.$$

Then (6.47) implies that

$$\sum_{k \in \mathbb{K}} |\hat{E}_b^k| \geq 2R - L_\delta \quad \text{with} \quad L_\delta = C_\delta + KC'_\delta + 2 \tag{6.49}$$

with K independent on R, bounded in (6.46).

Step 9: Conclusion
We can change the names of the integers, writing the set

$$\mathbb{K} = \left\{ \bar{k}_1, ..., \bar{k}_K \right\} = \{k_1, ..., k_K\}$$

where the integers k_i's are chosen such that the intervals

$$\tilde{I}_i := \widehat{E}_b^{k_i}$$

satisfy

$$\sup \tilde{I}_i \leq \inf \tilde{I}_{i+1}.$$

Then (6.48) and (6.49) imply (6.42).
Moreover, for $R \geq 1$ and $k_{K+1} = \ell + k_1$, we have (using (6.44))

$$\begin{aligned} 2M' &\geq TV\left(\beta(u_R), (-R, 3R)\right) \\ &\geq \inf_{(e_j)_j} \sum_{j=1}^{K} |\beta(k_{j+1} + e_{j+1}) - \beta(k_j + e_j)| \\ &\geq \sum_{j=1}^{K} \max\left(0, \ |\beta(k_{j+1}) - \beta(k_j)| - 2C_2\nu^2\delta^2\right) \\ &\geq \sum_{j=1}^{K} \max\left(0, \ \beta(1)|k_{j+1} - k_j| - 2C_2\nu^2\delta^2\right), \end{aligned}$$

where in the third line, we have used the fact that $|e_j| \leq \nu\delta$ and (4.31).
Because $|k_{j+1} - k_j| \geq 1$ for $1 \leq j \leq K-1$ and for $2C_2\nu^2\delta^2 \leq \frac{\beta(1)}{2}$, we get

$$K \leq 1 + \frac{4M'}{\beta(1)} =: \bar{N}. \tag{6.50}$$

Because $k_{K+1} := k_1 + \ell$, notice that we may have $k_{K+1} = k_K$ or not. In any cases, we get

$$\sum_{j=1}^{K} |k_{j+1} - k_j| \leq \frac{4M'}{\beta(1)} \leq \bar{N}. \tag{6.51}$$

Therefore (6.50), (6.51) and (6.45) imply (6.43), for

$$\delta_1 = \min\left(\frac{1}{2}, \delta_0, \sqrt{\frac{\beta(1)}{4C_2\nu^2}}\right).$$

This ends the proof of the proposition. □

7 Proof of Proposition 1.4 and the limit as R goes to infinity

This section is composed of two independent subsections. In a first subsection, we prove Proposition 1.4 and in a second subsection we study the limit of the solution u_R as R goes to infinity.

7.1 The energy $E(k)$ and proof of Proposition 1.4

Proof of Proposition 1.4
Proof of (i).
From (3.20), we have for all $\psi \in C_c^\infty(\mathbb{R})$

$$J(\psi) \quad \geq \quad \frac{c^2-1}{2}\int_{\mathbb{R}} (\psi')^2\,dx + \int_{\mathbb{R}} W(\psi)\,dx \geq 0 = J(0).$$

Therefore $0 = J(0) \leq \inf_{\psi \in C_c^\infty(\mathbb{R})} J(\psi) = E(0) \leq J(0)$ and then $E(0) = 0$.
Proof of (ii).
Let $k \in \mathbb{Z}$ and $\psi \in C_c^\infty(\mathbb{R})$. Notice that if $u \in H$, then

$$J(u(-\cdot)) = J(u) \qquad \text{and} \qquad J(u+l) = J(u) \quad \text{if} \quad l \in \mathbb{Z}.$$

Therefore

$$\begin{aligned} J\left(-k\varphi + \psi\right) &= J\left(-k\varphi(-\cdot) + \psi(-\cdot)\right) \\ &= J\left(k(-\varphi(-\cdot)) + \psi(-\cdot)\right) \\ &= J\left(k(\varphi - 1) + \psi(-\cdot))\right) \\ &= J\left(k\varphi + \psi(-\cdot)\right), \end{aligned}$$

where we have used (1.5), namely $\varphi(x) = 1 - \varphi(-x)$. Thus

$$E(-k) = \inf_{\psi} J(-k\varphi + \psi) = \inf_{\psi(-\cdot)} J(k\varphi + \psi(-\cdot)) = E(k).$$

Proof of (iii).
Let $k \in \mathbb{Z}$. We know that $E(k) = \inf_{\psi \in C_c^\infty(\mathbb{R})} J(k\varphi + \psi)$. Then for all $\eta > 0$ and $j \in \mathbb{Z}$ there exists $\psi_\eta^j \in C_c^\infty(\mathbb{R})$ such that

$$\left|E(j) - J(j\varphi + \psi_\eta^j)\right| \leq \eta. \tag{7.52}$$

Let

$$\psi^k = (k-p)\varphi(\cdot+n) + \psi_\eta^{k-p}(\cdot+n) + p\varphi(\cdot-n) + \psi_\eta^p(\cdot-n) - k\varphi \in C_c^\infty(\mathbb{R}),$$

