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A Class of Generalized Laplacians on Vector Bundles Devoted to Multi-Channels Image Processing

Thomas Batard and Nir Sochen

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Abstract In the context of fibre bundles theory, there exist some differential operators of order 2, called generalized Laplacians, acting on sections of vector bundles over Riemannian manifolds, and generalizing the Laplace-Beltrami operator. Such operators are determined by the choice of a covariant derivative on the vector bundle. In this paper, we construct a class of generalized Laplacians, devoted to multi-channels image processing, from the construction of optimal covariant derivatives. The key idea is to consider an image as a section of an associate bundle, that is a vector bundle related to a principal bundle through a group representation. In this context, covariant derivatives are determined by connection 1-forms on principal bundles. We construct optimal connection 1-forms by the minimization of a variational problem on principal bundles. From the heat equations of the generalized Laplacians induced by the corresponding optimal covariant derivatives, we obtain diffusions whose behaviours depend of the choice of the group representation. We provide experiments on color images.

Keywords Multi-Channels Image Processing · Diffusion · Heat equation · Generalized Laplacian · Differential Geometry

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1 Introduction

Laplacians are widely used for image processing/analysis and computer vision. For instance, linear space-scale theory [18] is related to the heat equation of the Euclidean Laplacian. It is a fact that the isotropic behaviour of the linear scale-space is not well-adapted in many problems like denoising or contrast enhancement. Since then, many techniques of anisotropic diffusion of images have been employed (see e.g. [1],[8],[23],[31] and references therein). Some of them are based on the Laplace-Beltrami operator, a generalization of the Euclidean Laplacian on Riemannian manifold (see e.g. [21] for details about this operator).

In [24], the authors introduced the Beltrami framework, an efficient tool for anisotropic regularization of fields. Mathematically speaking, it is related to the heat equation of the Laplace-Beltrami operator. The Beltrami framework has been applied on different kinds of fields: vector-valued fields (see e.g. [24],[25] for multi-channels image regularization), group-valued fields (see [14] for $SO(n)$ -valued field regularization), symmetric space-valued fields (see [15] for regularization of fields of symmetric definite positive matrices). It has also been applied as preprocessing for motion segmentation [22] or regularizing term for optical flow estimation [5]. There exist other uses of the Laplace-Beltrami operator. For instance, approximations of its spectrum have found recently applications in the context of shape analysis (see e.g. [7],[19],[20],[29]).

In this paper, we are interested in a generalization of the Euclidean Laplacian and Laplace-Beltrami operators on fiber bundles. Fiber bundles theory provides a geometric framework generalizing the notion of manifold (see [16] for an introduction to fiber bundle theory). In this context, there exist a class of differential oper-

ators of order 2 called generalized Laplacians, and acting on sections of vector bundles. They are determined by two geometric data: a Riemannian metric on the base manifold and a covariant derivative on the vector bundle. The Euclidean Laplacian and Laplace-Beltrami operators appear to be a generalized Laplacians on vector bundles of rank 1 equipped with the trivial connection. We refer to [6] for more details about generalized Laplacians. Applications of generalized Laplacians for vector-valued field regularization were introduced in [2],[3]. In [3], it is showed that some standard methods of color image regularization, i.e. the Beltrami flow in [25], trace-based PDEs [27], curvature-preserving PDEs [28], and divergence-based PDEs (see [31] and the references therein) may be viewed as heat equations of generalized Laplacians. In the second part of the paper, some covariant derivatives that take into account the geometry of the color space RGB were constructed. By the heat equations of the corresponding generalized Laplacians, new diffusions of color images were performed.

The aim of this paper is to construct covariant derivatives that take into account not only the geometry of the color space but also the local variations of images. In the context of associated bundles, that are vector bundles associated to principal bundles through a group representation (ρ, G) (see e.g. [30] for an introduction to groups representations theory), a covariant derivative on the vector bundle is determined by a connection 1-form on the principal bundle. The notions of covariant derivative and connection 1-form are at the heart of differential geometry of manifolds and fibre bundles [17],[26]. Whereas they play a fundamental role in the mathematical description of physics, as in general relativity and Gauge fields theories [13], they have been few investigated in the field of image processing. Besides [2],[3], we may refer to the works of Duits et al. where connections on principal bundles were constructed, with applications to crossing-preserving smoothing [12] and contour enhancement [10],[11].

In [4], we introduced the concept of (ρ, G) -equivariance on principal bundles in the context of image processing. We constructed a Polyakov action measuring the energy of the graph of functions on principal bundles. We showed that the function whose graph minimizes the functional satisfies a Laplace equation relatively to the Riemannian metric of the principal bundle. However, by the Riemannian metric we constructed on the principal bundle, the Beltrami flow did not preserve the (ρ, G) -equivariance property of the initial condition. In this paper, we follow the approach of [4] and define

an image as a section of an associate bundle or equivalently as a (ρ, G) -equivariant function on a principal bundle. We consider the dual representations of \mathbb{R}^{+*} , the group $DC(3)$ of 3×3 diagonal matrices of strictly positive coefficients, $SO(2)$ and $SO(3)$ on \mathbb{R}^3 dealing with color images. In this context, we construct covariant derivatives on associated bundles from connection 1-forms on principal bundles. The main idea is to construct a Riemannian metric on a principal bundle from a connection 1-form and a Riemannian metric on the base manifold. Then, we consider a Polyakov action measuring the energy of the graph of functions on the principal bundle, with an extra term given by the action of a zero order operator called Casimir operator. Minimizing the functional along a section with respect to the metric of the principal bundle under the assumption that the metric of the base manifold is fixed provides a optimal connection 1-form. By the minimization of the functional with respect to the embedding, we obtain a flow that preserves the equivariance property of the initial condition. Under the identification between (ρ, G) -equivariant functions on principal bundles and sections of associated bundles, we obtain the heat flow of the generalized Laplacian induced by the optimal connection 1-form constructed. Experiments show that the flow preserves more some edges than the heat flow of the Laplace-Beltrami operator, the type of edges depending of the Lie group involved.

This paper is organized as follows. In Sect. 2, we first show how to obtain a section of an associated bundle from an image. Then, we give an interpretation of the different groups representations used throughout this paper. Sect. 3 is devoted to the construction of the twisted Polyakov action and the minimization problem with respect to the embedding, that leads to a heat equation of generalized Laplacian. In Sect. 4, we construct optimal connection 1-forms by restricting the twisted Polyakov action along a section and minimizing the functional with respect to the metric of the principal bundle. We give the expressions of the corresponding covariant derivatives on the associated vector bundles for the group representations mentioned above. Sect. 5 is devoted to applications to image processing. We first construct Riemannian metrics on the base manifold and on the embedding spaces for the twisted Polyakov action, from which we obtain explicit expressions of the covariant derivatives. Then, we provide experiments on color images. In Appendix A, we give the definitions of the geometric concepts used throughout the paper. In Appendices B and C, we give some details of computations related to the construction of optimal connection 1-forms and the corresponding covariant derivatives.

2 Images as sections of associated bundles

We refer to the Appendix A for the definitions of principal/associated bundles, and groups representations.

2.1 Construction of the geometric framework

Let $I = (I^1, \dots, I^n): \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ be a n -channels image. Let (e_1, \dots, e_n) be a basis of the vector space \mathbb{R}^n . We write

$$I(x_1, x_2) = I^1(x_1, x_2)e_1 + \dots + I^n(x_1, x_2)e_n \quad (1)$$

Let (ρ, G) be a Lie group representation on \mathbb{R}^n , and e be the neutral element of G . Under its action on \mathbb{R}^n , the group G acts on the basis (e_1, \dots, e_n) . Let \mathcal{P} be the set of bases obtained by the transformations of (e_1, \dots, e_n) under the action of the group G , denoted by \cdot . The action of G on \mathcal{P} is transitive and free. Denoting by π the projection of $\Omega \times \mathcal{P}$ on Ω such that

$$\pi(x_1, x_2, g \cdot (e_1, \dots, e_n)) = (x_1, x_2),$$

the quadruplet $P = (\Omega \times \mathcal{P}, \pi, \Omega, G)$ is a trivial principal bundle, where the global diffeomorphism $\Phi: \Omega \times G \longrightarrow \Omega \times \mathcal{P}$ is given by

$$\Phi(x_1, x_2, g) = (x_1, x_2, g \cdot (e_1, \dots, e_n))$$

From the function I , we construct a (ρ, G) -equivariant function J on P defined by

$$J(x_1, x_2, g \cdot (e_1, \dots, e_n)) = \rho(g)I(x_1, x_2) \quad (2)$$

In particular, we have

$$J(x_1, x_2, (e_1, \dots, e_n)) = I(x_1, x_2) \quad (3)$$

since $\rho(e) = Id$ by property of a group representation.

We construct the associated bundle $E = P \times_{(\rho, G)} \mathbb{R}^n$. Under the correspondance between (ρ, G) -equivariant functions on P and sections of E , J is a section of E of the form $J(x_1, x_2) = [(x_1, x_2, (e_1, \dots, e_n)), I(x_1, x_2)]$, and I the realization of J under the trivializing section $s(x_1, x_2) = (x_1, x_2, (e_1, \dots, e_n)) = \Phi(x_1, x_2, e)$ of P .

