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March 28, 2012

Abstract

We prove, under the exterior geometric control condition, the Kato smoothing effect for solutions of an inhomogenous and damped Schrödinger equation on exterior domains.

Contents

Keywords

Schrödinger equation, Exterior domain, Smoothing effect, Regularized equation, Geometric control condition, Frequency localization.

1 Introduction and results

This paper is devoted to the study of a smoothing effect for a damped Schrödinger equation on exterior domain. In order to formulate the results, we shall begin by recalling some results for Schrödinger equation linking the regularity of solutions and the geometry of domain where these equations are posed.
It is well known that the free Schrödinger equation enjoys the property of the $\mathscr{C}^\infty$ smoothing effect, which can be described as follows: For any distribution u_0 of compact support, the solution of the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^d \\ u_{|t=0} = u_0, \end{cases}$$

is infinitely differentiable with respect to t and x when $t \neq 0$ and $x \in \mathbb{R}^d$.

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Another type of smoothing effect says that if $u_0 \in L^2(\mathbb{R}^d)$ then the solution of the Schrödinger equation satisfies the Kato $\frac{1}{2}$-smoothing effect ($H^{1/2}$-smoothing effect):

$$\int_{\mathbb{R}} \left\| \langle x \rangle^{-s} \Delta^{1/4} u \right\|^2_{L^2(\mathbb{R}^d)} \leq C \|u_0\|^2_{L^2}, \ s > 1/2.$$

This property of gain of regularity has been first observed in the case of $\mathbb{R}^d$ in the works of Constantin-Saut [12], Sjölin [31] and Vega [33] and it has been extended locally in time to variable coefficient operators with non trapping metric by Doi ([13, 15])).

In the case of domains with boundary Burq, Gérard and Tzvetkov [11] proved a local smoothing estimate for $\exp(it\Delta)$ in the exterior domains with non-trapping assumption. Using the $TT^\star$ argument, the proof of the smoothing effect with respect to initial data in [11] is reduced to the non-homogeneous bound which, by performing Fourier transform in time, can be deduced from the bounds on the cut-off resolvent:

$$\|\chi(\lambda^2 - \Delta)^{-1}\chi\|_{L^2 \to L^2} \leq C, \ \forall \lambda \gg 1.$$

The resolvent bound, for which the non-trapping assumption plays a crucial role, is proven for $|\lambda| >> 1$ in greater generality by Lax-Phillips [21], Melrose-Sjostrand [24, 25], Vainberg [32] and Vazy-Zworski [34]

The Kato-effect has been extended by Robbiano and Zuily in [30] to variable coefficients operators with unbounded potential in exterior domains with non trapping metric. The proof of their result is reduced to an estimate localized in frequency which has been established by contradiction using in a crucial way the semiclassical defect measure introduced by P. Gerard [17] (see also [22]). The use of the microlocal defect measure to prove an estimate by contradiction method (Wilcox [35]) go back to Lebeau [22]. This idea has been followed with success by several authors (see Burq [8, 9, 10] Aloui and Khenissi [3, 4, 20]).

In [10], Burq proved that the non trapping condition is necessary for the $H^{1/2}$ smoothing effect and showed, in the case of several convex obstacles satisfying certain assumptions, the smoothing effect with an $\varepsilon > 0$ loss:

$$\|\chi u\|_{L^2(H^{1/2-\varepsilon}(\Omega))} \leq C \|u_0\|_{L^2(\Omega)},$$

where χ is compactly supported.

On the other hand, the non-trapping assumption is also equivalent to the uniform decay of the local energy for the wave equation (see [21, 28, 23]). For the trapping domains, when no such decay is hoped, the idea of stabilization for the wave equation is to add a dissipative term to the equation to force the energy of the solution to decrease uniformly. There is a large literature on the problem of stabilization of wave equation. In the case of bounded domains, we quote essentially the work of J. Rauch and M. Taylor [29] and the one of C. Bardos, G. Lebeau and J. Rauch [6] whose introduced and developed the geometric control condition (GCC). This condition that asserts, roughly speaking, that every ray of geometric optics enters the region where the damping term is effective in a uniform time, turns out to be almost necessary and sufficient for the uniform exponential decay of waves. In [3], Aloui and Khenissi introduced the Exterior Geometric control condition (see below Definition 1.1) and hence extended the result of [6] to the case of exterior domains (see also [4]).

Recently, by analogy with the stabilization problem the first author [1, 2] has introduced the forced smoothing effect for Schrödinger equation in bounded domains; it consists to act on the equation to produce some smoothing effects. More precisely he considered the following equation

$$\begin{cases} i\partial_t u - \Delta_D u + ia(x)(-\Delta_D)^{\frac{1}{2}} a(x) u = 0 & \text{in} \quad]0, +\infty) \times \Omega, \\ u(0, .) = f & \text{in} \quad \Omega, \\ u|_{\mathbb{R}^+ \times \partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain and Δ_D is the Dirichlet-Laplace operator on Ω.

Using the strategy of [11], Aloui [2] proved a weak Kato -Smoothing effect:

$$\|v\|_{L^2([\varepsilon,T],H_D^{s+1}(\Omega))} \leq c \|v_0\|_{H_D^s(\Omega)}, \tag{1.2}$$

where $0 < \varepsilon < T < \infty$ and $v_0 \in H_D^s(\Omega)$, (See [2] for the definition of H_D^s).

By iteration of the last result, Aloui deduced also a $\mathscr{C}^\infty$-smoothing effect for the regularized Schrödinger equation (1.1). Recently, Aloui, Khenissi and Vodev [5] have proved that the Geometric control condition is not necessary to obtain the forced $\mathscr{C}^\infty$- smoothing effect.

On the other hand, using the arguments of [11], we can prove, for the equation (1.1) in exterior domains, the cut-off resolvent bound, which is sufficient to deduce the non-homogenous bound. But, unfortunately, the generator operator $\Delta_D - ia(x)(-\Delta_D)^{\frac{1}{2}}a(x)$ is not self-adjoint and then the $TT^\star$ argument fails. For this reason, we can not prove (with this strategy) the weak Kato-smoothing effect (1.2) for exterior domains.

The question now is the following:

Can we establish the Kato-smoothing effect for the regularized Schrödinger equation (1.1) for which the Geometric Control Condition is necessary? and if so, does this result still hold for exterior problems?

In this paper, we give an affirmative answer. Indeed, under the Exterior Geometric Control condition, we prove the Kato-smoothing effect and the non homogenous bound for the regularized Schrödinger equation in exterior domains. Notice that the case of bounded domains can be treated by the same method.

Our approach for deriving such results is to combine the strategies of Robbiano-Zuily in [30] and Aloui-Khenissi in [3], [20].

In order to state our results, we give several notations and assumptions.
Let K be a compact obstacle in $\mathbb{R}^d$ whose complement Ω an open set with $\mathscr{C}^\infty$ boundary $\partial\Omega$ and $\tilde{P}$ be a second-order differential operator of the form

$$\tilde{P} = \sum_{j,k=1}^{d} D_j(b^{jk}D_k) + V(x), \qquad D_j = \frac{\partial}{i\partial x_j}, \tag{1.3}$$

where coefficients b^{jk} and V are assumed to be in $\mathscr{C}^\infty(\mathbb{R}^d)$, real valued, and $b^{jk} = b^{kj}$, $1 \leq j, k \leq d$.