then for $p \in \mathbb{Z}$, we have

$$\begin{aligned} E(k) &\leq J(k\varphi + \psi^k) \\ &\leq J\left((k-p)\varphi(\cdot+n) + \psi_\eta^{k-p}(\cdot+n) + p\varphi(\cdot-n) + \psi_\eta^p(\cdot-n)\right) \\ &\leq J\left(k\varphi(\cdot+n) + \psi_\eta^{k-p}(\cdot+n)\right) + J\left(p\varphi(\cdot-n) + \psi_\eta^p(\cdot-n)\right) \\ &\leq E(k-p) + E(p) + 2\eta, \end{aligned}$$

where we choose $n \geq 2$ in the third line, large enough such that for $|a| \leq \frac{1}{2}$

$$\text{supp}\left(\psi_\eta^{k-p}((\cdot+n)+a)\right) \bigcap \text{supp}\left(\psi_\eta^{k-p}((\cdot-n)-a)\right) = \emptyset$$

and in the last line we applied (7.52). Letting η tend to zero in the last inequality, we deduce that

$$E(k) \leq E(k-p) + E(p).$$

□

7.2 The limit $R \to +\infty$

We first start this subsection with a simple passage to the limit in the function u_R, where the first properties of the limit are given in Proposition 7.1. A further stability property (whose the proof is more involved) is given in a second result (see Proposition 7.2).
We recall that the solution u_R given in Proposition 3.4 depends on ℓ through the definition of the space H_R in (6.51), namely

$$H_R := \left\{u \in H^1_{loc}(\mathbb{R}) \text{ such that } u(x+2R) - u(x) = \ell\right\}.$$

Proposition 7.1. ***(Limit of u_R)***
For any $\ell \in \mathbb{Z}\backslash\{0\}$, there is $s \in \mathbb{Z}\backslash\{0\}$ such that there exists a solution u of (1.3) satisfying

$$u(-\infty) = 0 \quad \textit{and} \quad u(+\infty) = s$$

and up to extract a subsequence, we have

$$u_R(\cdot + b_R) + m_R \to u \quad \textit{in } L^\infty_{loc}(\mathbb{R}) \textit{ as } \quad R \to +\infty$$

where u_R is given in Proposition 3.4, and for sequences of reals b_R and integers m_R.
Moreover, if $\ell \in 1 + 2\mathbb{Z}$, then we can choose $s \in 1 + 2\mathbb{Z}$.

Proof of Proposition 7.1.
We apply Proposition 6.1 to use the properties of u_R and do the proof in two steps.

Step 1: Normalization of u_R.
We fix $\ell \in \mathbb{Z}\backslash\{0\}$. We first notice that from Proposition 6.1 we have

$$\begin{cases} \left|\bigcup_{i=1}^K \tilde{I}_i\right| \geq 2R - L_\delta \quad \text{with} \quad \tilde{I}_i \subset \overline{\Omega}_R, \quad \text{and} \quad |\tilde{I}_i \cap \tilde{I}_j| = 0 \quad \text{for} \quad i \neq j, \\ |u - k_i| \leq C\delta \quad \text{on} \quad \tilde{I}_i, \quad \text{for} \quad i = 1, ..., K, \\ \ell = \sum_{i=1}^K (k_{i+1} - k_i) \quad \text{and} \quad \sum_{i=1}^K |k_{i+1} - k_i| \leq \bar{N} \end{cases} \tag{7.53}$$

where $k_{K+1} = k_1 + \ell$. Up to reduce the number K of intervals, up to change u_R in $u_R + n_R$ (for some integer n_R), and up to extract a subsequence, we can always assume that $K \geq 1$ is fixed and the values k_i for $i = 1, ..., K$ are fixed as R goes to infinity, and

$$|\tilde{I}_i| \to +\infty \quad \text{as} \quad R \to +\infty \quad \text{for each} \quad i = 1, ..., K.$$

In the particular case $K = 1$, notice that by periodicity $\tilde{I}_1 + 2R$ is another interval of length going to $+\infty$ as $R \to +\infty$, such that $|u - (k_1 + \ell)| \leq C\delta$ on $\tilde{I}_1 + 2R$. Calling z_1 the center of $\tilde{I}_1$ and shifting z_1 to $-R$, we get a solution still denoted by u_R like on Figure 2.

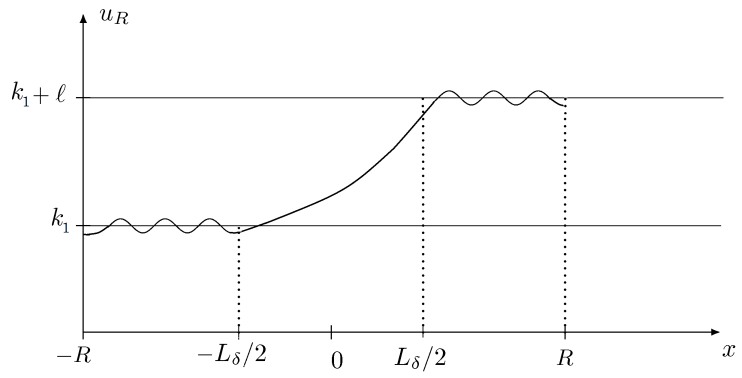

Figure 2: Schematic graph of the function u_R

Step 2: Definition of s

Therefore for any fixed choice $s = s_{i_0} := k_{i_0+1} - k_{i_0} \in \mathbb{Z}\backslash\{0\}$, up to replace $u_R(x)$ by $u_R(x+b_R)+m_R$ (for some real b_R and integer m_R), we have a solution still denoted by u_R like on Figure 3.

