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# A Class of Generalized Laplacians on Vector Bundles Devoted to Multi-Channel Image Processing

Thomas Batard and Nir Sochen

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**Abstract** In the context of fibre bundles theory, there exist some differential operators of order 2, called generalized Laplacians, acting on sections of vector bundles over Riemannian manifolds, and generalizing the Laplace-Beltrami operator. Such operators are determined by covariant derivatives on vector bundles. In this paper, we construct a class of generalized Laplacians, devoted to multi-channel image processing, from the construction of optimal covariant derivatives. The key idea is to consider an image as a section of an associate bundle, that is a vector bundle related to a principal bundle through a group representation. In this context, covariant derivatives are determined by connection 1-forms on principal bundles. We construct optimal connection 1-forms by the minimization of a variational problem on principal bundles. From the heat equations of the generalized Laplacians induced by the corresponding optimal covariant derivatives, we obtain diffusions whose behaviours depend of the choice of the group representation. We provide experiments on color images.

**Keywords** Multi-Channel Image Processing · Variational Methods · Heat equation · Generalized Laplacian · Differential Geometry

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T. Batard, N. Sochen  
Department of Applied Mathematics  
Tel Aviv University  
Ramat Aviv, Tel Aviv 69978, Israel

T. Batard (corresponding author)  
E-mail: tomas.batard@gmail.com

N. Sochen  
E-mail: sochen@post.tau.ac.il

## 1 Introduction

Laplacians are widely used in image processing/analysis and computer vision. For instance, linear scale-space theory [20] is related to the heat equation of the Euclidean Laplacian. It is a fact that the isotropic behaviour of the linear scale-space is not well-adapted in many problems like denoising or contrast enhancement. Since then, many techniques of anisotropic diffusion of images have been employed (see e.g. [1],[9],[26],[34] and references therein). Some of them are based on the Laplace-Beltrami operator, a generalization of the Euclidean Laplacian on Riemannian manifold (see e.g. [24] for details about this operator).

In [27] was introduced the Beltrami framework, an efficient tool for anisotropic regularization of fields. Mathematically speaking, it is related to the heat equation of the Laplace-Beltrami operator. The Beltrami framework has been applied on different kinds of fields: vector-valued field (see e.g. [27],[28] for multi-channel image regularization), group-valued field (see [15] for  $SO(n)$ -valued field regularization), symmetric space-valued field (see [16] for regularization of field of symmetric definite positive matrices). It has also been applied as pre-processing for motion segmentation [25] or regularizing term for optical flow estimation [6]. There exist other uses of the Laplace-Beltrami operator. For instance, techniques for approximating its spectrum have been employed recently in the context of shape analysis (see e.g. [8],[22],[23],[32]).

Fiber bundles theory provides a geometric framework generalizing the notion of manifold (see [17] for an introduction to fiber bundles theory). In this context, there exists a class of differential operators of order 2 called generalized Laplacians, and acting on sections of vector bundles. They are determined by two geometric data: a

Riemannian metric on the base manifold and a covariant derivative on the vector bundle. In particular, the Euclidean Laplacian and Laplace-Beltrami operators are generalized Laplacians on vector bundles of rank 1 equipped with the trivial connection. We refer to [7] for more details about generalized Laplacians. Applications of generalized Laplacians for vector-valued field regularization were introduced in [2],[3]. In [3], it is shown that some standard methods of color image regularization, i.e. the Beltrami flow in [28], trace-based PDEs [30], curvature-preserving PDEs [31], and divergence-based PDEs (see [34] and the references therein) may be viewed as heat equations of generalized Laplacians.

In [4], we introduced the concept of  $(\rho, G)$ -equivariance on principal bundles for color images, where  $(\rho, G)$  is a Lie group representation (see e.g. [33] for an introduction to groups representations theory). We considered a Polyakov action measuring the energy of the graph of functions on principal bundles. We performed applications to image processing from the minimization of the functional with respect to the embedding. However, by the Riemannian metric we constructed on the principal bundle, it is not guaranteed that the gradient descent flow preserves the  $(\rho, G)$ -equivariance property of the initial condition, i.e. the original image. In this paper, we modify the Polyakov action in [4] in order to obtain a gradient descent flow that preserves the  $(\rho, G)$ -equivariance property of the initial condition:

1. We introduce an extra term involving the Casimir operator.
2. We construct the metric of the principal bundle from a metric on the base manifold and a horizontal bundle.

Under the identification between sections of associate bundles and  $(\rho, G)$ -equivariant functions on principal bundles, we show that the gradient descent flow of the new functional (that we call twisted Polyakov action) with respect to the embedding corresponds to the heat equation of a generalized Laplacian. In particular, the covariant derivative is determined by the horizontal bundle (or equivalently the connection 1-form) involved in the metric of the principal bundle.

The modification 2. of the Polyakov action raises the issue of the choice of the horizontal bundle involved in the metric of the principal bundle. By definition, horizontal bundles satisfy a  $G$ -equivariance property, meaning that they are completely determined by their values along a section of the principal bundle. Taking this property into account, we restrict the twisted Polyakov action along a section and minimize it with respect to

the horizontal bundle for fixed embedding and metric of the base manifold. We obtain a unique solution. Then, by the choice of a group representation, we obtain the expression of the corresponding covariant derivative on the associated bundle.

The notions of covariant derivative and connection 1-form are at the core of the differential geometry of fibre bundles [18],[29]. Whereas they are involved in many mathematical models in theoretical physics, as in general relativity and Gauge fields theories [14], they are involved in very few works in image processing: besides [2],[3] and this paper, we may refer to the work of Duits et al. with applications to crossing-preserving smoothing [13] as well as contour enhancement and completion [11],[12]. Both approaches deal with evolution equations associated with operators constructed from covariant derivatives but the geometric contexts are different. In our work the original field is a section of a vector bundle over a Riemannian manifold and the operator is a generalized Laplacian. In the work of Duits et al., the original field is a function defined on the Lie group  $SE(2)$  [11],[12] or  $SE(3)$  [13], and the differential operator is determined by left-invariant tangent vector fields, the left-invariance being related with a suitable covariant derivative constructed on the tangent bundle of the manifold  $SE(2)$  or  $SE(3)$ .

This paper is organized as follows. We first remind the main points of our former work [4] in Sect. 2. We discuss the  $(\rho, G)$ -equivariance property of color images and introduce a Polyakov action on principal bundles. In Sect. 3, we introduce the twisted Polyakov action that extends the Polyakov action in [4]. We first consider the minimization problem with respect to the embedding and show that the gradient descent flow preserves the  $(\rho, G)$ -equivariance property. In particular, we show that it can be identified with the heat equation of a generalized Laplacian. Then, we construct optimal horizontal fields from the minimization of the twisted Polyakov action with respect to the horizontal bundle along a section of the principal bundle. We give the expressions of the corresponding covariant derivatives on the associated vector bundles for the groups representations presented in Sect. 2. Details of computations are given in Appendices B and C. In Sect. 4, we present applications to image processing. We first compare the heat diffusions of generalized Laplacians constructed in Sect. 3 with results obtained in [4]. Then, we focus on the Lie group  $SO(3)$ . We show that the heat diffusion of the generalized Laplacian preserves edges and colors better than the heat diffusion of the Laplace-Beltrami operator. Definitions of the geometric concepts used throughout the paper are given in Appendix A.

## 2 Images as $(\rho, G)$ -equivariant functions on principal bundles

### 2.1 Construction

The key idea in [4] was to consider an image as a  $(\rho, G)$ -equivariant function on a principal bundle. In this paper, we make use of the identification between  $(\rho, G)$ -equivariant functions on principal bundles and sections of associated bundles.

**Definition 1** A smooth **principal bundle** is a quadruplet  $(P, \pi, M, G)$  where  $M$  and  $P$  are two  $C^\infty$  manifolds,  $G$  is a Lie group,  $\pi: P \rightarrow M$  is a surjective map such that for all  $x \in M$ , the preimage  $\pi^{-1}(x)$  is diffeomorphic to  $G$  and there is an action  $\cdot$  of  $G$  on  $P$  satisfying:  
-  $\pi(p \cdot g) = \pi(p)$  for  $p \in \pi^{-1}(x)$  and  $g \in G$ .  
- the restriction  $\cdot: G \times \pi^{-1}(x) \rightarrow \pi^{-1}(x)$  is free and transitive.

The principal bundle is said to be **trivial** if there exists a diffeomorphism  $\Phi: M \times G \rightarrow P$  satisfying  $\pi \circ \Phi(x, g) = x$ , and such that the restriction  $\Phi_x: G \rightarrow \pi^{-1}(x)$  is compatible with the action  $\cdot: G \times \pi^{-1}(x) \rightarrow \pi^{-1}(x)$ .

See Appendix A.2 for more details about principal bundles.

**Definition 2** Let  $G$  be a Lie group and  $V$  a vector space. A **representation**  $\rho$  of  $G$  on  $V$  is a group morphism  $\rho: G \rightarrow GL(V)$ . The representation  $\rho$  determines a group representation  $\rho^C$  on the dual space  $V^*$  of  $V$  called the **dual representation** of  $\rho$ , and defined by  $\rho^C(g) = \overline{\rho(g^{-1})}$  where  $\overline{\phantom{x}}$  denotes the transpose.

**Definition 3** Let  $(P, \pi, M, G)$  be a principal bundle and  $\rho$  a representation of  $G$  on a finite dimensional vector space  $V$ . A function  $J: P \rightarrow V$  is called  **$(\rho, G)$ -equivariant** if it satisfies

$$J(p \cdot g) = \rho(g^{-1})J(p) \quad (1)$$

Given a  $n$ -channels image  $I: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$ , we construct a  $(\rho, G)$ -equivariant function on a principal bundle as follows.

Let  $(e_1, \dots, e_n)$  be a basis of the vector space  $\mathbb{R}^n$ . Let  $(\rho, G)$  be a Lie group representation on  $\mathbb{R}^n$ , and  $e$  be the neutral element of  $G$ . Under its action on  $\mathbb{R}^n$ , the group  $G$  acts on the basis  $(e_1, \dots, e_n)$ . Let  $\mathcal{P}$  be the set of bases obtained by the transformations of  $(e_1, \dots, e_n)$  under the action of the group  $G$ , denoted by  $\cdot$ . The action of  $G$  on  $\mathcal{P}$  is transitive and free. Denoting by  $\pi$  the projection of  $\Omega \times \mathcal{P}$  on  $\Omega$  such that

$$\pi(x_1, x_2, g \cdot (e_1, \dots, e_n)) = (x_1, x_2),$$

the quadruplet  $P = (\Omega \times \mathcal{P}, \pi, \Omega, G)$  is a trivial principal bundle, where the global diffeomorphism  $\Phi: \Omega \times G \rightarrow \Omega \times \mathcal{P}$  is given by

$$\Phi(x_1, x_2, g) = (x_1, x_2, g \cdot (e_1, \dots, e_n)). \quad (2)$$

Then, we construct a  $(\rho, G)$ -equivariant function  $J$  on  $P$  defined by

$$J(x_1, x_2, g \cdot (e_1, \dots, e_n)) = \rho(g^{-1})I(x_1, x_2) \quad (3)$$

In particular we have

$$J(x_1, x_2, (e_1, \dots, e_n)) = I(x_1, x_2) \quad (4)$$

since  $\rho(e) = Id$  by property of a group representation.

We construct the associated bundle  $E = P \times_{(\rho, G)} \mathbb{R}^n$ . Under the correspondance between  $(\rho, G)$ -equivariant functions on  $P$  and sections of  $E$ ,  $J$  is a section of  $E$  of the form

$$J(x_1, x_2) = [(x_1, x_2, (e_1, \dots, e_n)), I(x_1, x_2)], \quad (5)$$

and  $I$  the realization of  $J$  under the trivializing section  $s$  of  $P$  defined by

$$s(x_1, x_2) = (x_1, x_2, (e_1, \dots, e_n)) = \Phi(x_1, x_2, e). \quad (6)$$

See Appendix A.3 for details about associated bundles.

### 2.2 Interpretation of the $(\rho, G)$ -equivariance property for color images

Dealing with color images, we interpret the fomula (4) as the assignment of the basis  $(e_1, e_2, e_3)$  to the light source of the image  $I$ . Then, we assimilate a basis change, given by the action  $g \mapsto g \cdot (e_1, e_2, e_3)$ , to a modification of the light source. By (3), the representation  $\rho$  encodes the way the pixels of the image change under this basis change, namely under the corresponding modification of the light source. Hence, by the  $(\rho, G)$ -equivariant function  $J$  that we construct, we take into account that some modifications of the light source induce transformations of the pixels. Note that the use of the fiber bundle context allows the modification of the light source to change with respect to the points of  $\Omega$ .

Increase resp. decrease the intensity of the light source makes increase resp. decrease the intensity of the pixels. Under the identification between modifications of the light source and transformations of the basis  $(e_1, e_2, e_3)$  that we made above, it means that color pixels transformations follow the basis transformations. From a mathematical viewpoint, this interpretation makes the color pixels behave as covectors.

In what follows, we give some examples of groups representations acting on the dual  $\mathbb{R}^{3*}$ , i.e. the space of covectors, of the vector space  $\mathbb{R}^3$ .

