On the class of gamma-mixing graphs

Mikhail Isaev, K.V Isaeva

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Abstract

We study three mixing properties of a graph: large algebraic connectivity, large Cheeger constant (isoperimetric number) and large spectral gap from 1 for the second largest eigenvalue of the transition probability matrix of the random walk on the graph. We prove equivalence of this properties (in some sense). We give estimates for the probability for a random graph to satisfy these properties. In addition, we present asymptotic formulas for the numbers of Eulerian orientations and Eulerian circuits in an undirected simple graph.

1 Introduction

Let G be an undirected simple graph with vertex set VG and edge set EG. We define $n \times n$ matrix Q by

$$Q_{jk} = \begin{cases} -1, & \{v_j, v_k\} \in EG, \\ d_j, & j = k, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $n = |VG|$ and d_j denotes the degree of $v_j \in VG$. The matrix $Q = Q(G)$ is called the *Laplacian matrix* of the graph G. The eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ of the matrix Q are always non-negative real numbers and $\lambda_1 = 0$. The eigenvalue $\lambda_2 = \lambda_2(G)$ is called the *algebraic connectivity* of the graph G. The original theory related to algebraic connectivity was produced by Fiedler, see [5], [6]. The number $\lambda_2(G)$ is a discrete analogue of the smallest positive eigenvalue of the Laplace differential operator on the Riemannian manifold. (For more information on the spectral properties of the Laplace matrix see, for example, [13] and references therein.)

Let $\mathcal{F}_\gamma$ be the set of simple graphs G satisfying the following property:

Property 1. The algebraic connectivity $\lambda_2(G) \ge \gamma |VG|$.

For a subset of vertices $A \subseteq VG$ let ∂A denote the set of all edges connecting a vertex in A and a vertex outside A:

$$\partial A = \{\{u, v\} \in EG : u \in A, v \in VG \setminus A\}. \tag{2}$$

The *Cheeger constant* (or *isoperimetric number*) of G, denoted $i(G)$, is defined by

$$i(G) = \min\left\{ \frac{|\partial A|}{|A|} : A \subset VG,\ 0 < |A| \leq \frac{|VG|}{2} \right\}. \tag{3}$$

The number $i(G)$ is a discrete analogue of the (Cheeger) isoperimetric constant in the theory of Riemannian manifolds and it has many interesting interpretations (for more detailed information see, for example, [12] and references therein).

Let $\mathcal{C}_\gamma$ be the set of simple graphs G satisfying the following property:

Property 2. The Cheeger constant (isoperimetric number) $i(G) \geq \gamma |VG|$.

Let $P = P(G)$ be the transition probability matrix of the random walk on the graph G.

$$P_{jk} = \begin{cases} \frac{1}{d_j}, & \{v_j, v_k\} \in EG, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

The eigenvalues of P are such that

$$1 = \chi_1 \geq \chi_2 \ldots \geq \chi_n \geq -1. \tag{5}$$

The graph G is connected if and only if the random walk is an irreducible Markov chain. In this case, there exists a unique stationary distribution and multiplicity of the eigenvalue $\chi_1 = 1$ is equal to one. (For more information on random walks on graphs see, for example, [8] and references therein.)

Let $\mathcal{M}_\gamma$ be the set of simple graphs G satisfying the following property:

Property 3. The spectral gap $1 - \chi_2(G) \geq \gamma$ and $\min\limits_j d_j \geq \gamma |VG|$

Graphs of $\mathcal{F}_\gamma$, $\mathcal{C}_\gamma$, $\mathcal{M}_\gamma$ have strong mixing properties. We call $\mathcal{F}_\gamma \cap \mathcal{C}_\gamma \cap \mathcal{M}_\gamma$ as the class of γ-mixing graphs. Actually (see Section 2 of the present work), Properties 1-3 are equivalent in the following sense: if a graph satisfies one of these properties with $\gamma = \gamma_0 > 0$, then it satisfies all Properties 1-3 with some $\gamma > 0$ depending only on γ_0.

In Section 3 we estimate the probability of a random graph to be γ-mixing. We consider the following model (Gilbert's random graph model): every possible edge occurs independently with some fixed probability $0 < p < 1$. It turned out that in this model almost all graphs (asymptotically) are γ-mixing with some $\gamma > 0$ depending only on p.