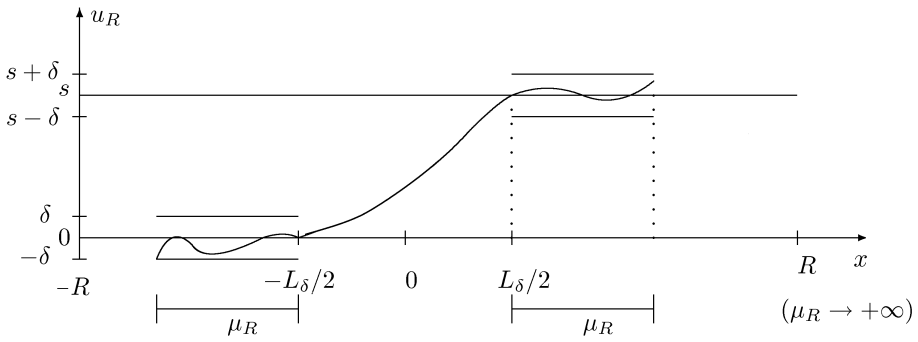

Figure 3: Graph of the "s-transition u_R"

with

$$\begin{cases} |u_R| \le C\delta & \text{on } (-\mu_R - L_\delta/2, -L_\delta/2) \\ |u_R - s| \le C\delta & \text{on } (L_\delta/2, L_\delta/2 + \mu_R) \\ |u_R| \le \bar{N} + C\delta & \text{on } (-\mu_R - L_\delta/2, \mu_R + L_\delta/2) \\ \text{with } \mu_R \to +\infty & \text{as } R \to +\infty. \end{cases} \tag{7.54}$$

Notice that because of the last line of (7.53), if $\ell = 1 \pmod 2$ then there exists $i_0 \in \{1, \dots, K\}$ such that $s = s_{i_0} = k_{i_0+1} - k_{i_0} = 1 \pmod 2$.
Therefore, we can pass to the limit in (1.3) and get a solution u of (1.3) satisfying (7.54) with $\mu_R = +\infty$.
From Proposition 2.1, we deduce (for $\delta > 0$ fixed but small enough) that $u(-\infty) = 0$ and $u(+\infty) = s$. This ends the proof of the proposition. □

Proposition 7.2. ***(ℓ is not stable)***
Let u be the function constructed in Proposition 7.1. Under the assumptions of Proposition 7.1, we have

$$J(u) = E(s), \tag{7.55}$$

and

$$E(s) + E(\ell - s) \le E(\ell). \tag{7.56}$$

Consequently if $s \neq \ell$, then ℓ is not stable.

In order to prove Proposition 7.2, we need the following lemma, whose proof is similar to the one of Lemma 3.1 (that is why we skip its proof).

Lemma 7.3. (Control on Du on a half line)
For every measurable function u such that $u' \in L^2(\mathbb{R})$, we have

$$\int_{-\infty}^{0} |Du|^2 \, dx \le \int_{-\infty}^{\frac{1}{2}} u'^2 \, dx.$$

Proof of Proposition 7.2
We will prove (7.56) in the six first steps and (7.55) in the seventh step.

Step 1: Preliminaries
Let φ be the function defined in the introduction (see Subsection 1.2). Then we set

$$\varphi_R(x) := \sum_{k \ge 0} \varphi(x - 2kR) + \sum_{k<0} \left(\varphi(x - 2kR) - 1\right), \tag{7.57}$$

and check that $\ell\varphi_R \in H_R$.

Step 2: $J_R(\ell\varphi_R + \psi) = J(\ell\varphi + \psi)$ for all $\psi \in C_c^\infty(\mathbb{R})$ with supp $\psi \subset [-R + \frac{1}{2}, R - \frac{1}{2}]$.
Straightforward, because $\varphi_R = \varphi$ on $[-R - 1, R + 1]$ for $R \ge 2$.

Step 3: Splitting of u_R in "two phase transitions" $\widetilde{u}_R$ and $\widehat{u}_R$
We use the same notations as in Step 2 of the proof of Proposition 7.1. In particular, we are in the situation of Figure 3.

Let $a_R = \frac{\mu_R + L_\delta}{2}$. Let $\xi \in C_c^\infty(\mathbb{R})$ such that

$$\xi = 1 \text{ on } [-a_R + \frac{1}{2}, a_R - \frac{1}{2}], \qquad \text{supp}(\xi) \subset [-a_R - \frac{1}{2}, a_R + \frac{1}{2}] \qquad \text{and} \qquad 0 \le \xi \le 1.$$

See in particular Figure 4 for the graph of ξ.