2.2 Some Lie groups representations on the color space RGB and their interpretations

2.2.1 RGB and HSL color spaces

Besides the RGB (Red,Green,Blue) color space we consider a HSL (Hue,Saturation,Luminance) color space defined as follows. We set first

$$\begin{pmatrix} Y \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

Then the luminance l , the saturation s and the hue h are respectively given by

$$\begin{aligned} l &= Y \\ s &= \sqrt{C_1^2 + C_2^2} \\ h &= \begin{cases} \arccos(C_2/s) & \text{if } C_2 > 0 \\ 2\pi - \arccos(C_2/s) & \text{otherwise} \end{cases} \end{aligned}$$

Embedding RGB into the quadratic space $(\mathbb{R}^3, \|\cdot\|_2)$ of basis (e_1, e_2, e_3) , we can decompose any color $\alpha = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ with respect to its projection $L(\alpha)$ and its rejection $V(\alpha)$ along the axis $(e_1 + e_2 + e_3)$. The axis $(e_1 + e_2 + e_3)$ is called the **luminance axis**, it encodes the luminance information of a color. Indeed, the luminance of α corresponds to the norm of its projection on the luminance axis. The orthogonal of the luminance axis in $(\mathbb{R}^3, \|\cdot\|_2)$ is called the **chrominance plane**, and encodes the saturation and hue information. It is generated by the vectors $(2e_1 - e_2 - e_3)$ and $(-e_2 + e_3)$. The saturation corresponds to the norm of its projection on the chrominance plane (up to multiplication by the scalar $3/2$), and the hue corresponds to the angle it forms with the projection of the vector e_1 on this plane.

2.2.2 Dual representation of \mathbb{R}^{+*}

Let $v \in \mathbb{R}^3$. The dual representation of \mathbb{R}^{+*} on \mathbb{R}^3 is the map

$$\begin{aligned} \mathbb{R}^{+*} &\longrightarrow GL(\mathbb{R}^3) \\ \rho^C: b &\longmapsto v \longmapsto 1/b v \end{aligned} \quad (4)$$

Simple computations lead to

$$l(\rho^C(b)\alpha) = \rho^C(b)l(\alpha) \quad (5)$$

$$s(\rho^C(b)\alpha) = \rho^C(b)s(\alpha) \quad (6)$$

$$h(\rho^C(b)\alpha) = h(\alpha) \quad (7)$$

2.2.3 Dual representations of $SO(2)$

Let $v \in \mathbb{R}^3$. The dual representations of $SO(2)$ on \mathbb{R}^3 are the maps

$$\begin{aligned} SO(2) &\longrightarrow GL(\mathbb{R}^3) \\ \rho_{\mathcal{P}}^C: \theta &\longmapsto \text{rotation of angle } -\theta \text{ in the plane } \mathcal{P} \end{aligned} \quad (8)$$

where \mathcal{P} is a 2-dimensional subspace of \mathbb{R}^3 .

For \mathcal{P} being the chrominance plane Chr , we have

$$\begin{aligned} l(\rho_{Chr}^C(\theta)\alpha) &= l(\alpha) & s(\rho_{Chr}^C(\theta)\alpha) &= s(\alpha) \\ h(\rho_{Chr}^C(\theta)\alpha) &= h(\alpha) - \theta \end{aligned}$$

2.2.4 Dual representation of $DC(3)$

The group $DC(3)$ is the set of 3×3 diagonal matrices of strictly positive coefficients. We denote by $diag(b_1, b_2, b_3)$ the element of $DC(3)$ of coefficients b_1, b_2, b_3 .

Let $v \in \mathbb{R}^3$. The dual representation of $DC(3)$ on \mathbb{R}^3 is the map

$$\begin{aligned} DC(3) &\longrightarrow GL(\mathbb{R}^3) \\ \rho^C: diag(b_1, b_2, b_3) &\longmapsto v \longmapsto diag(1/b_1, 1/b_2, 1/b_3)v \end{aligned} \quad (9)$$

The representation $(\rho^C, DC(3))$ on the color space RGB does not have interpretation in terms of the hue, saturation and luminance components.

2.2.5 Dual representations of $SO(3)$

Let $v \in \mathbb{R}^3$ and $R_{\theta_1, \theta_2, \theta_3}$ be the matrix of the rotation associated to the Euler angles $\theta_1, \theta_2, \theta_3$. The dual representation of $SO(3)$ on \mathbb{R}^3 is the map

$$\begin{aligned} SO(3) &\longrightarrow GL(\mathbb{R}^3) \\ \rho^C: R_{\theta_1, \theta_2, \theta_3} &\longmapsto v \longmapsto R_{\theta_1, \theta_2, \theta_3}^T v \end{aligned} \quad (10)$$

where T denotes the transpose.

The representation $(\rho^C, SO(3))$ on the color space RGB does not have interpretation in terms of the hue, saturation and luminance components neither.

2.3 Interpretation of the notion of (ρ, G) -equivariance

By the (ρ, G) -equivariant function J we construct, we take into account that some modifications of the light source induce transformations of the pixels. Indeed, we can interpret the fomula (3) as the assignement of the basis (e_1, \dots, e_n) to the light source of the original image I . Then, we assimilate a basis change given by the action $g \longmapsto g \cdot (e_1, \dots, e_n)$ to a modification of the light source. By (2), the representation ρ tells how the pixels of the image change under this basis change, and consequently under the corresponding modification of the light source. Note that the use of the fiber bundle context allows the transformation of the light source to change with respect to the points of Ω .

Taking ρ as the dual representation of a Lie group G on \mathbb{R}^n makes the (ρ, G) -equivariance encode that pixels transform in the same way as the basis does under the action of G . From a mathematical viewpoint, it means that pixels are treated as covectors. It is coherent with the interpretation above where we identified bases

changes with modifications of the light source since pixels clearly transform in the same way as the light source does.

The action of \mathbb{R}^{+*} on the basis (e_1, e_2, e_3) for color images might be interpreted as a homogeneous modification of the intensity of the light source, whereas the action of $DC(3)$ on the basis (e_1, e_2, e_3) might be interpreted as an inhomogeneous modification of the light source in the red, green and blue components. The actions of the groups $SO(3)$ and $SO(2)$ on the basis (e_1, e_2, e_3) are more difficult to interpret in terms of modification of the light source.

3 A variational problem related to heat equations of generalized Laplacians

We refer to the Appendix A. for the definitions of the concepts related to the differential geometry of fiber bundles.

3.1 Riemannian metric on principal bundle induced by a horizontal bundle

Let (P, π, M, G) be a principal bundle, where the base manifold M is of dimension m and the Lie group of structure G is of dimension n . Let g and \mathcal{B} be respectively a Riemannian metric on M and a bi-invariant Riemannian metric on G . Let HP be a horizontal bundle of P and VP be the vertical bundle of P .

We construct a Riemannian metric h on P from g, \mathcal{B} and HP as follows.

Let (e_1, \dots, e_m) be an orthonormal frame of TM , the tangent bundle of M , with respect to g . Let (e_1^h, \dots, e_m^h) be the horizontal lift of (e_1, \dots, e_m) on TP , the tangent bundle of P , with respect to HP . Let (X_1, \dots, X_n) be an orthonormal basis of $T_e G = \mathfrak{g}$ with respect to \mathcal{B} . Let (X_1^v, \dots, X_n^v) be the frame of VP , defined for $i = 1, \dots, n$ by

$$X_i^v(p) = \left(\frac{d}{dt} p \cdot \exp(tX_i) \right)_{|t=0}$$

where \cdot denotes the action of G on the fibers of P . Then, considering the metric h on P given by the Identity matrix field in the frame $(e_1^h, \dots, e_m^h, X_1^v, \dots, X_n^v)$ of TP turns (P, h) into a Riemannian manifold.

Remark 1 The construction makes the bundles HP and VP be orthogonal with respect to h .

Remark 2 By the G -equivariance property of the bundles HP and VP , the metric h is completely determined by its values along a section of P .

3.2 A twisted Polyakov action on principal bundles

Let (P, π, M, G) be a principal bundle, ρ a representation of G on a vector space V of dimension d equipped with the standard scalar product (\cdot, \cdot) . Let $J: P \rightarrow V$ and φ its graph. Let h be a Riemannian metric on P induced by a Riemannian metric g on M and a horizontal bundle HP (see the construction in Sect. 3.1). Let Q be a Riemannian metric on $P \times V$ of the form $\mathcal{H} \oplus \kappa \mathbb{I}_d$, for a Riemannian metric \mathcal{H} on P and strictly positive constant κ , where \mathbb{I}_d denotes the $d \times d$ Identity matrix.

Let $Cas \in \text{End}(V)$ be the Casimir operator defined by

$$Cas = \sum \rho_*(\mathfrak{g}^i)^2$$

for any orthonormal basis $(\mathfrak{g}^1, \dots, \mathfrak{g}^n)$ of the Lie algebra \mathfrak{g} of G , where $\rho_*: \mathfrak{g} \rightarrow \text{End}(V)$ is the Lie algebra representation induced by ρ .

Let S be the Polyakov functional of the embedding $\varphi: (P, h) \rightarrow (P \times V, Q)$ twisted by an action of the Casimir operator

$$S(h, \varphi, Q) = \int_P h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} + \frac{1}{2} (Cas J, J) dP \quad (11)$$

The role of the extra term $1/2(Cas J, J)$ in (11) will appear in the proof of the following Proposition.

Proposition 1 *The twisted Beltrami flow for reaching the function J_{opt} whose graph φ_{opt} minimizes the action (11) preserves the (ρ, G) -equivariance property.*

Proof Let ψ be the embedding of a Riemannian manifold (M_1, g_1) into a Riemannian manifold (M_2, g_2) , and X the corresponding Polyakov action, i.e.

$$X = \int_{M_1} g_1^{\mu\nu} \frac{\partial \psi^i}{\partial x_\mu} \frac{\partial \psi^j}{\partial x_\nu} g_2^{ij} dM_1 \quad (12)$$

From [24], we know that the Euler-Lagrange equations of (12) with respect to ψ are

$$-\frac{1}{2\sqrt{g_1}} g_2^{il} \frac{\partial X}{\partial \psi^l} = \Delta_{g_1} \psi^i + \Gamma_{jk}^i \partial_\mu \psi^j \partial_\nu \psi^k g_2^{\mu\nu} = 0 \quad (13)$$

where Δ_{g_1} is the Laplace-Beltrami on (M_1, g_1) and Γ_{jk}^i are the symbols of the Levi-Civita connection on (M_2, g_2) .