### 2.2.1 The HSL color space

Besides the RGB (Red,Green,Blue) color space, we consider a HSL (Hue,Saturation,Luminance) color space defined as follows. We set first

$$\begin{pmatrix} Y \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

Then the luminance  $l$ , the saturation  $s$  and the hue  $h$  components are respectively given by

$$\begin{aligned} l &= Y \\ s &= \sqrt{C_1^2 + C_2^2} \\ h &= \begin{cases} \arccos(C_2/s) & \text{if } C_2 > 0 \\ 2\pi - \arccos(C_2/s) & \text{otherwise} \end{cases} \end{aligned}$$

Embedding *RGB* into the quadratic space  $(\mathbb{R}^{3*}, \|\cdot\|_2)$  of basis  $(e^1, e^2, e^3)$ , we can decompose any color  $\alpha = \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3$  with respect to its projection  $L(\alpha)$  and its rejection  $V(\alpha)$  along the axis  $(e^1 + e^2 + e^3)$ . The axis  $(e^1 + e^2 + e^3)$  is called the **luminance axis**, it encodes the luminance information of a color. Indeed, the luminance of  $\alpha$  corresponds to the norm of its projection on the luminance axis. The orthogonal of the luminance axis in  $(\mathbb{R}^{3*}, \|\cdot\|_2)$  is called the **chrominance plane**, and encodes the saturation and hue information. The saturation of a color corresponds to the norm of its projection on the chrominance plane (up to multiplication by the scalar  $3/2$ ), and the hue corresponds to the angle it forms with the projection of the covector  $e^1$  on this plane.

### 2.2.2 Dual representation of $\mathbb{R}^{+*}$

Let  $v \in \mathbb{R}^{3*}$ . The dual representation of  $\mathbb{R}^{+*}$  on  $\mathbb{R}^{3*}$  is the map

$$\begin{aligned} \mathbb{R}^{+*} &\longrightarrow GL(\mathbb{R}^{3*}) \\ \rho^C: b &\longmapsto v \longmapsto 1/b v \end{aligned} \quad (7)$$

Simple computations lead to

$$l(\rho^C(b)\alpha) = \rho^C(b)l(\alpha) \quad (8)$$

$$s(\rho^C(b)\alpha) = \rho^C(b)s(\alpha) \quad (9)$$

$$h(\rho^C(b)\alpha) = h(\alpha) \quad (10)$$

The hue component is invariant under the action of the dual representation of  $\mathbb{R}^{+*}$  on  $\mathbb{R}^{3*}$  whereas the luminance and saturation components are modified.

### 2.2.3 Dual representations of $SO(2)$

Let  $v \in \mathbb{R}^{3*}$ . The dual representations of  $SO(2)$  on  $\mathbb{R}^{3*}$  are the maps

$$\begin{aligned} SO(2) &\longrightarrow GL(\mathbb{R}^{3*}) \\ \rho_{\mathcal{P}}^C: \theta &\longmapsto \text{rotation of angle } -\theta \text{ in the plane } \mathcal{P} \end{aligned} \quad (11)$$

where  $\mathcal{P}$  is a 2-dimensional subspace of  $\mathbb{R}^{3*}$ .

For  $\mathcal{P}$  being the chrominance plane *Chr*, we have

$$l(\rho_{Chr}^C(\theta)\alpha) = l(\alpha) \quad s(\rho_{Chr}^C(\theta)\alpha) = s(\alpha) \quad (12)$$

$$h(\rho_{Chr}^C(\theta)\alpha) = h(\alpha) - \theta \quad (13)$$

The luminance and saturation components are invariant under the action of the representation  $(\rho_{Chr}^C, SO(2))$ , whereas the hue component is modified.

### 2.2.4 Dual representation of $DC(3)$

The group  $DC(3)$  is the set of  $3 \times 3$  diagonal matrices of strictly positive coefficients. We denote by  $diag(b_1, b_2, b_3)$  the element of  $DC(3)$  of coefficients  $b_1, b_2, b_3$ .

Let  $v \in \mathbb{R}^{3*}$ . The dual representation of  $DC(3)$  on  $\mathbb{R}^{3*}$  is the map

$$\begin{aligned} DC(3) &\longrightarrow GL(\mathbb{R}^{3*}) \\ \rho^C: diag(b_1, b_2, b_3) &\longmapsto v \longmapsto diag(1/b_1, 1/b_2, 1/b_3)v \end{aligned} \quad (14)$$

All the components luminance, saturation and hue are modified under the action of the representation  $(\rho^C, DC(3))$  on  $\mathbb{R}^{3*}$ .

### 2.2.5 Dual representation of $SO(3)$

Let  $v \in \mathbb{R}^{3*}$  and  $R_{\theta_1, \theta_2, \theta_3}$  be the rotation associated to the Euler angles  $\theta_1, \theta_2, \theta_3$ . The dual representation of  $SO(3)$  on  $\mathbb{R}^{3*}$  is the map

$$\begin{aligned} SO(3) &\longrightarrow GL(\mathbb{R}^{3*}) \\ \rho^C: R_{\theta_1, \theta_2, \theta_3} &\longmapsto v \longmapsto R_{\theta_1, \theta_2, \theta_3} v \end{aligned} \quad (15)$$

All the components luminance, saturation and hue are modified under the action of the representation  $(\rho^C, SO(3))$  on  $\mathbb{R}^{3*}$ .

### 2.3 Polyakov action on principal bundles

We introduced in [4] a color image regularization method from variational problems on principal bundles. The geometric context is the following. Let  $(\Omega \times \mathcal{P}, \pi, \Omega, G)$  be a principal bundle constructed in Sect. 2.1, for some Lie group  $G$  studied in Sect. 2.2, and  $\rho^C$  be the dual representation of  $G$  on  $\mathbb{R}^{3^*}$ . Let  $h$  resp.  $Q$  be a Riemannian metric on  $\Omega \times \mathcal{P}$  resp.  $\Omega \times \mathcal{P} \times \mathbb{R}^{3^*}$ . Let  $J = (J^1, J^2, J^3): \Omega \times \mathcal{P} \rightarrow \mathbb{R}^{3^*}$  be a  $(\rho, G)$ -equivariant function and  $\varphi$  its graph. We considered the Polyakov action of the embedding  $\varphi: (\Omega \times \mathcal{P}, h) \rightarrow (\Omega \times \mathcal{P} \times \mathbb{R}^{3^*}, Q)$

$$S(h, \varphi, Q) = \int_{\Omega \times \mathcal{P}} h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} d(\Omega \times \mathcal{P}) \quad (16)$$

Assuming that  $Q$  is of the form  $\mathcal{H} \oplus \kappa \mathbb{I}_3$  for a Riemannian metric  $\mathcal{H}$  on  $\Omega \times \mathcal{P}$  and  $\kappa$  a strictly positive constant, where  $\mathbb{I}_3$  is the  $3 \times 3$  Identity matrix, we showed that the Euler-Lagrange equations of (16) with respect to the components  $J^i$  are

$$\Delta_h J^i = 0, \quad i = 1, 2, 3 \quad (17)$$

where  $\Delta_h$  is the Laplace-Beltrami operator of the Riemannian manifold  $(\Omega \times \mathcal{P}, h)$ .

If the embedded manifold is of dimension 2, it is well-known that the metric  $h$  minimizing the Polyakov action for fixed embedding  $\varphi$  and metric  $Q$  is the metric  $h^{ind}$  induced by  $Q$  on  $\Omega \times \mathcal{P}$  through  $\varphi$ . Based on this fact, we took the induced metric in our experiments, i.e. the metric  $h^{ind}$  given by

$$h_{\mu\nu}^{ind} = \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} \quad (18)$$

Then, we showed that the gradient descent flow

$$\frac{\partial J_t^i}{\partial t} = \Delta_{h^{ind}} J_t^i, \quad J_0^i = J^i \quad i = 1, 2, 3 \quad (19)$$

provides a regularization of the initial condition  $J$  whose behaviour depends of the chosen group representation.

The work in [4] raises the following two issues:

1. For embedded manifold of dimension greater than 2, is it not guaranteed that the induced metric minimizes the Polyakov action.
2. It is not guaranteed that the flow (19) preserves the  $(\rho^C, G)$ -equivariance property of the initial condition.

In this paper, we present the following three modifications of the work in [4].

- i. We introduce an extra term in the functional (16) involving the Casimir operator  $Cas \in \text{End}(\mathbb{R}^{3^*})$ .
- ii. We assume that the Riemannian metric  $h$  of the principal bundle  $\Omega \times \mathcal{P}$  is constructed from the data of a Riemannian metric  $g$  on  $\Omega$ , a horizontal bundle on  $\Omega \times \mathcal{P}$ , and a bi-invariant Riemannian metric  $\mathcal{B}$  on  $G$ .
- iii. The metrics  $h$  we consider in our experiments minimize a variational problem (Sect. 3.2).

The first two modifications make the gradient descent flow preserve the  $(\rho^C, G)$ -equivariance property of the initial condition (Sect. 3.1). The modification iii. makes the metrics involved in the experiments be optimal, unlike the ones used in [4].

### 3 Twisted Polyakov action on principal bundles

Let  $(P, \pi, M, G)$  be a principal bundle, where the base manifold  $M$  is of dimension  $m$  and the Lie group  $G$  is of dimension  $n$ . Let  $g$  and  $\mathcal{B}$  be respectively a Riemannian metric on  $M$  and a bi-invariant Riemannian metric on  $G$ . Let  $VP$  be the vertical bundle of  $P$  and  $HP$  be a horizontal bundle of  $P$  (see Definitions 10,11 in Appendix A.2). Let  $\rho$  be a representation of  $G$  on a finite dimensional vector space  $V$  of dimension  $d$  equipped with the standard scalar product  $(\cdot, \cdot)$ . Let  $J: P \rightarrow V$  be a  $(\rho, G)$ -equivariant function and  $\varphi$  its graph.

Let  $Cas \in \text{End}(V)$  be the Casimir operator, defined by

$$Cas = \sum \rho_*(\mathfrak{g}^i)^2 \quad (20)$$

for any orthonormal basis  $(\mathfrak{g}^1, \dots, \mathfrak{g}^n)$  of the Lie algebra  $\mathfrak{g}$  of  $G$ , where  $\rho_*: \mathfrak{g} \rightarrow \text{End}(V)$  is the Lie algebra representation induced by  $\rho$ .

We assume that  $P$  is equipped with a Riemannian metric  $h$  induced by a Riemannian metric  $g$  on  $M$ , a bi-invariant Riemannian metric  $\mathcal{B}$  on  $G$ , and a horizontal bundle  $HP$  (see the construction in Sect. 3.1.1). Moreover, we assume that  $P \times V$  is equipped with a Riemannian metric  $Q$  of the form  $\mathcal{H} \oplus \kappa \mathbb{I}_d$ , for a Riemannian metric  $\mathcal{H}$  on  $P$  and strictly positive constant  $\kappa$ , where  $\mathbb{I}_d$  denotes the  $d \times d$  Identity matrix.

We denote by  $S$  the Polyakov action of the embedding  $\varphi: (P, h) \rightarrow (P \times V, Q)$  twisted by an action of the Casimir operator

$$S(h, \varphi, Q) = \int_P h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} + \frac{1}{2} (Cas J, J) dP \quad (21)$$

In what follows, we consider two minimization problems derived from the functional (21).

### 3.1 Minimization with respect to the embedding

#### 3.1.1 Riemannian metric on principal bundle induced by a horizontal bundle

We construct a Riemannian metric  $h$  on  $P$  from  $g, \mathcal{B}$  and  $HP$  as follows.

Let  $(e_1, \dots, e_m)$  be an orthonormal frame of  $TM$ , the tangent bundle of  $M$ , with respect to  $g$ . Let  $(e_1^h, \dots, e_m^h)$  be the horizontal lift of  $(e_1, \dots, e_m)$  on  $TP$ , the tangent bundle of  $P$ , with respect to  $HP$  (see Definition 12 in Appendix A.2). Let  $(X_1, \dots, X_n)$  be an orthonormal basis of  $T_e G = \mathfrak{g}$  with respect to  $\mathcal{B}$ . Let  $(X_1^v, \dots, X_n^v)$  be the frame of  $VP$ , defined for  $i = 1, \dots, n$  by

$$X_i^v(p) = \left( \frac{d}{dt} p \cdot \exp(tX_i) \right)_{|t=0}$$

where  $\cdot$  denotes the action of  $G$  on the fibers of  $P$ . Then, considering the metric  $h$  on  $P$  given by the Identity matrix field in the frame  $(e_1^h, \dots, e_m^h, X_1^v, \dots, X_n^v)$  of  $TP$  turns  $(P, h)$  into a Riemannian manifold.

*Remark 1* The construction makes the bundles  $HP$  and  $VP$  be orthogonal with respect to  $h$ .

*Remark 2* By the  $G$ -equivariance property of the bundles  $HP$  and  $VP$  (see Definitions 10,11 in Appendix A.2), the metric  $h$  is completely determined by its values along a section of  $P$ .

#### 3.1.2 A gradient descent flow preserving the $(\rho, G)$ -equivariance property

Fixing  $h$  and  $Q$ , we aim at minimizing the functional (21) with respect to the  $(\rho, G)$ -equivariant function  $J$ . We obtain the following result.