Throughout this paper, $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$ and we denote by $S_\Omega(M,g)$ the Hörmander's class of symbols if M is a weight and the metric

$$g = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle \xi\rangle^2}.$$

We shall denote by p the principal symbol of $\tilde{P}$, namely

$$p(x,\xi) = \sum_{j,k=1}^{d} b^{jk}(x)\xi_j\xi_k,$$

and we assume that

$$\exists\ c > 0 : p(x,\xi) \geq c|\xi|^2, \quad \text{for } x \text{ in } \mathbb{R}^d \text{ and } \xi \text{ in } \mathbb{R}^d, \tag{1.4}$$

$$\begin{cases} (i)\ b^{jk} \in S_\Omega(1,g),\ \nabla_x b^{jk}(x) = o(\frac{1}{|x|}), \quad |x| \to +\infty, \quad 1 \leq j,\ k \leq d. \\ (ii)\ V \in S_\Omega(\langle x\rangle^2, g), \quad V \geq -C_0 \text{ for some positive constant } C_0. \end{cases} \tag{1.5}$$

Under the assumptions (1.4) and (1.5), the operator $\tilde{P}$ is essentially self-adjoint on $\mathscr{C}_0^\infty(\Omega)$ and we denote by P its self-adjoint extension.
Now we set

$$\Lambda = ((1+C_0)Id + P)^{1/2},$$

which is well defined by functional calculus of self-adjoint positive operators.
We consider the following regularized Schrödinger equation

$$\begin{cases} (D_t + P)u - ia\Lambda a u = f \text{ in }]0,+\infty) \times \Omega \\ u = 0 \text{ on } [0,+\infty) \times \partial\Omega, \\ u_{|t=0} = u_0, \end{cases} \tag{1.6}$$

where $(u_0, f) \in \mathscr{C}_0^\infty(\Omega) \times \mathscr{C}_0^\infty(]0, +\infty) \times \Omega)$ and $a \in \mathscr{C}_0^\infty(\overline{\Omega})$.

Let's recall the Exterior Geometric Control (E.G.C.) condition [3]

Definition 1.1 (E.G.C.)**.** Let $R > 0$ be such that $K \subset B_R = \{|x| < R\}$ and ω be a subset of Ω. We say that ω verifies the Exterior Geometric Control condition on B_R (E.G.C.) if there exists $T_R > 0$ such that every generalized bicharacteristic γ starting from B_R at time $t = 0$, is such that:

- γ leaves $\mathbb{R}^+ \times B_R$ before the time T_R, or
- γ meets $\mathbb{R}^+ \times \omega$ between the times 0 and T_R.

We assume also that the bicaracteristics have no contact of infinite order with the boundary (see, for a precise statement, Definition 2.11).

Under this condition on $\omega = \{x \in \Omega, a^2(x) > 0\}$, we can state our main result.

Theorem 1.2. *Let $T > 0$, $\alpha \in (-1/2, 1/2)$ and $s \in (1/2, 1]$. Let P defined by (1.3) satisfying the assumptions (1.4) and (1.5). Then under, the E.G.C on ω one can find a positive constant $C(T, \alpha, s) = C$ such that*

$$\int_0^T \left\| \Lambda^{\alpha+1/2} \langle x \rangle^{-s} u \right\|^2_{L^2(\Omega)} dt + \sup_{t \in [0,T]} \|\Lambda^\alpha u(t)\|^2_{L^2(\Omega)} \leq C \left(\|\Lambda^\alpha u_0\|^2_{L^2(\Omega)} + \int_0^T \left\| \Lambda^{\alpha-1/2} \langle x \rangle^s f \right\|^2_{L^2(\Omega)} dt \right) \tag{1.7}$$

for all u_0 in $\mathscr{C}_0^\infty(\Omega)$, f in $\mathscr{C}_0^\infty(\Omega \times \mathbb{R}^+)$, where u denotes the solution of (1.6).

Working with $\tilde{u} = e^{i(1+C_0)t} u$, one may assume $V \geq 1$ in (1.5) and $\Lambda = P^{1/2}$, which will be assumed in the sequel. It turns into the following equation

$$\begin{cases} (D_t + P)u - iaP^{1/2}au = f \text{ in } [0, +\infty) \times \Omega \\ u = 0 \text{ on } [0, \infty) \times \partial\Omega, \\ u_{|t=0} = u_0, \end{cases} \tag{1.8}$$

where $P \geq 1$.

Remarks 1.3.

1. When the obstacle is nontrapping, we obtain the result of Robbiano Zuily [30] by taking $a(x) = 0$ and moreover, we improve their result to non homogenous bound.
2. If we consider the equation in a bounded domain Ω of $\mathbb{R}^d$, and replace the exterior geometric condition (E.G.C) by the classical microlocal condition of Bardos-Lebeau-Rauch [6], we can still prove the Kato-effect and then we improve the result of Aloui [2].
3. If there is a trapped ray which does not intersect the regularized region, due to Burq [10], the Kato-effect does not hold. In this context, our result is thus optimal.

The rest of the paper is organized as follows: Section 2 is devoted to the proof of Theorem 1.2 while in the Section A we shall prove some Lemmata used in Section 2.

2 Proofs

Let's describe the strategy of the proof of theorem 1.2. In a first step, we reduce the estimate (1.7) to an analogue one localized in frequencies. By following a contradiction argument, we can construct an adapted microlocal defect measure. Our aim in the rest of the proof is to obtain a contradiction on this measure. First, we prove that this measure is not identically null. Next, we show that it is null on incoming set and on $\{a^2 > 0\}$. Finaly, using the geometrical assumption (E.G.C.) and that the support of this measure is propagated along the generalized flow, we conclude that the measure is identically null. This gives the contradiction.

2.1 Reduction to an estimate localized in frequency

We recall the Paley-Littlewood decomposition. Let $\Phi \in \mathscr{C}_0^\infty([0,+\infty))$ be a decreasing function such that

$$\Phi(s) = 1 \text{ if } \ s \leq 1/2, \quad \Phi(s) = 0 \ \text{ if } \ s \geq 1.$$

Let $\psi(s) = \Phi(4^{-1}s) - \Phi(s)$, $\psi(s) = 0$ if $s \leq 1/2$ or $s \geq 4$, $0 \leq \psi \leq 1$. For $s \geq 0$ we have

$$1 = \Phi(s) + \sum_{n=0}^{+\infty} \psi(4^{-n}s),$$

and using $P \geq 1$, we have

$$u = \sum_{n=0}^{+\infty} \psi(4^{-n}P)u.$$

For support reason

$$\psi(4^{-n}s)\psi(4^{-k}s) = 0 \text{ if } |k-n| \geq 2,$$

thus there exists $C > 0$ such that for all $u \in L^2(\Omega)$,

$$\|u\|^2_{L^2(\Omega)} \leq C \sum_{n=0}^{+\infty} \|\psi(4^{-n}P)u\|^2_{L^2(\Omega)} \leq C^2 \|u\|^2_{L^2(\Omega)}.$$

In the sequel we denote by $h_n = 2^{-n}$ and $u_n = u_{h_n} = \psi(h_n^2 P)u$.
If u satisfies

$$D_t u + Pu - iaP^{1/2}(au) = f, \tag{2.1}$$

thus u_n is a solution of the following semi-classical Schrödinger equation:

$$h_n^2(D_t + P)u_n - ih_n a(h_n^2 P)^{1/2}(au_n) = h_n g_n, \tag{2.2}$$

where

$$g_n = g_{h_n} = h_n \psi(h_n^2 P)f + i[\psi(h_n^2 P), a](h_n^2 P)^{1/2}(au) + ia(h_n^2 P)^{1/2}[\psi(h_n^2 P), a]u. \tag{2.3}$$