Hence, taking a metric Q of the form $\mathcal{H} \oplus \kappa \mathbb{I}_d$, for a Riemannian metric \mathcal{H} on P and strictly positive constant κ , the Euler-Lagrange equations of (11) with respect to J are

$$(\Delta_h \otimes 1 + 1 \otimes Cas) J = 0 \quad (14)$$

where Δ_h is the Laplace-Beltrami operator on (P, h) .

The twisted Beltrami flow for reaching the solution of (14) is

$$\frac{\partial J_t}{\partial t} = (\Delta_h \otimes 1 + 1 \otimes Cas) J_t, \quad J_0 = J \quad (15)$$

Under the identification between (ρ, G) -equivariant functions of P and sections of $P \times_{(\rho, G)} V$ (see Appendix A.3.1), the operator $(\Delta_h \otimes 1 + 1 \otimes Cas)$ writes

$$\Delta^A := \sum_{ij} g^{ij} \left(\nabla_{\partial_i}^A \nabla_{\partial_j}^A - \sum_k \Gamma_{ij}^k \nabla_{\partial_k}^A \right) \quad (16)$$

It is a generalized Laplacian on $P \times_{(\rho, G)} V$ where the covariant derivative ∇^A is induced by HP (see Appendix A.3.2). We refer to [6] for the proof of the identification between the operators $(\Delta_h \otimes 1 + 1 \otimes Cas)$ and Δ^A .

Let us assume that the initial condition J is (ρ, G) -equivariant. Then, the twisted Beltrami flow (15) may be identified to the heat flow of Δ^A

$$\frac{\partial J_t}{\partial t} = \Delta^A J_t, \quad J_0 = J \quad (17)$$

Finally, as the flow (17) preserves the structure of section of associated bundle, the flow (15) preserves the (ρ, G) -equivariance of the initial condition. \square

4 A class of covariant derivatives on associated bundles

4.1 From horizontal bundles on principal bundles to covariant derivatives on associated bundles

4.1.1 The general case

Let (P, π, M, G) be a principal bundle, ρ a representation of G on a finite dimensional vector space V , and $E = P \times_{(\rho, G)} V$ the associated bundle. Let $S \in \Gamma(E)$ and f_S the corresponding (ρ, G) -equivariant function. Let HP be a horizontal bundle of P . Let $X \in \Gamma(TM)$ and $X^h \in \Gamma(HP)$ be the horizontal lift of X with respect to HP .

Then, there exist a covariant derivative ∇^A on (E, π_E, M) such that the following correspondance holds

$$d_{X^h} f_S \longleftrightarrow \nabla_{X^h}^A S \quad (18)$$

The proof of (18) appears in the proof of the identification between $(\Delta_h \otimes 1 + 1 \otimes Cas)$ and Δ^A [6].

Let ω be the connection 1-form induced by HP . Let s be a section of P and A be the \mathfrak{g} -valued 1-form on M given by

$$A = s^*\omega$$

where $*$ denotes the pull-back map.

Then, relatively to the local frame of (E, π_E, M) induced by s , the covariant derivative is given by

$$\nabla_X^A = d_X + \rho_*(A)(X)$$

where d_X denotes the differential with respect to a tangent vector field X on M .

4.1.2 Covariant derivatives induced by optimal horizontal bundles

Let (P, π, M, G) be a principal bundle and Φ a trivialization of the bundle. Let ρ be a representation of G on a finite-dimensional vector space V . Let s be the section of (P, π, M, G) of the form $s(x_1, x_2) = \Phi(x_1, x_2, e)$ where e is the neutral element of G .

Let $J: P \rightarrow V$ and φ its graph. Let h be a Riemannian metric on P along s induced by a Riemannian metric g on M and a horizontal bundle HP of P (see Sect. 3.1 for the construction of h). Let Q be a Riemannian metric on $P \times V$.

Let $S|_s$ be the restriction along the section s of the twisted Polyakov action (11) of $\varphi: (P, h) \rightarrow (P \times V, Q)$

$$S|_s(h, \varphi, Q) = \int_s h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} + \frac{1}{2} (Cas J, J) \, dM \quad (19)$$

Minimizing the functional (19) with respect to the metric h under the assumption that g is fixed, we obtain the expression of optimal horizontal bundles HP^{opt} along the section s . Then, following the construction of Sect. 4.1.1, we obtain the expression of the corresponding covariant derivatives $\nabla^{A^{opt}}$ on $P \times_{(\rho, G)} V$, relatively to the frame induced by s .

4.1.3 A class of covariant derivatives derived from optimal horizontal bundles

Let ω^{opt} be a connection 1-form induced by an optimal horizontal bundle $H_s P^{opt}$ of Sect. 4.1.2. From ω^{opt} , we derive a class of connection 1-forms ω_f^{opt} such that $\omega_f^{opt} = f\omega^{opt}$ along the section s , for $f \in C^0(M)$.

Then, the \mathfrak{g} -valued 1-form A_f^{opt} on M induced by the section s , i.e. $s^*\omega_f^{opt}$, is fA^{opt} . The corresponding covariant derivatives $\nabla^{A_f^{opt}}$ on $P \times_{(\rho, G)} V$ are given by $\rho_*(A_f^{opt}) = f\rho_*(A^{opt})$ in the frame induced by the section s . Hence, if we denote by Υ_{ij}^k the symbols of $\nabla^{A^{opt}}$ in the frame induced by s , the symbols of $\nabla^{A_f^{opt}}$ are of the form $f\Upsilon_{ij}^k$ in this frame.

For $f \equiv 0$, we have the trivial covariant derivative, whereas for $f \equiv 1$, we obtain the optimal covariant derivatives of Sect. 4.1.2.

Remark 3 The role of the function f appears in the experiments (Sect. 5), where we test heat diffusions of generalized Laplacians of the form $\Delta_f^{A^{opt}}$ for different functions f . We observe that the behaviour of the diffusion depends locally of the values of f . For $f \equiv 0$, this is the heat diffusion of the Laplace-Beltrami operator.

4.2 Explicit expressions on some associated bundles

Let us assume that the base manifold M is of dimension 2. Let (x_1, x_2) be a coordinates system of M and (a_1, \dots, a_n) be a coordinates system of G . By definition, a horizontal bundle HP is a subbundle of rank 2 of the tangent bundle TP of P such that $TP \simeq HP \oplus \mathfrak{g}$. Hence, along the section s , it is of the form

$$H_s P = Vect \left\{ \alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{1k} \partial/\partial a_k, \right. \\ \left. \alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{2k} \partial/\partial a_k \right\} \quad (20)$$

for some functions $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \dots, \gamma_{2n}$ on M such that $\alpha_1\beta_2 - \alpha_2\beta_1$ never vanish.

In what follows, we give expressions of covariant derivatives induced by horizontal bundles minimizing (19). See Appendices B and C for details of computations.

4.2.1 Associated bundle given by the unique dual representation of \mathbb{R}^{+*} on \mathbb{R}^3

Proposition 2 *The field of horizontal spaces $H_s P^{opt}$ of $(P, \pi, M, \mathbb{R}^{+*})$ minimizing (19) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative $\nabla^{A^{opt}}$ in the associated bundle $P \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^3$ are given, relatively to the frame $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and the frame of $P \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^3$ induced by s , by*

$$\Upsilon_{lk}^k(x) = \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij} (x, e)$$

for $l = 1, 2$ and $k = 1, 2, 3$.

4.2.2 Associated bundles given by dual representations of $SO(2)$ on \mathbb{R}^3

Let \mathcal{P} be a 2-dimensional subspace of \mathbb{R}^3 . In the Clifford algebra context, \mathcal{P} might be described by a bivector $\gamma_{12}e_1e_2 + \gamma_{13}e_1e_3 + \gamma_{23}e_2e_3$, for some $\gamma_{12}, \gamma_{13}, \gamma_{23} \in \mathbb{R}$ such that $\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 = 1$.

Proposition 3 *The field of horizontal spaces $H_s P^{opt}$ of $(P, \pi, M, SO(2))$ minimizing (19) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative $\nabla^{A^{opt}}$ in the associated bundle $P \times_{(\rho_P^c, SO(2))} \mathbb{R}^3$ are given, relatively to the frame $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and the frame of $P \times_{(\rho_P^c, SO(2))} \mathbb{R}^3$ induced by s , by*

$$\begin{aligned} \Upsilon_{lp}^k(x) &= -\Upsilon_{lk}^p(x) \\ &= (-1)^{l+1} \gamma_{kp} \left(\frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij} (x, e) \end{aligned}$$

for $l = 1, 2$ and $k, p = 1, 2, 3$ with $k \neq p$.