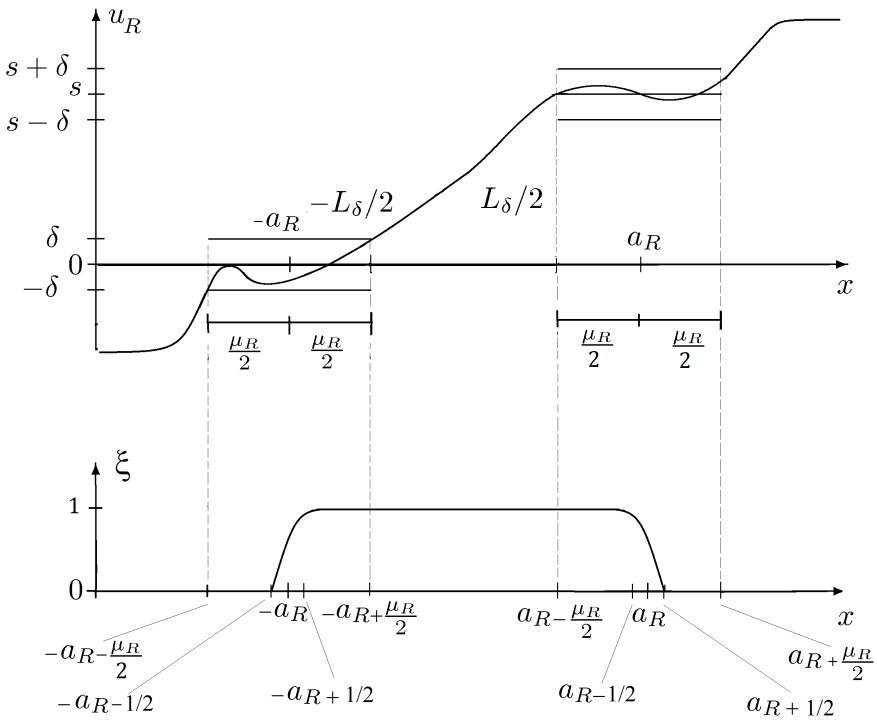

Figure 4: Possible graphs for u_R and ξ.

Set

$$\xi_R(x) = \sum_{k\ge 0} \xi(x - 2kR) + \sum_{k \le -1} \xi(x - 2kR)$$

and

$$\begin{cases} \widetilde{\varphi}_R^s &= s\varphi_R, \\ \widehat{\varphi}_R^{\ell-s} &= (\ell - s)\varphi_R(\cdot - R), \\ \varphi_R^\ell &= \widetilde{\varphi}_R^s + \widehat{\varphi}_R^{\ell-s}. \end{cases}$$

Then we can write u_R as

$$u_R = \widetilde{u}_R + \widehat{u}_R, \tag{7.58}$$

where

$$\widetilde{u}_R = \xi_R \left(u_R - \varphi_R^\ell\right) + \widetilde{\varphi}_R^s \qquad \text{and} \qquad \widehat{u}_R = (1 - \xi_R)\left(u_R - \varphi_R^\ell\right) + \widehat{\varphi}_R^{\ell-s}.$$

See Figure 5 for the graph of $\widetilde{u}_R$.

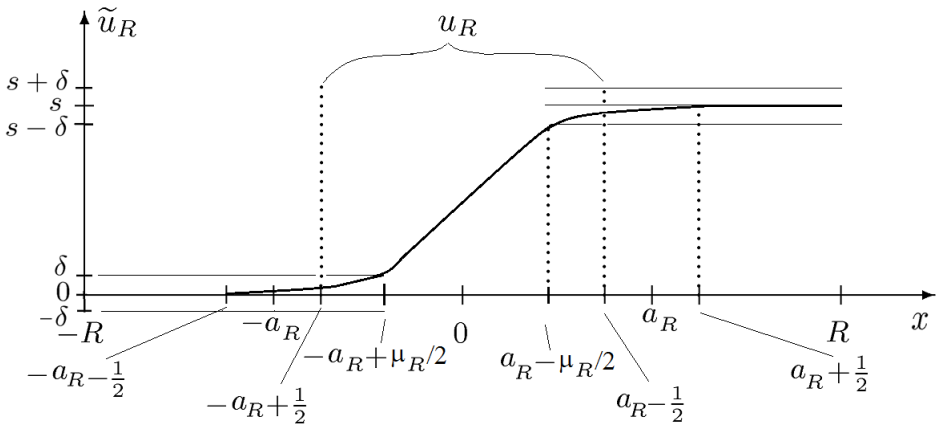

Figure 5: Schematic graph for $\widetilde{u}_R$.

Notice that we have in particular

$$\begin{cases} |\widetilde{u}_R| \le \delta & \text{on } [-a_R - \frac{\mu_R}{2}, -a_R + \frac{\mu_R}{2}], \\ \text{and} & \\ |\widetilde{u}_R - s| \le \delta & \text{on } [a_R - \frac{\mu_R}{2}, a_R + \frac{\mu_R}{2}]. \end{cases} \tag{7.59}$$

We set

$$\Lambda(v) := \frac{c^2 v'^2}{2} - \frac{1}{2}(Dv)^2 + W(v)$$

for the integrand arising in the definition of J. We have

$$\Lambda(\widetilde{u}_R + \widehat{u}_R) - \Lambda(\widetilde{u}_R) - \Lambda(\widehat{u}_R) = c^2 \widetilde{u}_R' \widehat{u}_R' - (D\widetilde{u}_R)(D\widehat{u}_R) + W(\widetilde{u}_R + \widehat{u}_R) - W(\widetilde{u}_R) - W(\widehat{u}_R),$$

with

$$\Big[\mathrm{supp}(\widetilde{u}_R') \bigcup \mathrm{supp}(D\widetilde{u}_R) \bigcup \mathrm{supp}(W(\widetilde{u}_R))\Big] \bigcap \overline{\Omega}_R \subset [-a_R - 1, a_R + 1] =: \widetilde{I}_R,$$

and

$$\Big[\mathrm{supp}(\widehat{u}_R') \bigcup \mathrm{supp}(D\widehat{u}_R) \bigcup \mathrm{supp}(W(\widehat{u}_R))\Big] \bigcap \overline{\Omega}_R \subset \overline{\Omega}_R \backslash [-a_R + 1, a_R - 1] =: \widehat{I}_R.$$