Proposition 2.1. *Let $s \in (1/2, 1]$, $T > 0$ and $\alpha \in (-1/2, 1/2)$. Assume there exists $C > 0$ such that for $u_n = \psi(h_n^2 P)u$ satisfying (2.2), we have, for all $n \geq 1$*

$$\|\langle x\rangle^{-s} u_n\|^2_{L^2([0,T]\times\Omega)} + h_n \sup_{t\in[0,T]} \|u_n(t)\|^2_{L^2(\Omega)} \leq C\left(h_n \|u_n(0)\|^2_{L^2(\Omega)} + \|\langle x\rangle^s g_n\|^2_{L^2([0,T]\times\Omega)}\right), \tag{2.4}$$

then there exists $C' > 0$ such that for all u satisfying (2.1) we have

$$\begin{aligned} \|P^{\alpha/2+1/4}\langle x\rangle^{-s} u\|^2_{L^2([0,T]\times\Omega)} + \sup_{t\in[0,T]} \|P^{\alpha/2}u(t)\|^2_{L^2(\Omega)} \\ \leq C'\left(\|P^{\alpha/2}u(0)\|^2_{L^2(\Omega)} + \|P^{\alpha/2-1/4}\langle x\rangle^s f\|^2_{L^2([0,T]\times\Omega)}\right). \end{aligned} \tag{2.5}$$

Proof. We multiply (2.4) by $h_n^{-2\alpha-1}$ and we sum over $n \in \mathbb{N}$, we obtain,

$$\begin{aligned} &\sum_{n\in\mathbb{N}} h_n^{-2\alpha-1} \|\langle x\rangle^{-s} u_n\|^2_{L^2([0,T]\times\Omega)} + \sum_{n\in\mathbb{N}} h_n^{-2\alpha} \sup_{t\in[0,T]} \|u_n(t)\|^2_{L^2(\Omega)} \\ &\leq C\left(\sum_{n\in\mathbb{N}} h_n^{-2\alpha} \|u_n(0)\|^2_{L^2(\Omega)} + \sum_{n\in\mathbb{N}} h_n^{-2\alpha-1} \|\langle x\rangle^s g_n\|^2_{L^2([0,T]\times\Omega)}\right). \end{aligned} \tag{2.6}$$

Now, let us estimate each term appearing in inequality (2.5). We have,

$$\begin{aligned} \sup_{t\in[0,T]} \|P^{\alpha/2}u(t)\|^2_{L^2(\Omega)} &\leq C \sup_{t\in[0,T]} \sum_{n\in\mathbb{N}} \|\psi(h_n^2 P)P^{\alpha/2}u(t)\|^2_{L^2(\Omega)} \\ &\leq C \sup_{t\in[0,T]} \sum_{n\in\mathbb{N}} h_n^{-2\alpha} \|\psi_0(h_n^2 P)u(t)\|^2_{L^2(\Omega)} \text{ where } \psi_0(\sigma) = \sigma^{\alpha/2}\psi(\sigma) \\ &\leq C \sum_{n\in\mathbb{N}} h_n^{-2\alpha} \sup_{t\in[0,T]} \|\psi(h_n^2 P)u(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.7}$$

We have also with $\psi_1(\sigma) = \sigma^{\alpha/2+1/4}\psi(\sigma)$,

$$\begin{aligned}\|P^{\alpha/2+1/4}\langle x\rangle^{-s}u\|^2_{L^2([0,T]\times\Omega)} &\leq C\sum_{n\in\mathbb{N}} h_n^{-2\alpha-1}\|\psi_1(h_n^2P)\langle x\rangle^{-s}u\|^2_{L^2([0,T]\times\Omega)}\\ &\leq C\sum_{n\in\mathbb{N}} h_n^{-2\alpha-1}\|\langle x\rangle^{-s}\psi(h_n^2P)u\|^2_{L^2([0,T]\times\Omega)}\text{ (by Lemma A.8)}\\ &\leq C\sum_{n\in\mathbb{N}} h_n^{-2\alpha-1}\|\langle x\rangle^{-s}u_n\|^2_{L^2([0,T]\times\Omega)}. \end{aligned} \tag{2.8}$$

Now we can estimate, with $\psi_2(\sigma) = \sigma^{-\alpha/2}\psi(\sigma)$,

$$\begin{aligned}\sum_{n\in\mathbb{N}} h_n^{-2\alpha}\|u_n(0)\|^2_{L^2(\Omega)} &\leq C\sum_{n\in\mathbb{N}}\|\psi_2(h_n^2P)P^{\alpha/2}u(0)\|^2_{L^2(\Omega)}\\ &\leq C\|P^{\alpha/2}u(0)\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.9}$$

The term g_n contains three terms (see (2.3)). For the first, we have, with $\psi_3(\sigma) = \sigma^{-\alpha/2+1/4}\psi(\sigma)$,

$$\begin{aligned}\sum_{n\in\mathbb{N}} h_n^{-2\alpha+1}\|\langle x\rangle^s\psi(h_n^2P)f\|^2_{L^2([0,T]\times\Omega)} &\leq \sum_{n\in\mathbb{N}} h_n^{-2\alpha+1}\|\psi(h_n^2P)\langle x\rangle^s f\|^2_{L^2([0,T]\times\Omega)}\\ &\leq C\sum_{n\in\mathbb{N}}\|\psi_3(h_n^2P)P^{\alpha/2-1/4}\langle x\rangle^s f\|^2_{L^2([0,T]\times\Omega)}\\ &\leq C\|P^{\alpha/2-1/4}\langle x\rangle^s f\|^2_{L^2([0,T]\times\Omega)}. \end{aligned} \tag{2.10}$$

For the second and the third terms of g_n we can apply the Lemmata A.9 and A.11, to obtain with (2.10),

$$\sum_{n\in\mathbb{N}} h_n^{-2\alpha-1}\|\langle x\rangle^s g_n\|^2_{L^2([0,T]\times\Omega)} \leq C\|P^{\alpha/2-1/4}\langle x\rangle^s f\|^2_{L^2([0,T]\times\Omega)} + C\|P^{\alpha/2}u\|^2_{L^2([0,T]\times\Omega)}. \tag{2.11}$$

Then following (2.6) (2.7), (2.8), (2.9) and (2.11), we obtain

$$\begin{aligned}&\|P^{\alpha/2+1/4}\langle x\rangle^{-s}u\|^2_{L^2([0,T]\times\Omega)} + \sup_{t\in[0,T]}\|P^{\alpha/2}u(t)\|^2_{L^2(\Omega)}\\ &\leq C\left(\|P^{\alpha/2}u(0)\|^2_{L^2(\Omega)} + \|P^{\alpha/2-1/4}\langle x\rangle^s f\|^2_{L^2([0,T]\times\Omega)} + \|P^{\alpha/2}u\|^2_{L^2([0,T]\times\Omega)}\right).\end{aligned}$$

By Gronwall's Lemma, we can remove the last term in the previous inequality and we obtain (2.5). ∎

2.2 Construction of microlocal defect measure

In this section we will prove the localized frequency estimate (2.4) by a contradiction argument and using microlocal defect measure.

More precisely, let u_h solution of

$$h^2(D_t + P)u_h - iha(h^2P)^{1/2}(au_h) = hg_h. \tag{2.12}$$

We will prove by contradiction the following estimate,

$$\|\langle x\rangle^{-s}u_h\|^2_{L^2([0,T]\times\Omega)} + h\sup_{t\in[0,T]}\|u_h(t)\|^2_{L^2(\Omega)} \leq Ch\|u_h(0)\|^2_{L^2(\Omega)} + C\|\langle x\rangle^s g_h\|^2_{L^2([0,T]\times\Omega)}. \tag{2.13}$$

Assuming it is false. Taking $C = k \in \mathbb{N}$, we deduce sequences $h_k \underset{k\to+\infty}{\to} 0$, $u_k^0 = u_{h_k}(0) \in L^2(\Omega)$ and $g_k = g_{h_k} \in L^2(\Omega)$ such that,

$$h_k\left\|u_k^0\right\|^2_{L^2(\Omega)} \underset{k\to+\infty}{\to} 0,\ \ \left\|\langle x\rangle^s g_k\right\|^2_{L^2([0,T]\times\Omega)} \underset{k\to+\infty}{\to} 0. \tag{2.14}$$