4.2.3 Associated bundle given by the unique dual representation of $DC(3)$ on \mathbb{R}^3

Proposition 4 *The field of horizontal spaces $H_s P^{opt}$ of $(P, \pi, M, DC(3))$ minimizing (19) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative $\nabla^{A^{opt}}$ in the associated bundle $P \times_{(\rho^c, DC(3))} \mathbb{R}^3$ are given, relatively to the frame $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and the frame of $P \times_{(\rho^c, DC(3))} \mathbb{R}^3$ induced by s , by*

$$\begin{aligned} \Upsilon_{l1}^1(x) &= \frac{1}{c_1 c_2 c_3} \left[c_2 c_3 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right] (x, e) \\ \Upsilon_{l2}^2(x) &= \frac{1}{c_1 c_2 c_3} \left[c_1 c_3 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right] (x, e) \\ \Upsilon_{l3}^3(x) &= \frac{1}{c_1 c_2 c_3} \left[c_1 c_2 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right] (x, e) \end{aligned}$$

for $l = 1, 2$, where

$$c_k = \frac{\partial \varphi^i}{\partial a_k} \frac{\partial \varphi^j}{\partial a_k} Q_{ij}, \quad k = 1, 2, 3$$

4.2.4 Associated bundle given by the unique dual representation of $SO(3)$ on \mathbb{R}^3

Proposition 5 *The field of horizontal spaces $H_s P^{opt}$ of $(P, \pi, M, SO(3))$ minimizing (19) is unique. Moreover, the nonzero symbols of the corresponding covariant derivative $\nabla^{A^{opt}}$ in the associated bundle $P \times_{(\rho^c, SO(3))} \mathbb{R}^3$ are given, relatively to the frame $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and the frame of $P \times_{(\rho^c, SO(3))} \mathbb{R}^3$ induced by s , by*

$$\begin{aligned} \gamma_{l2}^1(x) &= -\gamma_{l1}^2(x) \\ &= \lambda \left[(be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \gamma_{l3}^1(x) &= -\gamma_{l1}^3(x) \\ &= \lambda \left[(ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \gamma_{l3}^2(x) &= -\gamma_{l2}^3(x) \\ &= \lambda \left[(fd - e^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

for $l = 1, 2$, where

$$\begin{aligned} a &= \frac{\partial \varphi^i}{\partial \theta_1} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} & b &= \frac{\partial \varphi^i}{\partial \theta_1} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \\ c &= \frac{\partial \varphi^i}{\partial \theta_1} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} & d &= \frac{\partial \varphi^i}{\partial \theta_2} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \\ e &= \frac{\partial \varphi^i}{\partial \theta_2} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} & f &= \frac{\partial \varphi^i}{\partial \theta_3} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \end{aligned}$$

and

$$\lambda = \frac{1}{2bce + adf - ae^2 - b^2f - c^2d}$$

5 Experiments

5.1 Construction of diffusion process for color image

Let $I: \Omega \rightarrow RGB \subset \mathbb{R}^3$ be a color image. Following Sect. 2, we construct principal bundles $(P_1, \pi_1, \Omega, \mathbb{R}^{+*})$, $(P_2, \pi_2, \Omega, DC(3))$, $(P_3, \pi_3, \Omega, SO(3))$, and corresponding associated bundles $P_1 \times_{(\rho^c, \mathbb{R}^{+*})} \mathbb{R}^3$, $P_2 \times_{(\rho^c, DC(3))} \mathbb{R}^3$ and $P_3 \times_{(\rho^c, SO(3))} \mathbb{R}^3$, as well as a (ρ, G) -equivariant function J such that $J \circ s = I$ on each of these bundles.

We equip the base manifold Ω of the Riemannian metric g given by the following matrix representation

$$\begin{pmatrix} 1 + \kappa \sum_{k=1}^3 I_{x_1}^k{}^2 & \kappa \sum_{k=1}^3 I_{x_1}^k I_{x_2}^k \\ \kappa \sum_{k=1}^3 I_{x_1}^k I_{x_2}^k & 1 + \kappa \sum_{k=1}^3 I_{x_2}^k{}^2 \end{pmatrix} \quad (21)$$

in the frame $(\partial/\partial x_1, \partial/\partial x_2)$, for some strictly positive constant κ .

We equip the embedding space $P_1 \times \mathbb{R}^3$ of the metric

$$Q = \mathbb{I}_2 \oplus \delta \oplus \kappa \mathbb{I}_3, \quad (22)$$

the embedding space $P_2 \times \mathbb{R}^3$ of the metric

$$Q = \mathbb{I}_2 \oplus \delta \mathbb{I}_3 \oplus \kappa \mathbb{I}_3, \quad (23)$$

and the embedding space $P_3 \times \mathbb{R}^3$ of the metric

$$Q = \mathbb{I}_2 \oplus \begin{pmatrix} \delta & 0 & \delta \sin \theta_2 \\ 0 & \delta & 0 \\ \delta \sin \theta_2 & 0 & \delta \end{pmatrix} \oplus \kappa \mathbb{I}_3, \quad (24)$$

for some strictly positive constant δ .

From Sect. 4.2, we construct covariant derivatives $\nabla^{A^{opt}}$ on $P_1 \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^3$, $P_2 \times_{(\rho^C, DC(3))} \mathbb{R}^3$ and $P_3 \times_{(\rho^C, SO(3))} \mathbb{R}^3$. The explicit expressions (under the trivializing section s) of the covariant derivatives with respect to the metrics (22), (23) and (24) are respectively given in Appendix B.1.2, B.2.2 and B.2.3. Then, we construct the covariant derivatives $\nabla^{A_f^{opt}}$ (see Sect. 4.1.3) for the following functions f :

- $f_0 \equiv 0$: this is the trivial covariant derivative.
- $f_1 \equiv 1$: these are the covariant derivatives $\nabla^{A^{opt}}$.
- f_2 defined by

$$f_2 = \begin{cases} 3 & \text{if } \sqrt{\lambda^+ - \lambda^-} \geq 0.5 \\ 0 & \text{if } \sqrt{\lambda^+ - \lambda^-} \leq 0.49 \\ 400 \sqrt{\lambda^+ - \lambda^-} - 196 & \text{otherwise} \end{cases} \quad (25)$$

where λ^+ and λ^- are respectively the highest and lowest eigenvalue fields of (21).

At last, following Sect. 3, we obtain heat equations of the corresponding generalized Laplacians $\Delta^{A_f^{opt}}$

$$\frac{\partial J_t}{\partial t} + \Delta^{A_f^{opt}} J_t = 0, \quad J_0 = J$$

whose solutions are approximated with the Euler scheme

$$J_{t+dt} = J_t + dt \Delta^{A_f^{opt}} J_t, \quad J_0 = J$$

Under the trivializing section s , it gives the scheme

$$I_{t+dt} = I_t + dt \Delta^{A_f^{opt}} I_t, \quad I_0 = I \quad (26)$$

which provides a diffusion of the initial condition I .

5.2 Interpretation of the results

We compare the diffusions induced by generalized Laplacians of the form $\Delta^{A_{f_1}^{opt}}$ and $\Delta^{A_{f_2}^{opt}}$ with the diffusion induced by the Laplace-Beltrami operator $\Delta^{A_{f_0}^{opt}}$. By the chosen metric g (21) of the base manifold, the diffusion induced by the Laplace-Beltrami operator tends to preserve the strong edges and smooth the small ones.

On Fig. 1, we test the diffusion induced by $\Delta^{A_{f_1}^{opt}}$ relatively to the group $SO(3)$ on the color image 'Barbara' (Fig. 1(c)). We apply the scheme (26) with the following parameters: 20 iterations, $dt = 0.1$, $\kappa = 0.001$, and $\delta = 1$. We observe that the diffusion has a similar behaviour than the one induced by the Laplace-Beltrami operator $\Delta^{A_{f_0}^{opt}}$ (Fig. 1(b)), in the sense that it preserves the strong edges and smooth the small ones too. However, it seems that the chromatic information (saturation, hue) of the image is less smoothed by the diffusion induced by $\Delta^{A_{f_1}^{opt}}$, and consequently some edges are better preserved (compare the back of the chair on Fig. 1(b) and Fig. 1(c) with the original one on Fig. 1(a)).

On Fig. 2, we test the diffusion induced by $\Delta^{A_{f_1}^{opt}}$ relatively to the group \mathbb{R}^{+*} on the color image 'Lena' (Fig. 2(c)). We apply the scheme (26) with the following parameters: 50 iterations, $dt = 0.1$, $\kappa = 0.001$, and $\delta = 50$. The parameters were chosen in such a way that the resulting image gets the same memory size than the diffusion induced by the Laplace-Beltrami operator of parameters 20 iterations, $dt = 0.1$, $\kappa = 0.001$ (Fig. 2(b)). Both diffusions tend to preserve the strong edges and smooth the low ones. However, the texture of the hat is better preserved by the diffusion induced by $\Delta^{A_{f_1}^{opt}}$.

From the experiments on Fig. 1 and Fig. 2, we conclude that diffusions induced by $\Delta^{A_{f_1}^{opt}}$ tend to better preserve some edges of the images than the diffusion induced by the Laplace-Beltrami operator, the type of edges depending of the Lie group involved. However, more experiments are required in order to get general properties of the diffusions induced by these operators.

On Fig. 1 and Fig. 2, we have tested diffusions related to covariant derivatives of the form $\nabla^{A_f^{opt}}$ with $f \equiv 1$ for optimal covariant derivatives and $f \equiv 0$ for the trivial covariant derivative. Then, it seems natural to test diffusions related with covariant derivatives of the form $\nabla^{A_f^{opt}}$ with f being a constant strictly greater than 1. Some experiments we made showed that the corresponding diffusions enhance the contrast of the original image; the more f is high, the more the enhancement

is high. However, the drawback is that the noise component is enhanced too.

In order to control the enhancement of the noise, we propose to make f varying with the point, in such a way that the diffusion reduces noise while enhancing the contrast. If the noise is small enough, it might be assimilated with the small local variations of the image, and consequently the function $\sqrt{\lambda^+ - \lambda^-}$ might distinguish noise from relevant information. Following this analysis, we construct the function f_2 (25) in order to perform a diffusion that reduces noise and enhance contrast.

