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A Class of generalized Laplacians on vector bundles devoted to multi-channels image processing

Thomas Batard and Nir Sochen

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Abstract In the context of fibre bundles theory, there exist some differential operators of order 2, called generalized Laplacians, acting on sections of vector bundles over Riemannian manifolds, and generalizing the Laplace-Beltrami operator. Such operators are determined by the choice of a covariant derivative on the vector bundle. In this paper, we construct a class of generalized Laplacians, devoted to multi-channels image processing, from the construction of particular covariant derivatives. The construction requires to deal with the notion of associated bundle, that relates principal and vector bundles by the choice of a group representation. In particular, covariant derivatives are determined by connection 1-forms on principal bundles. We consider a minimization problem to construct particular connection 1-forms. Then, from the heat equation of the corresponding Laplacians, we obtain a class of diffusions whose behaviours depend of the choice of the group representation. We provide experiments on grey-level and color images.

Keywords Multi-channels Image Processing · Diffusion · Heat equation · Generalized Laplacian · Differential Geometry

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1 Introduction

The Laplace-Beltrami operator is the natural differential operator acting on functions of a Riemannian manifold. It is involved widely in image processing/analysis and computer vision, as for image regularization from the corresponding heat equation [10],[11] and shape analysis from the approximation of its spectrum [4],[8],[9].

In [1],[2] Batard introduced generalized Laplacians in the context of image and more generally vector-valued field regularization. They are second order differential operators acting on sections of vector bundles over Riemannian manifolds. They are determined by two geometric data, i.e. a Riemannian metric on the base manifold and a covariant derivative on the vector bundle. In this setting, the Laplace-Beltrami operator appears to be a generalized Laplacian on a vector bundle of rank 1 equipped with the trivial connection. We refer to [3] for more details about these operators. In [2], it is showed that the methods developed in [11],[13],[14],[15],[16] for color image regularization may be viewed as heat equations associated to generalized Laplacians. Then, particular covariant derivatives that take into account the specificity of color images were constructed. By the heat equations of the corresponding generalized Laplacians, new diffusions of color images were performed. However, the constructed covariant derivatives were trivial in the sense that they were constant with respect to the point.

The notions of covariant derivative and connection 1-form are at the heart of differential geometry of manifolds and fibre bundles. They play also a fundamental role in the mathematical description of physics, as in general relativity [5] and Gauge fields [7] theories.

However, such notions have been few investigated in the field of computer vision. Let us cite for example Duits et al. [6] where a Cartan connection is constructed on the tangent bundle of $SE(3)$ yielding a left-invariant diffusion for the purpose of crossing-preserving smoothing of HARDI images.

In this paper, we construct a class of non trivial covariant derivatives and the subsequent generalized Laplacians (up to the choice of a metric on the base manifold). They are parametrized by a group representation (ρ, G) acting on the acquisition space of the image, and a positive function f on the base manifold. Then we consider the corresponding heat equations and show that they are suitable for applications to multi-channels image processing. The function f determines the local behaviour of the diffusion. More precisely, the diffusion acts as a Beltrami flow when $f = 0$ and highlights data when $f > 0$. Then, the group representation (ρ, G) determines the nature of highlights, depending of its unitary character. In this paper, we consider the dual ρ^C of natural representations ρ of the groups \mathbb{R}^{+*} on \mathbb{R} dealing with grey-level images, and of the groups \mathbb{R}^{+*} , $DC(3)$ of 3x3 diagonal matrices of strictly positive coefficients, $SO(2)$ and $SO(3)$ on \mathbb{R}^3 for color images.

To perform the construction of non-trivial covariant derivatives, we deal with the notion of associated bundles, that are vector bundles associated to principal bundles through a group representation (ρ, G) . In this context, a covariant derivative is completely determined by a connection 1-form on the principal bundle. Moreover, sections of associated bundles may be identified with (ρ, G) -equivariant functions on principal bundles. From the data of a connection 1-form on a principal bundle and a Riemannian metric on the base manifold, we construct a Riemannian metric on the total space of the principal bundle by lifting up the metric of the base manifold into the kernel of the connection 1-form, which is called horizontal bundle. Then, we consider a minimization problem in order to construct optimal connection 1-forms. We deal with a Polyakov action on the graph of functions J on a principal bundle (P, π, M, G) twisted by the Casimir operator acting on J . Minimizing this functional with respect to the metric of the principal bundle under the constraint that the metric of the base manifold is fixed provides optimal connection 1-forms ω . Then we derive the family of 1-forms $f\omega$, from which follows a family of covariant derivatives on an associated bundle $P \times_{(\rho^C, G)} V$. We show that the function J whose graph minimizing the functional satisfies a generalized Laplace equation, i.e. an equation of the form $\Delta^{f\omega} J = 0$, where $\Delta^{f\omega}$ is the

connection Laplacian on $P \times_{(\rho^C, G)} V$ induced by the 1-form $f\omega$. Then, the method of gradient descent to reach the solution may be considered as the Euler scheme of a PDE

$$\frac{\partial J_t}{\partial t} = \Delta^{f\omega} J_t, \quad J_0 = J \quad (1)$$

and consequently provides a diffusion of the original section J .

This paper is organized as follows. In Sect. 2, we construct the minimization problems leading to heat equations of generalized Laplacians, whose corresponding covariant derivatives are induced by optimal connection 1-forms. We first detail the method to construct a Riemannian metric on the total space of a principal bundle from the data of a Riemannian metric on the base manifold and a horizontal bundle. We give explicit expressions for the Lie groups involved in this paper, i.e. \mathbb{R}^{+*} , $SO(2)$, $DC(3)$ and $SO(3)$. Then we present the functional we are dealing with, and the minimization problems with respect to the connection 1-form and function on principal bundle. In Sect. 3, we give the expressions of optimal connection 1-forms for the different group presented in this paper, from which we give the expressions of the corresponding covariant derivatives on the associated vector bundles for dual representations ρ^C . At last, in Sect. 4, we propose applications to grey-level and color image processing. We first give a physical interpretation of the notion of (ρ^C, G) -equivariance for images. Then, we show experiments by computing the discrete approximation of the PDE (1) for different functions f and for the Lie group representations mentioned above.

2 Kernels of generalized Laplacians as solution spaces of a minimization problem

For the reader who is not familiar with the notions related to the differential geometry of fiber bundles, we refer to [12] for a complete introduction.