Therefore

$$J_R(u_R) - J_R(\widetilde{u}_R) - J_R(\widehat{u}_R) = \int_{\widetilde{I}_R \bigcap \widehat{I}_R} \left\{ c^2 \widetilde{u}_R' \widehat{u}_R' - (D\widetilde{u}_R)(D\widehat{u}_R) + W(\widetilde{u}_R + \widehat{u}_R) - W(\widetilde{u}_R) - W(\widehat{u}_R) \right\}.$$

We have

$$\widetilde{I}_R \bigcap \widehat{I}_R = I_- \bigcup I_+ \quad \text{with} \quad I_- = [-a_R - 1, a_R + 1], \quad I_+ = [a_R - 1, a_R + 1]$$

From (7.54) and using the PDE (1.3) satisfied by u_R, we get that

$$|u_R'|,\ |Du_R|,\ \sqrt{|W(u_R)|} \le C'\delta \qquad \text{on } I_\pm.$$

Notice that it is also possible (but not necessary) to use the exponential decay property of u_R on $[-L_\delta/2 - \mu_R, -L_\delta/2]$ and on $[L_\delta/2, L_\delta/2 + \mu_R]$, (see Proposition 2.1 (iii)).

Therefore, we can easily get

$$J_R(\widetilde{u}_R) + J_R(\widehat{u}_R) \leq J_R(u_R) + C\delta^2. \tag{7.60}$$

Step 4: $E(s) \leq J(\widetilde{u})$
We define

$$\widetilde{u}(x) = \begin{cases} \widetilde{u}_R(-R) = 0 & \text{for } x \leq -R, \\ \widetilde{u}_R(x) & \text{for } -R \leq x \leq R, \\ \widetilde{u}_R(R) = s & \text{for } x \geq R. \end{cases}$$

Notice that $\widetilde{u}$ is constant on $(-\infty, -R+1]$ and on $[R-1, +\infty)$. Therefore, $J(\widetilde{u}) = J_R(u_R)$.

Moreover $\widetilde{u} \in s\varphi + C_c^\infty(\mathbb{R})$. Therefore

$$E(s) \leq J(\widetilde{u}).$$

Step 5: $E(\ell - s) \leq J(\widehat{u})$
We define

$$\widehat{u}(x) = \begin{cases} \widehat{u}_R(-R+R) = 0 & \text{for } x \leq -R, \\ \widehat{u}_R(x+R) & \text{for } -R \leq x \leq R, \\ \widehat{u}_R(R+R) = \ell - s & \text{for } x \geq R. \end{cases}$$

Similarly, we have

$$E(\ell - s) \leq J(\widehat{u}) = \int_0^{2R} \Lambda(\widehat{u}_R)(x)\, dx = J_R(\widehat{u}_R),$$

because $\widehat{u}_R(x) - (\ell - s)x/(2R)$ is $2R$-periodic.
Step 6: Conclusion
From (7.60) and Steps 4 and 5, we have for $\psi \in C_c^\infty(\mathbb{R})$ with supp $\psi \subset [-R + \frac{1}{2}, R - \frac{1}{2}]$

$$\begin{aligned} E(s) + E(\ell - s) &\leq J(\widetilde{u}) + J(\widehat{u}) \\ &= J_R(\widetilde{u}_R) + J_R(\widehat{u}_R) \\ &\leq J_R(u_R) + C\delta^2 \\ &\leq J_R(\ell\varphi_R + \psi) + C\delta^2 \\ &= J(\ell\varphi + \psi) + C\delta^2. \end{aligned}$$

Because $\delta > 0$ is arbitrarily small, and then R arbitrarily large, (see Proposition 6.1), we get for all $\psi \in C_c^\infty(\mathbb{R})$

$$E(s) + E(\ell - s) \leq J(\ell\varphi + \psi). \tag{7.61}$$

Consequently

$$E(s) + E(\ell - s) \leq E(\ell).$$

Step 7: Proof of (7.55)
Step 7.1: $J_R(u_R) \leq E(s) + E(\ell - s) + \eta$ **for** $R \geq R_0(\eta)$
For every $\eta > 0$ there exists $\psi_\eta^s \in C_c^\infty(\mathbb{R})$ such that

$$J(s\varphi + \psi_\eta^s) \leq E(s) + \eta,$$

and there exists $\psi_\eta^{\ell-s} \in C_c^\infty(\mathbb{R})$ such that

$$J((\ell-s)\varphi + \psi_\eta^{\ell-s}) \leq E(\ell-s) + \eta.$$

For $R > 0$ large enough (with in particular $R \geq 2$)

$$\text{supp } \psi_\eta^s \subset [-R/4, R/4],$$

and

$$\text{supp } \psi_\eta^{\ell-s} \subset [-R/4, R/4].$$

We set

$$\widetilde{u}^s(x) = s\varphi_R(x) + \sum_{k\in\mathbb{Z}} \psi_\eta^s(x-2kR),$$

$$\widehat{u}^{\ell-s}(x+R) = (\ell-s)\varphi_R(x) + \sum_{k\in\mathbb{Z}} \psi_\eta^{\ell-s}(x-2kR),$$