We normalize by the left term in (2.13), thus

$$\left\| \langle x\rangle^{-s} u_k \right\|^2_{L^2([0,T]\times\Omega)} + h_k \sup_{t\in[0,T]} \|u_k(t)\|^2_{L^2(\Omega)} = 1,$$

where, for simplicity, we have denoted $u_{h_k} = u_k$. By the Lemma A.1 we have

$$h_k \sup_{t\in[0,T]} \|u_k(t)\|^2_{L^2(\Omega)} \underset{k\to+\infty}{\to} 0, \tag{2.15}$$

then

$$\left\| \langle x\rangle^{-s} u_k \right\|^2_{L^2([0,T]\times\Omega)} \underset{k\to+\infty}{\to} 1. \tag{2.16}$$

The sequence (u_k) is bounded in $L^2_{loc}(\mathbb{R}_t, L^2_{loc}(\Omega))$. Indeed, for $R > 0$, there exists $c > 0$ such that $\langle x\rangle^{-2s} \geq c,\ \forall x \in B(0,R)$ and then we have

$$\int_0^T \int_{\Omega\cap B_R} |u_k|^2 dtdx \leq \frac{1}{c} \int_0^T \int_{\Omega\cap B_R} \langle x\rangle^{-2s} |u_k|^2 dtdx \leq \frac{1}{c}. \tag{2.17}$$

We set

$$\begin{cases} w_k = 1_\Omega u_k(t) \\ W_k = 1_{[0,T]} w_k. \end{cases} \tag{2.18}$$

It follows from (2.17) that the sequence (W_k) is bounded in $L^2(\mathbb{R}_t, L^2_{loc}(\mathbb{R}^d))$.
We associate to a symbol $b = b(x,t,\xi,\tau) \in \mathscr{C}^\infty_0(T^*\mathbb{R}^{d+1})$ the semiclassical pseudo-differential operator (pdo) by the formula

$$\mathcal{O}p(b)(y,s,hD_x,h^2D_t)v(x,t) = \frac{1}{(2\pi h)^{d+1}} \iint e^{i\left(\frac{x-y}{h}\xi + \frac{t-s}{h^2}\tau\right)} \varphi(y) b(x,t,\xi,\tau) v(y,s) dyds d\xi d\tau,$$

where $\varphi \in \mathscr{C}^\infty_0(\mathbb{R}^d)$ is equal to one on a neighborhood of the x-projection of the support of b. As in [30] we can associate to (W_k) a semi-classical measure μ. More precisely,

Proposition 2.2. *There exists a subsequence $(W_{\sigma(k)})$ and a Radon measure μ on $T^*\mathbb{R}^{d+1}$ such that for every $b \in \mathscr{C}^\infty_0(T^*\mathbb{R}^{d+1})$ one has*

$$\lim_{k\to+\infty} \left(\mathcal{O}p(b) \left(x,t,h_{\sigma(k)}D_x, h^2_{\sigma(k)}D_t \right) W_{\sigma(k)}, W_{\sigma(k)} \right)_{L^2(\mathbb{R}^{d+1})} = \langle \mu, b\rangle .$$

We prove first that the measure μ satisfies the following property.

Proposition 2.3. *The support of μ is contained in the characteristic set of the operator $D_t + P$*

$$\Sigma = \{(x,t,\xi,\tau) \in T^*\mathbb{R}^{d+1} : x \in \overline{\Omega}, t \in [0,T] \text{ and } \tau + p(x,\xi) = 0\}. \tag{2.19}$$

Proof. According to (2.18), it is obvious that

$$\operatorname{supp}\mu \subset \{(x,t,\xi,\tau) \in T^*\mathbb{R}^{d+1} : x \in \overline{\Omega}, t \in [0,T]\}.$$

Therefore it remains to show that if $m_0 = (x_0,t_0,\xi_0,\tau_0)$ with $x_0 \in \overline{\Omega}, t_0 \in [0,T]$, and $\tau_0 + p(x_0,\xi_0) \neq 0$ then $m_0 \notin \operatorname{supp}\mu$. For simplicity, we shall denote the sequence $W_{\sigma(k)}$ by W_k.

Case 1. Assume that $x_0 \in \Omega$.

Let $\varepsilon > 0$ be such that $B(x_0,\varepsilon) \subset \Omega$, $\varphi \in \mathscr{C}^\infty_0(B(x_0,\varepsilon))$, $\varphi = 1$ on $B(x_0,\frac{\varepsilon}{2})$ and $\tilde{\varphi} \in \mathscr{C}^\infty_0(\Omega)$, $\tilde{\varphi} = 1$ on $\operatorname{supp}\varphi$. Let $b \in \mathscr{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ such that $\pi_x \operatorname{supp} b \subset B(x_0,\frac{\varepsilon}{2})$ and $\chi \in \mathscr{C}^\infty_0(\mathbb{R}_t \times \mathbb{R}_\tau)$. Recall that we have $W_k = 1_{[0,T]} 1_\Omega u_k$ and that (u_k) is bounded sequence in $L^2([0,T], L^2_{loc}(\Omega))$. We set

$$I_k = (b(x,h_kD_x)\,\chi(t,h_k^2D_t)\varphi(x)h_k^2(D_t + P(x,D_x))W_k, \tilde{\varphi}W_k)_{L^2(\mathbb{R}^{d+1})}.$$

As in [30] we have

$$\lim_{k\to+\infty} I_k = \langle \mu, (\tau+p)b\chi\rangle\,. \tag{2.20}$$

On the other hand, since we have

$$h_k^2(D_t+P(x,D_x))u_k = h_k ia(h_k^2P)^{1/2}au_k + h_k g_k,$$

and $\varphi\in\mathscr{C}_0^\infty(\Omega)$,

$$\varphi(h_k^2D_t+h_k^2P(x,D_x))W_k = \varphi(ih_k a(h_k^2P)^{1/2}au_k+h_kg_k)+h_k^2\varphi(u_k(0)\delta_{t=0}-h_k^2u_k(T)\delta_{t=T}). \tag{2.21}$$

Then I_k is a sum of four terms,

$$I_k = I_k^1+I_k^2+I_k^3+I_k^4,$$

$$\begin{aligned}
I_k^1 &= ih_k\,(b(x,h_kD_x)\,\chi(t,h_k^2D_t)\varphi(x)a(h_k^2P)^{1/2}au_k\,,\tilde\varphi W_k)_{L^2(\mathbb{R}^{d+1})}\\
I_k^2 &= h_k\,(b(x,h_kD_x)\,\chi(t,h_k^2D_t)\varphi(x)g_k,\tilde\varphi W_k)_{L^2(\mathbb{R}^{d+1})}\\
I_k^3 &= (b(x,h_kD_x)\,\chi(t,h_k^2D_t)h_k^2\varphi(x)u_k(0)\delta_{t=0},\tilde\varphi W_k)_{L^2(\mathbb{R}^{d+1})}\\
I_k^4 &= -(b(x,h_kD_x)\chi(t,h_k^2D_t)h_k^2\varphi(x)u_k(T)\delta_{t=T},\tilde\varphi W_k)_{L^2(\mathbb{R}^{d+1})}.
\end{aligned}$$

For the first term I_k^1, we use the Lemma A.6, we have,

$$\left\|(h_k^2P)^{1/2}au_k\right\|^2_{L^2(\Omega)} \le Ch_k^2\|u_k\|^2_{L^2(\Omega)}+C\|au_k\|^2_{L^2(\Omega)}, \tag{2.22}$$

and we deduce,

$$|I_k^1| \le c(h_k^2\sup_{t\in[0,T]}\|u_k\|^2_{L^2(\Omega)}+h_k\sup_{t\in[0,T]}\|u_k\|^2_{L^2(\Omega)}). \tag{2.23}$$