2.1 Riemannian metric on principal bundle induced by a horizontal bundle

2.1.1 The general construction

Let (P, π, M, G) be a principal bundle, where the base manifold M is of dimension m , the Lie group G is of dimension n , and $\pi: P \rightarrow M$ is the projection from the total manifold P to the base manifold M . For the sake of shortness, we assimilate sometimes P and

(P, π, M, G) , and more generally any fiber bundle with its total manifold. Let g and \mathcal{B} be respectively a Riemannian metric on M and a bi-invariant Riemannian metric on G . Let HP be a horizontal subbundle of the tangent bundle TP of P .

We construct a Riemannian metric h on P from g, \mathcal{B} and HP as follows.

Let (e_1, \dots, e_m) be an orthonormal frame of TM with respect to g . Let (e_1^h, \dots, e_m^h) be the horizontal lift of (e_1, \dots, e_m) on HP . Let (X_1, \dots, X_n) be a set of orthogonal vectors of $T_e G = \mathfrak{g}$ with respect to \mathcal{B} . Let (X_1^v, \dots, X_n^v) be the frame of VP , the vertical subbundle of TP , defined for $i = 1, \dots, n$ by

$$X_i^v(p) = \left(\frac{d}{dt} p \cdot \exp(tX_i) \right)_{|t=0}$$

where \cdot denotes the action of G on the fibers of P . Then, considering the metric h on TP given by the Identity matrix field in the frame $(e_1^h, \dots, e_m^h, X_1^v, \dots, X_n^v)$ of TP turns (P, h) into a Riemannian manifold.

Remark 1 The subbundles HP and VP are orthogonal with respect to h .

Remark 2 By the G -equivariance of the subbundles HP and VP , the metric h is completely determined by its values along a section of P .

2.1.2 A particular case: base manifold of dimension 2 and matrix Lie groups

For the purpose of applications to image processing, we focus on the particular case where the base manifold M is of dimension 2 and the Lie group G is a matrix Lie group. Let (x_1, x_2) and (a_1, \dots, a_n) be respectively coordinates systems of M and G . In what follows, we compute the matrix representation of the metric h in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \dots, \partial/\partial a_n)$ of TP .

For matrix Lie groups, the linear tangent map TR_{g_0} of the right translation R_{g_0} is given by

$$TR_{g_0}(A) = A \cdot g_0$$

where \cdot denotes the matrix multiplication.

Let p_0 be the section of P of the form $p_0(x_1, x_2) = \Phi(x_1, x_2, e)$ where Φ is a trivialization of the bundle and e is the neutral element of G . For the sake of clarity, we omit the trivialization Φ from now on. By definition of a subbundle HP of P , there exist functions

$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \dots, \gamma_{1n}, \gamma_{21}, \dots, \gamma_{2n}$ on M such that

$$H_{p_0}P = Vect \left\{ \alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{1k} \partial/\partial a_k, \right. \\ \left. \alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{2k} \partial/\partial a_k \right\}$$

and $T_{p_0}P = H_{p_0}P \oplus Vect\{\partial/\partial a_1, \dots, \partial/\partial a_n\}$. Then, by the G -equivariance property of HP , we have

$$H_{p_0 \cdot g_0}P = \\ Vect \left\{ T_{p_0} \widetilde{R}_{g_0} \left(\alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{1k} \partial/\partial a_k \right), \right. \\ \left. T_{p_0} \widetilde{R}_{g_0} \left(\alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2 + \sum_{k=1}^n \gamma_{2k} \partial/\partial a_k \right) \right\} \\ = Vect \left\{ \alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2 + \left(\sum_{k=1}^n \gamma_{1k} \partial/\partial a_k \right) \cdot g_0, \right. \\ \left. \alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2 + \left(\sum_{k=1}^n \gamma_{2k} \partial/\partial a_k \right) \cdot g_0 \right\}$$

Finally, the data of smooth functions

$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \dots, \gamma_{1n}, \gamma_{21}, \dots, \gamma_{2n}$ on M such that $\alpha_1 \beta_2 - \alpha_2 \beta_1$ never vanishes completely determines the subbundle HP .

In the frame $(\alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2, \alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2)$, we have

$$\partial/\partial x_1 = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix} \\ \partial/\partial x_2 = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix}$$

Then, denoting respectively by $\partial/\partial x_1^h$ and $\partial/\partial x_2^h$ the liftings of $\partial/\partial x_1$ and $\partial/\partial x_2$ in HP , we have

$$\partial/\partial x_1^h = \\ \partial/\partial x_1 + \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \left(\sum_{k=1}^n [(\beta_2 \gamma_{1k} - \beta_1 \gamma_{2k})] \partial/\partial a_k \right) \cdot g_0$$

$$\partial/\partial x_2^h = \\ \partial/\partial x_2 + \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \left(\sum_{k=1}^n [(\alpha_1 \gamma_{2k} - \alpha_2 \gamma_{1k})] \partial/\partial a_k \right) \cdot g_0$$

We put

$$A = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \left(\sum_{k=1}^n (\beta_2 \gamma_{1k} - \beta_1 \gamma_{2k}) \partial/\partial a_k \right)$$

$$B = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1} \left(\sum_{k=1}^n (\alpha_1\gamma_{2k} - \alpha_2\gamma_{1k}) \partial/\partial a_k \right)$$

Following Sect. 2.1.1, we construct a Riemannian metric h on P of matrix representation

$$H = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \oplus C$$

in the frame $(\partial/\partial x_1^h, \partial/\partial x_2^h, \partial/\partial a_1, \dots, \partial/\partial a_n)$, where $\{g_{ij}\}$ is the matrix representation of g in the frame $(\partial/\partial x_1, \partial/\partial x_2)$ and $C = \{C_{lk}\}$ is the matrix representation of \mathcal{B} in the frame $(\partial/\partial a_1, \dots, \partial/\partial a_n)$.

Then, the matrix representation of h in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \dots, \partial/\partial a_n)$ is given by

$${}^T Q^{-1} H Q^{-1}$$

where Q is the change frame field from $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \dots, \partial/\partial a_n)$ to $(\partial/\partial x_1^h, \partial/\partial x_2^h, \partial/\partial a_1, \dots, \partial/\partial a_n)$.