**Proposition 1** *The gradient descent flow for reaching the function  $J_{opt}$  whose graph  $\varphi_{opt}$  minimizes the functional (21) preserves the  $(\rho, G)$ -equivariance property of the initial condition.*

*Proof* Let  $\psi$  be the embedding of a Riemannian manifold  $(M_1, g_1)$  into a Riemannian manifold  $(M_2, g_2)$ , and  $X$  the corresponding Polyakov action, i.e.

$$X = \int_{M_1} g_1^{\mu\nu} \frac{\partial \psi^i}{\partial x_\mu} \frac{\partial \psi^j}{\partial x_\nu} g_2^{ij} dM_1 \quad (22)$$

From [27], we know that the Euler-Lagrange equations of (22) with respect to  $\psi$  are

$$-\frac{1}{2\sqrt{g_1}} g_2^{i\ell} \frac{\partial X}{\partial \psi^\ell} = \Delta_{g_1} \psi^i + \Gamma_{jk}^i \partial_\mu \psi^j \partial_\nu \psi^k g_2^{\mu\nu} = 0 \quad (23)$$

where  $\Delta_{g_1}$  is the Laplace-Beltrami operator on  $(M_1, g_1)$  and  $\Gamma_{jk}^i$  are the symbols of the Levi-Civita connection on  $(M_2, g_2)$  (see Appendix A.1 for details about vector bundles, and Definition 7 in particular for the definition of covariant derivative, the case of the Levi-Civita connection being treated in Example 4).

Hence, taking a metric  $Q$  of the form  $\mathcal{H} \oplus \kappa \mathbb{I}_d$ , the Euler-Lagrange equations of (21) with respect to  $J$  are

$$(-\Delta_h \otimes 1 + 1 \otimes Cas) J = 0 \quad (24)$$

The gradient descent flow for reaching the solution of (24) is

$$\frac{\partial J_t}{\partial t} = (\Delta_h \otimes 1 - 1 \otimes Cas) J_t \quad (25)$$

Under the identification between  $(\rho, G)$ -equivariant functions on  $P$  and sections of  $P \times_{(\rho, G)} V$  (see Appendix A.3.1), the operator  $(\Delta_h \otimes 1 - 1 \otimes Cas)$  writes

$$\Delta^A := \sum_{ij} g^{ij} \left( \nabla_{\partial_{x_i}}^A \nabla_{\partial_{x_j}}^A - \sum_k \Gamma_{ij}^k \nabla_{\partial_{x_k}}^A \right) \quad (26)$$

The operator  $\Delta_A$  is a generalized Laplacian on  $P \times_{(\rho, G)} V$ , where the covariant derivative  $\nabla^A$  is induced by  $HP$  (see Appendix A.3.2). We refer to Prop. 5.6 p. 172 in [7] for the proof of the identification between the operators  $(\Delta_h \otimes 1 - 1 \otimes Cas)$  and  $\Delta^A$ .

Then, taking the  $(\rho, G)$ -equivariant function  $J$  as initial condition, the gradient descent flow (25) may be identified with the heat flow of  $\Delta^A$

$$\frac{\partial J_t}{\partial t} = \Delta^A J_t, \quad J_0 = J \quad (27)$$

As the flow (27) preserves the structure of section of associated bundle of the initial condition, we deduce that the flow (25) preserves the  $(\rho, G)$ -equivariance property of the initial condition.  $\square$

### 3.2 Minimization with respect to the horizontal bundle

Fixing  $J, Q, g$  and  $\mathcal{B}$ , we aim at computing the horizontal bundles  $H_s P^{opt}$  minimizing the energy (21) along a section  $s$  of  $P$ , i.e. minimizing the energy

$$S|_s(h, \varphi, Q) = \int_s h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} + \frac{1}{2} (Cas J, J) dM \quad (28)$$

Then,  $H_s P^{opt}$  extends to a horizontal bundle  $HP^{opt}$  by  $G$ -equivariance property. However, this construction does not prove that  $HP^{opt}$  minimizes the energy (21).

In what follows, we first remind the relation between horizontal bundle and covariant derivative on associated bundle. Then, we restrict to the case of base manifolds of dimension 2 and  $s$  being the section defined by  $s(x) = \Phi(x, e)$  where  $\Phi$  is a trivialization of  $P$ . We give explicit expressions of the covariant derivatives  $\nabla^{A^{opt}}$  induced by  $H_s P^{opt}$  for the dual representations of the group  $\mathbb{R}^{+*}$ ,  $SO(2)$ ,  $DC(3)$  and  $SO(3)$  on  $\mathbb{R}^{3*}$  (see Sect. 2.2). We refer to Appendices B and C for details of computations.

### 3.2.1 From horizontal bundles to covariant derivatives on associated bundles: the general case

Let  $(P, \pi, M, G)$  be a principal bundle,  $\rho$  a representation of  $G$  on a finite dimensional vector space  $V$ , and  $E = P \times_{(\rho, G)} V$  the associated bundle. Let  $S \in \Gamma(E)$  and  $f_S$  be the corresponding  $(\rho, G)$ -equivariant function on  $P$ . Let  $HP$  be a horizontal bundle of  $P$ . Let  $X \in \Gamma(TM)$  and  $X^h \in \Gamma(TP)$  be the horizontal lift of  $X$  with respect to  $HP$ .

Then, there exist a covariant derivative  $\nabla^A$  on  $E$  such that the following correspondance holds

$$d_{X^h} f_S \longleftrightarrow \nabla_X^A S \quad (29)$$

A proof of the correspondance (29) can be found p. 24 in [7].

Let  $\omega$  be the connection 1-form induced by  $HP$  (see Definition 13 Appendix A.2). Let  $s$  be a section of  $P$  and  $A$  be the  $\mathfrak{g}$ -valued 1-form on  $M$  given by

$$A = s^* \omega$$

where  $*$  denotes the pull-back map.

Then, relatively to the local frame of  $E$  induced by  $s$ , the covariant derivative is given by

$$\nabla_X^A = d_X + \rho_*(A)(X)$$

where  $d_X$  denotes the differential with respect to  $X \in \Gamma(TM)$ .

### 3.2.2 Explicit expressions of covariant derivatives induced by optimal horizontal fields

Let  $(x_1, x_2)$  resp.  $(a_1, \dots, a_n)$  be a coordinates system of  $M$  resp.  $G$ .

#### Dual representation of $\mathbb{R}^{+*}$ on $\mathbb{R}^{3*}$

**Proposition 2** *The field of horizontal spaces  $H_s P^{opt}$  of  $(P, \pi, M, \mathbb{R}^{+*})$  minimizing (21) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative  $\nabla^{A^{opt}}$  in the associated bundle  $P \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^{3*}$  are given, relatively to the frame  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and the frame of  $P \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^{3*}$  induced by  $s$ , by*

$$\Upsilon_{lk}^k(x) = \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) \quad (30)$$

for  $l = 1, 2$  and  $k = 1, 2, 3$ .

#### Dual representations of $SO(2)$ on $\mathbb{R}^{3*}$

The dual representations of  $SO(2)$  on  $\mathbb{R}^{3*}$  are parametrized by 2-dimensional subspaces of  $\mathbb{R}^{3*}$  (see Sect. 2.2.3). In the Clifford algebras context, a 2-dimensional subspace  $\mathcal{P}$  of  $\mathbb{R}^{3*}$  can be described by a bivector  $\gamma_{12}e^1e^2 + \gamma_{13}e^1e^3 + \gamma_{23}e^2e^3$ , for some  $\gamma_{12}, \gamma_{13}, \gamma_{23} \in \mathbb{R}$  such that  $\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 = 1$ .

**Proposition 3** *The field of horizontal spaces  $H_s P^{opt}$  of  $(P, \pi, M, SO(2))$  minimizing (21) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative  $\nabla^{A^{opt}}$  in the associated bundle  $P \times_{(\rho_P^C, SO(2))} \mathbb{R}^{3*}$  are given, relatively to the frame  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and the frame of  $P \times_{(\rho_P^C, SO(2))} \mathbb{R}^{3*}$  induced by  $s$ , by*

$$\begin{aligned} \Upsilon_{lp}^k(x) &= -\Upsilon_{lk}^p(x) \\ &= (-1)^{l+1} \gamma_{kp} \left( \frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) \end{aligned} \quad (31)$$

for  $l = 1, 2$  and  $k, p = 1, 2, 3$  with  $k \neq p$ .

#### Dual representation of $DC(3)$ on $\mathbb{R}^{3*}$

**Proposition 4** *The field of horizontal spaces  $H_s P^{opt}$  of  $(P, \pi, M, DC(3))$  minimizing (21) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative  $\nabla^{A^{opt}}$  in the associated bundle  $P \times_{(\rho^C, DC(3))} \mathbb{R}^{3*}$  are given, relatively to the frame  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and the frame of  $P \times_{(\rho^C, DC(3))} \mathbb{R}^{3*}$  induced by  $s$ , by*

$$\Upsilon_{l1}^1(x) = \frac{1}{c_1 c_2 c_3} \left[ c_2 c_3 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right](x, e) \quad (32)$$

$$\Upsilon_{l2}^2(x) = \frac{1}{c_1 c_2 c_3} \left[ c_1 c_3 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right](x, e) \quad (33)$$

$$\Upsilon_{l3}^3(x) = \frac{1}{c_1 c_2 c_3} \left[ c_1 c_2 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right](x, e) \quad (34)$$

for  $l = 1, 2$ , where

$$c_k = \frac{\partial \varphi^i}{\partial a_k} \frac{\partial \varphi^j}{\partial a_k} Q_{ij}, \quad k = 1, 2, 3 \quad (35)$$



Dual representation of  $SO(3)$  on  $\mathbb{R}^{3^*}$

**Proposition 5** *The field of horizontal spaces  $H_s P^{opt}$  of  $(P, \pi, M, SO(3))$  minimizing (21) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative  $\nabla^{A^{opt}}$  in the associated bundle  $P \times_{(\rho^C, SO(3))} \mathbb{R}^{3^*}$  are given, relatively to the frame  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and the frame of  $P \times_{(\rho^C, SO(3))} \mathbb{R}^{3^*}$  induced by  $s$ , by*

$$\begin{aligned} \Upsilon_{12}^1(x) &= -\Upsilon_{11}^2(x) \\ &= \lambda \left[ (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned} \quad (36)$$

$$\begin{aligned} \Upsilon_{13}^1(x) &= -\Upsilon_{11}^3(x) \\ &= \lambda \left[ (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned} \quad (37)$$

$$\begin{aligned} \Upsilon_{13}^2(x) &= -\Upsilon_{12}^3(x) \\ &= \lambda \left[ (fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned} \quad (38)$$

for  $l = 1, 2$ , where

$$a = \frac{\partial \varphi^i}{\partial \theta_1} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} \quad b = \frac{\partial \varphi^i}{\partial \theta_1} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \quad (39)$$

$$c = \frac{\partial \varphi^i}{\partial \theta_1} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \quad d = \frac{\partial \varphi^i}{\partial \theta_2} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \quad (40)$$

$$e = \frac{\partial \varphi^i}{\partial \theta_2} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \quad f = \frac{\partial \varphi^i}{\partial \theta_3} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \quad (41)$$

and

$$\lambda = \frac{1}{2bce + adf - ae^2 - b^2f - c^2d} \quad (42)$$

## 4 Experiments

4.1 Some heat diffusions applied on color images

### 4.1.1 The geometric framework

Let  $I: \Omega \longrightarrow RGB \subset \mathbb{R}^{3^*}$  be a color image, and  $(\rho, G)$  be a Lie group representation on  $\mathbb{R}^{3^*}$ . Following Sect. 2.1, we construct a  $(\rho, G)$ -equivariant function  $J$  on a trivial principal bundle diffeomorphic to  $(\Omega \times G, \pi, \Omega, G)$  such that  $J(x_1, x_2, e) = I(x_1, x_2)$  (up to the trivialization  $\Phi$ ). Moreover, we construct a Riemannian metric on the space  $\Omega \times G \times \mathbb{R}^{3^*}$ .

We consider the following three groups representations and Riemannian metrics:

i. the dual representation  $\rho^C$  of  $\mathbb{R}^{+*}$  on  $\mathbb{R}^{3^*}$ : we have

$$\begin{aligned} J: (x_1, x_2, \exp(a)) &\longmapsto \\ &(\exp(a)I^1(x_1, x_2), \exp(a)I^2(x_1, x_2), \exp(a)I^3(x_1, x_2)) \end{aligned} \quad (43)$$

The graph of the function  $J$ , defined by

$$\varphi: (x_1, x_2, \exp(a)) \longmapsto (x_1, x_2, \exp(a), J(x_1, x_2, \exp(a))), \quad (44)$$

realizes an embedding of  $\Omega \times \mathbb{R}^{+*}$  into  $\Omega \times \mathbb{R}^{+*} \times \mathbb{R}^{3^*}$ .

We equip the embedding space  $\Omega \times \mathbb{R}^{+*} \times \mathbb{R}^{3^*}$  of the Riemannian metric  $Q$  given by

$$\mathbb{I}_2 \oplus \delta \oplus \kappa \mathbb{I}_3, \quad \delta, \kappa > 0 \quad (45)$$

in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a, \partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3)$ .

ii. the dual representation  $\rho^C$  of  $DC(3)$  on  $\mathbb{R}^{3^*}$ : we have

$$\begin{aligned} J: (x_1, x_2, \exp(a_1), \exp(a_2), \exp(a_3)) &\longmapsto \\ &(\exp(a_1)I^1(x_1, x_2), \exp(a_2)I^2(x_1, x_2), \exp(a_3)I^3(x_1, x_2)) \end{aligned} \quad (46)$$

The graph of the function  $J$ , defined by

$$\begin{aligned} \varphi: (x_1, x_2, \exp(a_1), \exp(a_2), \exp(a_3)) &\longmapsto (x_1, x_2, \exp(a_1), \\ &(\exp(a_2), \exp(a_3), J(x_1, x_2, \exp(a_1), \exp(a_2), \exp(a_3))), \end{aligned} \quad (47)$$

realizes an embedding of  $\Omega \times DC(3)$  into  $\Omega \times DC(3) \times \mathbb{R}^{3^*}$ .