On Fig. 3 and Fig. 4, we test the diffusions induced by $\Delta^{A_{f_2}^{opt}}$ relatively to the groups \mathbb{R}^{+*} , $\text{DC}(3)$ and $\text{SO}(3)$ on the color images 'Barbara' (Fig. 3) and 'Lake' (Fig. 4). The experiments confirm what we were expected. Indeed, the noise is reduced (see for instance the floor on Fig. 3(c,d,e)), and the contrast of the image is increased. Once again, the nature of the enhancement depends of the Lie group involved. Dealing with the group $\text{SO}(3)$, it is mainly a chromatic enhancement (see for instance the back of the chair Fig. 3(c)), whereas it is more a enhancement in the luminance component for the group \mathbb{R}^{+*} . Regarding the group $\text{DC}(3)$, the enhancement is mainly in the luminance component, but taking more into account the chromatic information than the group \mathbb{R}^{+*} (compare the color of leaves between Fig. 4(d) and Fig. 4(e)).

6 Conclusion

We constructed a class of covariant derivatives on associated bundles from connection 1-forms on principal bundles and Lie groups representations. The connection 1-forms we constructed are optimal in the sense that they are solution of a minimization problem. The covariant derivatives we obtained take into account the local geometry of the images, as expected. The Lie groups representations we considered are the dual representations of groups acting on the color space RGB : \mathbb{R}^{+*} , $\text{SO}(2)$, $\text{DC}(3)$ and $\text{SO}(3)$. By the heat equations of the generalized Laplacians induced by the covariant derivatives, we obtained diffusions that preserve strong edges and smooth the small ones. Comparisons with the Beltrami flow showed that our new approach better preserve some edges, the type of edges depending of the Lie group involved. Then, we constructed covariant derivatives derived from the 'optimal' covariant derivatives. By the corresponding heat equations, we obtained diffusions that enhance the contrast of the initial image. Once again, the characteristics of the enhancement depend of the Lie group involved. Further work will be

devoted to perform a theoretical study of the generalized Laplacians constructed in this paper and their heat equations. For instance, we are interested in the analysis of the exact role of the Lie group involved.

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Appendix A. Differential Geometry of Associated Bundles

A.1. Differential geometry of vector bundles

Definition 1 A smooth **vector bundle** of rank n is a triplet (E, π, M) where M and E are two C^∞ manifolds, and $\pi: E \rightarrow M$ is a surjective map such that the preimage $\pi^{-1}(x)$ of $x \in M$ is endowed with a structure of vector space of dimension n . M is called the **base manifold** and E the **total space** of the vector bundle. The set $\pi^{-1}(x)$ is called the **fiber** over x , and is denoted by E_x .

The vector bundle is said to be **locally trivial** if the following conditions hold: for each $x \in M$, there is a neighborhood U of x and a diffeomorphism $\phi: U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$ satisfying $\pi \circ \phi(x, f) = x \forall f \in \mathbb{R}^n$, and such that the restriction $\phi_x: \mathbb{R}^n \rightarrow E_x$ is a vector space isomorphism. The couple (U, ϕ) is called a **local trivialization**.

The vector bundle is said to be **trivial** if there exists a diffeomorphism $\Phi: M \times \mathbb{R}^n \rightarrow E$ satisfying $\pi \circ \Phi(x, f) = x$, and such that the restriction $\Phi_x: \mathbb{R}^n \rightarrow E_x$ is a vector space isomorphism.

For shortness, (E, π, M) is sometimes denoted by E .

Example 1 Let M be a C^∞ manifold of dimension m . The disjoint union of tangent spaces $TM := \bigsqcup T_x M$ for $x \in M$, is the total space of a vector bundle (TM, π, M) of rank m called the **tangent bundle of M** .

Definition 2 A metric h on a vector bundle is the assignment of a scalar product h_x on each fiber $\pi^{-1}(x)$.

Example 2 A Riemannian metric on a manifold is a positive definite metric on its tangent bundle.

Definition 3 A **section** of a smooth vector bundle (E, π, M) is a differentiable map $S: M \rightarrow E$ such that $\pi \circ S = \text{Id}_M$.

Let (f_1, \dots, f_n) be a basis of \mathbb{R}^n . In a local trivialization (U, ϕ) of (E, π, M) , any section may be written

$$S(x) = \sum_{i=1}^n s_i(x) \phi(x, f_i)$$

for some functions $s_i \in C^\infty(U)$.

The set $\{\phi(\cdot, f_1), \dots, \phi(\cdot, f_n)\}$ is called a **local frame** of (E, π, M) .

The set of sections of (E, π, M) is denoted by $\Gamma(E)$.

Example 3 Tangent vector fields on M are the sections of the tangent bundle (TM, π, M) .

Definition 4 A **covariant derivative** on (E, π, M) is a map $\nabla^E: \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E)$ satisfying the following axioms:

$$\begin{aligned} -\nabla_{fX+gY}^E S &= f\nabla_X^E S + g\nabla_Y^E S \\ -\nabla_X^E fS &= (d_X f)S + f\nabla_X^E S \end{aligned}$$

for $f, g \in C^\infty(M)$, $X, Y \in \Gamma(TM)$ and $S \in \Gamma(E)$.

Hence, a covariant derivative on (E, π, M) may be written as $d + \omega$ for some $\omega \in \Gamma(T^*M \otimes \text{End}(E))$.

In local frames (e_1, \dots, e_n) of E and (X_1, \dots, X_m) of TM , a covariant derivative is determined by $n^2 \times m$ functions Υ_{ij}^k such that

$$\nabla_{X_i}^E e_j = \sum_{k=1}^n \Upsilon_{ij}^k e_k$$

Example 4 The **Levi-Cevita connection** is the covariant derivative on the tangent bundle of a Riemannian manifold (M, g) determined by the m^3 functions

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij})$$

with respect to the local frame $(\partial/\partial x_1, \dots, \partial/\partial x_m)$ of TM given by a local coordinates system (x_1, \dots, x_m) on M .

A.2. Differential geometry of principal bundles

Definition 5 A smooth **principal bundle** is a quadruplet (P, π, M, G) where M and P are two C^∞ manifolds, G is a Lie group, $\pi: P \rightarrow M$ is a surjective map such that for all $x \in M$, the preimage $\pi^{-1}(x)$ is diffeomorphic to G and there is an action \cdot of G on P satisfying:

- $\pi(p \cdot g) = \pi(p)$ for $p \in \pi^{-1}(x)$ and $g \in G$.
- the restriction $\cdot: G \times \pi^{-1}(x) \rightarrow \pi^{-1}(x)$ is free and transitive.

M is called the **base manifold**, P the **total space** and G the **Lie group structure** of the principal bundle. The set $\pi^{-1}(x)$ is called the **fiber** over x , and is denoted by P_x .

The principal bundle is said to be **locally trivial** if the following conditions hold: for each $x \in M$, there is a neighborhood U of x and a diffeomorphism $\phi: U \times G \rightarrow \pi^{-1}(U)$ satisfying $\pi \circ \phi(x, g) = x$, and such that the map $\phi_x: G \rightarrow P_x$ is compatible with the restriction $\cdot: G \times P_x \rightarrow P_x$, i.e. $\phi_x(ga) = \phi_x(g) \cdot a$. The couple (U, ϕ) is called a **local trivialization**.

The principal bundle is said to be **trivial** if there exists a diffeomorphism $\Phi: M \times G \rightarrow P$ satisfying $\pi \circ \Phi(x, g) = x$, and such that the restriction $\Phi_x: G \rightarrow P_x$ is compatible with the action $\cdot: G \times P_x \rightarrow P_x$.

For shortness, (P, π, M, G) is sometimes denoted by P .

Example 5 Let (M, g) be a Riemannian manifold of dimension m and TM its tangent bundle. The disjoint union of oriented orthonormal bases of $T_x M$ with respect to $g(x)$, for $x \in M$, forms the total space of a principal bundle $PSO(TM)$ over M of structure Lie group $SO(n)$ called the **bundle of orthonormal frames** of TM .

Definition 6 A **section** of a smooth principal bundle (P, π, M, G) is a differentiable map $S: M \rightarrow P$ such that $\pi \circ S = Id_M$.

Let (U, ϕ) be a local trivialization of (P, π, M, G) , and $s: U \rightarrow P$ defined by $s(x) = \phi(x, e)$. The section s is called a **local trivializing section** of P and any local section S of P takes the form

$$S(x) = s(x) \cdot g(x)$$

for some function $g: U \rightarrow G$.

The function g is called the **local trivialization of the section** S .

The set of sections of (P, π, M, G) is denoted by $\Gamma(P)$.

Definition 7 Let (P, π, M, G) be a principal bundle. Let $T_p \pi: T_p P \rightarrow T_{\pi(p)} M$ be the linear tangent map of π at $p \in P$. A tangent vector $X \in T_p P$ is called **vertical** if $T_p \pi(X) = 0$. We denote by V_p the set of vertical vectors at the point p . It is the vector space generated by vectors X_p^v of the form

$$X_p^v = \left(\frac{d}{dt} p \cdot \exp(tX) \right)_{|t=0}, \text{ for } X \in \mathfrak{g}.$$

The set $VP = \bigsqcup_{p \in P} V_p$ forms the total space of a vector bundle over P called the **vertical bundle** of P . It is a subbundle of the tangent bundle of P .

Let $\tilde{R}_g: p \mapsto p \cdot g$. The map $T_p \tilde{R}_g: T_p P \rightarrow T_{p \cdot g} P$ maps V_p on $V_{p \cdot g}$. This property is called **G -equivariance**.

Definition 8 Let (P, π, M, G) be a principal bundle and VP the vertical bundle bundle of P . Let $x \in M$ and $p \in P_x$. Let H_p be a subspace of $T_p P$ such that $T_p P = H_p \oplus V_p$. A vector $X_p^h \in H_p$ is called **horizontal**. We extend H_p on P_x by setting a G -equivariance property, i.e. we put

$$H_{p \cdot g} = T_p \tilde{R}_g H_p$$

Repeating the construction for each $x \in M$ (under an assumption of smoothness of the sets H_p with respect to x) turns the set $HP: = \bigsqcup_{p \in P} H_p$ into the total space of a vector bundle over P called a **horizontal bundle** of P . It is a subbundle of the tangent bundle of P .