We have $Q^{-1} =$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ (A \cdot g_0)_1 & (B \cdot g_0)_1 & & & \\ \vdots & \vdots & & Id_n & \\ \vdots & \vdots & & & \\ (A \cdot g_0)_n & (B \cdot g_0)_n & & & \end{pmatrix}$$

Hence, we obtain

$$h_{11} = g_{11} + \sum_{l=1}^n \sum_{k=1}^n C_{lk} (A \cdot g_0)_k (A \cdot g_0)_l$$

$$h_{12} = h_{21} = g_{12} + \sum_{l=1}^n \sum_{k=1}^n C_{lk} (A \cdot g_0)_l (B \cdot g_0)_k$$

$$h_{1l} = h_{l1} = \sum_{k=1}^n C_{kl-2} (A \cdot g_0)_k, \quad l \geq 3$$

$$h_{22} = g_{22} + \sum_{l=1}^n \sum_{k=1}^n C_{lk} (B \cdot g_0)_k (B \cdot g_0)_l$$

$$h_{2l} = h_{l2} = \sum_{k=1}^n C_{kl-2} (B \cdot g_0)_k, \quad l \geq 3$$

$$h_{ij} = h_{ji} = C_{i-2j-2}, \quad i, j \geq 3$$

in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \dots, \partial/\partial a_n)$.

2.1.3 Example 1: The Lie Group \mathbb{R}^{+*}

The map $\psi: a \mapsto \exp(a)$ determines a chart of \mathbb{R}^{+*} .

The metric \mathcal{B} on \mathbb{R}^{+*} determined by a strictly positive scalar δ in the frame $(\partial/\partial a)$ is a bi-invariant Riemannian metric. Hence, in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a)$, h is given by the matrix representation

$$h_{11} = g_{11} + \delta \frac{(\beta_2\gamma_1 - \beta_1\gamma_2)^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}, \quad h_{22} = g_{22} + \delta \frac{(\alpha_1\gamma_2 - \alpha_2\gamma_1)^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}$$

$$h_{12} = h_{21} = g_{12} + \delta \frac{(\beta_2\gamma_1 - \beta_1\gamma_2)(\alpha_1\gamma_2 - \alpha_2\gamma_1)}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}, \quad h_{33} = \delta$$

$$h_{13} = h_{31} = \delta \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \quad h_{23} = h_{32} = \delta \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}$$

2.1.4 Example 2: The Lie Group $SO(2)$

The map $\psi: \theta \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ determines a local chart of $SO(2)$.

The metric \mathcal{B} on $SO(2)$ determined by a strictly positive scalar δ in the frame $(\partial/\partial \theta)$ is a bi-invariant Riemannian metric. As a consequence, in the local frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \theta)$, h is given by the matrix field

$$h_{11} = g_{11} + \delta \frac{(\beta_2\gamma_1 - \beta_1\gamma_2)^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}, \quad h_{22} = g_{22} + \delta \frac{(\alpha_1\gamma_2 - \alpha_2\gamma_1)^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}$$

$$h_{12} = h_{21} = g_{12} + \delta \frac{(\beta_2\gamma_1 - \beta_1\gamma_2)(\alpha_1\gamma_2 - \alpha_2\gamma_1)}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}, \quad h_{33} = \delta$$

$$h_{13} = h_{31} = \delta \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \quad h_{23} = h_{32} = \delta \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}$$

2.1.5 Example 3: The Lie Group $DC(3)$

The map $\psi: (a_1, a_2, a_3) \mapsto$

$$\begin{pmatrix} \exp(a_1) & 0 & 0 \\ 0 & \exp(a_2) & 0 \\ 0 & 0 & \exp(a_3) \end{pmatrix}$$

determines a chart of $DC(3)$.

The metric \mathcal{B} on $DC(3)$ given by the matrix field $C = \delta I_3$ in the frame $(\partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3)$, where δ is a strictly positive scalar, is a bi-invariant Riemannian metric.

It follows that the matrix representation of h in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3)$ is given by

$$\begin{aligned}
h_{11} &= g_{11} + \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \left(\sum_{k=1}^3 (\beta_2\gamma_{1k} - \beta_1\gamma_{2k})^2 \right) \\
h_{12} &= h_{21} = g_{12} + \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \left(\sum_{k=1}^3 (\beta_2\gamma_{1k} - \beta_1\gamma_{2k})(\alpha_1\gamma_{2k} - \alpha_2\gamma_{1k}) \right) \\
h_{22} &= g_{22} + \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \left(\sum_{k=1}^3 (\alpha_1\gamma_{2k} - \alpha_2\gamma_{1k})^2 \right) \\
h_{13} &= h_{31} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\beta_2\gamma_{11} - \beta_1\gamma_{21}) \\
h_{14} &= h_{41} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\beta_2\gamma_{12} - \beta_1\gamma_{22}) \\
h_{15} &= h_{51} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\beta_2\gamma_{13} - \beta_1\gamma_{23}) \\
h_{23} &= h_{32} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\alpha_1\gamma_{21} - \alpha_2\gamma_{11}) \\
h_{24} &= h_{42} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\alpha_1\gamma_{22} - \alpha_2\gamma_{12}) \\
h_{25} &= h_{52} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\alpha_1\gamma_{23} - \alpha_2\gamma_{13}) \\
h_{33} &= h_{44} = h_{55} = \delta \\
h_{34} &= h_{43} = h_{35} = h_{53} = h_{45} = h_{54} = 0
\end{aligned}$$

2.1.6 Example 4: The Lie Group $SO(3)$

The map $\psi: (\theta_1, \theta_2, \theta_3) \mapsto R_{(\theta_1, \theta_2, \theta_3)} =$

$$\begin{pmatrix}
\cos \theta_2 \cos \theta_3 & \cos \theta_2 \sin \theta_3 & \sin \theta_2 \\
-\cos \theta_1 \sin \theta_3 & \cos \theta_3 \cos \theta_1 & \cos \theta_2 \sin \theta_1 \\
-\cos \theta_3 \sin \theta_2 \sin \theta_1 & -\sin \theta_2 \sin \theta_3 \sin \theta_1 & \\
-\cos \theta_3 \cos \theta_1 \sin \theta_2 & -\cos \theta_1 \sin \theta_2 \sin \theta_3 & \cos \theta_2 \cos \theta_1 \\
+\sin \theta_3 \sin \theta_1 & -\cos \theta_3 \sin \theta_1 &
\end{pmatrix}$$

determines a local chart of $SO(3)$.

The metric \mathcal{B} on $SO(3)$ given by the matrix field

$$C = \begin{pmatrix} \delta & 0 & \delta \sin \theta_2 \\ 0 & \delta & 0 \\ \delta \sin \theta_2 & 0 & \delta \end{pmatrix}$$

in the frame $(\partial/\partial\theta_1, \partial/\partial\theta_2, \partial/\partial\theta_3)$, where δ is a strictly positive scalar, is a bi-invariant Riemannian metric.