We equip the embedding space  $\Omega \times \text{DC}(3) \times \mathbb{R}^{3*}$  of the Riemannian metric  $Q$ , given by

$$\mathbb{I}_2 \oplus \delta \mathbb{I}_3 \oplus \kappa \mathbb{I}_3, \quad \delta, \kappa > 0 \quad (48)$$

in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3, \partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3)$ .

iii. the dual representation  $\rho^C$  of  $\text{SO}(3)$  on  $\mathbb{R}^{3*}$ : we have

$$J(x_1, x_2, \theta_1, \theta_2, \theta_3) = R_{\theta_1, \theta_2, \theta_3}^T (I^1(x_1, x_2), I^2(x_1, x_2), I^3(x_1, x_2))^T \quad (49)$$

where  $R_{\theta_1, \theta_2, \theta_3} =$

$$\begin{pmatrix} \cos \theta_2 \cos \theta_3 & \cos \theta_2 \sin \theta_3 & \sin \theta_2 \\ -\cos \theta_1 \sin \theta_3 & \cos \theta_3 \cos \theta_1 & \cos \theta_2 \sin \theta_1 \\ -\cos \theta_3 \sin \theta_2 \sin \theta_1 & -\sin \theta_2 \sin \theta_3 \sin \theta_1 \\ -\cos \theta_3 \cos \theta_1 \sin \theta_2 & -\cos \theta_1 \sin \theta_2 \sin \theta_3 & \cos \theta_2 \cos \theta_1 \\ +\sin \theta_3 \sin \theta_1 & -\cos \theta_3 \sin \theta_1 \end{pmatrix}$$

The graph of the function  $J$ , defined by

$$\varphi: (x_1, x_2, \theta_1, \theta_2, \theta_3) \mapsto (x_1, x_2, \theta_1, \theta_2, \theta_3, J(x_1, x_2, \theta_1, \theta_2, \theta_3)), \quad (50)$$

realizes an embedding of  $\Omega \times \text{SO}(3)$  into  $\Omega \times \text{SO}(3) \times \mathbb{R}^{3*}$ .

We equip the embedding space  $\Omega \times \text{SO}(3) \times \mathbb{R}^{3*}$  of the Riemannian metric  $Q$  of the form

$$Q = \mathbb{I}_2 \oplus \begin{pmatrix} \delta & 0 & \delta \sin \theta_2 \\ 0 & \delta & 0 \\ \delta \sin \theta_2 & 0 & \delta \end{pmatrix} \oplus \kappa \mathbb{I}_3, \quad \delta, \kappa > 0 \quad (51)$$

in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \theta_1, \partial/\partial \theta_2, \partial/\partial \theta_3, \partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3)$ .

Then, for each group representation  $(\rho^C, G)$  and metric  $Q$  on  $\Omega \times G \times \mathbb{R}^{3*}$  given above, we compute the gradient descent flows for reaching the functions  $J$  minimizing the following functionals with respect to the graph  $\varphi$ :

1. The Polyakov action (16) where the metric  $h$  of the principal bundle is induced by the metric  $Q$  of the embedding space. It corresponds to what it is done in [4].
2. The twisted Polyakov action (21) where the metric  $h$  of the principal bundle is induced by the metric  $Q$  of the embedding space.
3. The twisted Polyakov action (21) where the metric  $h$  of the principal bundle is constructed from a metric  $g$  on the base manifold and an optimal horizontal field  $H_s P^{opt}$  along the section  $s$  (see Sect. 3.2).

Details of computations are given in what follows.

#### 4.1.2 Polyakov action with metric $h$ induced by the metric of the embedding space

Taking a metric  $Q$  constructed in Sect. 4.1.1, the minimization of the Polyakov action (16) with respect to the graph  $\varphi$  yields the heat equations

$$\frac{\partial J_t^k}{\partial t} = \Delta_h J_t^k, \quad k = 1, 2, 3 \quad (52)$$

(see Sect. 2.3 for details). These are heat equations of the Laplace-Beltrami operator  $\Delta_h$ , defined by

$$\Delta_h := \frac{1}{\sqrt{h}} \partial_\mu \left( \sqrt{h} h^{\mu\nu} \partial_\nu \right), \quad \mu, \nu = 1, \dots, 2 + \dim G$$

The general expression for the induced metric  $h^{ind}$  is

$$h_{\mu\nu}^{ind} = \frac{\partial \varphi^i}{\partial x^\mu} \frac{\partial \varphi^j}{\partial x^\nu} Q_{ij} \quad (53)$$

for  $\mu, \nu = 1, \dots, 2 + \dim G$  and  $i, j = 1, \dots, 5 + \dim G$ .

The explicit expressions of  $h^{ind}$  for the groups representations and metrics  $Q$  of Sect. 4.1.1 are given in [4].

We perform an experiment in the end of the paper where we compute a discrete approximation of the solutions of (52) for  $h = h^{ind}$ , at the points  $(x_1, x_2, e)$ , with the following scheme

$$J_{t+dt}^k(x_1, x_2, e) = J_t^k(x_1, x_2, e) + dt \Delta_{h^{ind}} J_t^k(x_1, x_2, e) \quad (54)$$

of initial condition  $J_0^k(x_1, x_2, e) = I^k(x_1, x_2)$ .

In order to discretize the terms  $\Delta_{h^{ind}} J^k$  in (54), we extend the method in [27] to manifolds of dimension greater than 2. This method holds for any metric  $h$ . In the case of the manifold is of dimension 3, with coordinates system  $(x_1, x_2, x_3)$ , the algorithm is the following:

1. Compute the matrices  $p^k, q^k, r^k$  given by

$$p^k = (h_{22}h_{33} - (h_{23})^2) \partial_{x_1} J_t^k + (h_{13}h_{23} - h_{12}h_{33}) \partial_{x_2} J_t^k$$

$$+ (h_{12}h_{33} - h_{13}h_{22}) \partial_{x_3} J_t^k$$

$$q^k = (h_{13}h_{23} - h_{12}h_{33}) \partial_{x_1} J_t^k + (h_{11}h_{33} - (h_{13})^2) \partial_{x_2} J_t^k$$

$$+ (h_{13}h_{12} - h_{11}h_{23}) \partial_{x_3} J_t^k$$

$$r^k = (h_{12}h_{23} - h_{13}h_{22}) \partial_{x_1} J_t^k + (h_{12}h_{13} - h_{11}h_{23}) \partial_{x_2} J_t^k$$

$$+ (h_{11}h_{22} - (h_{12})^2) \partial_{x_3} J_t^k$$

2. Compute  $\Delta_h J_t^k :=$

$$\frac{1}{h} (\partial_{x_1} p^k + \partial_{x_2} q^k + \partial_{x_3} r^k) - \frac{1}{2h^2} (p^k \partial_{x_1} h + q^k \partial_{x_2} h + r^k \partial_{x_3} h)$$

See Fig. 1 for results and Sect. 4.2.1 for an interpretation of the results.

#### 4.1.3 Twisted Polyakov action with metric $h$ induced by the metric of the embedding space

Taking a metric  $Q$  constructed in Sect. 4.1.1, the minimization of the twisted Polyakov action (21) with respect to the graph  $\varphi$  yields the gradient descent flow

$$\frac{\partial J_t}{\partial t} = (H_h - Cas)J_t \quad (55)$$

where  $H_h$  stands for the operator acting as the Laplace-Beltrami operator  $\Delta_h$  on each component of the section. Formula (55) is a direct consequence of formula (52).

The definition of the Casimir operator is given in (20).

In what follows we give the explicit expressions of the Casimir operators for the Lie groups representations of Sect. 4.1.1.

*Dual representation of  $\mathbb{R}^{++}$  on  $\mathbb{R}^{3*}$*  Let  $\mathcal{B}$  be a Riemannian metric on  $\mathbb{R}^{++}$  given by the strictly positive constant  $\delta$  in the frame  $(\partial/\partial a)$  at the point 1. Let  $E \in T_1\mathbb{R}^{++}$  of component  $1/\sqrt{\delta}$  in the frame  $(\partial/\partial a)$ . We have  $\|E\|_{\mathcal{B}} = 1$  and

$$\rho_*^C(E) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It gives

$$Cas := \rho_*^C(E)^2 = (1/\delta)Id \quad (56)$$

*Dual representation of  $DC(3)$  on  $\mathbb{R}^{3*}$*  Let  $\mathcal{B}$  be a Riemannian metric on  $DC(3)$  given by  $\delta\mathbb{I}_3$  in the frame  $(\partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3)$  at the neutral element  $e$  of  $DC(3)$ . Let  $E_1$  resp.  $E_2$  resp.  $E_3 \in T_e DC(3)$  of components  $(1/\sqrt{\delta}, 0, 0)$  resp.  $(0, 1/\sqrt{\delta}, 0)$  resp.  $(0, 0, 1/\sqrt{\delta})$  in the frame  $(\partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3)$ . The set  $(E_1, E_2, E_3)$  forms an orthonormal basis of  $T_e DC(3)$  for the metric  $\mathcal{B}$  and we have

$$\rho_*^C(E_1) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho_*^C(E_2) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rho_*^C(E_3) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It gives

$$Cas := \rho_*^C(E_1)^2 + \rho_*^C(E_2)^2 + \rho_*^C(E_3)^2 = (1/\delta)Id \quad (57)$$

*Dual representation of  $SO(3)$  on  $\mathbb{R}^{3*}$*  Let  $\mathcal{B}$  be a Riemannian metric on  $SO(3)$  given by  $\delta\mathbb{I}_3$  in the frame  $(\partial/\partial\theta_1, \partial/\partial\theta_2, \partial/\partial\theta_3)$  at the neutral element  $e$  of  $SO(3)$ . Let  $E_1$  resp.  $E_2$  resp.  $E_3 \in T_e SO(3)$  of components  $(1/\sqrt{\delta}, 0, 0)$  resp.  $(0, 1/\sqrt{\delta}, 0)$  resp.  $(0, 0, 1/\sqrt{\delta})$  in the frame  $(\partial/\partial\theta_1, \partial/\partial\theta_2, \partial/\partial\theta_3)$ . The set  $(E_1, E_2, E_3)$  forms an orthonormal basis of  $T_e SO(3)$  for the metric  $\mathcal{B}$  and we have

$$\rho_*^C(E_1) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \rho_*^C(E_2) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rho_*^C(E_3) = \frac{1}{\sqrt{\delta}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It gives

$$Cas := \rho_*^C(E_1)^2 + \rho_*^C(E_2)^2 + \rho_*^C(E_3)^2 = (-2/\delta)Id \quad (58)$$

We perform an experiment in the end of the paper where we compute a discrete approximation of the solutions of (55) for  $h = h^{ind}$  (see (53)), at the points  $(x_1, x_2, e)$ , with the following scheme

$$J_{t+dt}(x_1, x_2, e) = J_t(x_1, x_2, e) + dt(H_{h^{ind}} - Cas)J_t(x_1, x_2, e) \quad (59)$$

of initial condition  $J_0(x_1, x_2, e) = I(x_1, x_2)$ .

We discretize the term  $H_{h^{ind}}$  following the method in Sect. 4.1.2.

See Fig. 1 for results and Sect. 4.2.1 for an interpretation of the results.

#### 4.1.4 Twisted Polyakov action with metric $h$ constructed from a metric on the base manifold and an optimal horizontal field

Taking a metric  $Q$  constructed in Sect. 4.1.1, the minimization of the twisted Polyakov action (21) with respect to the graph  $\varphi$  yields the heat equation

$$\frac{\partial J_t}{\partial t} = \Delta^{A^{opt}} J_t \quad (60)$$

of the generalized Laplacian  $\Delta^{A^{opt}}$  associated to the covariant derivative  $\nabla^{A^{opt}}$  induced by the optimal horizontal field  $H_s P^{opt}$  (see Sect. 3).

We perform experiments in the end of the paper where we compute a discrete approximation of the solution of (60) in the frame induced by the trivializing section  $s$ , taking  $I$  as initial condition. We consider the scheme

$$J_{t+dt} = J_t + dt \Delta^{A^{opt}} J_t, \quad J_0 = I \quad (61)$$

We compute the term  $\Delta^{A^{opt}} J_t$  in (61) from the formula

$$\Delta^{A^{opt}} := g^{ij} \left( \nabla_{\partial_{x_i}}^{A^{opt}} \nabla_{\partial_{x_j}}^{A^{opt}} - \sum_{k=1}^2 \Gamma_{ij}^k \nabla_{\partial_{x_k}}^{A^{opt}} \right), \quad i, j = 1, 2 \quad (62)$$

and the terms in (62) as follows:

1. We compute the covariant derivatives  $\nabla_{\partial_{x_l}}^{A^{opt}} J_t, l = 1, 2$ , from the explicit expressions of the symbols  $\Upsilon_{ij}^k$ , given in the Appendix:

- dual representation of  $\mathbb{R}^{+*}$  on  $\mathbb{R}^{3*}$ : Appendix B.1.2
- dual representation of  $\text{DC}(3)$  on  $\mathbb{R}^{3*}$ : Appendix B.2.2
- dual representation of  $\text{SO}(3)$  on  $\mathbb{R}^{3*}$ : Appendix B.2.3.

and the formula

$$\nabla_{\partial_{x_l}}^{A^{opt}} J_t = \sum_{i=1}^3 \sum_{k=1}^3 \partial_{x_l} J_t^i e^i + J_t^i \Upsilon_{li}^k e^k \quad (63)$$

2. We take  $g$  given by the matrix representation

$$\begin{pmatrix} 1 + \kappa \sum_{k=1}^3 I_{x_1}^k & \kappa \sum_{k=1}^3 I_{x_1}^k I_{x_2}^k \\ \kappa \sum_{k=1}^3 I_{x_1}^k I_{x_2}^k & 1 + \kappa \sum_{k=1}^3 I_{x_2}^k \end{pmatrix} \quad (64)$$

in the frame  $(\partial/\partial x_1, \partial/\partial x_2)$ , where  $\kappa$  is the constant that appears in the expression of  $Q$  (see Sect. 4.1.1).