In Section 4 we construct some general family of graphs, satisfying Properties 1-3 (see Example 3 and Remark 4.1). For example, the family of complete bipartite graphs $\{K_{n,n}\}$ is the special case of our general family. We also give some other examples.

In addition, we consider two enumeration problems: counting the number of Eulerian orientations (EO) and counting the number of Eulerian circuits (EC) in an undirected simple graph. It is known that both of these problems is complete for the class $\#P$, see [2], [9].

Recently, in [3], [4] the asymptotic behaviour of the numbers of Eulerian orientations and Eulerian circuits was determined for γ-mixing graphs (more precisely, for graphs satisfying Property 1).

In Section 5 we present the asymptotic formulas for EO, EC and compare them against the exact values for small graphs. Actually, if the graph G is γ-mixing then for any $\varepsilon > 0$ the error term $|\delta(G)| \le Cn^{-1/2+\varepsilon}$, where $C > 0$ depends only on ε and γ. We plan to give the proofs of these formulas in a subsequent paper.

2 Equivalence of Properties 1-3

We recall that for a simple graph G with n vertices:

$$\lambda_2(G) \le \frac{n}{n-1} \min_j d_j, \tag{6a}$$

$$\lambda_2(G) \ge 2 \min_j d_j - n + 2, \tag{6b}$$

$$\frac{\lambda_2(G)}{2} \le i(G) \le \sqrt{\lambda_2(G)(2 \max_j d_j - \lambda_2(G))}, \tag{7}$$

$$\lambda_2(G) \le \lambda_2(G_1) + 1, \tag{8a}$$

$$\lambda_2(G) \le \lambda_2(G'), \tag{8b}$$

where G_1 arises from G by removing one vertex and all adjacent edges, G' is an arbitrary graph such that $VG' = VG$ and $EG \subset EG'$.

Estimates (6), (8) were obtained in [5]. Estimates (7) were given in [12].

Using (7) and the inequality $d_j \le n$, we find that for any $\gamma_0 > 0$

$$\mathcal{F}_{\gamma_0} \subset \mathcal{C}_{\gamma_1} \quad \text{and} \quad \mathcal{C}_{\gamma_0} \subset \mathcal{F}_{\gamma_1}, \tag{9}$$

where $\gamma_1 > 0$ depends only on γ_0.

In order to complete the proof of the equivalence of Properties 1-3 we need the following lemma. For $\vec{x} \in \mathbb{R}^n$ and $n \times n$ matrix M let us denote

$$\|\vec{x}\| = \sqrt{\vec{x}^T \vec{x}}, \qquad \|M\| = \sup_{\vec{x} \in \mathbb{R}^n,\ \|\vec{x}\|=1} \|M\vec{x}\|. \tag{10}$$

Lemma 1. *Let $a, b_1, b_2 > 0$. Let A be symmetric positive semidefinite $n \times n$ matrix such that for some $\vec{w} \in \mathbb{R}^n$, $\vec{w} \ne \vec{0}$,*

$$A\vec{w} = 0 \tag{11a}$$

and for any $\vec{u} \in \mathbb{R}^n$ such that $\vec{u}^T \vec{w} = 0$

$$\|A\vec{u}\| \ge a\|\vec{u}\|. \tag{11b}$$

Then for any symmetric $n \times n$ matrix B such that

$$\|B\| \leq b_1 \quad \textit{and} \quad \vec{w}^T B \vec{w} \geq b_2 \|\vec{w}\|^2 \tag{11c}$$

the following statement holds:

$$\begin{cases} \det(A - \lambda B) = 0, \\ \lambda \neq 0 \end{cases} \Longrightarrow \lambda \geq \rho \tag{12}$$

for some $\rho = \rho(a, b_1, b_2) > 0$.

The proof of Lemma 1 is given at the end of this section.

Note that

$$P = I - D^{-1}Q, \tag{13}$$

where Q and P are the same as in (1) and (4), respectively, I denotes the identity matrix and D is the diagonal matrix defined by $D_{jj} = d_j$.