Hence, in the frame $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial\theta_1, \partial/\partial\theta_2, \partial/\partial\theta_3)$, the matrix representation of the metric h is given by

$$h_{11} = g_{11} + \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \left(\sum_{k=1}^3 (\beta_2\gamma_{1k} - \beta_1\gamma_{2k})^2 \right)$$

$$\begin{aligned}
h_{12} &= h_{21} = g_{12} + \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \left(\sum_{k=1}^3 (\beta_2\gamma_{1k} - \beta_1\gamma_{2k})(\alpha_1\gamma_{2k} - \alpha_2\gamma_{1k}) \right) \\
h_{22} &= g_{22} + \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2} \left(\sum_{k=1}^3 (\alpha_1\gamma_{2k} - \alpha_2\gamma_{1k})^2 \right) \\
h_{13} &= h_{31} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\cos \theta_2 \cos \theta_3 (\beta_1\gamma_{21} - \beta_2\gamma_{11}) \\
&\quad - \cos \theta_2 \sin \theta_3 (\beta_1\gamma_{22} - \beta_2\gamma_{12}) + \sin \theta_2 (\beta_2\gamma_{13} - \beta_1\gamma_{23})) \\
h_{14} &= h_{41} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\sin \theta_3 (\beta_2\gamma_{11} - \beta_1\gamma_{21}) \\
&\quad + \cos \theta_3 (\beta_2\gamma_{12} - \beta_1\gamma_{22})) \\
h_{15} &= h_{51} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\beta_2\gamma_{13} - \beta_1\gamma_{23}) \\
h_{23} &= h_{32} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\cos \theta_2 \cos \theta_3 (\alpha_1\gamma_{21} - \alpha_2\gamma_{11}) \\
&\quad - \cos \theta_2 \sin \theta_3 (\alpha_1\gamma_{22} - \alpha_2\gamma_{12}) + \sin \theta_2 (\alpha_1\gamma_{23} - \alpha_2\gamma_{13})) \\
h_{24} &= h_{42} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\sin \theta_3 (\alpha_1\gamma_{21} - \alpha_2\gamma_{11}) \\
&\quad + \cos \theta_3 (\alpha_1\gamma_{22} - \alpha_2\gamma_{12})) \\
h_{25} &= h_{52} = \frac{\delta}{(\alpha_1\beta_2 - \alpha_2\beta_1)} (\alpha_1\gamma_{23} - \alpha_2\gamma_{13}) \\
h_{33} &= h_{44} = h_{55} = \delta \\
h_{34} &= h_{43} = h_{45} = h_{54} = 0 \quad h_{35} = h_{53} = \delta \sin \theta_2
\end{aligned}$$

2.2 A twisted Polyakov action on principal bundles

Let (P, π, M, G) be a principal bundle, ρ a representation of G on a vector space V equipped with the standard scalar product (\cdot, \cdot) . Let $J: P \rightarrow V$ be a smooth function. We denote by φ the graph of J .

Let h be a Riemannian metric on P induced by a Riemannian metric g on M and a horizontal subbundle HP of TP following the construction of Sect. 2.1. Let Q be a Riemannian metric on $P \times V$.

We consider the following functional which measures the energy of the embedding $\varphi: (P, h) \rightarrow (P \times V, Q)$ twisted by the action of the Casimir operator

$$S = \int_P h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} + \frac{1}{2} (Cas J, J) dP \quad (2)$$

where the Casimir operator $Cas \in \text{End}(V)$ is defined by

$$Cas = \sum \rho_*(\mathfrak{g}^i)^2$$

for any orthonormal basis $(\mathfrak{g}^1, \dots, \mathfrak{g}^n)$ of the Lie algebra \mathfrak{g} of G , where $\rho_*: \mathfrak{g} \rightarrow \text{End}(V)$ is the Lie algebra representation induced by ρ .

The role of the extra term $1/2(CasJ, J)$ in (2) will appear in the proof of the following Proposition.

Proposition 1 *Assuming that $Q = \mathcal{H} \oplus \kappa I_{\dim V}$, for a Riemannian metric \mathcal{H} on P and strictly positive constant κ , then the function J_{opt} whose graph φ_{opt} minimizes the action (2) may be approximated by a (ρ, G) -equivariant function.*

Proof Let φ be an embedding of a Riemannian manifold (M, g) into a Riemannian manifold (N, h) . It is well-known that the Euler-Lagrange equations of the Polyakov action

$$X = \int_M g^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} h_{ij} dM$$

with respect to φ are

$$-\frac{1}{2\sqrt{g}} h^{il} \frac{\partial X}{\partial \varphi^l} = \Delta_g \varphi^i + \Gamma_{jk}^i \partial_\mu \varphi^j \partial_\nu \varphi^k h^{\mu\nu} = 0 \quad (3)$$

Then, considering the functional (2) with the assumption that $Q = \mathcal{H} \oplus \kappa I_{\dim V}$, the Euler-Lagrange equations with respect to J are

$$(\Delta_h \otimes 1 + Cas) J = 0 \quad (4)$$

Under the identification between (ρ, G) -equivariant functions on P and sections of $P \times_{(\rho, G)} V$, the equation (4) may be rewritten $\Delta^A J = 0$ where Δ^A is the operator

$$\sum_{ij} g^{ij} \left(\nabla_{\partial_i}^A \nabla_{\partial_j}^A - \sum_k \Gamma_{ij}^k \nabla_{\partial_k}^A \right)$$

that is the connection Laplacian associated with the covariant derivative ∇^A induced by HP on $P \times_{(\rho, G)} V$. We refer to [3] for the proof of the identification between the operators $(\Delta_h \otimes 1 + Cas)$ and Δ^A .

The gradient descent

$$J_{t+dt} = J_t + dt \Delta^A J_t, \quad J_0 = J \quad (5)$$

for reaching the solution preserves the structure of section of associated bundle, and consequently the scheme

$$J_{t+dt} = J_t + dt (\Delta_h \otimes 1 + Cas) J_t \quad (6)$$

preserves the (ρ, G) -equivariance property. Hence, taking as initial condition a (ρ, G) -equivariant function, the scheme (6) provides a set of (ρ, G) -equivariant functions approximating the solution of (4). \square

Remark 3 The gradient descent (5) may be viewed as the Euler scheme of a PDE

$$\frac{\partial J_t}{\partial t} = \Delta^A J_t, \quad J_0 = J \quad (7)$$

that is a heat equation of a generalized Laplacian. Hence, the scheme (5) provides a diffusion of the section J .