and

$$\overline{u} = \widetilde{u}^s + \widehat{u}^{\ell-s} \in H_R.$$

Notice that by construction, we have $(\widetilde{u}^s)'(x+a)(\widehat{u}^{\ell-s})'(x) = 0$ for every $a \in [0,1]$. Then we compute:

$$\begin{aligned}
J_R(u_R) &\leq J_R(\overline{u}) \\
&= J_R(\widetilde{u}^s) + J_R(\widehat{u}^{\ell-s}) \\
&= J_R(\widetilde{u}^s) + J_R(\widehat{u}^{\ell-s}(\cdot + R)) \\
&= J_R(s\varphi + +\psi_\eta^s) + J_R((\ell-s)\varphi + +\psi_\eta^{\ell-s}) \\
&\leq E(s) + E(\ell-s) + 2\eta,
\end{aligned}$$

for any $R \geq R_0(\eta)$. This shows the statement of Step 7.1.

Step 7.2: $J_R(\widetilde{u}_R) \leq E(s) + \eta$ **for** $R \geq R_0(\eta)$
We recall that

$$J_R(\widetilde{u}_R) + J_R(\widehat{u}_R) \leq J_R(u_R) + C\delta^2,$$

for u_R given in Proposition 6.1- Proposition 7.1 for some $R \geq R_1(\delta)$.
This implies that

$$J_R(\widetilde{u}_R) + J_R(\widehat{u}_R) \leq E(s) + E(\ell-s) + 2\eta + C\delta^2,$$

for $R \geq \max(R_1(\delta), R_0(\eta))$. On the other hand, we have

$$E(s) \leq J(\widetilde{u}) = J_R(\widetilde{u}_R) \qquad \text{and} \qquad E(\ell-s) \leq J(\widehat{u}) = J_R(\widehat{u}_R).$$

This implies

$$0 \leq J_R(\widetilde{u}_R) - E(s) \leq 2\eta + C\delta^2$$

and

$$0 \leq J_R(\widehat{u}_R) - E(\ell-s) \leq 2\eta + C\delta^2.$$

Step 7.3: Conclusion: $J(u) \leq E(s)$
For any A such that $a_R - 1 \geq A \geq L_\delta/2 + 2$, we get:

$$\begin{aligned}
J_R(\widetilde{u}_R) &\geq J_A(\widetilde{u}_R) + \int_{[-\infty,-A]\bigcup[A,+\infty]} \left[\frac{c^2}{2}(\widetilde{u}_R')^2 - \frac{1}{2}(D\widetilde{u}_R)^2\right] \\
&\geq J_A(\widetilde{u}_R) - \int_{[-A-\frac{1}{2},-A]\bigcup[A,A+\frac{1}{2}]} \frac{1}{2}(D\widetilde{u}_R)^2 \\
&\geq J_A(\widetilde{u}_R) - C\delta^2,
\end{aligned}$$

where we have used $c \geq 1$ and Lemma 7.3 in the second line. In the third line, we used (see (7.59)) the fact that $|\widetilde{u}_R| \leq C\delta$ on $[-A-1, -A+\frac{1}{2}]$ and $|\widetilde{u}_R - s| \leq C\delta$ on $[A-\frac{1}{2}, A+1]$.
Therefore

$$J_A(\widetilde{u}_R) \leq E(s) + 2\eta + C'\delta^2,$$

$$\widetilde{u}_R = u_R \qquad \text{on } [-A, A]$$

and

$$D(\widetilde{u}_R) = D(u_R) \qquad \text{on } [-A, A].$$

This implies that

$$J_A(u_R) \leq E(s) + 2\eta + C'\delta^2.$$

Passing to the limit $R \to +\infty$, we get

$$J_A(u) \leq E(s) + 2\eta + C'\delta^2.$$

Because $\eta > 0$ is arbitrarily small, we get

$$J_A(u) \leq E(s) + C'\delta^2.$$

This implies (with both $\delta \to 0$ and $A \to \infty$) that

$$J(u) \leq E(s).$$

On the other hand, it is easy to show that

$$E(s) \leq J(u)$$

using an approximation of u by $s\varphi + \psi$ with $\psi \in C_c^\infty(\mathbb{R})$. This shows that $J(u) = E(s)$, which ends the proof of the Proposition. □

8 Stability

In this section, we give the proofs of the main results (Theorems 1.10, 1.8 and 1.9).
In order to do the proof of Theorem 1.10, we need the following lemma, whose proof is similar to the one of Lemma 3.1 (that is why we skip its proof).

Lemma 8.1. (Control on Du on the whole line)
For every function $u \in H^1_{loc}(\mathbb{R})$, such that $u' \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |Du|^2 \, dx \leq \int_{\mathbb{R}} (u')^2 \, dx.$$

Proof of Theorem 1.10
To show that the integer 1 is stable, we have to prove the following inequality

$$E(1) < E(s) + E(1-s) \qquad \text{for every } s \in \mathbb{Z}\backslash\{0,1\}. \tag{8.62}$$

Step 1: Lower bound on $E(s)$
Using Lemma 8.1, we get for any $u \in s\varphi + C_c^\infty(\mathbb{R})$

$$J(u) \geq \frac{(c^2-1)}{2} \int_{\mathbb{R}} \left(u'(x)\right)^2 \, dx + \int_{\mathbb{R}} W(u(x)) \, dx.$$