Let $A_1 = \frac{1}{n}Q$, $B_1 = \frac{1}{n}D$ and $\vec{w}_1 = [1, \ldots, 1]^T$. Using (13), we find that:

$$\det(A_1 - \lambda I) = 0 \Longleftrightarrow \det(Q - \lambda n I) = 0; \tag{14}$$

$$\det(A_1 - \lambda B_1) = 0 \Longleftrightarrow \det(P - (1 - \lambda) I) = 0; \tag{15}$$

for any $\vec{u} \in \mathbb{R}^n$ such that $\vec{u}^T \vec{w}_1 = 0$

$$\vec{u}^T A_1 \vec{u} \geq \frac{1}{n} \lambda_2(G) \vec{u}^T \vec{u}; \tag{16}$$

$$\|B_1\| \leq \frac{1}{n} \max_j d_j \leq 1, \quad \vec{w}_1^T B_1 \vec{w}_1 \geq \frac{1}{n} \min_j d_j \, \vec{w}_1^T \vec{w}_1. \tag{17}$$

Combining Property 1, (6a), (14)-(17) and Lemma 1, we get that for any $\gamma_0 > 0$

$$\mathcal{F}_{\gamma_0} \subset \mathcal{M}_{\gamma_2}, \tag{18}$$

where $\gamma_2 > 0$ depends only on γ_0.

Let $A_2 = D^{-\frac{1}{2}} Q D^{-\frac{1}{2}}$, $B = nD^{-1}$ and $\vec{w}_2 = D^{\frac{1}{2}} \vec{w}_1$, where D^s is the diagonal matrix defined by $D^s_{jj} = (d_j)^s$. Using (13), we find that:

$$\det(A_2 - \lambda I) = 0 \Longleftrightarrow \det(P - (1 - \lambda) I) = 0; \tag{19}$$

$$\det(A_2 - \lambda B_2) = 0 \Longleftrightarrow \det(Q - \lambda n I) = 0; \tag{20}$$

for any $\vec{u} \in \mathbb{R}^n$ such that $\vec{u}^T \vec{w}_2 = 0$

$$\vec{u}^T A_2 \vec{u} \geq (1 - \chi_2(G)) \vec{u}^T \vec{u}; \tag{21}$$

$$\|B_2\| \leq \frac{n}{\min_j d_j}, \quad \vec{w}_2^T B_1 \vec{w}_2 = n \vec{w}_1^T \vec{w}_1 \geq \vec{w}_2^T \vec{w}_2. \tag{22}$$

Combining Property 3, (19)-(22) and Lemma 1, we get that for any $\gamma_0 > 0$

$$\mathcal{M}_{\gamma_0} \subset \mathcal{F}_{\gamma_3}, \tag{23}$$

where $\gamma_3 > 0$ depends only on γ_0.

Putting together (9), (18) and (23), we obtain the desired assertion:

Theorem 1. *Let $\mathcal{F}_\gamma, \mathcal{C}_\gamma, \mathcal{M}_\gamma$ be defined as in Section 1. Then for any $\gamma_0 > 0$*

$$\mathcal{F}_{\gamma_0} \cup \mathcal{C}_{\gamma_0} \cup \mathcal{M}_{\gamma_0} \subset \mathcal{F}_\gamma \cap \mathcal{C}_\gamma \cap \mathcal{M}_\gamma, \tag{24}$$

where $\gamma > 0$ depends only on γ_0.

Now it remains to prove Lemma 1.

Proof of Lemma 1. Let $\det(A - \lambda B) = 0$. Then for some $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq 0$,

$$A\vec{v} = \lambda B\vec{v}. \tag{25}$$

Let $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$, where $\vec{v}_\parallel \parallel \vec{w}$ and $\vec{v}_\perp^T \vec{w} = 0$. Due to (11a), we have that

$$\vec{v}_\parallel^T A\vec{v} = 0. \tag{26}$$

Since $\lambda \neq 0$, using (25), we get that

$$\vec{v}_\parallel^T B\vec{v}_\parallel = -\vec{v}_\parallel^T B\vec{v}_\perp. \tag{27}$$

Using (11c), (27) and the CauchySchwarz inequality, we find that

$$b_1\|\vec{v}_\parallel\|\|\vec{v}_\perp\| \geq \|\vec{v}_\parallel\|\|B\vec{v}_\perp\| \geq |\vec{v}_\parallel^T B\vec{v}_\perp| = |\vec{v}_\parallel^T B\vec{v}_\parallel| \geq b_2\|\vec{v}_\parallel\|^2. \tag{28}$$