3 Some optimal connection 1-forms and corresponding covariant derivatives

We restrict the functional (2) at the points of the form $\Phi(x, e)$ for $x \in M$, i.e.

$$S = \int_{\Phi(M \times \{e\})} h^{\mu\nu} \frac{\partial \varphi^i}{\partial x_\mu} \frac{\partial \varphi^j}{\partial x_\nu} Q_{ij} + \frac{1}{2} (Cas J, J) dM \quad (8)$$

and minimize it with respect to horizontal bundles along the section $s: x \mapsto \Phi(x, e)$. Then we obtain the expression of optimal horizontal bundles along the section s , from which we derive the expression of optimal connection 1-forms along s . By the lifting down procedure, we obtain a \mathfrak{g} -valued 1-form A on M , given by

$$A = s^* \omega$$

from which we obtain the expression of the corresponding covariant derivative ∇^A , relatively to the trivializing section s , on the associated bundle $P \times_{(\rho^C, G)} V$ by

$$\nabla_X^A J = d_X J + \rho_*(A)(X)J$$

for X being a tangent vector field on M , and ρ^C a dual representation of G on V .

In what follows, when dealing with the Lie groups G of Sect. 2, we assume that Q is of the form $I_2 \oplus C \oplus \kappa I_n$ in the frame induced by the standard coordinates system on $P \times V$. We also assume that J is a (ρ^C, G) -equivariant function, and we denote by $I: M \rightarrow V$ the function defined by

$$I(x) = J \circ \Phi(x, e)$$

3.1 The case of Lie groups of dimension 1

3.1.1 Construction of optimal connection 1-forms

Following Sect. 2.1.2, the horizontal bundle HP along the section s is of the form

$$\text{Vect} \{ \alpha_1 \partial / \partial x_1 + \beta_1 \partial / \partial x_2 + \gamma_1 \partial / \partial a,$$

$$\alpha_2 \partial / \partial x_1 + \beta_2 \partial / \partial x_2 + \gamma_2 \partial / \partial a \}$$

for some functions $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ on M .

We assume that the functions $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constant. Then, computing the Euler-Lagrange equations of (8) with respect to the functions γ_1 and γ_2 , we obtain a unique solution, given for $k = 1, 2$ by

$$\gamma_k =$$

$$\left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \left[\alpha_k \left(\frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) + \beta_k \left(\frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \right]$$

It follows the expression of the horizontal bundle HP along s .

3.1.2 Lifting down of the connection 1-form on M

Let X be a tangent vector field on P along the section s of the form $\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + \gamma \partial/\partial a$ for some functions α, β, γ on M . The connection 1-form ω induced by HP is given along s by $\omega(X) = \left[\gamma + \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \right. \partial/\partial a$

$$\times \left(-\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a} Q_{ij} - \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right) \partial/\partial a$$

By the lifting down procedure induced by s , we obtain a 1-form A on M given by $A(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2)(x) = \omega(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + 0 \partial/\partial a)(x, e)$.

3.1.3 Corresponding covariant derivative for the unique dual representation ρ^C of \mathbb{R}^{+*} on \mathbb{R}^n

Writing the 1-form A as $A_1 \partial/\partial a$, we have $\rho_*^C(A) = -A_1 I_n$, and the nonzero symbols of the corresponding covariant derivative ∇^A are the terms Υ_{lk}^k given by

$$\begin{aligned} \Upsilon_{lk}^k(x) &= (\rho_*^C A)_{kk}(\partial/\partial x_l)(x) \\ &= \left(\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) \end{aligned}$$

Simple computations give

$$\frac{\partial \varphi^i}{\partial a} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) = \delta + \kappa \left(\sum_{p=1}^n I^{p^2} \right)(x)$$

and

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a} Q_{ij}(x, e) = -\kappa \left(\sum_{p=1}^n I_{x_l}^p I^p \right)(x)$$

3.1.4 Corresponding covariant derivative for a class of unitary dual representations ρ^C of $SO(2)$ on \mathbb{R}^3

The maps

$$\rho_p^C : \theta \longmapsto \text{rotation of angle } -\theta \text{ in the plane } \mathcal{P}$$

where \mathcal{P} is a 2-plan in \mathbb{R}^3 are unitary representations of $SO(2)$ on \mathbb{R}^3 .

Writing the 1-form A as $A_1 \partial/\partial \theta$, we can show that $\rho_*^C(A)$ is of the form

$$\rho_{p*}^C(A) = A_1 \begin{pmatrix} 0 & -\gamma_{12} & -\gamma_{13} \\ \gamma_{12} & 0 & -\gamma_{23} \\ \gamma_{13} & \gamma_{23} & 0 \end{pmatrix}$$

for some $\gamma_{12}, \gamma_{13}, \gamma_{23} \in \mathbb{R}$ such that $\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 = 1$, and the nonzero symbols of the corresponding covariant derivative ∇^A are the terms Υ_p^k given by

$$\begin{aligned} \Upsilon_{lp}^k(x) &= (\rho_{p*}^C A)_{kp}(\partial/\partial x_l)(x) = -\Upsilon_{lk}^p(x) \\ &= (-1)^{l+1} \gamma_{kp} \left(\frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij} \right)^{-1} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) \end{aligned}$$

for $k \neq p$. Moreover, we have

$$\begin{aligned} \frac{\partial \varphi^i}{\partial \theta} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) &= \delta + \\ \kappa \left((\gamma_{12} I^2 + \gamma_{13} I^3)^2 + (\gamma_{12} I^1 - \gamma_{23} I^3)^2 + (\gamma_{13} I^1 + \gamma_{23} I^2)^2 \right) &(x) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta} Q_{ij}(x, e) &= -\kappa \times \\ \left(I_{x_l}^1 (-\gamma_{12} I^2 - \gamma_{13} I^3) + I_{x_l}^2 (\gamma_{12} I^1 - \gamma_{23} I^3) + I_{x_l}^3 (\gamma_{13} I^1 + \gamma_{23} I^2) \right) &(x) \end{aligned}$$

3.2 The case of Lie groups of dimension 3

3.2.1 Construction of optimal connection 1-forms

Following Sect. 2.1.1, the horizontal bundle HP along the section s is of the form

$$\text{Vect} \left\{ \alpha_1 \partial/\partial x_1 + \beta_1 \partial/\partial x_2 + \sum_{k=1}^3 \gamma_{1k} \partial/\partial a_k, \right.$$

$$\left. \alpha_2 \partial/\partial x_1 + \beta_2 \partial/\partial x_2 + \sum_{k=1}^3 \gamma_{2k} \partial/\partial a_k \right\}$$

for some functions $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{22}, \gamma_{23}$ on M .