Then we obtain, that I_k^1 goes to zero by (2.15). For the second term I_k^2,

$$\begin{aligned}
\left|I_k^2\right| &\le h_k\,\|g_k\|_{L^2([0,T],B(x_0,\varepsilon))}\,\|\tilde\varphi W_k\|_{L^2(\mathbb{R}^{d+1})}\\
&\le Ch_k\,\|\langle x\rangle^s g_k\|_{L^2([0,T]\times\Omega)}\left\|\langle x\rangle^{-s}u_k\right\|_{L^2([0,T]\times\Omega)}.
\end{aligned}$$

Using (2.14) and (2.16), we deduce that

$$\lim_{k\to+\infty} I_k^2 = 0. \tag{2.24}$$

The third and fourth terms in (2.21) have the following form,

$$J_k = \left(b(x,h_kD_x)\chi(t,h_k^2D_t)\varphi h_k^2u_k(s)\delta_{t=s},\tilde\varphi W_k\right)_{L^2(\mathbb{R}^{d+1})}, \quad s=0 \text{ or } T.$$

Since $(\tilde\varphi W_k)$ is bounded in $L^2(\mathbb{R}^{d+1})$, we see that

$$|J_k|^2 \le c\,\|b\varphi w_k(s)\|^2_{L^2(\mathbb{R}^d)}\left\|h_k^2\chi(t,h_k^2D_t)\delta_{t=s}\right\|^2_{L^2(\mathbb{R})}\sup_{t\in[0,T]}\|u_k(t)\|^2_{L^2(\Omega)},$$

so, using [30, Lemma A.5] with $p=2$ and $l=2$, we deduce that,

$$|J_k|^2 \le ch_k^2\,\|u_k(s)\|^2_{L^2(\Omega)}\sup_{t\in[0,T]}\|u_k(t)\|^2_{L^2(\Omega)} \le c\,h_k^2\sup_{t\in[0,T]}\|u_k(t)\|^4_{L^2(\Omega)}. \tag{2.25}$$

It follows from (2.23), (2.24), (2.25) and (2.15) that

$$\lim_{k\to\infty} I_k = 0. \tag{2.26}$$

As the linear combination of $\chi(t,\tau)b(x,\xi)$ are dense in $\mathscr{C}_0^\infty(T^\star(\mathbb{R}^{d+1}))$, using (2.20) and (2.26), we deduce that $m_0=(x_0,t_0,\xi_0,\tau_0)\notin\operatorname{supp}\mu$.

Case 2. Assume that $x_0 \in \partial\Omega$.

We would like to show that one can find a neighborhood U_{x_0} of x_0 in $\mathbb{R}^d$ such that for any $b \in \mathscr{C}_0^\infty(U_{x_0} \times \mathbb{R}_t \times \mathbb{R}^d_\xi \times \mathbb{R}_\tau)$, we have

$$\langle \mu, (\tau + p)b \rangle = 0. \tag{2.27}$$

Indeed this will imply that the point $m_0(x_0, t_0, \xi_0, \tau_0)$ (with $\tau_0 + (x_0, \xi_0) \neq 0$) does not belong to the support of μ as claimed. Formula (2.27) will be implied, by

$$\begin{cases} \lim\limits_{k\to+\infty} I_k = 0 \text{ where} \\ I_k = \left(b(x, t, h_k D_x, h_k^2 D_t)\varphi h_k^2 (D_t + P)W_k, W_k\right)_{L^2(\mathbb{R}^{d+1})}. \end{cases} \tag{2.28}$$

where $\varphi \in \mathscr{C}_0^\infty(U_{x_0}), \varphi = 1$ on $\pi_x \operatorname{supp} b$. Let U_{x_0} a neighborhood of x_0 such that there exists a $\mathscr{C}^\infty$ diffeomorphisme F from U_{x_0} to a neighborhood U_0 of the origin in $\mathbb{R}^d$ satisfying,

$$\begin{cases} F(U_{x_0} \cap \Omega) = \{y \in U_0 : y_1 > 0\} \\ F(U_{x_0} \cap \partial\Omega) = \{y \in U_0 : y_1 = 0\} \\ (P(x, D)W_k) \circ F^{-1} = (D_1^2 + R(y, D') + L(x, D))(W_k \circ F^{-1}), \end{cases} \tag{2.29}$$

where R is a second-order differential operator, $D' = (D_2, ..., D_d)$ and $L(x, D)$ a first order differential operator. Let us set

$$v_k = u_k \circ F^{-1}, \quad V_k = 1_{[0,T]} 1_{y_1>0} v_k, \tag{2.30}$$

then we will have

$$\begin{cases} \left(D_t + D_1^2 + R(y, D') + L(x, D)\right) v_k = iaP^{1/2}(au_k) \circ F^{-1} + h_k^{-1} g_k \circ F^{-1} := f_k \\ v_k|_{y_1=0} = 0. \end{cases} \tag{2.31}$$

Making the change of variable $x = F^{-1}(y)$ on the right-hand side of the second line of (2.28), we see that

$$I_k = \left(\tilde{b}(y, t, h_k D_y, h_k^2 D_t)\psi h_k^2 (D_t + D_1^2 + R(y, D') + L(x, D))V_k, V_k\right)_{L^2(\mathbb{R}^{d+1})},$$

where $\tilde{b} \in \mathscr{C}_0^\infty(U_0 \times \mathbb{R}_t \times \mathbb{R}^d_\eta \times \mathbb{R}_\tau)$, and $\psi \in \mathscr{C}_0^\infty(U_0)$, $\psi = 1$ on $\pi_y \operatorname{supp} \tilde{b}$. To prove (2.28) it is sufficient to prove that,

$$\lim_{k\to+\infty} J_k = \lim_{k\to+\infty} \left(T\psi_0(y_1)\psi_1(y') h_k^2 (D_t + D_1^2 + R(y, D') + L(x, D))V_k, V_k\right)_{L^2(\mathbb{R}^{d+1})} = 0,$$

where $T = \theta(y_1, h_k D_1)\Phi(y', h_k D')\chi(t, h_k^2 D_t)$, $\theta\Phi\chi \in \mathscr{C}_0^\infty(U_0 \times \mathbb{R}_t \times \mathbb{R}^d_\eta \times \mathbb{R}_\tau)$, $\psi_0\psi_1 \in \mathscr{C}_0^\infty(U_0)$, $\psi_0\psi_1 = 1$ on $\pi_y \operatorname{supp} \theta\Phi\chi$; According to (2.31) we have,

$$\begin{aligned} (D_t + D_1^2 + R(y, D') + L(x, D))V_k = f_k - i1_{y_1>0} v_k(0, .)\delta_{t=0} + i1_{y_1>0} v_k(T, .)\delta_{t=T} \\ - i1_{[0,T]}(D_1 v_k|_{y_1=0}) \otimes \delta_{y_1=0}. \end{aligned}$$

Therefore (2.28) will be proved if we can prove that

$$\begin{cases} \lim\limits_{k\to+\infty} A_k^j = 0, \quad j = 1, 2, 3, \text{ where} \\ A_k^1 = \left(\theta(y_1, h_k D_1)\Phi(y', h_k D')\chi(t, h_k^2 D_t)\psi_0\psi_1 h_k^2 1_{y_1>0} v_k(s, .)\delta_{t=s}, V_k\right), \ s = 0, \ T, \\ A_k^2 = \left(\theta(y_1, h_k D_1)\Phi(y', h_k D')\chi(t, h_k^2 D_t)\psi_0\psi_1 h_k^2 1_{[0,T]}(D_1 v_k|_{y_1=0}) \otimes \delta_{y_1=0}, V_k\right), \\ A_k^3 = \left(\theta(y_1, h_k D_1)\Phi(y', h_k D')\chi(t, h_k^2 D_t)\psi_0\psi_1 h_k^2 f_k, V_k\right). \end{cases} \tag{2.32}$$