Thus we have that

$$\|\vec{v}_\perp\| \geq \frac{b_2}{\sqrt{b_1^2 + b_2^2}}\|\vec{v}\|. \tag{29}$$

Using (11b), (11c) and (29), we find that:

$$\begin{aligned} &\|A\vec{v}\| \geq a\|\vec{v}_\perp\|, \\ &\|B\vec{v}\| \leq b_1\|\vec{v}\| \leq \frac{b_1\sqrt{b_1^2 + b_2^2}}{b_2}\|\vec{v}_\perp\|. \end{aligned} \tag{30}$$

Combining (25) and (30), we obtain that

$$\lambda \geq \frac{ab_2}{b_1\sqrt{b_1^2 + b_2^2}}. \tag{31}$$

■

3 Probability for a random graph to be γ-mixing

Let ξ be a random variable belonging to the binomial distribution $B(M, p)$:

$$\Pr(\xi = k) = \frac{M!}{k!(M-k)!}p^k(1-p)^{M-k}, \quad 0 < p < 1,\ M \in \mathbb{N}. \tag{32}$$

Note that

$$\Pr(\xi \leq \alpha M) \leq c^{-M} \tag{33}$$

for some $\alpha > 0$, $c > 1$ depending only on p. This follows, for example, from the following estimate: for $1 \le k \le \frac{p(M+1)}{p+2}$

$$\frac{\Pr(\xi = k)}{\Pr(\xi = k-1)} = \frac{M-k+1}{k}\,\frac{p}{1-p} \ge \frac{\frac{2}{p+2}(M+1)}{\frac{p}{p+2}(M+1)}\,\frac{p}{1-p} \ge 2. \tag{34}$$

Let G be a random graph belonging to Gilbert's random graph model $G(n,p)$:

$$\begin{gathered}\forall_{1\le i<j\le n}\ \Pr(\{v_i, v_j\} \in EG) = p, \quad 0 < p < 1, \\ \text{(independently for each } \{i,j\}).\end{gathered} \tag{35}$$

For a subset of vertices $A \subset VG$, using (33), we find that

$$\Pr(|\partial A| \le \alpha |A|(n - |A|)) \le c^{-|A|(n-|A|)}. \tag{36}$$

Using (36), we get that

$$\begin{aligned}\Pr(i(G) \le \frac{\alpha n}{2}) &\le \sum_{A\subset VG,\ 0<|A|\le \frac{n}{2}} \Pr(|\partial A| \le \alpha |A| \frac{n}{2}) \le \\ &\le \sum_{k=1}^{n/2} \sum_{|A|=k} \Pr(|\partial A| \le \alpha |A|(n-|A|)) \le \\ &\le \sum_{k=1}^{n/2} \frac{n!}{k!(n-k)!}\, c^{-k(n-k)} \le \sum_{k=1}^{n/2} \frac{n!}{k!(n-k)!}\, (c^{-n/2})^k \le \\ &\le (1 + c^{-n/2})^n - 1 \le \beta^{-n}\end{aligned} \tag{37}$$

for some $\beta > 1$ depending only on p.

Due to (37) and Theorem 1, we obtain that probability for a random graph (in Gilbert's model $G(n,p)$) to be γ-mixing is at least $1-\beta^{-n}$, where $\gamma = \gamma(p) > 0$ and $\beta = \beta(p) > 1$.

4 Some basic properties and examples

We note that, due to (6b) and Theorem 1,

$$\begin{gathered}\text{if } \min_j d_j \ge \sigma |VG| \\ \text{for some } \sigma > 1/2\end{gathered} \Longrightarrow \begin{gathered}\text{the graph } G \text{ is } \gamma\text{-mixing} \\ \text{for some } \gamma = \gamma(\sigma) > 0.\end{gathered} \tag{38}$$

Example 1. Let K_n and $\tilde{K}_n$ be two complete graphs with n vertices. We define $G_n^{(1)}$ by

$$\begin{aligned}VG_n^{(1)} &= VK_n \cup V\tilde{K}_n, \\ EG_n^{(1)} &= EK_n \cup E\tilde{K}_n \cup E^+, \\ &\qquad \text{where } E^+ = \{\{v_i, \tilde{v}_i\}, i = 1, \dots n\}.\end{aligned} \tag{39}$$