Definition 9 Let (P, π, M, G) be a principal bundle, VP its vertical bundle and HP a horizontal bundle of P . Let $x \in M$ and $p \in P_x$. From the surjectivity of the projection π , the map $T_p \pi$ is surjective too. Moreover, as $V_p = \text{Ker } T_p \pi$ and $\dim H_p P = \dim T_x M$, we deduce that the map

$$T_p \pi: H_p \rightarrow T_x M$$

is an isomorphism. Given $X_x \in T_x M$, the unique vector $X_p^h \in H_p$ such $T_p \pi(X_p^h) = X_x$ is called **horizontal lift** of X_x with respect to HP .

Definition 10 Let (P, π, M, G) be a principal bundle, VP its vertical bundle and HP a horizontal bundle of P . The \mathfrak{g} -valued 1-form ω defined on P by

$$\begin{cases} \omega_p(X_p) = 0 & \text{for } X_p \in H_p P \\ \omega_p(X_p) = A & \text{for } X_p = \left(\frac{d}{dt} p \cdot \exp(tA) \right)_{|t=0} \in V_p P \end{cases}$$

is called a **connection 1-form**.

As a consequence of the G -equivariance of the vertical and horizontal bundle, the connection 1-form satisfies

$$\omega_{p \cdot g}(T_p \tilde{R}_g X_p) = g^{-1} \omega_p(X_p) g$$

when G is a matrix Lie group.

A.3. From principal to vector bundles

A.3.1. Definitions

Definition 11 Let G be a Lie group and V a vector space. A **representation** ρ of G on V is a group morphism $\rho: G \rightarrow GL(V)$. The representation ρ determines a group representation ρ^C on the dual space V^* of V called the **dual representation** of ρ , and defined by $\rho^C(g) = \overline{\rho(g^{-1})}$ where $\overline{}$ denotes the transpose.

Definition 12 Let (P, π, M, G) be a principal bundle, ρ a representation of G on V of dimension n , and $E = (P \times V)/G$, i.e. a point in E is of the form

$$[p, f] := \{(p \cdot g, \rho(g^{-1})f), g \in G\}$$

where $p \in P$ and $f \in V$. Let $\pi_E: E \rightarrow M$ given by $\pi_E[p, f] = \pi(p)$. Then, the triplet (E, π_E, M) forms a vector bundle of rank n , called **vector bundle associated** to P and denoted by $P \times_{(\rho, G)} V$.

Let (U, ϕ) be a local trivialization of (P, π, M, G) and s the local trivializing section of P given by $s(x) = \phi(x, e)$. Then, the map

$$\begin{aligned} \chi_s: U \times V &\rightarrow \pi_E^{-1}(U) \\ (x, f) &\mapsto [s(x), f] \end{aligned}$$

is a **local trivialization** of $P \times_{(\rho, G)} V$.

Let (f_1, \dots, f_n) be a basis of V .

Then, the set $(\chi_s(\cdot, f_1), \dots, \chi_s(\cdot, f_n))$ determines a local frame of (E, π_E, M) .

Example 6 Let (M, g) be a Riemannian manifold of dimension m , and $(PSO(TM), \pi, M, SO(m))$ be the principal bundle of orthonormal frames of TM . The tangent bundle of M is a vector bundle associated to $PSO(TM)$ under the natural representation ρ of $SO(m)$ on \mathbb{R}^m , and might be written $PSO(TM) \times_{(\rho, SO(m))} \mathbb{R}^m$.

Definition 13 Let (P, π, M, G) be a principal bundle and ρ a representation of G on a finite-dimensional vector space V . A function $J: P \rightarrow V$ is called **(ρ, G) -equivariant** if it satisfies

$$J(p \cdot g) = \rho(g)J(p)$$

We denote by $C^\infty(P, V)^{(\rho, G)}$ the set of smooth (ρ, G) -equivariant functions on P .

There is a correspondance between sections of associated bundles and (ρ, G) -equivariant functions on principal bundles. Indeed, given $f \in C^\infty(P, V)^{(\rho, G)}$, we put

$S_f(x) = [p, f(p)]$ for any $p \in \pi^{-1}(x)$. By the (ρ, G) -equivariance property of f , the map S_f is independent of the choice of p . Hence, S_f is a section of $E = P \times_{(\rho, G)} V$. Conversely, for $S \in \Gamma(E)$, we put $f_S(p) = v$ such that $S \circ \pi(p) = [p, v]$. We observe that f_S is (ρ, G) -equivariant.

A.3.2. From horizontal bundles on principal bundles to covariant derivatives on associated bundles

Let (P, π, M, G) be a principal bundle, ρ a representation of G on a finite-dimensional vector space V , and $E = P \times_{(\rho, G)} V$ the associated bundle. Let $S \in \Gamma(E)$ and f_S the corresponding (ρ, G) -equivariant function. Let HP be a horizontal bundle of P . Let $X \in \Gamma(TM)$ and $X^h \in \Gamma(TP)$ be the horizontal lift of X with respect to HP . Then, there exist a covariant derivative ∇^A on (E, π_E, M) such that the following correspondence holds

$$d_{X^h} f_S \longleftrightarrow \nabla_X^A S$$

Let ω be the connection 1-form induced by HP . Let s be a section of P and A be the \mathfrak{g} -valued 1-form on M given by

$$A = s^* \omega$$

where $*$ denotes the pull-back map.

Then, relatively to the local frame of (E, π_E, M) induced by s , the covariant derivative is given by

$$\nabla_X^A = d_X + \rho_*(A)(X)$$

where $\rho_* : \mathfrak{g} \rightarrow \text{End}(V)$ is the Lie algebra representation induced by ρ .

Appendix B. Optimal connection 1-forms and corresponding covariant derivatives

B.1. The case of Lie groups of dimension 1

B.1.1. Construction of optimal connection 1-forms

Computing the Euler-Lagrange equations of (19) with respect to the functions $\alpha_k, \beta_k, \gamma_k$ for $k = 1, 2$, we obtain $\gamma_k =$

$$\left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[\alpha_k \left(\frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_k \left(\frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right].$$

It defines a unique horizontal field $H_s P$ along s whose expression is obtained from (20). We refer to Appendix C.1 for more details.

Let X be a tangent vector field on P along the section s

of the form $\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + \gamma \partial/\partial a$ for some functions α, β, γ on M . The connection 1-form ω induced by HP is given along s by $\omega(X) = \left[\gamma + \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \right.$

$$\left. \times \left(-\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} - \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right] \partial/\partial a$$

The 1-form $A = s^* \omega$ on M is given by $A(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2)(x) = \omega(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + 0 \partial/\partial a)(x, e)$.

B.1.2. Corresponding covariant derivative for the unique dual representation ρ^C of \mathbb{R}^{+*} on \mathbb{R}^3

Writing the 1-form A as $A_1 \partial/\partial a$, we have $\rho_*^C(A) = -A_1 \mathbb{I}_3$.

Hence, the nonzero symbols of the corresponding covariant derivative ∇^A are given in the frames $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$ of $P \times_{(\rho^C, \mathbb{R}^{+*})} \mathbb{R}^3$ by

$$\begin{aligned} \Upsilon_{lk}^k(x) &= (\rho_*^C A)_{kk}(\partial/\partial x_l)(x) \\ &= \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) \end{aligned}$$

for $l = 1, 2$ and $k = 1, 2, 3$.

Taking a metric Q of the form $\mathbb{I}_2 \oplus \delta \oplus \kappa \mathbb{I}_3$ in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a, e_1, e_2, e_3)$, we obtain

$$\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) = \delta + \kappa \left(\sum_{k=1}^3 I^{k^2} \right)(x)$$

and

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) = -\kappa \left(\sum_{k=1}^3 I_{x_l}^k I^k \right)(x)$$

B.1.3. Corresponding covariant derivative for a class of unitary dual representations ρ^C of $SO(2)$ on \mathbb{R}^3

Let \mathcal{P} be a 2-dimensional subspace of \mathbb{R}^3 , and the corresponding bivector $\gamma_{12}e_1e_2 + \gamma_{13}e_1e_3 + \gamma_{23}e_2e_3$ for $\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 = 1$. Writing the 1-form A as $A_1 \partial/\partial \theta$, we have

$$\rho_{\mathcal{P}}^C(A) = A_1 \begin{pmatrix} 0 & -\gamma_{12} & -\gamma_{13} \\ \gamma_{12} & 0 & -\gamma_{23} \\ \gamma_{13} & \gamma_{23} & 0 \end{pmatrix}$$

Hence, the nonzero symbols of the corresponding covariant derivative ∇^A are given in the frames

$(\partial/\partial x_1, \partial/\partial x_2)$ of TM and $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$ of $P \times_{(\rho^c, sO(2))} \mathbb{R}^3$ by

$$\begin{aligned} \Upsilon_{lp}^k(x) &= (\rho_{p^*}^c A)_{kp}(\partial/\partial x_l)(x) = -\Upsilon_{lk}^p(x) \\ &= (-1)^{l+1} \gamma_{kp} \left(\frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) \end{aligned}$$

for $l = 1, 2$ and $k, p = 1, 2, 3$ with $k \neq p$.