We assume that the functions $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constant. Then, computing the Euler-Lagrange equations of (8) with respect to the functions γ_{ij} , we obtain a unique solution given by

$$\begin{aligned} \gamma_{11} &= \lambda \left[\alpha_1 \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ &\quad \left. \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_1 \left((fd - e^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \\ &\quad \left. \left. + (ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \end{aligned}$$

$$\gamma_{12} = \lambda \left[\alpha_1 \left((ce - fb) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right.$$

$$+(bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \Big) + \beta_l \left((ce - fb) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \\ \left. + (af - c^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \Big]$$

$$\gamma_{l3} = \lambda \left[\alpha_l \left((be - dc) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right. \right. \\ \left. \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) + \beta_l \left((be - dc) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right. \right. \\ \left. \left. + (bc - ae) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + (ad - b^2) \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right]$$

for $l = 1, 2$, where $\lambda = 1/(2cbe + fad - e^2a - fb^2 - c^2d)$ and

$$a = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \quad b = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \quad c = \frac{\partial \varphi^i}{\partial a_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \\ d = \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \quad e = \frac{\partial \varphi^i}{\partial a_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \quad f = \frac{\partial \varphi^i}{\partial a_3} \frac{\partial \varphi^j}{\partial a_3} Q_{ij}$$

It follows the expression of the horizontal bundle HP along s .

3.2.2 Lifting down of the connection 1-form on M

Let X be a tangent vector field on P along the section s of the form $\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + \gamma_1 \partial/\partial a_1 + \gamma_2 \partial/\partial a_2 + \gamma_3 \partial/\partial a_3$ for some functions $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3$ on M . The connection 1-form ω induced by HP is given along s by $\omega(X) =$

$$\left[\gamma_1 - \lambda \left[(fd - e^2) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ \left. \left. + (ce - fb) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (be - dc) \times \right. \right. \\ \left. \left. \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_1 \\ + \left[\gamma_2 - \lambda \left[(ce - fb) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ \left. \left. + (af - c^2) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (bc - ae) \times \right. \right. \\ \left. \left. \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_2 \\ + \left[\gamma_3 - \lambda \left[(be - dc) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right) \right. \right. \\ \left. \left. + (bc - ae) \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right) + (ad - b^2) \times \right. \right. \\ \left. \left. \left(\alpha \frac{\partial \varphi^i}{\partial x_1} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} + \beta \frac{\partial \varphi^i}{\partial x_2} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right) \right] \right] \partial/\partial a_3$$

By the lifting down procedure induced by s , we obtain a 1-form A given by $A(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2)(x) =$

$$\omega(\alpha \partial/\partial x_1 + \beta \partial/\partial x_2 + 0 \partial/\partial a_1 + 0 \partial/\partial a_2 + 0 \partial/\partial a_3)(x, e)$$

3.2.3 Corresponding covariant derivative for the unique dual ρ^C representation of $DC(3)$ on \mathbb{R}^3

Writing the 1-form A as $A_1 \partial/\partial a_1 + A_2 \partial/\partial a_2 + A_3 \partial/\partial a_3$, we have $\rho_*^C(A) = -diag(A_1, A_2, A_3)$. and the nonzero symbols of the corresponding covariant derivative ∇^A are

$$\Upsilon_{l1}^1(x) = (\rho_* A)_{11}(\partial/\partial x_l)(x) \\ = \lambda \left[c_2 c_3 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_1} Q_{ij} \right] (x, e)$$

$$\Upsilon_{l2}^2(x) = (\rho_* A)_{22}(\partial/\partial x_l)(x) \\ = \lambda \left[c_1 c_3 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_2} Q_{ij} \right] (x, e)$$

$$\Upsilon_{l3}^3(x) = (\rho_* A)_{33}(\partial/\partial x_l)(x) \\ = \lambda \left[c_1 c_2 \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_3} Q_{ij} \right] (x, e)$$

Simple computations give

$$\frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial a_k} Q_{ij}(x, e) = -\kappa I_{x_l}^k I^k(x)$$

for $l = 1, 2, k = 1, 2, 3$, and

$$c_p(x, e) = \delta + \kappa I^{p2}(x), \quad p = 1, 2, 3$$

3.2.4 Corresponding covariant derivative for the unique dual representation ρ^C of $SO(3)$ on \mathbb{R}^3

Writing the 1-form A as $A_1 \partial/\partial \theta_1 + A_2 \partial/\partial \theta_2 + A_3 \partial/\partial \theta_3$, we have

$$\rho_*^C(A) = \begin{pmatrix} 0 & -A_3 & -A_2 \\ A_3 & 0 & -A_1 \\ A_2 & A_1 & 0 \end{pmatrix}$$

and the nonzero symbols of the corresponding covariant derivative ∇^A are

$$\gamma_{l2}^1(x) = (\rho_*^C A)_{12}(\partial/\partial x_l)(x) = -\gamma_{l1}^2(x) \\ = \lambda \left[(be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ \left. + (ad - b^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e)$$

$$\gamma_{l3}^1(x) = (\rho_*^C A)_{13}(\partial/\partial x_l)(x) = -\gamma_{l1}^3(x) \\ = \lambda \left[(ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (af - c^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right]$$

$$+(bc - ae) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \Big] (x, e)$$

$$\begin{aligned} \gamma_{l3}^2(x) &= (\rho_*^C A)_{23}(\partial/\partial x_l)(x) = -\gamma_{l2}^3(x) \\ &= \lambda \left[(fd - e^2) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij} + (ce - fb) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij} \right. \\ &\quad \left. + (be - dc) \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij} \right] (x, e) \end{aligned}$$

for $l = 1, 2$. Simple computations show that

$$\begin{aligned} \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_1} Q_{ij}(x, e) &= -\kappa(I_{x_l}^2 I^3 - I_{x_l}^3 I^2)(x) \\ \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_2} Q_{ij}(x, e) &= -\kappa(I_{x_l}^1 I^3 - I_{x_l}^3 I^1)(x) \\ \frac{\partial \varphi^i}{\partial x_l} \frac{\partial \varphi^j}{\partial \theta_3} Q_{ij}(x, e) &= -\kappa(I_{x_l}^1 I^2 - I_{x_l}^2 I^1)(x) \end{aligned}$$

and

$$\begin{aligned} a(x, e) &= \delta + \kappa(I^{22} + I^{32})(x) & b(x, e) &= \kappa I^1 I^2(x) \\ c(x, e) &= -\kappa I^1 I^3(x) & d(x, e) &= \delta + \kappa(I^{12} + I^{32})(x) \\ e(x, e) &= \kappa I^2 I^3(x) & f(x, e) &= \delta + \kappa(I^{12} + I^{22})(x) \end{aligned}$$