For $G = G_n^{(1)}$ we have that for all $j = 1, \ldots, 2n$

$$d_j = n + 1 > \frac{1}{2}|VG_n^{(1)}|, \tag{40}$$

but

$$\frac{i(G_n^{(1)})}{n} \leq \frac{|E^+|}{n|VK_n|} \to 0, \quad \text{as } n \to \infty. \tag{41}$$

Thus the family $\{G_n^{(1)}\}$ does not satisfy Property 2 (and hence Properties 1,3). We note also that even the vertex and the edge connectivity is large for this family of graphs.

Example 2. Let K_n be the complete graph with n vertices. We define $G_n^{(2)}$ by

$$\begin{aligned} VG_n^{(2)} &= VK_n \cup v_{n+1}, \\ EG_n^{(2)} &= EK_n \cup \{v_n, v_{n+1}\}. \end{aligned} \tag{42}$$

For $G = G_n^{(2)}$ we have that $\min\limits_j d_j = 1$, but one can show that $\chi_2(G_n^{(2)}) = 1/\sqrt{n}$. Note also that Examples 1 and 2 show, in particular, that both conditions of Property 3 are necessary.

Example 3. Let G^0 be a connected simple graph with $m > 1$ vertices. Let $c_1, c_2, \ldots, c_m$ be some natural numbers. We define $G_n^{(3)}$ by

$$\begin{aligned} VG_n^{(3)} &= \{v_j^i : \ i = 1, \ldots, nc_j, \ j = 1, \ldots, m\}, \\ \{v_{j_1}^{i_1}, v_{j_2}^{i_2}\} \in EG_n^{(3)} &\Longleftrightarrow \{v_{j_1}, v_{j_2}\} \in EG^0. \end{aligned} \tag{43}$$

We estimate the Cheeger constant (isoperimetric number) $i(G_n^{(3)})$. Let

$$c_0 = \min_{1 \leq j \leq m} c_j, \quad C = \sum_{j=1}^{m} c_j \ . \tag{44}$$

We have that

$$\text{the degree of each vertex of } VG_n^{(3)} \text{ at least } c_0 n \tag{45a}$$

and

$$|VG_n^{(3)}| = Cn. \tag{45b}$$

Let $A \subset VG_n^{(3)}$, $|A| \leq Cn/2$.

- Case 1. $|A| \leq c_0 n/2$. Using (45), we find that

$$|\delta A| \geq c_0 n|A| - |A|^2 \geq \frac{c_0 n}{2}|A| = \frac{c_0 |VG_n^{(3)}|}{2C}|A| \tag{46}$$

- Case 2. $|A| > c_0 n/2$. Let $V_1, V_2 \subset VG_0$ such that

$$v_j \in V_1 \Longleftrightarrow |\left\{v_j^i : v_j^i \in A\right\}| \geq \frac{c_0 n}{2m}, \tag{47}$$

$$v_j \in V_2 \Longleftrightarrow |\left\{v_j^i : v_j^i \notin A\right\}| \geq \frac{c_j n}{2}. \tag{48}$$

Due to $c_0 n/2 < |A| \leq Cn/2$, we have that

$$V_1 \cup V_2 = VG_0, \quad |V_1| > 0, \quad |V_2| > 0. \tag{49}$$

Since G^0 is connected, we can find $v_{j_1} \in V_1$, $v_{j_2} \in V_2$ such that $\{v_{j_1}, v_{j_2}\} \in EG_0$. Estimating the number of edges in ∂A, matching these vertices, we obtain that

$$|\delta A| \geq \frac{c_0 c_{j_2} n^2}{4m} \geq \frac{c_0^2 n}{2mC}|A| = \frac{c_0^2 |VG_n^{(3)}|}{2mC^2}|A|. \tag{50}$$

Combining (46), (50) and Theorem 1, we obtain that the family $\left\{VG_n^{(3)}\right\}$ satisfies Properties 1-3 with $\gamma > 0$, depending only on G_0, $c_1, \ldots, c_m$.

We note that Example 3 can be modified so that the constants $c_1, \ldots, c_m > 0$ are not necessarily natural numbers. We assumed that just for simplicity of the proof.