As in [30, A.18]

$$\lim_{k\to+\infty} A_k^1 = 0. \tag{2.33}$$

To estimate the term A_k^2 we need a Lemma. With U_0 introduced in (2.29), we set $U_0^+ = \{y \in U_0 : y_1 > 0\}$. We consider a smooth solution of the problem:

$$\begin{cases} \left(D_t + D_1^2 + R(y, D') + L(x, D)\right) u = g \ \text{ in } \ U_0^+ \times \mathbb{R}_t \\ u|_{y_1=0} = 0 \end{cases} \tag{2.34}$$

Lemma 2.4. *Let $\chi \in \mathscr{C}_0^\infty(U_0)$ and $\chi_1 \in \mathscr{C}_0^\infty(U_0)$ $\chi_1 = 1$ on $\operatorname{supp}\chi$. There exists $C > 0$ such that for any solution u of (2.34) and all h in $]0,1]$, we have*

$$\int_0^T \left\| (\chi h\partial_1 u)_{|y_1=0}(t) \right\|_{L^2}^2 dt \le C \left(\int_0^T \sum_{|\alpha|\le 1} \|\chi_1 (hD)^\alpha u(t)\|^2_{L^2(U_0^+)}\, dt \right.$$
$$+ \left\| h^{\frac{1}{2}} \chi u(0) \right\|_{L^2(U_0^+)} \left\| h^{\frac{1}{2}} (h\partial_1 u)(0) \right\|_{L^2(U_0^+)}$$
$$\left. + \left\| h^{\frac{1}{2}} \chi u(T) \right\|_{L^2(U_0^+)} \left\| h^{\frac{1}{2}} (h\partial_1 u)(T) \right\|_{L^2(U_0^+)} + \|\chi_1 hg\|^2_{L^2} \right).$$

Proof of the Lemma. It is analogue to the proof of [30, Lemma A.6]. ∎

We replace in the previous Lemma g by $iaP^{1/2}(au_k)\circ F^{-1} + h_k^{-1} g_k \circ F^{-1}$ and by (2.30), we obtain easily the following corollary.

Corollary 2.5. *One can find a constant $C > 0$ such that*

$$\int_0^T \left\| (\chi h_k \partial_1 v_k)_{|y_1=0}(t) \right\|_{L^2}^2 dt \le C \left(\int_0^T \|\tilde\chi u_k(t)\|^2_{L^2(\Omega)}\, dt + \left\| h_k^{1/2} u_k(0) \right\|^2_{L^2(\Omega)} dt \right.$$
$$\left. + \int_0^T \left(\left\| \tilde\chi a (h_k^2 P)^{1/2} a u_k \right\|_{L^2}^2 + \|\tilde\chi g_k\|^2_{L^2} \right) dt \right)$$
$$\le C,$$

where v_k has been defined in (2.30) and $\tilde\chi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$.

Let us go back to the estimate of A_k^2 defined in (2.32). We have

$$\left|A_k^2\right|^2 \le C h_k^2 \left\|\theta(y_1, h_k D_1)\delta_{y_1=0}\right\|^2_{L^2(\mathbb{R})} \left\|(\psi_2 V_k)\right\|^2_{L^2(\mathbb{R}^{d+1})} \int_0^T \left\| (\psi_1 h_k D_1 v_k)_{|y_1=0}(t) \right\|^2_{L^2(\mathbb{R}^{d-1})} dt.$$

Applying (2.17), [30, Lemma A.5] with $p = 2$, $l = 1$ and corollary 2.5, we obtain

$$\left|A_k^2\right| \le c h_k \longrightarrow 0. \tag{2.35}$$

The term $\left|A_k^3\right|$ can be treated as the first and the second term in the case 1.
Using (2.33) and (2.35), we deduce (2.32), which implies (2.28) thus (2.27). The proof of Proposition 2.3 is complete. ∎

2.3 The microlocal defect measure does not vanish identically

First let us prove that the sequence (u_k) have mass in a compact domain.

Lemma 2.6. *There exists a subsequence k_ν, there exists $R > 0$ such that*

$$\int_0^T \|u_{k_\nu}(t)\|^2_{L^2(x\in\Omega,\ |x|<R)} dt \ge 1/2.$$

Proof of Lemma. We prove the Lemma by contradiction. Assume that

$$\forall R > R_0,\ \limsup_k \int_0^T \|u_k(t)\|^2_{L^2(x\in\Omega,\ |x|\le 2R+1)} dt \le 3/4, \tag{2.36}$$

where R_0 is large enough such that $\operatorname{supp} a \subset \{|x| \le R_0/2\}$.

Let $\chi \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $\chi = 1$ for $|x| > 2$ and $\chi = 0$ for $|x| < 1$. We set $\chi_R(x) = \chi(x/R)$ and by the choice of R_0 we have $a\chi_R = \chi_R a = 0$. The function $v_k := \chi_R u_k$ satisfies

$$D_t v_k + P v_k = h_k^{-1} \chi_R g_k + [P, \chi_R] u_k.$$

From [16, Theorem 2.8], we have

$$\int_0^T \|\langle x\rangle^{-s} v_k\|^2_{L^2(\mathbb{R}^d)} \leq C(\|E_{-\frac{1}{2}} v_k(0)\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \|\langle x\rangle^s E_{-1}\left(h_k^{-1}\chi_R g_k + [P,\chi_R]u_k\right)\|^2_{L^2(\mathbb{R}^d)} dt), \tag{2.37}$$

where E_s is the pseudo-differential operator with symbol $e_s = (1 + p(x,\xi) + |x|^2)^{\frac{s}{2}}$ which belongs to $S((|\xi| + < x >)^s, g)$.

For the first term of the right hand side of (2.37) we have, where $(\cdot,\cdot)$ means the scalar product in $L^2(\Omega)$,

$$\begin{aligned}
\|E_{-\frac{1}{2}} v_k(0)\|^2_{L^2} &= h_k \|E_{-\frac{1}{2}} \chi_R P^{\frac{1}{4}} (h_k^2 P)^{-\frac{1}{4}} \psi_1(h_k^2 P)\psi(h_k^2 P) u(0)\|^2_{L^2}, \\
&= h_k (S\psi_2(h_k^2 P)u_k(0), S\psi_2(h_k^2 P)u_k(0)), \text{ where } S = E_{-\frac{1}{2}}\chi_R P^{\frac{1}{4}}, \text{ and } \psi_2(t) = t^{-\frac{1}{4}}\psi_1 \\
&= h_k (\psi_2(h_k^2 P) S^\star S \psi_2(h_k^2 P) u_k(0), u_k(0)) \\
&= h_k (\psi_2(h_k^2 P)(h_k^2 P)^{-\frac{1}{4}} Q \chi_R (h_k^2 P)^{\frac{1}{4}} \psi_2(h_k^2 P) u_k(0), u_k(0)) \\
&\leq C h_k \|u_k(0)\|^2_{L^2},
\end{aligned}$$

where $\psi_1 \in \mathscr{C}_0^\infty(0,+\infty)$ and $\psi_1 = 1$ on $\mathrm{supp}(\psi)$, $S^\star S = P^{-\frac{1}{4}} Q \chi_R P^{\frac{1}{4}}$, $Q = P^{\frac{1}{2}}\chi_R A_{-1}$, and $A_{-1} = E^\star_{-\frac{1}{2}} E_{-\frac{1}{2}}$. We have used that the operator Q is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\Omega)$ (see [30, Lemma 4.2]).
Then from (2.15), we deduce that

$$\lim_{k\to+\infty} \|E_{-\frac{1}{2}} v_k(0)\|^2_{L^2} = 0. \tag{2.38}$$

Concerning the term $\displaystyle\int_0^T \|\langle x\rangle^s E_{-1} h_k^{-1}\chi_R g_k\|^2_{L^2} dt$, we will prove that it tends to zero.
Let $\psi_1 \in \mathscr{C}_0^\infty(\mathbb{R})$, such that $\psi_1 = 1$ on $\mathrm{supp}\,\psi$.
Since $\psi_1(h_k^2 P)u_k = u_k$ then applying $1 - \psi_1(h_k^2 P)$ to Formula (2.12), we obtain

$$h_k^{-1} g_k = h_k^{-1}\psi_1(h_k^2 P) g_k - i h_k^{-1} a(h_k^2 P)^{1/2} a \psi_1(h_k^2 P) u_k + i h_k^{-1}\psi_1(h_k^2 P) a (h_k^2 P)^{1/2} a u_k.$$