4 Application to image processing

4.1 From images to sections of associated bundles

Let $I: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be a n -channels image given with its coordinates (I^1, \dots, I^n) in the acquisition space that we embed into \mathbb{R}^n . We endow \mathbb{R}^n of a vector space structure and basis (e_1, \dots, e_n) , and write

$$I(x_1, x_2) = I^1(x_1, x_2)e_1 + \dots + I^n(x_1, x_2)e_n$$

Let (ρ, G) be a Lie group representation on \mathbb{R}^n . Under its action on \mathbb{R}^n , the group G acts on the basis (e_1, \dots, e_n) . Let \mathcal{P} be the set of basis obtained by the transformations of (e_1, \dots, e_n) under the action of the group G , denoted by \cdot . The action of G on \mathcal{P} is transitive and free. Denoting by π the projection of $\Omega \times \mathcal{P}$ on Ω such that

$$\pi(x_1, x_2, g \cdot (e_1, \dots, e_n)) = (x_1, x_2),$$

the quadruplet $P = (\Omega \times \mathcal{P}, \pi, \Omega, G)$ forms a trivial principal bundle. The global diffeomorphism $\Phi: \Omega \times G \rightarrow \Omega \times \mathcal{P}$ is given by

$$\Phi(x_1, x_2, g) = (x_1, x_2, g \cdot (e_1, \dots, e_n))$$

We construct the associated bundle $E = P \times_{(\rho, G)} \mathbb{R}^n$.

From the function I , we construct a (ρ, G) -equivariant function J on P defined by

$$J(x_1, x_2, g \cdot (e_1, \dots, e_n)) = \rho(g)I(x_1, x_2) \quad (9)$$

In particular, we have

$$J(x_1, x_2, (e_1, \dots, e_n)) = I(x_1, x_2) \quad (10)$$

since $\rho(e) = Id$ by property of a group representation.

Under the correspondance between (ρ, G) -equivariant functions of P and sections of E , J is a section of E of the form $J(x_1, x_2) = [(x_1, x_2, (e_1, \dots, e_n)), I(x_1, x_2)]$, and I the realization of J under the trivializing section $s(x_1, x_2) = (x_1, x_2, (e_1, \dots, e_n)) = \Phi(x_1, x_2, e)$ of P .

4.2 Interpretation of the (ρ, G) -equivariance for images

In this paper, we consider grey-level and color images. We embed the grey-level space into \mathbb{R} and the *RGB* color space into \mathbb{R}^3 .

By the (ρ, G) -equivariant function J we construct, we take into account that some transformations of the light source induce modifications of the pixels. Indeed, by (10) we assign the basis (e_1, \dots, e_n) to the light source of the original image I . Then, we assimilate a basis change given by the action $g \mapsto g \cdot (e_1, \dots, e_n)$ to a modification of the light source. By (9), the representation ρ tells how the pixels of the image change under this basis change, and consequently under the corresponding modification of the light source. By the use of the fiber bundle context, we allow that the transformation of the light source change with respect to the points of Ω .

Taking ρ as the dual of a natural representation of a Lie group G on \mathbb{R}^n makes the (ρ, G) -equivariance encode that pixels transform in the same way as the basis does under the action of G . From a mathematical viewpoint, it means that pixels are treated as covectors. It is coherent with the interpretation above where we identified basis change with modification of the light source since pixels clearly transform in the same way as the light source does.

The action of \mathbb{R}^{+*} on the basis e_1 for grey-level images and (e_1, e_2, e_3) for color images might be interpreted as a homogeneous modification of the intensity of the light source, whereas the action of $DC(3)$ on the basis (e_1, e_2, e_3) might be interpreted as an inhomogeneous modification of the light source in the red, green and

blue components. The actions of the groups $\text{SO}(3)$ and $\text{SO}(2)$ on the basis (e_1, e_2, e_3) are more difficult to interpret in terms of modification of the light source.

4.3 A class of generalized Laplacians derived from optimal connection 1-forms

From the optimal connection 1-forms constructed in Sect. 3.1.2 for Lie groups of dimension 1 and Sect. 3.2.2 for Lie groups of dimension 3, we derive a family of generalized Laplacians.

Let ω be such an optimal connection 1-form, s be the section $\Phi(x, e)$ and $A = s^*\omega$ the lift down of ω on the base manifold Ω induced by s . From ω we derive the family $\omega_f: = f\omega$, $f \in C^0(\Omega)$ positive, of connection 1-forms. Then, we construct a family of \mathfrak{g} -valued 1-forms on the base manifold, defined by

$$A_f: = s^*\omega_f = f s^*\omega = f A$$

from which follows a family of covariant derivatives ∇^{A_f} on $P \times_{(\rho, G)} V$ given by $\rho_*(A_f) = f \rho_*(A)$.

Then the minimization problem (2) with respect to the embedding yields a family of connection Laplacians Δ^{A_f} associated with the covariant derivatives ∇^{A_f} .

Remark 4 For $f \equiv 0$, the operator Δ^{A_f} acts as the Beltrami operator with respect to the metric of the base manifold on each components of the sections.

4.4 Experiments

We equip the base manifold Ω of the Riemannian metric g given by the following matrix representation

$$g = \begin{pmatrix} 1 + \kappa \sum_{k=1}^n I_{x_1}^k{}^2 & \kappa \sum_{k=1}^n I_{x_1}^k I_{x_2}^k \\ \kappa \sum_{k=1}^n I_{x_1}^k I_{x_2}^k & 1 + \kappa \sum_{k=1}^n I_{x_2}^k{}^2 \end{pmatrix} \quad (11)$$

in the frame $(\partial/\partial x_1, \partial/\partial x_2)$, for some constant strictly positive κ , where $n = 1$ dealing with grey-valued images and $n = 3$ for color images.