Remark 1. Combining (8), Theorem 1 and Example 3, one can prove Properties 1-3 for a large number of classic examples (including $\{K_n\}$, $\{K_{n,n}\}$ and many others).

5 Asymptotic estimates for γ-mixing graphs

An *Eulerian orientation* of G is an orientation of its edges such that for every vertex the number of incoming edges and outgoing edges are equal. We denote $EO(G)$ the number of Eulerian orientations. Eulerian orientations of the complete graph K_n are called regular tournaments.

An *Eulerian circuit* in G is a closed walk which uses every edge of G exactly once. Let $EC(G)$ denote the number of these up to cyclic equivalence.

We recall that $EO(G) = EC(G) = 0$, if the degree of at least one vertex of G is odd (for more information see, for example, [1]). In this section we always assume that every vertex has even degree.

Let consider two enumeration problems: counting the number of Eulerian orientations and counting the number of Eulerian circuits in an undirected simple graph. It is known that both of these problems is complete for the class $\#P$, see [2], [9].

The results presented in this section are based on estimates of [3], [4]. We plan to give detailed proofs in a subsequent paper.

5.1 Eulerian orientations

We recall that the problem of counting Eulerian orientations can be reduced to counting perfect matching for some class of bipartitute graphs for which it can be done approximately with high probability in polynomial time, see [9]. However, the degree of the polynomial is large, so, in fact, these algorithms have a very big work time for the error term $O(n^{-1/2})$.

For γ-mixing graphs we have the following asymptotic formula:

Proposition 1. *Let G be an undirected simple graph with n vertices $v_1, v_2, \ldots, v_n$ having even degrees. Let G be a γ-mixing graph for some $\gamma > 0$. Then*

$$\begin{gathered} EO(G) = (1+\delta(G)) \left(2^{|EG|+\frac{n-1}{2}} \pi^{-\frac{n-1}{2}} \frac{1}{\sqrt{t(G)}} \prod_{\{v_j,v_k\}\in EG} P_{jk} \right), \\ P_{jk} = 1 - \frac{1}{4(d_j+1)^2} - \frac{1}{2(d_j+1)(d_k+1)} - \frac{1}{4(d_k+1)^2}, \end{gathered} \tag{51}$$

where d_j denotes the degree of vertex v_j, $t(G)$ denotes the number of spanning trees of the graph G and for any $\varepsilon > 0$

$$|\delta(G)| \le C n^{-1/2+\varepsilon}, \tag{52}$$

where constant $C > 0$ depends only on γ and ε.

Remark 2. We note that, according to the Kirchhoff Matrix-Tree-Theorem, see [7], we have that

$$t(G) = \frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n = \det M_{11}, \tag{53}$$

where M_{11} results from deleting the first row and the first column of Q.

Remark 3. For the complete graph $\lambda_2(K_n) = n$, $EK_n = \frac{n(n-1)}{2}$, $t(K_n) = n^{n-2}$,

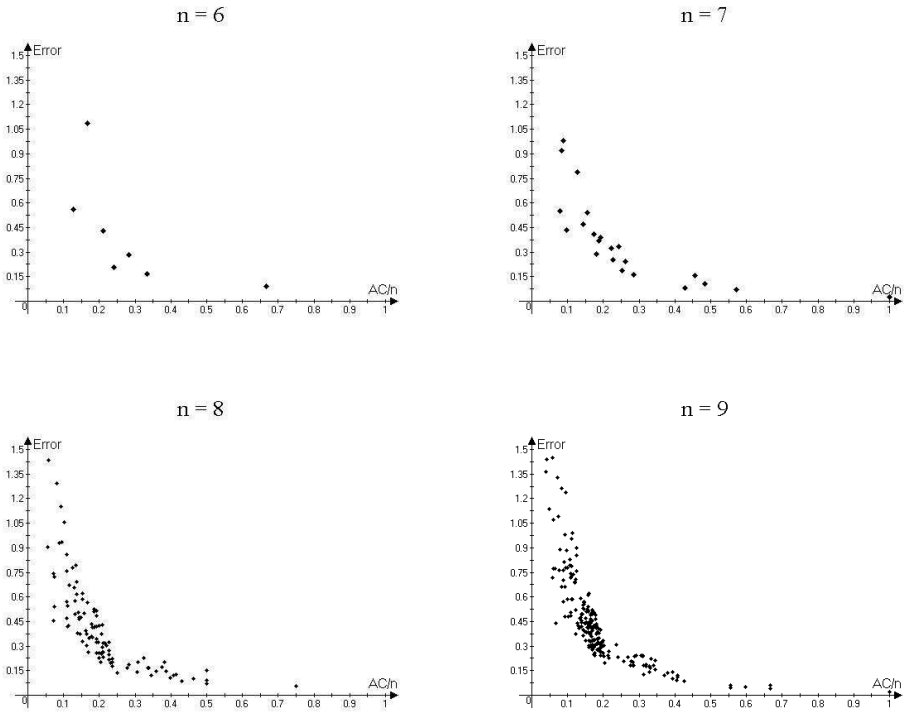
$$\begin{gathered} \prod_{\{v_j,v_k\}\in EK_n} P_{jk} = \left(1 - \frac{1}{4n^2} - \frac{1}{2n^2} - \frac{1}{4n^2}\right)^{\frac{n(n-1)}{2}} = \\ = \left(e^{\ln\left(1-\frac{1}{n^2}\right)}\right)^{\frac{n(n-1)}{2}} = e^{-1/2} + O(n^{-1}). \end{gathered} \tag{54}$$