Taking a metric Q of the form $\mathbb{I}_2 \oplus \delta \oplus \kappa \mathbb{I}_3$ in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \theta, e_1, e_2, e_3)$, we obtain

$$\begin{aligned} \frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) &= \delta + \\ \kappa \left((\gamma_{12} I^2 + \gamma_{13} I^3)^2 + (\gamma_{12} I^1 - \gamma_{23} I^3)^2 + (\gamma_{13} I^1 + \gamma_{23} I^2)^2 \right) &(x) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) &= -\kappa \times \left(I_{x_l}^1 (-\gamma_{12} I^2 - \gamma_{13} I^3) + \right. \\ &\left. I_{x_l}^2 (\gamma_{12} I^1 - \gamma_{23} I^3) + I_{x_l}^3 (\gamma_{13} I^1 + \gamma_{23} I^2) \right) (x) \end{aligned}$$

B.2. The case of Lie groups of dimension 3

B.2.1. Construction of optimal connection 1-forms

Computing the Euler-Lagrange equations of (19) with respect to the functions $\alpha_l, \beta_l, \gamma_{lj}$ for $l = 1, 2$ and $j = 1, 2, 3$, we obtain

$$\begin{aligned} \gamma_{11} &= \lambda \left[\alpha_l \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\quad \left. \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \\ &\quad \left. \left. + (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \end{aligned}$$

$$\begin{aligned} \gamma_{12} &= \lambda \left[\alpha_l \left((ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\quad \left. \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left((ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \\ &\quad \left. \left. + (af - c^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \end{aligned}$$

$$\begin{aligned} \gamma_{13} &= \lambda \left[\alpha_l \left((be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\quad \left. \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left((be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \end{aligned}$$

$$\left. \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right]$$

where $\lambda = 1/(2cbe + fad - e^2a - fb^2 - c^2d)$ and

$$a = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \quad b = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \quad c = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij}$$

$$d = \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \quad e = \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \quad f = \frac{\partial \varphi^i}{\partial a_3} \frac{\partial \varphi^j}{\partial a_3} Q_{ij}$$

It defines a unique horizontal field $H_s P$ along s whose expression is obtained from (20). We refer to Appendix C.2 for more details.

Let X be a tangent vector field on P along the section s of the form $\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + \sum_{k=1}^3 \gamma_k \partial/\partial a_k$ for some functions α, β, γ_k on M . The connection 1-form ω induced by HP is given along s by $\omega(X) =$

$$\begin{aligned} &\left[\gamma_1 - \lambda \left[(fd - e^2) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ &\quad \left. \left. + (ce - fb) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (be - dc) \times \right. \right. \\ &\quad \left. \left. \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_1 \\ &+ \left[\gamma_2 - \lambda \left[(ce - fb) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ &\quad \left. \left. + (af - c^2) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (bc - ae) \times \right. \right. \\ &\quad \left. \left. \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_2 \\ &+ \left[\gamma_3 - \lambda \left[(be - dc) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ &\quad \left. \left. + (bc - ae) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (ad - b^2) \times \right. \right. \\ &\quad \left. \left. \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_3 \end{aligned}$$

The 1-form $A = s^* \omega$ on M is given by $A(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2)(x) =$

$$\omega(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + 0 \partial/\partial a_1 + 0 \partial/\partial a_2 + 0 \partial/\partial a_3)(x, e)$$

B.2.2. Corresponding covariant derivative for the unique dual representation ρ^C of $DC(3)$ on \mathbb{R}^3

Writing the 1-form A as $A_1\partial/\partial a_1 + A_2\partial/\partial a_2 + A_3\partial/\partial a_3$, we have $\rho_*^C(A) = -\text{diag}(A_1, A_2, A_3)$.

Hence, the nonzero symbols of the corresponding covariant derivative ∇^A are given in the frames $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$ of $P \times_{(\rho^C, DC(3))} \mathbb{R}^3$ by

$$\begin{aligned} \Upsilon_{11}^1(x) &= (\rho_* A)_{11}(\partial/\partial x_1)(x) \\ &= \lambda \left[d f \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \Upsilon_{12}^2(x) &= (\rho_* A)_{22}(\partial/\partial x_1)(x) \\ &= \lambda \left[a f \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \Upsilon_{13}^3(x) &= (\rho_* A)_{33}(\partial/\partial x_1)(x) \\ &= \lambda \left[a d \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right] (x, e) \end{aligned}$$

for $l = 1, 2$.

Taking a metric Q of the form $\mathbb{I}_2 \oplus \delta \mathbb{I}_3 \oplus \kappa \mathbb{I}_3$ in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3, e_1, e_2, e_3)$, we obtain

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_k} Q_{ij}(x, e) = -\kappa I_{x_l}^k I^k(x)$$

for $k = 1, 2, 3$, and

$$\begin{aligned} a(x, e) &= \delta + \kappa I^1(x) & d(x, e) &= \delta + \kappa I^2(x) \\ f(x, e) &= \delta + \kappa I^3(x) \end{aligned}$$

B.2.3. Corresponding covariant derivative for the unique dual representation ρ^C of $SO(3)$ on \mathbb{R}^3

Writing the 1-form A as $A_1\partial/\partial \theta_1 + A_2\partial/\partial \theta_2 + A_3\partial/\partial \theta_3$, we have

$$\rho_*^C(A) = \begin{pmatrix} 0 & -A_3 & -A_2 \\ A_3 & 0 & -A_1 \\ A_2 & A_1 & 0 \end{pmatrix}$$

Hence the nonzero symbols of the corresponding covariant derivative ∇^A are given in the frames $(\partial/\partial x_1, \partial/\partial x_2)$ of TM and $(\chi_s(\cdot, e_1), \chi_s(\cdot, e_2), \chi_s(\cdot, e_3))$ of $P \times_{(\rho^C, SO(3))} \mathbb{R}^3$ by

$$\begin{aligned} \gamma_{12}^1(x) &= (\rho_*^C A)_{12}(\partial/\partial x_1)(x) = -\gamma_{11}^2(x) \\ &= \lambda \left[(be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \gamma_{13}^1(x) &= (\rho_*^C A)_{13}(\partial/\partial x_1)(x) = -\gamma_{11}^3(x) \\ &= \lambda \left[(ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

$$\begin{aligned} \gamma_{13}^2(x) &= (\rho_*^C A)_{23}(\partial/\partial x_1)(x) = -\gamma_{12}^3(x) \\ &= \lambda \left[(fd - e^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

for $l = 1, 2$.

Taking a metric Q of the form

$$\mathbb{I}_2 \oplus \begin{pmatrix} \delta & 0 & \delta \sin \theta_2 \\ 0 & \delta & 0 \\ \delta \sin \theta_2 & 0 & \delta \end{pmatrix} \oplus \kappa \mathbb{I}_3$$

in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \theta_1, \partial/\partial \theta_2, \partial/\partial \theta_3, e_1, e_2, e_3)$, we obtain

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij}(x, e) = -\kappa (I_{x_l}^2 I^3 - I_{x_l}^3 I^2)(x)$$

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij}(x, e) = -\kappa (I_{x_l}^1 I^3 - I_{x_l}^3 I^1)(x)$$

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij}(x, e) = -\kappa (I_{x_l}^1 I^2 - I_{x_l}^2 I^1)(x)$$

and

$$\begin{aligned} a(x, e) &= \delta + \kappa (I^2{}^2 + I^3{}^2)(x) & b(x, e) &= \kappa I^1 I^2(x) \\ c(x, e) &= -\kappa I^1 I^3(x) & d(x, e) &= \delta + \kappa (I^1{}^2 + I^3{}^2)(x) \\ e(x, e) &= \kappa I^2 I^3(x) & f(x, e) &= \delta + \kappa (I^1{}^2 + I^2{}^2)(x) \end{aligned}$$

Appendix C. Solutions of Euler-Lagrange Equations

C.1. The case of Lie groups of dimension 1

The Euler-Lagrange equations of the functional (19) with respect to $H_s P$ are

$$*\frac{\partial S|_s}{\partial \alpha_1} = 0$$

$$\begin{aligned} \iff & (\beta_2\gamma_1 - \beta_1\gamma_2) \left[(\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_2 - g_{22}\beta_2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_2 - g_{11}\alpha_2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_2 - g_{12}\beta_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_2 - g_{12}\alpha_2) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \alpha_2} = 0$$

$$\begin{aligned} \iff & (\beta_2\gamma_1 - \beta_1\gamma_2) \left[(\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \beta_1} = 0$$

$$\begin{aligned} \iff & (\alpha_2\gamma_1 - \alpha_1\gamma_2) \left[(\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_2 - g_{22}\beta_2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_2 - g_{11}\alpha_2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_2 - g_{12}\beta_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_2 - g_{12}\alpha_2) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \beta_2} = 0$$

$$\begin{aligned} \iff & (\alpha_2\gamma_1 - \alpha_1\gamma_2) \left[(\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right. \\ & \left. + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right] = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \gamma_1} = 0$$

$$\begin{aligned} \iff & (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ & + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \end{aligned}$$

$$*\frac{\partial S|_s}{\partial \gamma_2} = 0$$

$$\begin{aligned} \iff & (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ & + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \end{aligned}$$

Finally, the Euler-Lagrange equations reduce to the equations $\frac{\partial S|_s}{\partial \gamma_1} = 0$ and $\frac{\partial S|_s}{\partial \gamma_2} = 0$, i.e. to the following system

$$\left\{ \begin{array}{l} (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_1 - g_{22}\beta_1) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_1 - g_{11}\alpha_1) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_1 - g_{12}\beta_1) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_1 - g_{12}\alpha_1) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \\ (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\alpha_2 - g_{22}\beta_2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} + (\alpha_1\beta_2 - \alpha_2\beta_1)(g_{12}\beta_2 - g_{11}\alpha_2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ + \left((\alpha_1\gamma_2 - \alpha_2\gamma_1)(g_{11}\alpha_2 - g_{12}\beta_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)(g_{22}\beta_2 - g_{12}\alpha_2) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} = 0 \end{array} \right.$$