Following the proof of Lemma 4.2, we get (with β defined in Lemma 4.2)

$$\begin{aligned} J(u) &\geq \sqrt{2(c^2-1)} \int_{-\infty}^{+\infty} |u'(x)| \sqrt{W(u(x))}\, dx \\ &\geq \sqrt{2(c^2-1)} \int_0^{|s|} |\beta'(y)|\, dy \\ &= |s| \sqrt{2(c^2-1)} \beta(1). \end{aligned}$$

Therefore

$$E(s) \geq |s| \sqrt{2(c^2-1)} \beta(1).$$

Step 2: Upper bound on $E(1)$
Let $\phi \in C^2(\mathbb{R})$ be a function satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = 1$, and $\frac{c^2}{2} (\phi'(x))^2 = W(\phi(x))$. This function exists because of our assumption (1.2).
Then

$$\begin{aligned} J(\phi) &< \frac{c^2}{2} \int_{\mathbb{R}} \left(\phi'(x)\right)^2 \, dx + \int_{\mathbb{R}} W(\phi(x)) \\ &= c\sqrt{2} \int_{-\infty}^{+\infty} |\phi'(x)| \sqrt{W(\phi(x))}\, dx = c\sqrt{2} \int_0^1 \beta'(u)\, du \\ &= c\sqrt{2}\beta(1). \end{aligned}$$

Using the fact that ϕ can be approximated by $\varphi + \psi$ with $\psi \in C_c^\infty(\mathbb{R})$, we deduce that for any $\eta > 0$, there exists $\psi_\eta \in C_c^\infty(\mathbb{R})$ such that

$$|J(\phi) - J(\varphi + \psi_\eta)| \leq \eta.$$

Therefore

$$\begin{aligned} E(1) &\leq J(\varphi + \psi_\eta) \\ &\leq J(\phi) + \eta, \end{aligned}$$

Taking the limit $\eta \to 0$, we get

$$E(1) \leq J(\phi) < c\sqrt{2}\beta(1).$$

Step 3. Conclusion
Equality in (8.62) never holds for $s = -1, 0, 1, 2$. Therefore, inequality (8.62) is satisfied if

$$c\sqrt{2}\beta(1) \leq (|s| + |1-s|)\, \beta(1) \sqrt{2(c^2-1)} \quad \text{for} \quad s \in \mathbb{Z} \backslash \{-1, 0, 1, 2\}$$

i.e. if $c \leq 5\sqrt{c^2-1}$. Consequently (8.62) is true if $c \geq \sqrt{\frac{25}{24}}$. □

We will show the following result

Theorem 8.2. ***(Relation between stability and transition)***
If $\ell \in \mathbb{Z}$ is stable, then the solution u constructed in Proposition 7.1, is a ℓ-transition.

Proof of Theorem 1.8.
Apply Theorem 8.2.

Proof of Theorem 8.2.
Let $\ell \in \mathbb{Z}\backslash\{0\}$ be stable. As usual, we consider u_R such that $u_R(x+2R) = u_R(x) + \ell$. By Proposition 7.1 and Proposition 7.2 there exists a solution u which satisfies:

$$\begin{cases} u(-\infty) = 0, \quad u(+\infty) = s, \\ u \quad \text{solves (1.3)}, \\ J(u) = E(s) \end{cases}$$

We also have $s = \ell$, because ℓ is stable. Moreover, from the exponential decay property (Theorem 1.2) and the PDE (1.3), we deduce that

$$u \in H \cap C^2(\mathbb{R}).$$

Therefore u is a ℓ-transition. □

In order to prove Theorem 1.9, we introduce the set

$$I = \arg \inf_{k \in 1+2\mathbb{Z}} E(k).$$

Note that by Step 1 of the proof of Theorem 1.10, we have $I \neq \emptyset$.

Lemma 8.3. ***(Stability of the elements of I)***
k is stable for every $k \in I$.

Proof of Lemma 8.3
Let $k \in I$, we want to show that for any $\ell \in \mathbb{Z}\backslash\{0, k\}$

$$E(k) < E(k-\ell) + E(\ell). \tag{8.63}$$

By definition of I, for any $k' \in 1+2\mathbb{Z}$, we have

$$E(k') \geq \inf_{p \in 1+2\mathbb{Z}} E(p) = E(k). \tag{8.64}$$

Let $\ell \in \mathbb{Z}\backslash\{0\}$.

Case 1: $\ell \in 2\mathbb{Z}\backslash\{0\}$
Then $E(\ell) > 0$ and taking $k' = k - \ell$ in (8.64), we get $E(k-\ell) \geq E(k)$. This implies (8.63).

Case 2: $\ell \in 1+2\mathbb{Z}\backslash\{k\}$
Then $E(k-\ell) > 0$ and taking $k' = \ell$ in (8.64), we get $E(\ell) \geq E(k)$. Similarly this implies (8.63), and then it shows that k is stable. □

Proof of Theorem 1.9.
We simply apply Lemma 8.3. □

We can also get some results for even integers as follows. Set

$$I' = \arg \inf_{k \in 2\mathbb{Z}\backslash\{0\}} E(k).$$

Lemma 8.4. ***(Stability of elements of I')***
If

$$\inf_{k \in 2\mathbb{Z}\backslash\{0\}} E(k) < 2\left(\inf_{j \in 1+2\mathbb{Z}} E(j)\right), \tag{8.65}$$

then any $k \in I'$ is stable.