Using that $\chi_R a = 0$, we have

$$h_k^{-1}\chi_R g_k = h_k^{-1}\chi_R \psi_1(h_k^2 P) g_k + i h_k^{-1}\chi_R \psi_1(h_k^2 P) a (h_k^2 P)^{1/2} a u_k.$$

And then

$$\begin{aligned}
&\int_0^T \|\langle x\rangle^s E_{-1} h_k^{-1}\chi_R g_k\|^2_{L^2} dt \\
&\quad\leq \int_0^T \|\langle x\rangle^s E_{-1}\chi_R h_k^{-1}\psi_1(h_k^2 P) g_k\|^2 dt + \int_0^T \|\langle x\rangle^s E_{-1}\chi_R h_k^{-1}\psi_1(h_k^2 P) a (h_k^2 P)^{1/2} a u_k\|^2 dt \\
&\quad\leq \int_0^T \|\langle x\rangle^s E_{-1}\chi_R P^{1/2}\psi_2(h_k^2 P) g_k\|^2 dt + \int_0^T \|\langle x\rangle^s E_{-1}\chi_R h_k^{-1}\psi_1(h_k^2 P) a (h_k^2 P)^{1/2} a u_k\|^2 dt,
\end{aligned}$$

where $\psi_2(t) = t^{-1/2}\psi_1(t)$. We have,

$$\int_0^T \|\langle x\rangle^s E_{-1}\chi_R P^{1/2}\psi_2(h_k^2 P) g_k\|^2 dt \leq I + II, \tag{2.39}$$

where

$$I = \int_0^T \|\langle x\rangle^s E_{-1}\langle x\rangle^{-s}\chi_R P^{1/2}\psi_2(h_k^2 P)\langle x\rangle^s g_k\|^2 dt$$

and

$$II = h_k^{-2}\int_0^T \|\langle x\rangle^s E_{-1}\chi_R [(h_k^2 P)^{1/2}\psi_2(h_k^2 P), \langle x\rangle^{-s}]\langle x\rangle^s g_k\|^2 dt.$$

It follows that the symbol of $\langle x\rangle^s E_{-1}\langle x\rangle^{-s}$ belongs to $S((|\xi|+\langle x\rangle)^{-1})$ then $\langle x\rangle^s E_{-1}\langle x\rangle^{-s}\chi_R P^{1/2}$ is bounded on $L^2(\Omega)$ (see [30, Lemma 4.2]) and we have

$$I \leq C\int_0^T \|\langle x\rangle^s g_k\|^2 dt,$$

According to Lemma A.4, $h_k^{-1}\langle x\rangle^s[(h_k^2P)^{1/2}\psi_2(h_k^2P),\langle x\rangle^{-s}]$ is bounded on $L^2(\Omega)$ and we get

$$II \leq C\int_0^T \|\langle x\rangle^s g_k\|^2 dt.$$

To estimate

$$\int_0^T \|\langle x\rangle^s E_{-1}\chi_R h_k^{-1}\psi_1(h_k^2P)a(h_k^2P)^{1/2}au_k\|^2 dt,$$

we have with $\psi_2(s)=s^{-1}\psi_1(s)$ and $\tilde\chi$ a smooth function such that, $\tilde\chi=1$ for $|x|\geq 1$ and $\tilde\chi=0$ for $|x|\leq 1/2$, $\tilde\chi_R(x)=\tilde\chi(x/R)$,

$$\begin{aligned}\langle x\rangle^s E_{-1}\chi_R h_k^{-1}\psi_1(h_k^2P)a &= \langle x\rangle^s E_{-1}\chi_R P h_k\psi_2(h_k^2P)a = \langle x\rangle^s E_{-1}\chi_R P\tilde\chi_R h_k\psi_2(h_k^2P)a\\ &= \langle x\rangle^s E_{-1}\langle x\rangle^{-s}\chi_R P^{1/2}(h_k^2P)^{1/2}\langle x\rangle^s[\tilde\chi_R,\psi_2(h_k^2P)]a\\ &\quad+\langle x\rangle^s E_{-1}\langle x\rangle^{-s}\chi_R[\langle x\rangle^s,P]\tilde\chi_R h_k[\psi_2(h_k^2P),a],\end{aligned}\tag{2.40}$$

where we have used $a\tilde\chi_R=0$ if R large enough.

By the [30, Lemma A.5] and Lemma A.3 the first term of (2.40) is bounded on $L^2(\Omega)$ by Ch_k. As $[\langle x\rangle^s,P]$ is a sum of term $\alpha\partial_{x_j}$ where α is bounded, $\langle x\rangle^s E_{-1}\langle x\rangle^{-s}\chi_R[\langle x\rangle^s,P]$ is bounded on $L^2(\Omega)$, and $[\psi_2(h_k^2P),a]$ is bounded on $L^2(\Omega)$ by [30, Lemma 6.3]. Then the second term of (2.40) is bounded on $L^2(\Omega)$ by Ch_k. Finally, we yield by Lemma A.6,

$$\begin{aligned}\int_0^T \|\langle x\rangle^s E_{-1}\chi_R h_k^{-1}\psi_1(h_k^2P)a(h_k^2P)^{1/2}au_k\|^2 dt &\leq C_R h_k^2\int_0^T \|(h_k^2P)^{1/2}au_k\|^2 dt\\ &\leq C_R h_k^2\sup_{t\in[0,T]}\|u_k(t,.)\|^2.\end{aligned}\tag{2.41}$$

According to (2.14) and (2.15), we conclude that the second term of the right hand side of (2.37) goes to zeros when k tend to $+\infty$

$$\lim_{k\to\infty}\int_0^T \|\langle x\rangle^s E_{-1}h_k^{-1}\chi_R g_k\|_{L^2}^2 dt = 0.\tag{2.42}$$

Now we estimate the term $\displaystyle\int_0^T \|\langle x\rangle^s E_{-1}[P,\chi_R]u_k\|_{L^2}^2 dt$.

Let $\chi_1\in\mathscr{C}_0^\infty(R-1<|x|<2R+1),\chi_1\geq 0,\chi_1=1$ on $\mathrm{supp}(\nabla\chi_R)$,

$$\begin{aligned}\int_0^T \|\langle x\rangle^s E_{-1}[P,\chi_R]u_k\|_{L^2}^2)dt &\leq \int_0^T \|\langle x\rangle^s\chi_1 E_{-1}[P,\chi_R]\chi_1 u_k]\|_{L^2(\Omega)}^2 dt\\ &\quad+\int_0^T \|\langle x\rangle^s(1-\chi_1)E_{-1}[P,\chi_R]\chi_1 u_k]\|_{L^2(\Omega)}^2 dt,\\ &\leq CR^{2(s-1)}\int_0^T \|u_k\|_{L^2(R-1<|x|<2R+1)}^2 dt \leq CR^{2(s-1)},\end{aligned}\tag{2.43}$$

where we have used, first that $E_{-1}\partial_x$ is bounded on L^2, $\langle x\rangle^s$ is estimate by CR^s on support of χ_1 and $\partial_x\chi_R$ is the product of a bounded function by R^{-1}, second, the symbol of $\langle x\rangle^s(1-\chi_1)E_{-1}[P,\chi_R]$ is uniformly bounded in $R^{-1}S((\langle x\rangle+|\xi|)^{-N},g)$ for all N. The last inequality uses the contradiction assumption (2.36).