We obtain a linear system with respect to γ_1 and γ_2 of the form

$$D \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = (\alpha_1\beta_2 - \alpha_2\beta_1) \times \begin{pmatrix} (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \end{pmatrix} \quad (27)$$

where the coefficients of the matrix D are

$$\begin{aligned} D_{11} &= \left(\alpha_2(\alpha_2 g_{11} + \beta_2 g_{12}) + \beta_2(\beta_2 g_{22} + \alpha_2 g_{12}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ D_{12} &= - \left(\alpha_1(\beta_2 g_{12} + \alpha_2 g_{11}) + \beta_1(\alpha_2 g_{12} + \beta_2 g_{22}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ D_{21} &= - \left(\alpha_1(\beta_2 g_{12} + \alpha_2 g_{11}) + \beta_1(\alpha_2 g_{12} + \beta_2 g_{22}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \\ D_{22} &= \left(\alpha_1(\alpha_1 g_{11} + \beta_1 g_{12}) + \beta_1(\beta_1 g_{22} + \alpha_1 g_{12}) \right) \frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \end{aligned}$$

The solution of the system (27) is

$$\begin{aligned} \gamma_1 &= \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[\alpha_1 \left(\frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_1 \left(\frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right] \\ \gamma_2 &= \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[\alpha_2 \left(\frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_2 \left(\frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right] \end{aligned}$$

Conclusion: $H_s P =$

$$\begin{aligned} &Vect \left\{ \alpha_1 \partial / \partial x_1 + \beta_1 \partial / \partial x_2 + \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[-\alpha_1 \left(\frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} \right) + \beta_1 \left(\frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} \right) \right] \partial / \partial a, \right. \\ &\quad \left. \alpha_2 \partial / \partial x_1 + \beta_2 \partial / \partial x_2 + \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[-\alpha_2 \left(\frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} \right) + \beta_2 \left(\frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} \right) \right] \partial / \partial a \right\} \end{aligned}$$

C.2. The case of Lie groups of dimension 3

As for the case of Lie groups of dimension 1, the Euler Lagrange equations of the functional (19) with respect to $H_s P$ reduce to the equations $\frac{\partial S_l}{\partial \gamma_{ij}} = 0$, for $l = 1, 2, j = 1, 2, 3$, and we obtain a linear system with respect to the functions γ_{ij} of the form

$$D \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{pmatrix} = (\alpha_1 \beta_2 - \alpha_2 \beta_1) \times \begin{pmatrix} (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \\ (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \\ (\alpha_2 g_{12} + \beta_2 g_{22}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} - (\beta_2 g_{12} + \alpha_2 g_{11}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \\ (\alpha_1 g_{11} + \beta_1 g_{12}) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} - (\beta_1 g_{22} + \alpha_1 g_{12}) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \end{pmatrix} \quad (28)$$

where

$$D = \begin{pmatrix} aG & bG & cG & aF & bF & cF \\ bG & dG & eG & bF & dF & eF \\ cG & eG & fG & cF & eF & fF \\ aF & bF & cF & aE & bE & cE \\ bF & dF & eF & bE & dE & eE \\ cF & eF & fF & cE & eE & fE \end{pmatrix}$$

and

$$\begin{aligned} a &= \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} & b &= \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} & c &= \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ d &= \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} & e &= \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} & f &= \frac{\partial \varphi^i}{\partial a_3} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ E &= \alpha_1(g_{11}\alpha_1 + g_{12}\beta_1) + \beta_1(g_{22}\beta_1 + g_{12}\alpha_1) \\ F &= -[\alpha_1(g_{12}\beta_2 + g_{11}\alpha_2) + \beta_1(g_{12}\alpha_2 + g_{22}\beta_2)] \\ G &= \alpha_2(g_{11}\alpha_2 + g_{12}\beta_2) + \beta_2(g_{22}\beta_2 + g_{12}\alpha_2) \end{aligned}$$

The solution of the system (28) is

$$\begin{aligned} \gamma_{11} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[\alpha_1 \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \\ &\quad \left. + \beta_1 \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \\ \gamma_{12} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[\alpha_1 \left((ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \\ &\quad \left. + \beta_1 \left((ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \\ \gamma_{13} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[\alpha_1 \left((be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \\ &\quad \left. + \beta_1 \left((be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \\ \gamma_{21} &= \frac{1}{2cbe + fad - e^2a - fb^2 - c^2d} \left[\alpha_2 \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right. \end{aligned}$$

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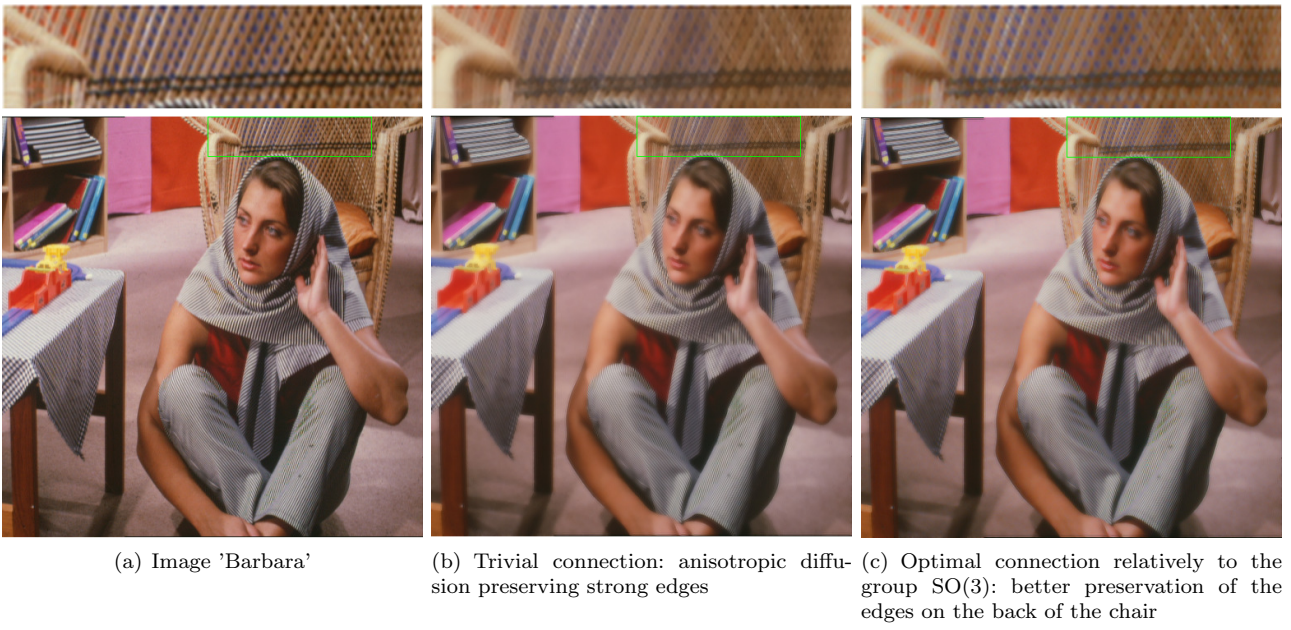


Fig. 1 Heat diffusions of the color image 'Barbara' induced by generalized Laplacians associated with connection 1-forms



Fig. 2 Heat diffusions of the color image 'Lena' induced by generalized Laplacians associated with connection 1-forms



(a) Image 'Barbara'



(b) Trivial connection: no enhancement of the contrast



(c) Connection relatively to the group $SO(3)$: enhancement of the contrast in the chrominance component



(d) Connection relatively to the group \mathbb{R}^{+*} : enhancement of the contrast in the luminance component

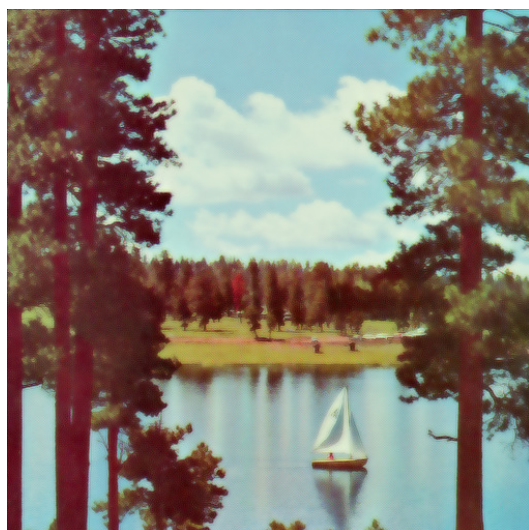


(e) Connection relatively to the group $DC(3)$: enhancement of the contrast in luminance and chrominance components

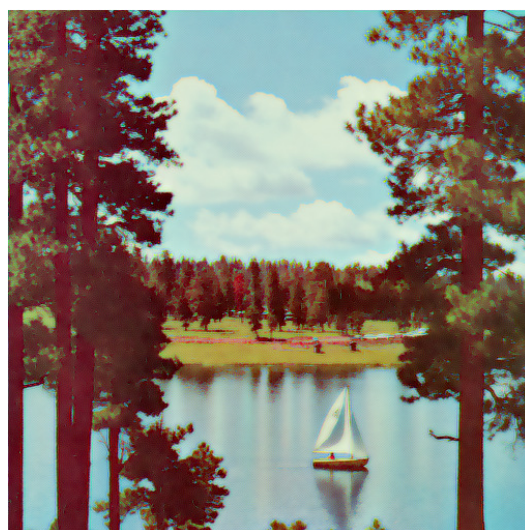
Fig. 3 Heat diffusions of the color image 'Barbara' induced by generalized Laplacians associated with connection 1-forms



(a) Image 'Lake'



(b) Trivial connection: no enhancement of the contrast

(c) Connection relatively to the group $SO(3)$: enhancement of the contrast in the chrominance component(d) Connection relatively to the group \mathbb{R}^{+*} : enhancement of the contrast in the luminance component(e) Connection relatively to the group $DC(3)$: enhancement of the contrast in luminance and chrominance components**Fig. 4** Heat diffusions of the color image 'Lake' induced by generalized Laplacians associated with connection 1-forms