3. We compute the symbols  $\Gamma_{ij}^k$  of the Levi-Civita connection associated to  $g$  from the formula

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_{x_j} g_{li} + \partial_{x_i} g_{lj} - \partial_{x_l} g_{ij}) \quad (65)$$

See Fig. 2,3,4 for results and Sect. 4.2.2,4.2.3,4.2.4 for interpretations of the results.

## 4.2 Interpretations of the results

### 4.2.1 Induced metrics and Casimir operators

As mentioned in Sect. 2.3, we provide two modifications of the work in [4]. The first one concerns the introduction of a term involving the Casimir operator, that gives the twisted Polyakov functional (21).

On Fig. 1, we compare the gradient descent flow (55) of the twisted Polyakov action (21) with the gradient descent flow (52) of the Polyakov action (16). We compute 20 iterations of the numerical schemes (59) and (54), for  $\kappa = 0.0001$ ,  $\delta = 2$ ,  $dt = 0.1$ , tested on the image 'kodim07' (Fig. 1(a)) taken in the kodak database

[19]. As we can observe in [4], the diffusion (54) combines the anisotropic smoothing of the heat diffusion of the Laplace-Beltrami operator  $\Delta_g$ , for  $g$  given in (64) (see Fig. 1(b)), and a global transformation of colors that depends of the Lie group involved. More precisely, these global transformations are related with the actions of the representations  $\rho^C$  on the color space  $RGB$  (see Sect. 2.2). Indeed, for  $G = \mathbb{R}^{+*}$ , both components luminance and saturation are increased. This is typically what happens when  $\mathbb{R}^{+*}$  acts on  $RGB$  with a parameter greater than 1. The reddish aspect of the diffusion for  $G = \text{SO}(3)$  (see Fig. 1(g)) results from the fact that rotations in  $RGB$  modify the hue of colors. At last, as the action of  $\text{DC}(3)$  on  $RGB$  is transitive, the behaviour in (Fig. 1(e)) might be described by an element of  $\text{DC}(3)$ . We claim that such behaviours come from the choice of the metrics  $h$  in the heat diffusions (54), which are induced metrics through the graphs of  $(\rho^C, G)$ -equivariant functions.

We observe that the Casimir operator has an impact on the global behaviour of the diffusion but not on its local behaviour. This is a consequence of the fact that the Casimir operators are zero order operators for the Lie groups representations of Sect. 4.1.1. In particular, the contrast decreases for  $G = \mathbb{R}^{+*}$  (see Fig. 1(d)) and  $G = \text{DC}(3)$  (see Fig. 1(f)) because the corresponding Casimir operators are of the form  $k Id$  for  $k > 0$  (see expressions (56) and (57)), and the contrast increases for  $G = \text{SO}(3)$  (see Fig. 1(h)) because the corresponding Casimir operator is of the form  $k Id$  for  $k < 0$  (see expression (58)).

### 4.2.2 Optimal covariant derivatives

The second modification concerns the metric of the principal bundle. We replace the induced metric by a metric constructed from an optimal connection 1-form and a metric on the base manifold in the twisted Polyakov action (21).

On Fig. 2, we compute the gradient descent flow (60) of the twisted Polyakov action (21). We compute 200 iterations of the numerical scheme (61) with  $\kappa = 0.0001$ ,  $dt = 1$  and  $\delta = 1$ .

Unlike to the case of the induced metric (see Fig. 1 and comments in Sect. 4.2.1), the heat diffusions for metrics given by optimal connection 1-forms do not provide global actions of the Lie groups on the colors. However, we observe that the behaviour of a diffusion depends of the Lie group involved too. Indeed, for  $G = \mathbb{R}^{+*}$  (see Fig. 2(c,d)), the edges of the origi-

nal images (Fig. 2(a,b)) are still well preserved, despite the large number of iterations. However, the saturation components of the initial conditions, i.e. the original images, have decreased. Dealing with the group  $SO(3)$  (see Fig. 2(g,h)), the edges of the original images have been smoothed, but their saturation components have been preserved. The heat diffusion for the group  $DC(3)$  might be viewed as an intermediate between the two others ones (see Fig. 2(e,f)).

#### 4.2.3 Optimal covariant derivative for the group $SO(3)$

The heat diffusion of the generalized Laplacian associated to the optimal covariant derivative for the group representation  $(\rho^C, SO(3))$  is the only diffusion on Fig. 2 that behaves like standard regularization methods, e.g. heat diffusions of the Laplace-Beltrami operator and Euclidean Laplacian. We claim that this is a typical behaviour of heat diffusion of generalized Laplacian where the connection 1-form is  $\mathfrak{so}(n)$ -valued. Indeed, in a very recent paper of one of the author [5], the following result has been used (this is a direct consequence of Prop. 8.1 p.154 [21]):

**Proposition 6** *Let  $E$  be a vector bundle over a compact Riemannian manifold  $(X, g)$ . Let  $h$  be a definite positive metric on  $E$ , and  $H$  be the definite positive metric on the space  $\Gamma(T^*X \otimes E)$  determined by  $g$  and  $h$ . Let  $\nabla^A$  be a covariant derivative on  $E$  whose connection 1-form is  $\mathfrak{so}(h)$ -valued. Then, the Euler-Lagrange equation of the functional*

$$E(I) = \int_X \|\nabla^A I\|_H^2 dX \quad (66)$$

is  $\Delta^A I = 0$  where  $\Delta^A$  is the generalized Laplacian associated to the covariant derivative  $\nabla^A$ . Moreover,  $\Delta^A I = 0$  if and only if  $\nabla^A I = 0$ .

Hence, heat equations of generalized Laplacians whose connection 1-forms are  $\mathfrak{so}(n)$ -valued are gradient descent flows for reaching the solution of the variational problem (66).

Taking the covariant derivative  $\nabla^A$  of connection 1-form  $A \equiv 0$  in (66), we obtain the minimization problem of the Riemannian gradient L2 norm, whose gradient descent flow is the heat equation of the Laplace-Beltrami operator. Moreover, we obtain the Euclidean gradient and the heat equation of the Euclidean Laplacian if the metric  $g$  of the base manifold is Euclidean. We claim that the heat diffusion of the generalized Laplacian associated to the optimal covariant derivative for the group representation  $(\rho^C, SO(3))$  behaves like standard regularization methods because it is the

gradient descent flow for the minimization problem of a (generalized) gradient L2 norm too.

On Fig. 3, we compare the heat diffusion of the generalized Laplacian  $\Delta^{A^{opt}}$  associated to the optimal covariant derivative  $\nabla^{A^{opt}}$  for the group representation  $(\rho^C, SO(3))$  (see Fig. 3 (c,f)) with the heat diffusion of the Laplace-Beltrami operator  $\Delta_g$  (Fig. 3(b,e)). The parameters we take for both numerical schemes are  $\kappa = 0.0005$ ,  $dt = 0.1$  and 50 iterations. Moreover, we take  $\delta = 1$ . Results show that the heat diffusion of the "optimal" generalized Laplacian preserves more colors and edges of the original images (Fig. 3(a,d)) than the heat diffusion of the Laplace-Beltrami operator .

By Prop. 6, both diffusions tend to sections  $J_\infty$  satisfying  $\nabla^A J_\infty = 0$  for the covariant derivatives  $\nabla^A$  associated to the diffusions. As the Laplace-Beltrami operator is given by the covariant derivative of connection 1-form 0, the heat diffusion of the Laplace-Beltrami operator tends to a constant function. It explains why the diffusion smoothes edges (Fig. 3(b,e)). On the other hand, it is not guaranteed that the heat diffusion of the "optimal" generalized Laplacian tends to a constant section since it is not guaranteed that the sections  $J_\infty$  satisfying  $\nabla^{A^{opt}} J_\infty = 0$  are constant. This might explain why this heat diffusion preserves colors and edges better (Fig. 3 (c,f)).

#### 4.2.4 The parameter $\delta$

Dealing with the heat diffusion of the Laplace-Beltrami operator  $\Delta_g$ , for  $g$  given in (64), it is well-known that the parameter  $\kappa$  controls the anisotropy of the diffusion. In particular, for  $\kappa = 0$ , we obtain the isotropic diffusion of the Euclidean Laplacian.

The heat diffusions of the generalized Laplacians constructed in this paper involve an extra parameter  $\delta$  that controls the influence of the Lie group in the diffusion. Indeed,  $\delta$  appears in the expression of the Casimir operator (see Sect. 4.1.3) and in the Lie group component of the metric  $Q$  (see (45),(48),(51)). From the explicit expressions of the symbols  $\mathcal{Y}_{ij}^k$  in the Appendix B., we have  $\lim_{\delta \rightarrow +\infty} \mathcal{Y}_{ij}^k = 0$ . Hence

$$\lim_{\delta \rightarrow +\infty} \Delta^{A^{opt}} = \Delta_g, \quad (67)$$

which means that the diffusions tend to the heat diffusion of the Laplace-Beltrami operator when  $\delta \rightarrow +\infty$ .

On Fig. 4, we compare the heat diffusions of the generalized Laplacians constructed in this paper with the

heat diffusion of the Laplace-Beltrami operator, for the following values of the parameter  $\delta : 1, 50$ . We compute 50 iterations of the scheme (61) with  $\kappa = 0.001$  and  $dt = 0.1$ . The heat diffusion of the Laplace-Beltrami operator (Fig. 4(b)) smoothes the original image (Fig. 4(a)) by preserving the strong edges. Taking  $\delta = 1$ , the heat diffusion only preserves the strong edges for  $G = \text{SO}(3)$  (see Fig. 4(e)), whereas more edges are preserved for  $G = \mathbb{R}^{+*}$  (see Fig. 4(c)) and  $G = \text{DC}(3)$  (see Fig. 4(d)). As expected, the diffusions are getting close to the Beltrami diffusion taking  $\delta = 50$ . However, the speed of the convergence varies with the Lie group involved. Indeed, the smoothing is stronger for  $G = \text{DC}(3)$  (see Fig. 4(g)) than for  $G = \mathbb{R}^{+*}$  (Fig. 4(f)). For  $G = \text{SO}(3)$ , the diffusion (Fig. 4(h)) is close to the Beltrami diffusion (Fig. 4(b)).

## 5 Conclusion

In this paper, we extended our former work from a two-fold modification of the Polyakov action in [4]. First, we added an extra term involving the Casimir operator. Then, we assumed that the metric on the embedded manifold is constructed from an optimal horizontal field. We showed that the gradient descent flow for reaching the functions minimizing the functional corresponds to the heat equation of a generalized Laplacian on an associated bundle relatively to a group representation  $(\rho, G)$ . Whereas the behaviours of the diffusions in [4] are related with the actions of the representations  $(\rho, G)$  on the color space, the metrics constructed in this paper make the groups affect the behaviours of the diffusions in a different way. In particular, for  $G = \text{SO}(3)$ , the diffusion has a smoothing behaviour comparable to the Laplace-Beltrami heat diffusion, but preserving more colors and edges. We showed that the heat equation of the corresponding generalized Laplacian corresponds to the gradient descent flow for reaching the sections minimizing the L2 norm of a generalized gradient. Hence, it is a straightforward generalization of the Riemannian approach. We are currently working on the minimization of generalized gradients L1 norm for applications to image denoising.

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## Appendix A. Differential Geometry of Associated Bundles

### A.1. Differential geometry of vector bundles

**Definition 4** A smooth **vector bundle** of rank  $n$  is a triplet  $(E, \pi, M)$  where  $M$  and  $E$  are two  $C^\infty$  manifolds, and  $\pi: E \rightarrow M$  is a surjective map such that the preimage  $\pi^{-1}(x)$  of  $x \in M$  is endowed with a structure of vector space of dimension  $n$ .  $M$  is called the **base manifold** and  $E$  the **total space** of the vector bundle. The set  $\pi^{-1}(x)$  is called the **fiber** over  $x$ , and is denoted by  $E_x$ .

The vector bundle is said to be **locally trivial** if the following conditions hold: for each  $x \in M$ , there is a neighborhood  $U$  of  $x$  and a diffeomorphism  $\phi: U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$  satisfying  $\pi \circ \phi(x, f) = x \forall f \in \mathbb{R}^n$ , and such that the restriction  $\phi_x: \mathbb{R}^n \rightarrow E_x$  is a vector space isomorphism. The couple  $(U, \phi)$  is called a **local trivialization**.

The vector bundle is said to be **trivial** if there exists a diffeomorphism  $\Phi: M \times \mathbb{R}^n \rightarrow E$  satisfying  $\pi \circ \Phi(x, f) = x$ , and such that the restriction  $\Phi_x: \mathbb{R}^n \rightarrow E_x$  is a vector space isomorphism.