Following Sect. 3, we construct optimal connection 1-forms ω for the dual of the natural representations of \mathbb{R}^{+*} on \mathbb{R} and \mathbb{R}^3 , $\text{DC}(3)$ and $\text{SO}(3)$ on \mathbb{R}^3 , as well as $\text{SO}(2)$ on \mathbb{R}^3 parametrized by the oriented 2-plan $(e_1 + e_2 + e_3) \wedge e_1$. Note that this oriented 2-plan encodes red hue and its non oriented counterpart encodes both

red and cyan hue. Then, we consider different functions f : $f_0 \equiv 0$, $f_1 \equiv 4$ and f_2 defined as follows

$$f_2 = \begin{cases} 4 & \text{if } \sqrt{\lambda^+ - \lambda^-} \geq 0.5 \\ 0 & \text{if } \sqrt{\lambda^+ - \lambda^-} \leq 0.49 \\ 400\sqrt{\lambda^+ - \lambda^-} - 196 & \text{otherwise} \end{cases}$$

where λ^+ and λ^- are respectively the highest and lowest eigenvalue fields of (11). Following Sect. 4.1 and 4.3 we construct the corresponding operators Δ^{A_f} on the associated bundles.

At last, we consider the heat equations

$$\frac{\partial I_t}{\partial t} + \Delta^{A_f} I_t = 0, \quad I_0 = I$$

whose solutions are approximated with the Euler scheme

$$I_{t+dt} = I_t + dt \Delta^{A_f} I_t, \quad I_0 = I$$

We apply the scheme on a grey-valued image (Fig. 1(a)) and a color image (Fig. 2(a), Fig. 3(a)). The following parameters were taken: 20 iterations, $dt = 0.1$, $\delta = 1$ and $\kappa = 0.001$.

For $f = f_0$, the diffusion corresponds to a Beltrami flow on each component of the image. By the chosen metric g in (11), the diffusion tends to preserve the high edges and smooth the low ones. In particular, the texture and noise components are smoothed (Fig. 1(b) for grey-valued image and Fig. 2(b), Fig. 3(b) for color image).

For $f = f_1$, the behaviour of the diffusion follows the action of the dual representation of the group on the acquisition space, that we described in Sect. 4.1. We can distinguish two types of behaviour, depending of the unitary character of the representation. Indeed, dealing with the groups \mathbb{R}^{+*} and $\text{DC}(3)$, whose dual representations are non unitary, we observe an increase of the contrast, which is homogeneous in the components R, G, B for the group \mathbb{R}^{+*} (Fig. 2(c)) (see also Fig. 1(c) for grey-valued image) and inhomogeneous for the group $\text{DC}(3)$ (Fig. 2(e)). Dealing with the dual representation of $\text{SO}(3)$ and $\text{SO}(2)$ parametrized by the red-cyan hues, we obtain an increase of the contrast too, but in the chrominance part of the image. We also observe that the diffusions induced by such groups make appear false colors, of several hues for the case of the group $\text{SO}(3)$ (Fig. 3(c)), of red and cyan hue for the case of the group $\text{SO}(2)$ (Fig. 3(e)). False colors appear because there exist no non trivial rotation that preserve all the hues in the RGB color space. However, whatever the nature of

the group representation, we observe that the use of the function f_1 make highlight the edges, textures and noise of the images.

Combining the functions f_0 and f_1 , we present a diffusion that highlights edges and textures, and smoothes noise, i.e. a diffusion behaving like the diffusion constructed for $f = f_0$ with noise and like the diffusion constructed for $f = f_1$ with edges and textures. As we observe on Fig. 1(d) for the group \mathbb{R}^{+*} on grey-valued image and Fig. 2(d) for color image, on Fig. 2(f) for the group $DC(3)$, on Fig. 3(d) for the group $SO(3)$ and on Fig. 3(f) for the group $SO(2)$, the diffusion induced by the function f_2 constructed above fullfills these properties. Indeed, we use the function $\sqrt{\lambda^+ - \lambda^-}$ as a threshold for separating edges and textures from noise. Such a function is widely used in image segmentation methods. However, as textures and edges are not separated, false colors still appear on the textures dealing with the groups $SO(3)$ and $SO(2)$.

5 Conclusion

In this paper, we constructed a class of covariant derivatives, parametrized by a positive function f on the base manifold, and showed that the corresponding connection Laplacians are suitable operators in the context of image processing (up to a choice of a suitable Riemannian metric on the base manifold). Such a construction required to deal with the notion of bundle associated to principal bundle through a group representation (ρ, G) . In this context, an image may be viewed as (ρ, G) -equivariant function on the principal bundle. We considered the dual representations of the group \mathbb{R}^{+*} on \mathbb{R} for grey-level images, and of the groups \mathbb{R}^{+*} , $DC(3)$, $SO(2)$, $SO(3)$ on \mathbb{R}^3 for color images. Such representations make the pixels of an image be treated as covectors. We showed that the behaviour of the diffusion depends of the unitary character of the representation. Further work will be devoted to construct new functions f in order to perform new kinds of diffusion of multi-channels images, and to provide experiments on other kinds of vector-valued data.

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Fig. 1 Grey-valued image diffusions from heat equations of generalized Laplacians parametrized by a function f and the unique dual representation of \mathbb{R}^{+*} on \mathbb{R} .



(a) Color image

(b) \mathbb{R}^{++} , DC(3) for $f = f_0$ (c) \mathbb{R}^{++} for $f = f_1$ (d) \mathbb{R}^{++} for $f = f_2$ (e) DC(3) for $f = f_1$ (f) DC(3) for $f = f_2$

Fig. 2 Color image diffusions from heat equations of generalized Laplacians parametrized by a function f and the unique dual representations of the Lie groups \mathbb{R}^{++} and DC(3) on \mathbb{R}^3 .



(a) Color image

(b) SO(3), SO(2) for $f = f_0$ (c) SO(3) for $f = f_1$ (d) SO(3) for $f = f_2$ (e) SO(2) parametrized by $\mathcal{P} = (e_1 + e_2 + e_3) \wedge e_1$ and $f = f_1$ (f) SO(2) parametrized by $\mathcal{P} = (e_1 + e_2 + e_3) \wedge e_1$ and $f = f_2$

Fig. 3 Color image diffusions from heat equations of generalized Laplacians parametrized by a function f and dual representations of the groups SO(3) and SO(2) on \mathbb{R}^3 (unique for SO(3) and parametrized by $(e_1 + e_2 + e_3) \wedge e_1$ for SO(2)).