The result of Proposition 1 for this case is reduced to the result of [11] on counting regular tournaments in the complete graph.

We plan to give the proof of Proposition 1 in a subsequent paper. In the present work we just compare the answers given by formula (51) against the exact values for small graphs. Let

$$Error(G) = \frac{EO_{approx}(G) - EO(G)}{EO(G)}, \tag{55}$$

where $EO_{approx}(G)$ is taken according to right-hand side of (51). The following charts show the dependence of $Error(G)$ on the ratio $\lambda_2(G)/n$, where $\lambda_2(G)$ is the algebraic connectivity and $n = |VG| = 6, 7, 8, 9$:



The charts show, in particular, that *Error* decreases significantly with respect to the ratio $\lambda_2(G)/n$.

5.2 Eulerian circuits

To our knowledge approximate polynomial algorithms for counting the number of Eulerian circuits have not yet been obtained in the literature (in contrast to the orientations). However, we have the formula for $EC(G)$ similar to (51) for γ-mixing graphs. This formula is more complicated, so we need some additional notations. Let

$$W = \hat{Q}^{-1} = (Q + J)^{-1}, \tag{56}$$

where Q is the Laplacian matrix and J denotes the matrix with every entry 1. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ be defined by

$$\alpha_j = W_{jj} \tag{57}$$

Let $\vec{\beta} = Q\vec{\alpha}$ and

$$C_1 = \prod_{j=1}^{n-1} \left(1 - \beta_j \sum_{k=j+1}^{n} W_{jk}\beta_k\right), \tag{58}$$

$$C_2 = \prod_{j=1}^{n} \left(1 - \frac{\beta_j^2}{2(d_j+1)}\right), \tag{59}$$

where d_j is the degree of $v_j \in VG$. Let

$$R(\vec{\theta}) = \mathrm{tr}(\Lambda(\vec{\theta})W\Lambda(\vec{\theta})W), \tag{60}$$

where $\mathrm{tr}(\cdot)$ is the trace fucntion, $\Lambda(\vec{\theta})$ denotes the diagonal matrix whose diagonal elements are equal to components of the vector $Q\vec{\theta}$. Let $\vec{e}^{(k)} = (e_1^{(k)}, \ldots, e_n^{(k)}) \in \mathbb{R}^n$ be defined by $e_j^{(k)} = \delta_{jk}$, where δ_{jk} is the Kronecker delta. Let $r_k = R(\vec{e}^{(k)})$,

$$C_3 = \prod_{j=1}^{n} \left(1 + \frac{r_j^2}{2(d_j+1)}\right). \tag{61}$$

Finally, let

$$C_4 = \prod_{\{v_j, v_k\} \in EG} P_{jk}, \tag{62}$$

where P_{jk} is the same as in (51).