Proof of Lemma 8.4
Let $k \in I'$ and $l \in \mathbb{Z}$, then we have two possibilities.

Case 1: $l \in 2\mathbb{Z}\backslash\{0, k\}$
This implies that $0 < E(k) \leq E(l)$. On the other hand, we have also $k - l \in 2\mathbb{Z}\backslash\{0\}$. Hence

$$E(k) < E(k-l) + E(l).$$

Case 2: $l \in 1 + 2\mathbb{Z}$
Then $k - l \in 1 + 2\mathbb{Z}\backslash\{0\}$ and

$$E(l) \geq E(k') \text{ and } E(k-l) \geq E(k') \qquad \text{for all } k' \in I. \tag{8.66}$$

If

$$E(k) = E(k-l) + E(l), \tag{8.67}$$

then by equations (8.66), we have $E(k) \geq 2E(k')$ for all $k' \in I$. This contradicts assumption (8.65). Consequently, from Proposition 1.4, we deduce that $E(k) < E(k-l) + E(l)$.
This shows that k is stable. □

Acknowledgments:
RM thanks Yannick Sire for discussions about the literature. This work was supported by LASeR (Lebanese Association for Scientific Research).

References

[1] D. Bonheure, *Multitransition kinks and pulses for fourth order equations with a bistable nonlinearity*, Ann. I. H. Poincaré - AN 21 (2004), 319-340.

[2] D. Bonheure, L. Sanchez, M. Tarallo, S. Terracini, *Heteroclinic connections between nonconsecutive equilibria of a fourth order differential equation*, Calc. Var. 17 (2003), 341-356.

[3] O.M. Braun, Y.S. Kivshar, *Nonlinear dynamics of the Frenkel-Kontorova model.* Phys. Rep. 306 (1-2) (1998), 1-108.

[4] O.M. Braun, Y.S. Kivshar, *The Frenkel-Kontorova model. Concepts, methods, and applications.* Texts and Monographs in Physics. Springer-Verlag, Berlin, (2004).

[5] R. Calleja, Y. Sire, *Travelling waves in discrete nonlinear systems with non-nearest neighbour interactions.* Nonlinearity 22 (11) (2009), 2583-2605.

[6] L. C. Evans, *Partial Differential Equations.* Second edition. Graduate Studies in Mathematics 19. American Mathematical Society, Providence, RI, (2010).

[7] E. Fermi, J.Pasta, S. Ulam, *Studies of nonlinear problems.* Los Almos Scientific Laboratory report LA-1940, (1955). Reprinted in: Lect. in Appl. Math. 15 (1974), 143-156.

[8] J. Frenkel, T. Kontorova, *On the theory of plastic deformation and twinning.* Acad. Sci. U.S.S.R.J. Phys. 1 (1939), 137-149.

[9] G. Friesecke, J. Wattis, *Existence theorem for solitary waves on lattices*, Commun. Math. Phys. 161 (2) (1994), 391-418.

[10] M. Herrmann, *Periodic travelling waves in convex Klein-Gordon chains.* Nonlinear Anal. 71 (11) (2009), 5501-5508.

[11] G. Iooss, *Travelling waves in the Fermi-Pasta-Ulam lattice.* Nonlinearity 13 (3) (2000), 849-866.

[12] G. Iooss, K. Kirchgässner, *Travelling waves in a chain of coupled nonlinear oscillators*, Commun. Math. Phys. 211 (2) (2000), 439-464.

[13] W.D. Kalies, J. Kwapisz, R.C.A.M. VanderVorst, *Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria*, Comm. Math. Phys. 193 (1998), 337-371.

[14] C.-F. Kreiner, J. Zimmer, *Travelling wave solutions for the discrete sine-Gordon equation with nonlinear pair interaction*, Nonlinear Analysis 70 (2009), 3146-3158.

[15] C.-F. Kreiner, J. Zimmer, *Existence of subsonic heteroclinic waves for the Frenkel-Kontorova model with piecewise quadratic on-site potential.* Nonlinearity 24 (4) (2011), 1137-1163.

[16] O. Kresse, L. Truskinovsky, *Mobility of lattice defects: disrete and continuum approaches.* J. Mech. Phys. Solids 51 (7) (2003), 1305-1332.

[17] P. L., Lions, *The concentration-compactness principle in the calculus of variations. The locally case. I.* Ann. Inst. H. Poincaré, Anal. Non Linéaire 1 (2) (1984), 109-145.

[18] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. 98 (2) (1987), 123-142.

[19] L. Modica, S. Mortola, *Un esempio di* Γ*-convergenza*, Boll. Unione Mat. Ital. B(5) 14 (1) (1977), 285-299.

[20] A. A. Pankov, K. Pflüger, *Traveling Waves in Lattices Dynamical Systems.* Math. Meth. Appl. Sci. 23 (14) (2000), 1223-1235.

[21] W.-X. Qin, *Uniform sliding states in the undamped Frenkel-Kontorova model*, J. Differential Equations 249 (2010), 1764-1776.

[22] D. Smets, M. Willem, *Solitary Waves with Prescribed Speed on Infinite Lattices.* Journal of Functional Analysis 149 (1) (1997), 266-275.

[23] M. Toda, *Theory of Nonlinear Lattices*, Springer-Verlag, Berlin, (1989).