For shortness,  $(E, \pi, M)$  is sometimes denoted by  $E$ .

*Example 1* Let  $M$  be a  $C^\infty$  manifold of dimension  $m$ . The disjoint union of tangent spaces  $TM := \bigsqcup T_x M$  for  $x \in M$ , is the total space of a vector bundle  $(TM, \pi, M)$  of rank  $m$  called the **tangent bundle of  $M$** .

**Definition 5** A metric  $h$  on a vector bundle is the assignment of a scalar product  $h_x$  on each fiber  $\pi^{-1}(x)$ .

*Example 2* A Riemannian metric on a manifold is a positive definite metric on its tangent bundle.

**Definition 6** A **section** of a smooth vector bundle  $(E, \pi, M)$  is a differentiable map  $S: M \rightarrow E$  such that  $\pi \circ S = Id_M$ .

Let  $(f_1, \dots, f_n)$  be a basis of  $\mathbb{R}^n$ . In a local trivialization  $(U, \phi)$  of  $(E, \pi, M)$ , any section may be written

$$S(x) = \sum_{i=1}^n s_i(x) \phi(x, f_i)$$

for some functions  $s_i \in C^\infty(U)$ .

The set  $\{\phi(\cdot, f_1), \dots, \phi(\cdot, f_n)\}$  is called a **local frame** of  $(E, \pi, M)$ .

The set of sections of  $(E, \pi, M)$  is denoted by  $\Gamma(E)$ .

*Example 3* Tangent vector fields on  $M$  are the sections of the tangent bundle  $(TM, \pi, M)$ .

**Definition 7** A **covariant derivative** on  $(E, \pi, M)$  is a map  $\nabla^E: \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E)$  satisfying the following axioms:

$$\begin{aligned} -\nabla_{fX+gY}^E S &= f\nabla_X^E S + g\nabla_Y^E S \\ -\nabla_X^E fS &= (d_X f)S + f\nabla_X^E S \end{aligned}$$

for  $f, g \in C^\infty(M)$ ,  $X, Y \in \Gamma(TM)$  and  $S \in \Gamma(E)$ .

Hence, a covariant derivative on  $(E, \pi, M)$  may be written as  $d + \omega$  for some  $\omega \in \Gamma(T^*M \otimes \text{End}(E))$ .

In local frames  $(e_1, \dots, e_n)$  of  $E$  and  $(X_1, \dots, X_m)$  of  $TM$ , a covariant derivative is determined by  $n^2 \times m$  functions  $\Upsilon_{ij}^k$  such that

$$\nabla_{X_i}^E e_j = \sum_{k=1}^n \Upsilon_{ij}^k e_k$$

*Example 4* The **Levi-Cevita connection** is the covariant derivative on the tangent bundle of a Riemannian manifold  $(M, g)$  determined by the  $m^3$  functions

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij})$$

with respect to the local frame  $(\partial/\partial x_1, \dots, \partial/\partial x_m)$  of  $TM$  given by a local coordinates system  $(x_1, \dots, x_m)$  on  $M$ .

## A.2. Differential geometry of principal bundles

**Definition 8** A smooth **principal bundle** is a quadruplet  $(P, \pi, M, G)$  where  $M$  and  $P$  are two  $C^\infty$  manifolds,  $G$  is a Lie group,  $\pi: P \rightarrow M$  is a surjective map such that for all  $x \in M$ , the preimage  $\pi^{-1}(x)$  is diffeomorphic to  $G$  and there is an action  $\cdot$  of  $G$  on  $P$  satisfying:  
-  $\pi(p \cdot g) = \pi(p)$  for  $p \in \pi^{-1}(x)$  and  $g \in G$ .  
- the restriction  $\cdot: G \times \pi^{-1}(x) \rightarrow \pi^{-1}(x)$  is free and transitive.

$M$  is called the **base manifold**,  $P$  the **total space** and  $G$  the **Lie group structure** of the principal bundle. The set  $\pi^{-1}(x)$  is called the **fiber** over  $x$ , and is denoted by  $P_x$ .

The principal bundle is said to be **locally trivial** if the following conditions hold: for each  $x \in M$ , there is a neighborhood  $U$  of  $x$  and a diffeomorphism  $\phi: U \times G \rightarrow \pi^{-1}(U)$  satisfying  $\pi \circ \phi(x, g) = x$ , and such that the map  $\phi_x: G \rightarrow P_x$  is compatible with the restriction  $\cdot: G \times P_x \rightarrow P_x$ , i.e.  $\phi_x(ga) = \phi_x(g) \cdot a$ . The couple

$(U, \phi)$  is called a **local trivialization**.

The principal bundle is said to be **trivial** if there exists a diffeomorphism  $\Phi: M \times G \rightarrow P$  satisfying  $\pi \circ \Phi(x, g) = x$ , and such that the restriction  $\Phi_x: G \rightarrow P_x$  is compatible with the action  $\cdot: G \times P_x \rightarrow P_x$ .

For shortness,  $(P, \pi, M, G)$  is sometimes denoted by  $P$ .

*Example 5* Let  $(M, g)$  be a Riemannian manifold of dimension  $m$  and  $TM$  its tangent bundle. The disjoint union of oriented orthonormal bases of  $T_x M$  with respect to  $g(x)$ , for  $x \in M$ , forms the total space of a principal bundle  $PSO(TM)$  over  $M$  of structure Lie group  $SO(n)$  called the **bundle of orthonormal frames** of  $TM$ .

**Definition 9** A **section** of a smooth principal bundle  $(P, \pi, M, G)$  is a differentiable map  $S: M \rightarrow P$  such that  $\pi \circ S = Id_M$ .

Let  $(U, \phi)$  be a local trivialization of  $(P, \pi, M, G)$ , and  $s: U \rightarrow P$  defined by  $s(x) = \phi(x, e)$ . The section  $s$  is called a **local trivializing section** of  $P$  and any local section  $S$  of  $P$  takes the form

$$S(x) = s(x) \cdot g(x)$$

for some function  $g: U \rightarrow G$ .

The function  $g$  is called **the local trivialization of the section  $S$** .

The set of sections of  $(P, \pi, M, G)$  is denoted by  $\Gamma(P)$ .

**Definition 10** Let  $(P, \pi, M, G)$  be a principal bundle. Let  $T_p \pi: T_p P \rightarrow T_{\pi(p)} M$  be the linear tangent map of  $\pi$  at  $p \in P$ . A tangent vector  $X \in T_p P$  is called **vertical** if  $T_p \pi(X) = 0$ . We denote by  $V_p$  the set of vertical vectors at the point  $p$ . It is the vector space generated by vectors  $X_p^v$  of the form

$$X_p^v = \left( \frac{d}{dt} p \cdot \exp(tX) \right)_{|t=0}, \text{ for } X \in \mathfrak{g}.$$

The set  $VP = \bigsqcup_{p \in P} V_p$  forms the total space of a vector bundle over  $P$  called the **vertical bundle** of  $P$ . It is a subbundle of the tangent bundle of  $P$ .

Let  $\tilde{R}_g: p \mapsto p \cdot g$ . The map  $T_p \tilde{R}_g: T_p P \rightarrow T_{p \cdot g} P$  maps  $V_p$  on  $V_{p \cdot g}$ . This property is called  **$G$ -equivariance**.

**Definition 11** Let  $(P, \pi, M, G)$  be a principal bundle and  $VP$  the vertical bundle of  $P$ . Let  $x \in M$  and  $p \in P_x$ . Let  $H_p$  be a subspace of  $T_p P$  such that  $T_p P =$

$H_p \oplus V_p$ . A vector  $X_p^h \in H_p$  is called **horizontal**. We extend  $H_p$  on  $P_x$  by setting a  $G$ -equivariance property, i.e. we put

$$H_{p \cdot g} = T_p \widetilde{R}_g H_p$$

Repeating the construction for each  $x \in M$  (under an assumption of smoothness of the sets  $H_p$  with respect to  $x$ ) turns the set  $HP := \bigsqcup_{p \in P} H_p$  into the total space of a vector bundle over  $P$  called a **horizontal bundle** of  $P$ . It is a subbundle of the tangent bundle of  $P$ .

**Definition 12** Let  $(P, \pi, M, G)$  be a principal bundle,  $VP$  its vertical bundle and  $HP$  a horizontal bundle of  $P$ . Let  $x \in M$  and  $p \in P_x$ . From the surjectivity of the projection  $\pi$ , the map  $T_p \pi$  is surjective too. Moreover, as  $V_p = \text{Ker } T_p \pi$  and  $\dim H_p P = \dim T_x M$ , we deduce that the map

$$T_p \pi: H_p \longrightarrow T_x M$$

is an isomorphism. Given  $X_x \in T_x M$ , the unique vector  $X_p^h \in H_p$  such  $T_p \pi(X_p^h) = X_x$  is called **horizontal lift** of  $X_x$  with respect to  $HP$ .

**Definition 13** Let  $(P, \pi, M, G)$  be a principal bundle,  $VP$  its vertical bundle and  $HP$  a horizontal bundle of  $P$ . The  $\mathfrak{g}$ -valued 1-form  $\omega$  defined on  $P$  by

$$\begin{cases} \omega_p(X_p) = 0 & \text{for } X_p \in H_p P \\ \omega_p(X_p) = A & \text{for } X_p = \left( \frac{d}{dt} p \cdot \exp(tA) \right)_{|t=0} \in V_p P \end{cases}$$

is called a **connection 1-form**.

As a consequence of the  $G$ -equivariance of the vertical and horizontal bundle, the connection 1-form satisfies

$$\omega_{p \cdot g}(T_p \widetilde{R}_g X_p) = g^{-1} \omega_p(X_p) g$$

when  $G$  is a matrix Lie group.

### A.3. From principal to vector bundles

#### A.3.1. Definitions

**Definition 14** Let  $G$  be a Lie group and  $V$  a vector space. A **representation**  $\rho$  of  $G$  on  $V$  is a group morphism  $\rho: G \longrightarrow GL(V)$ . The representation  $\rho$  determines a group representation  $\rho^C$  on the dual space  $V^*$  of  $V$  called the **dual representation** of  $\rho$ , and defined by  $\rho^C(g) = \overline{\rho(g^{-1})}$  where  $\overline{\phantom{x}}$  denotes the transpose.

**Definition 15** Let  $(P, \pi, M, G)$  be a principal bundle,  $\rho$  a representation of  $G$  on  $V$  of dimension  $n$ , and  $E = (P \times V)/G$ , i.e. a point in  $E$  is of the form

$$[p, f] := \{(p \cdot g, \rho(g^{-1})f), g \in G\}$$

where  $p \in P$  and  $f \in V$ . Let  $\pi_E: E \longrightarrow M$  given by  $\pi_E[p, f] = \pi(p)$ . Then, the triplet  $(E, \pi_E, M)$  forms a vector bundle of rank  $n$ , called **vector bundle associated** to  $P$  and denoted by  $P \times_{(\rho, G)} V$ .

Let  $(U, \phi)$  be a local trivialization of  $(P, \pi, M, G)$  and  $s$  the local trivializing section of  $P$  given by  $s(x) = \phi(x, e)$ . Then, the map

$$\begin{aligned} \chi_s: U \times V &\longrightarrow \pi_E^{-1}(U) \\ (x, f) &\longmapsto [s(x), f] \end{aligned}$$

is a **local trivialization** of  $P \times_{(\rho, G)} V$ .

Let  $(f_1, \dots, f_n)$  be a basis of  $V$ .

Then, the set  $(\chi_s(\cdot, f_1), \dots, \chi_s(\cdot, f_n))$  determines a local frame of  $(E, \pi_E, M)$ .

*Example 6* Let  $(M, g)$  be a Riemannian manifold of dimension  $m$ , and  $(PSO(TM), \pi, M, SO(m))$  be the principal bundle of orthonormal frames of  $TM$ . The tangent bundle of  $M$  is a vector bundle associated to  $PSO(TM)$  under the natural representation  $\rho$  of  $SO(m)$  on  $\mathbb{R}^m$ , and might be written  $PSO(TM) \times_{(\rho, SO(m))} \mathbb{R}^m$ .

**Definition 16** Let  $(P, \pi, M, G)$  be a principal bundle and  $\rho$  a representation of  $G$  on a finite dimensional vector space  $V$ . A function  $J: P \longrightarrow V$  is called  **$(\rho, G)$ -equivariant** if it satisfies

$$J(p \cdot g) = \rho(g^{-1})J(p)$$

We denote by  $C^\infty(P, V)^{(\rho, G)}$  the set of smooth  $(\rho, G)$ -equivariant functions on  $P$ .

There is a correspondance between sections of associated bundles and  $(\rho, G)$ -equivariant functions on principal bundles. Indeed, given  $f \in C^\infty(P, V)^{(\rho, G)}$ , we put  $S_f(x) = [p, f(p)]$  for any  $p \in \pi^{-1}(x)$ . By the  $(\rho, G)$ -equivariance property of  $f$ , the map  $S_f$  is independent of the choice of  $p$ . Hence,  $S_f$  is a section of  $E = P \times_{(\rho, G)} V$ . Conversely, for  $S \in \Gamma(E)$ , we put  $f_S(p) = v$  such that  $S \circ \pi(p) = [p, v]$ . We observe that  $f_S$  is  $(\rho, G)$ -equivariant.