Proposition 2. *Let G be an undirected simple graph with n vertices $v_1, v_2, \ldots, v_n$ having even degrees. Let G be a γ-mixing graph for some $\gamma > 0$. Then*

$$EC(G) = (1 + \delta'(G)) \left(\prod_{j=1}^{n} \left(\frac{d_j}{2} - 1\right)! \, 2^{|EG| - \frac{n-1}{2}} \pi^{-\frac{n-1}{2}} \sqrt{t(G)} C_1 C_2 C_3 C_4\right), \tag{63}$$

where C_1, C_2, C_3, C_4 are defined according to (58), (59), (61), (62)*, respectively, d_j is the degree of vertex v_j, $t(G)$ is the number of spanning trees of G and for any $\varepsilon > 0$*

$$|\delta'(G)| \leq C' n^{-1/2+\varepsilon}, \tag{64}$$

where constant $C' > 0$ depends only on γ and ε.

Remark 4. One can obtain for the case of $G = K_n$ that

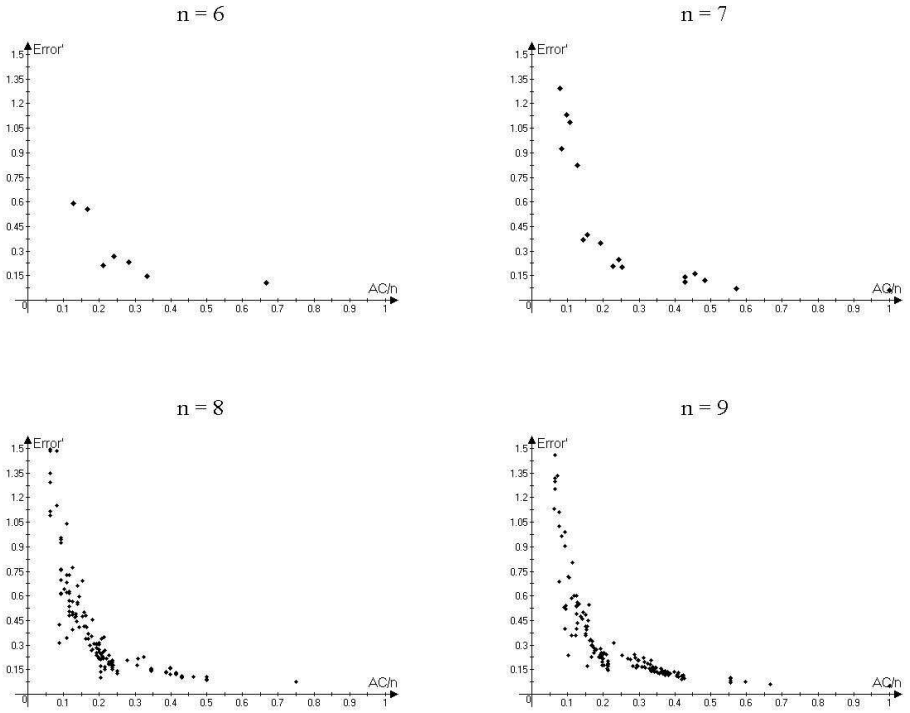
$$C_1 C_2 C_3 C_4 = 1 + O(n^{-1}) \tag{65}$$

Using Remark 5.2 and Stirling's formula for factorials, the result of Proposition 2 for this case can be reduced to the result of [10] on counting Eulerian circuits in the complete graph.

We plan to give the proof of Proposition 2 in a subsequent paper. In the present work we just compare the answers given by formula (63) against the exact values for small graphs. Let

$$Error'(G) = \frac{EC_{approx}(G) - EC(G)}{EC(G)}, \tag{66}$$

where $EC_{approx}(G)$ is taken according to right-hand side of (63). The following charts show the dependence of $Error'(G)$ on the ratio $\lambda_2(G)/n$, where $\lambda_2(G)$ is the algebraic connectivity and $n = |VG| = 6, 7, 8, 9$:



The charts show, in particular, that $Error'$ decreases significantly with respect to the ratio $\lambda_2(G)/n$.

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M.I. Isaev
Moscow Institute of Physics and Technology,
141700 Dolgoprudny, Russia
Centre de Mathématiques Appliquées, Ecole Polytechnique,
91128 Palaiseau, France
e-mail: Isaev.M.I@gmail.com

K.V. Isaeva
Moscow Institute of Physics and Technology,
141700 Dolgoprudny, Russia
e-mail: Isaeva.K.V@gmail.com