#### A.3.2. From horizontal bundles on principal bundles to covariant derivatives on associated bundles

Let  $(P, \pi, M, G)$  be a principal bundle,  $\rho$  a representation of  $G$  on a finite dimensional vector space  $V$ , and  $E = P \times_{(\rho, G)} V$  the associated bundle. Let  $S \in \Gamma(E)$  and  $f_S$  the corresponding  $(\rho, G)$ -equivariant function. Let  $HP$  be a horizontal bundle of  $P$ . Let  $X \in \Gamma(TM)$  and  $X^h \in \Gamma(HP)$  be the horizontal lift of  $X$  with respect to  $HP$ . Then, there exist a covariant derivative



$\nabla^A$  on  $(E, \pi_E, M)$  such that the following correspondence holds

$$d_{X^h} f_S \longleftrightarrow \nabla_X^A S$$

Let  $\omega$  be the connection 1-form induced by  $HP$ . Let  $s$  be a section of  $P$  and  $A$  be the  $\mathfrak{g}$ -valued 1-form on  $M$  given by

$$A = s^* \omega$$

where  $*$  denotes the pull-back map.

Then, relatively to the local frame of  $(E, \pi_E, M)$  induced by  $s$ , the covariant derivative is given by

$$\nabla_X^A = d_X + \rho_*(A)(X)$$

where  $\rho_* : \mathfrak{g} \rightarrow \text{End}(V)$  is the Lie algebra representation induced by  $\rho$ .

## Appendix B. Optimal connection 1-forms and corresponding covariant derivatives

Let us assume that the base manifold  $M$  is of dimension 2. Let  $(x_1, x_2)$  be a coordinates system of  $M$  and  $(a_1, \dots, a_n)$  be a coordinates system of  $G$ . By definition, a horizontal bundle  $HP$  is a subbundle of rank 2 of the tangent bundle  $TP$  of  $P$  such that  $TP \simeq HP \oplus \mathfrak{g}$ . Hence, along the section  $s$ , it is of the form

$$H_s P = \text{Vect} \left\{ \alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{1k} \partial/\partial a_k, \right. \\ \left. \alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{2k} \partial/\partial a_k \right\} \quad (68)$$

for some functions  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \dots, \gamma_{2n}$  on  $M$  such that  $\alpha_1 \beta_2 - \alpha_2 \beta_1$  never vanish.

### B.1. The case of Lie groups of dimension 1

#### B.1.1. Construction of optimal connection 1-forms

Computing the Euler-Lagrange equations of (28) with respect to the functions  $\alpha_k, \beta_k, \gamma_k$  for  $k = 1, 2$ , we obtain  $\gamma_k =$

$$\left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[ \alpha_k \left( \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_k \left( \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right].$$

It defines a unique horizontal field  $H_s P$  along  $s$  whose expression is obtained from (68). We refer to Appendix C.1 for more details.

Let  $X$  be a tangent vector field on  $P$  along the section  $s$

of the form  $\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + \gamma \partial/\partial a$  for some functions  $\alpha, \beta, \gamma$  on  $M$ . The connection 1-form  $\omega$  induced by  $HP$  is given along  $s$  by  $\omega(X) = \left[ \gamma + \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \right.$

$$\left. \times \left( -\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} - \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right] \partial/\partial a$$

The 1-form  $A = s^* \omega$  on  $M$  is given by  $A(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2)(x) = \omega(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + 0 \partial/\partial a)(x, e)$ .

#### B.1.2. Corresponding covariant derivative for the unique dual representation $\rho^C$ of $\mathbb{R}^{+*}$ on $\mathbb{R}^3$

Writing the 1-form  $A$  as  $A_1 \partial/\partial a$ , we have  $\rho_*^C(A) = -A_1 \mathbb{I}_3$ .

Hence, the nonzero symbols of the corresponding covariant derivative  $\nabla^A$  are given in the frames  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and  $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$  of  $P \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^3$  by

$$\Upsilon_{lk}^k(x) = (\rho_*^C A)_{kk}(\partial/\partial x_l)(x) \\ = \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e)$$

for  $l = 1, 2$  and  $k = 1, 2, 3$ .

Taking a metric  $Q$  of the form  $\mathbb{I}_2 \oplus \delta \oplus \kappa \mathbb{I}_3$  in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a, e_1, e_2, e_3)$ , we obtain

$$\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) = \delta + \kappa \left( \sum_{k=1}^3 I^{k^2} \right)(x)$$

and

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) = -\kappa \left( \sum_{k=1}^3 I_{x_l}^k I^k \right)(x)$$

#### B.1.3. Corresponding covariant derivative for a class of unitary dual representations $\rho^C$ of $SO(2)$ on $\mathbb{R}^3$

Let  $\mathcal{P}$  be a 2-dimensional subspace of  $\mathbb{R}^3$ , and the corresponding bivector  $\gamma_{12} e_1 e_2 + \gamma_{13} e_1 e_3 + \gamma_{23} e_2 e_3$  for  $\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 = 1$ . Writing the 1-form  $A$  as  $A_1 \partial/\partial \theta$ , we have

$$\rho_{\mathcal{P}*}^C(A) = A_1 \begin{pmatrix} 0 & -\gamma_{12} & -\gamma_{13} \\ \gamma_{12} & 0 & -\gamma_{23} \\ \gamma_{13} & \gamma_{23} & 0 \end{pmatrix}$$

Hence, the nonzero symbols of the corresponding covariant derivative  $\nabla^A$  are given in the frames

$(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and  $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$  of  $P \times_{(\rho^c, sO(2))} \mathbb{R}^3$  by

$$\begin{aligned} \Upsilon_{lp}^k(x) &= (\rho_{p^*}^c A)_{kp}(\partial/\partial x_l)(x) = -\Upsilon_{lk}^p(x) \\ &= (-1)^{l+1} \gamma_{kp} \left( \frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) \end{aligned}$$

for  $l = 1, 2$  and  $k, p = 1, 2, 3$  with  $k \neq p$ .

Taking a metric  $Q$  of the form  $\mathbb{I}_2 \oplus \delta \oplus \kappa \mathbb{I}_3$  in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \theta, e_1, e_2, e_3)$ , we obtain

$$\begin{aligned} \frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) &= \delta + \\ \kappa \left( (\gamma_{12} I^2 + \gamma_{13} I^3)^2 + (\gamma_{12} I^1 - \gamma_{23} I^3)^2 + (\gamma_{13} I^1 + \gamma_{23} I^2)^2 \right) &(x) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) &= -\kappa \times \left( I_{x_l}^1 (-\gamma_{12} I^2 - \gamma_{13} I^3) + \right. \\ &\left. I_{x_l}^2 (\gamma_{12} I^1 - \gamma_{23} I^3) + I_{x_l}^3 (\gamma_{13} I^1 + \gamma_{23} I^2) \right) (x) \end{aligned}$$

## B.2. The case of Lie groups of dimension 3

### B.2.1. Construction of optimal connection 1-forms

Computing the Euler-Lagrange equations of (28) with respect to the functions  $\alpha_l, \beta_l, \gamma_{lj}$  for  $l = 1, 2$  and  $j = 1, 2, 3$ , we obtain

$$\begin{aligned} \gamma_{11} &= \lambda \left[ \alpha_l \left( (fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\left. \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left( (fd - e^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \\ &\left. \left. + (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \end{aligned}$$

$$\begin{aligned} \gamma_{12} &= \lambda \left[ \alpha_l \left( (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\left. \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left( (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \\ &\left. \left. + (af - c^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \end{aligned}$$

$$\begin{aligned} \gamma_{13} &= \lambda \left[ \alpha_l \left( (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\left. \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left( (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \end{aligned}$$

$$\left. \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right]$$

where  $\lambda = 1/(2cbe + fad - e^2a - fb^2 - c^2d)$  and

$$a = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \quad b = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \quad c = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij}$$

$$d = \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \quad e = \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \quad f = \frac{\partial \varphi^i}{\partial a_3} \frac{\partial \varphi^j}{\partial a_3} Q_{ij}$$

It defines a unique horizontal field  $H_s P$  along  $s$  whose expression is obtained from (68). We refer to Appendix C.2 for more details.

Let  $X$  be a tangent vector field on  $P$  along the section  $s$  of the form  $\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + \sum_{k=1}^3 \gamma_k \partial/\partial a_k$  for some functions  $\alpha, \beta, \gamma_k$  on  $M$ . The connection 1-form  $\omega$  induced by  $HP$  is given along  $s$  by  $\omega(X) =$

$$\begin{aligned} &\left[ \gamma_1 - \lambda \left[ (fd - e^2) \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ &\left. \left. + (ce - fb) \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (be - dc) \times \right. \right. \\ &\left. \left. \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_1 \\ &+ \left[ \gamma_2 - \lambda \left[ (ce - fb) \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ &\left. \left. + (af - c^2) \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (bc - ae) \times \right. \right. \\ &\left. \left. \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_2 \\ &+ \left[ \gamma_3 - \lambda \left[ (be - dc) \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ &\left. \left. + (bc - ae) \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (ad - b^2) \times \right. \right. \\ &\left. \left. \left( \alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_3 \end{aligned}$$

The 1-form  $A = s^* \omega$  on  $M$  is given by  $A(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2)(x) =$

$$\omega(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + 0 \partial/\partial a_1 + 0 \partial/\partial a_2 + 0 \partial/\partial a_3)(x, e)$$

*B.2.2. Corresponding covariant derivative for the unique dual representation  $\rho^C$  of  $DC(3)$  on  $\mathbb{R}^3$*

Writing the 1-form  $A$  as  $A_1\partial/\partial a_1 + A_2\partial/\partial a_2 + A_3\partial/\partial a_3$ , we have  $\rho_*^C(A) = -\text{diag}(A_1, A_2, A_3)$ .

Hence, the nonzero symbols of the corresponding covariant derivative  $\nabla^A$  are given in the frames  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and  $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$  of  $P \times_{(\rho^C, DC(3))} \mathbb{R}^3$  by

$$\begin{aligned} \Upsilon_{11}^1(x) &= (\rho_* A)_{11}(\partial/\partial x_1)(x) \\ &= \lambda \left[ d f \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \Upsilon_{12}^2(x) &= (\rho_* A)_{22}(\partial/\partial x_1)(x) \\ &= \lambda \left[ a f \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \Upsilon_{13}^3(x) &= (\rho_* A)_{33}(\partial/\partial x_1)(x) \\ &= \lambda \left[ a d \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right] (x, e) \end{aligned}$$

for  $l = 1, 2$ .

Taking a metric  $Q$  of the form  $\mathbb{I}_2 \oplus \delta \mathbb{I}_3 \oplus \kappa \mathbb{I}_3$  in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3, e_1, e_2, e_3)$ , we obtain

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_k} Q_{ij}(x, e) = -\kappa I_{x_l}^k I^k(x)$$

for  $k = 1, 2, 3$ , and

$$\begin{aligned} a(x, e) &= \delta + \kappa I^1(x) & d(x, e) &= \delta + \kappa I^2(x) \\ f(x, e) &= \delta + \kappa I^3(x) \end{aligned}$$

*B.2.3. Corresponding covariant derivative for the unique dual representation  $\rho^C$  of  $SO(3)$  on  $\mathbb{R}^3$*

Writing the 1-form  $A$  as  $A_1\partial/\partial \theta_1 + A_2\partial/\partial \theta_2 + A_3\partial/\partial \theta_3$ , we have

$$\rho_*^C(A) = \begin{pmatrix} 0 & -A_3 & -A_2 \\ A_3 & 0 & -A_1 \\ A_2 & A_1 & 0 \end{pmatrix}$$

Hence the nonzero symbols of the corresponding covariant derivative  $\nabla^A$  are given in the frames  $(\partial/\partial x_1, \partial/\partial x_2)$  of  $TM$  and  $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$  of  $P \times_{(\rho^C, SO(3))} \mathbb{R}^3$  by

$$\begin{aligned} \Upsilon_{12}^1(x) &= (\rho_*^C A)_{12}(\partial/\partial x_1)(x) = -\Upsilon_{11}^2(x) \\ &= \lambda \left[ (be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \Upsilon_{13}^1(x) &= (\rho_*^C A)_{13}(\partial/\partial x_1)(x) = -\Upsilon_{11}^3(x) \\ &= \lambda \left[ (ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \Upsilon_{13}^2(x) &= (\rho_*^C A)_{23}(\partial/\partial x_1)(x) = -\Upsilon_{12}^3(x) \\ &= \lambda \left[ (fd - e^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

for  $l = 1, 2$ .

Taking a metric  $Q$  of the form

$$\mathbb{I}_2 \oplus \begin{pmatrix} \delta & 0 & \delta \sin \theta_2 \\ 0 & \delta & 0 \\ \delta \sin \theta_2 & 0 & \delta \end{pmatrix} \oplus \kappa \mathbb{I}_3$$

in the frame  $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \theta_1, \partial/\partial \theta_2, \partial/\partial \theta_3, e_1, e_2, e_3)$ , we obtain

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij}(x, e) = -\kappa (I_{x_l}^2 I^3 - I_{x_l}^3 I^2)(x)$$

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij}(x, e) = -\kappa (I_{x_l}^1 I^3 - I_{x_l}^3 I^1)(x)$$

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij}(x, e) = -\kappa (I_{x_l}^1 I^2 - I_{x_l}^2 I^1)(x)$$

and

$$\begin{aligned} a(x, e) &= \delta + \kappa (I^2{}^2 + I^3{}^2)(x) & b(x, e) &= \kappa I^1 I^2(x) \\ c(x, e) &= -\kappa I^1 I^3(x) & d(x, e) &= \delta + \kappa (I^1{}^2 + I^3{}^2)(x) \\ e(x, e) &= \kappa I^2 I^3(x) & f(x, e) &= \delta + \kappa (I^1{}^2 + I^2{}^2)(x) \end{aligned}$$

## Appendix C. Solutions of Euler-Lagrange Equations

### C.1. The case of Lie groups of dimension 1

The Euler-Lagrange equations of the functional (28) with respect to  $H_s P$  are

$$*\frac{\partial S|_s}{\partial \alpha_1} = 0$$

$$\begin{aligned} \iff & (\beta_2\gamma_1 - \beta_1\gamma_2) \left[ (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_2 - g_{22}\beta_2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_2 - g_{11}\alpha_2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_2 - g_{12}\beta_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_2 - g_{12}\alpha_2) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \alpha_2} = 0$$

$$\begin{aligned} \iff & (\beta_2\gamma_1 - \beta_1\gamma_2) \left[ (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \beta_1} = 0$$

$$\begin{aligned} \iff & (\alpha_2\gamma_1 - \alpha_1\gamma_2) \left[ (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_2 - g_{22}\beta_2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_2 - g_{11}\alpha_2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_2 - g_{12}\beta_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_2 - g_{12}\alpha_2) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \beta_2} = 0$$

$$\begin{aligned} \iff & (\alpha_2\gamma_1 - \alpha_1\gamma_2) \left[ (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \gamma_1} = 0$$

$$\begin{aligned} \iff & (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ & + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \gamma_2} = 0$$

$$\begin{aligned} \iff & (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ & + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \end{aligned}$$

Finally, the Euler-Lagrange equations reduce to the equations  $\frac{\partial S|_s}{\partial \gamma_1} = 0$  and  $\frac{\partial S|_s}{\partial \gamma_2} = 0$ , i.e. to the following system

$$\left\{ \begin{array}{l} (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \\ (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_2 - g_{22}\beta_2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_2 - g_{11}\alpha_2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ + \left( (\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_2 - g_{12}\beta_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_2 - g_{12}\alpha_2) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \end{array} \right.$$

We obtain a linear system with respect to  $\gamma_1$  and  $\gamma_2$  of the form

$$D \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = (\alpha_1\beta_2 - \alpha_2\beta_1) \times \begin{pmatrix} (\alpha_2g_{12} + \beta_2g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} - (\beta_2g_{12} + \alpha_2g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ (\alpha_1g_{11} + \beta_1g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} - (\beta_1g_{22} + \alpha_1g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \end{pmatrix} \quad (69)$$

where the coefficients of the matrix  $D$  are

$$\begin{aligned} D_{11} &= \left( \alpha_2(\alpha_2g_{11} + \beta_2g_{12}) + \beta_2(\beta_2g_{22} + \alpha_2g_{12}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ D_{12} &= - \left( \alpha_1(\beta_2g_{12} + \alpha_2g_{11}) + \beta_1(\alpha_2g_{12} + \beta_2g_{22}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ D_{21} &= - \left( \alpha_1(\beta_2g_{12} + \alpha_2g_{11}) + \beta_1(\alpha_2g_{12} + \beta_2g_{22}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ D_{22} &= \left( \alpha_1(\alpha_1g_{11} + \beta_1g_{12}) + \beta_1(\beta_1g_{22} + \alpha_1g_{12}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \end{aligned}$$

The solution of the system (69) is

$$\begin{aligned} \gamma_1 &= \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[ \alpha_1 \left( \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_1 \left( \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right] \\ \gamma_2 &= \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[ \alpha_2 \left( \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_2 \left( \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right] \end{aligned}$$

Conclusion:  $H_s P =$

$$\begin{aligned} &Vect \left\{ \alpha_1 \partial / \partial x_1 + \beta_1 \partial / \partial x_2 + \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[ -\alpha_1 \left( \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} \right) + \beta_1 \left( \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} \right) \right] \partial / \partial a, \right. \\ &\quad \left. \alpha_2 \partial / \partial x_1 + \beta_2 \partial / \partial x_2 + \left( \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[ -\alpha_2 \left( \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} \right) + \beta_2 \left( \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} \right) \right] \partial / \partial a \right\} \end{aligned}$$

## C.2. The case of Lie groups of dimension 3

As for the case of Lie groups of dimension 1, the Euler Lagrange equations of the functional (28) with respect to  $H_s P$  reduce to the equations  $\frac{\partial S_l}{\partial \gamma_{ij}} = 0$ , for  $l = 1, 2$ ,  $j = 1, 2, 3$ , and we obtain a linear system with respect to the functions  $\gamma_{ij}$  of the form

$$D \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{pmatrix} = (\alpha_1 \beta_2 - \alpha_2 \beta_1) \times \begin{pmatrix} (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \\ (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \\ (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \end{pmatrix} \quad (70)$$

where

$$D = \begin{pmatrix} aG & bG & cG & aF & bF & cF \\ bG & dG & eG & bF & dF & eF \\ cG & eG & fG & cF & eF & fF \\ aF & bF & cF & aE & bE & cE \\ bF & dF & eF & bE & dE & eE \\ cF & eF & fF & cE & eE & fE \end{pmatrix}$$

and

$$\begin{aligned} a &= \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} & b &= \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} & c &= \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ d &= \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} & e &= \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} & f &= \frac{\partial \varphi^i}{\partial a_3} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ E &= \alpha_1(g_{11}\alpha_1 + g_{12}\beta_1) + \beta_1(g_{22}\beta_1 + g_{12}\alpha_1) \\ F &= -[\alpha_1(g_{12}\beta_2 + g_{11}\alpha_2) + \beta_1(g_{12}\alpha_2 + g_{22}\beta_2)] \\ G &= \alpha_2(g_{11}\alpha_2 + g_{12}\beta_2) + \beta_2(g_{22}\beta_2 + g_{12}\alpha_2) \end{aligned}$$

The solution of the system (70) is

$$\begin{aligned} \gamma_{11} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[ \alpha_1 \left( (fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \\ &\quad \left. + \beta_1 \left( (fd - e^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \\ \gamma_{12} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[ \alpha_1 \left( (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \\ &\quad \left. + \beta_1 \left( (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \\ \gamma_{13} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[ \alpha_1 \left( (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \\ &\quad \left. + \beta_1 \left( (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \\ \gamma_{21} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[ \alpha_2 \left( (fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \end{aligned}$$



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(a) Original image 'kodim07'



(b) heat diffusion of the LBO : smoothing preserving strong edges

(c) Polyakov action,  $G = \mathbb{R}^{++}$ : smoothing preserving strong edges, increase of luminance and saturation components(d) Twisted Polyakov action,  $G = \mathbb{R}^{++}$ : decrease of the contrast in the image (c)(e) Polyakov action,  $G = DC(3)$ : smoothing preserving strong edges, increase of the luminance and decrease of the saturation(f) Twisted Polyakov action,  $G = DC(3)$ : decrease of the contrast in the image (e)(g) Polyakov action,  $G = SO(3)$ : smoothing preserving strong edges, decrease of luminance component and reddish aspect(h) Twisted Polyakov action,  $G = SO(3)$ : increase of the contrast in the image (g)

**Fig. 1** Gradient descent flows for reaching the functions minimizing the Polyakov action (16) and twisted Polyakov action (21), for induced metrics and different groups  $G$ . Comparison with the heat diffusion of the Laplace-Beltrami operator (LBO).

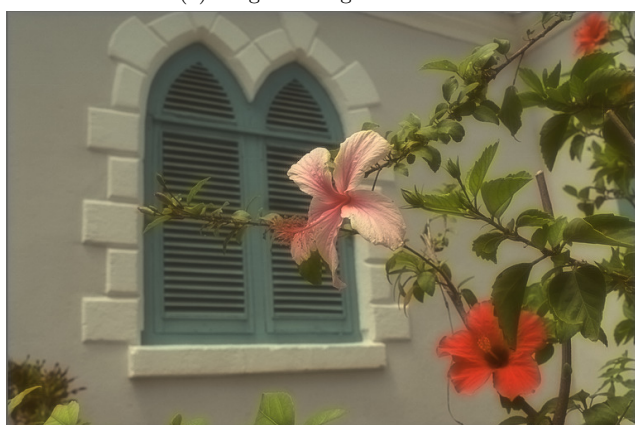
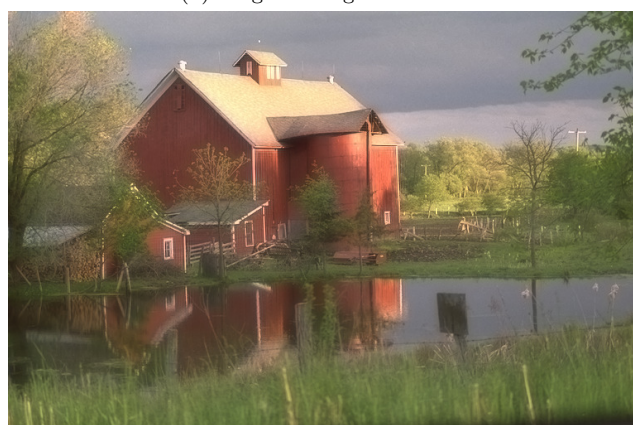
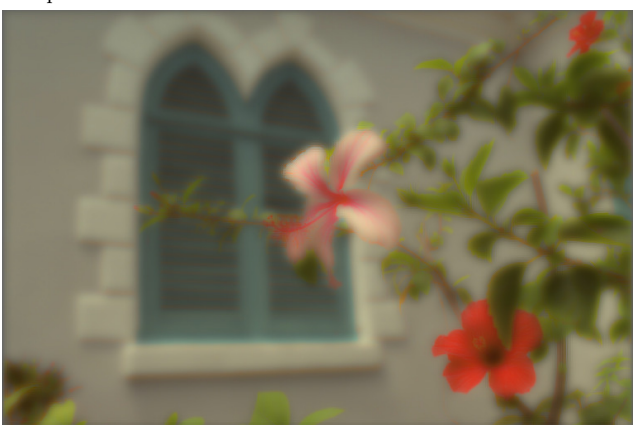
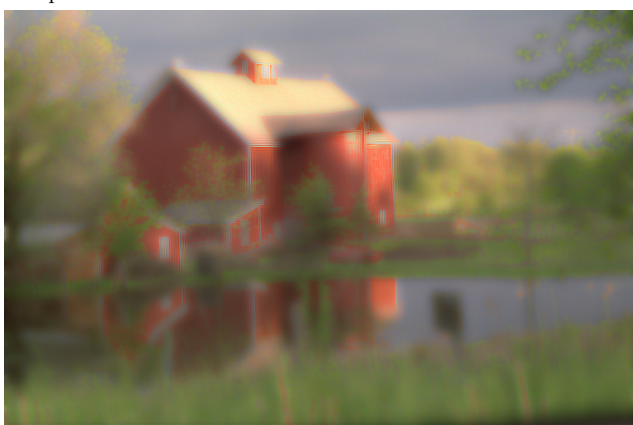




(a) Original image 'kodim07'



(b) Original image 'kodim22'

(c)  $G = \mathbb{R}^{+*}$ : weak smoothing of edges, decrease of the saturation component(d)  $G = \mathbb{R}^{+*}$ : weak smoothing of edges, decrease of the saturation component(e)  $G = \text{DC}(3)$ : smoothing of edges, preservation of the saturation component(f)  $G = \text{DC}(3)$ : smoothing of edges, preservation of the saturation component(g)  $G = \text{SO}(3)$ : strong smoothing of edges, preservation of the saturation component(h)  $G = \text{SO}(3)$ : strong smoothing of edges, preservation of the saturation component**Fig. 2** Comparison between the heat diffusions of generalized Laplacians  $\Delta^{A^{opt}}$  for different Lie groups  $G$ .





(a) Original image 'kodim07'

(b) Laplace-Beltrami operator: smoothing

(c) Operator  $\Delta^{A^{opt}}$  for  $G=SO(3)$ : better preservation of colors and edges

(d) Original image 'kodim22'

(e) Laplace-Beltrami operator: smoothing

(f) Operator  $\Delta^{A^{opt}}$  for  $G=SO(3)$ : better preservation of colors and edges

**Fig. 3** Heat diffusions of generalized Laplacians. Comparison between the operator  $\Delta^{A^{opt}}$  for  $G=SO(3)$  and the Laplace-Beltrami operator  $\Delta_g$ .





(a) Original image 'Barbara'

(b) Laplace-Beltrami operator : smoothing preserving strong edges



(c)  $G = \mathbb{R}^{+*}$ ,  $\delta = 1$ : weak smoothing, decrease of the saturation

(d)  $G = DC(3)$ ,  $\delta = 1$ : weak smoothing, preservation of the saturation

(e)  $G=SO(3)$ ,  $\delta = 1$ : smoothing preserving strong edges and the saturation



(f)  $G = \mathbb{R}^{+*}$ ,  $\delta = 50$ : increase of the smoothing in (c)

(g)  $G = DC(3)$ ,  $\delta = 50$ : increase of the smoothing in (d)

(h)  $G=SO(3)$ ,  $\delta = 50$ : Decrease of the saturation in (e), result close to (b)

**Fig. 4** Heat diffusions of the generalized Laplacians  $\Delta^{A^{opt}}$  for different Lie groups  $G$  and values of the parameter  $\delta$ . Comparison with the heat diffusion of the Laplace-Beltrami operator  $\Delta_g$  satisfying  $\Delta_g = \lim_{\delta \rightarrow +\infty} \Delta^{A^{opt}}$ .