Following (2.37), (2.38), (2.42) and (2.43), we have,

$$\int_0^T \|\langle x\rangle^{-s} u_k\|^2_{L^2(|x|>2R)}dt \le \int_0^T \|\langle x\rangle^{-s} v_k\|^2_{L^2(\mathbb{R}^d)} \le C_R\delta_k + CR^{2(s-1)},$$

where $\delta_k \to 0$ when $k \to +\infty$, C is independent of R and C_R may depend of R. Then we have

$$\begin{aligned}\int_0^T \|u_k\|^2_{L^2(x\in\Omega,\ |x|<2R)} &\ge \int_0^T \|\langle x\rangle^{-s} u_k\|^2_{L^2(x\in\Omega,\ |x|<2R)}\\ &\ge \int_0^T \|\langle x\rangle^{-s} u_k\|^2_{L^2(x\in\Omega)} - \int_0^T \|\langle x\rangle^{-s} u_k\|^2_{L^2(|x|>2R)}\\ &\ge \int_0^T \|\langle x\rangle^{-s} u_k\|^2_{L^2(x\in\Omega)} - C_R\delta_k - CR^{2(s-1)}.\end{aligned}$$

This with (2.16) implies a contradiction with (2.36) and proves the Lemma. ∎

In the sequel, for simplicity, we shall denote the sequence u_{k_ν} found in Lemma 2.6 by u_k. Thus there exist $R_0 > 0$, $k_0 > 0$ such that

$$\int_0^T \|u_k(t)\|^2_{L^2(|x|<R)}dt \ge \frac{1}{2},$$

when $R > R_0$ and $k > k_0$.
We consider $\chi_1 \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ such that

$$0 \le \chi_1 \le 1,\ \chi_1(x) = 1 \text{ if } |x| \le R_1 + 2 \text{ and } \operatorname{supp}\chi_1 \subset \{|x| \le R_1 + 3\},$$

with $R_1 > R_0$.
Let $A \ge 1$, $R \ge 1$, $\psi_A \in \mathscr{C}_0^\infty(\mathbb{R})$, $\phi_R \in \mathscr{C}_0^\infty(\mathbb{R})$ be such that $0 \le \psi_A,\ \phi_R \le 1$ and

$$\psi_A(\tau) = 1 \text{ if } |\tau| \le A, \phi_R(t) = 1 \text{ if } |t| \le R.$$

We recall that $w_k(t) = 1_\Omega u_k(t)$.

Proposition 2.7. *There exist positive constants A_0, R_0, k_0 such that*

$$\int_{\mathbb{R}} \|\psi_A(h_k^2 D_t)\phi_R(h_k^2\Delta)1_{[0,T]}\chi_1 w_k(t)\|^2_{L^2(\mathbb{R}^d)}dt \ge \frac{1}{4},$$

when $A \ge A_0$, $R \ge R_0$, $k \ge k_0$.

Corollary 2.8. *The measure μ does not vanish identically.*

Proof of Proposition . Set $I = (Id - \psi_A(h_k^2 D_t))1_{[0,T]}\chi_1 u_k$ and $\widetilde{\psi}(\tau) = \dfrac{1-\psi_A(\tau)}{\tau}$. It is easy to see that $\widetilde{\psi} \in L^\infty(\mathbb{R})$ and $|\widetilde{\psi}(\tau)| \le \frac{1}{A}$ for all $\tau \in \mathbb{R}$.

We have

$$\begin{aligned}I &= \widetilde{\psi}_A(h_k^2 D_t)h_k^2 D_t(1_{[0,T]}\chi_1 w_k)\\ &= \frac{h_k^2}{i}\widetilde{\psi}_A(h_k^2 D_t)\chi_1(u_k(0)\delta_{t=0} - u_k(T)\delta_{t=T})\\ &\quad \widetilde{\psi}_A(h_k^2 D_t)\chi_1 1_{[0,T]}(-h_k^2 P u_k + ih_k a(h_k^2 P)^{1/2} a u_k + h_k g_k)\\ &= B_k^1 + B_k^2 + B_k^3 + B_k^4.\end{aligned}$$

From [30, See the proof of Proposition 6.1] we know that $\|\widetilde{\psi}_A(h_k^2 D_t)\delta_{t=a}\|_{L^2(\mathbb{R})} \le Ch_k^{-1}$, then we deduce that

$$\lim_{k\to+\infty}\int_{\mathbb{R}} \|B_k^1\|^2_{L^2(\Omega)}dt \le \lim_{k\to+\infty} Ch_k^4 h_k^{-2}(\|u_k(0)\|^2_{L^2(\Omega)} + \|u_k(T)\|^2_{L^2(\Omega)}) = 0.$$

Using (2.22) and (2.15), we can prove easily that

$$\lim_{k\to+\infty}\int_{\mathbb{R}}\|B_k^3\|^2_{L^2(\Omega)}dt \le C\lim_{k\to+\infty}\int_0^T h_k\|(h_k^2P)^{1/2}au_k\|^2_{L^2(\Omega)}dt = 0.$$

From (2.14) we can see that

$$\lim_{k\to+\infty}\int_{\mathbb{R}}\|B_k^4\|^2_{L^2(\Omega)}dt \le C\lim_{k\to+\infty}\int_0^T \|\chi_1 g_k\|^2_{L^2(\Omega)}dt = 0.$$

Now, for B_k^2 we argue as in [30, See the proof of Proposition 6.1]. Let $\tilde\theta \in \mathscr{C}_0^\infty(0,+\infty)$ such $\tilde\theta = 1$ on the support of ψ and let $\tilde\theta_1(s) = s\tilde\theta(s)$. We have

$$\begin{aligned}B_k^2 &= -\widetilde{\psi}_A(h_k^2D_t)\chi_1 1_{[0,T]}h_k^2P\widetilde{\theta}(h_k^2P)u_k\\ &= -\widetilde{\psi}_A(h_k^2D_t)1_{[0,T]}[\chi_1,\widetilde{\theta}_1(h_k^2P)]u_k - \widetilde{\psi}_A(h_k^2D_t)1_{[0,T]}\widetilde{\theta}_1(h_k^2P)\chi_1u_k.\end{aligned}$$

Using Lemma 6.3 in [30] and the fact that

$$\|\widetilde{\psi}_A(h_k^2D_t)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = O\left(\frac{1}{A}\right),\ \|\widetilde{\theta}_1(h_k^2P)\|_{L^2(\Omega)\to L^2(\Omega)} = O(1),$$

uniformly in k, we deduce that

$$\int_{\mathbb{R}}\|B_k^2\|^2_{L^2(\Omega)}dt \le C(h_k^2\sup_{t\in[0,T]}\|u_k(t)\|^2_{L^2(\Omega)}dt + \frac{1}{A}\int_0^T\|\chi_1u_k\|^2_{L^2(\Omega)}dt).$$

Taking k and A sufficiently large we obtain

$$\int_{\mathbb{R}}\|\psi_A(h_k^2D_t)1_{[0,T]}\chi_1w_k(t)\|^2_{L^2(\mathbb{R}^d)}dt \ge \frac{1}{3}. \tag{2.44}$$

Now, we set

$$\mathrm{II} = (Id - \phi_R(h_k^2\Delta))\psi_A(h_k^2D_t)1_{[0,T]}\chi_1w_k.$$

It is proved in [30] that

$$\int_{\mathbb{R}}\|\mathrm{II}\|^2_{L^2(\mathbb{R}^d)}dt \le \frac{C_{R_1}}{R}(1+h_k^2), \tag{2.45}$$

where C_{R_1} depends on R_1 and The proof does not depend on the equation, so it remains valid in our case. Nevertheless we recall the proof in the sequel for the convenience of the reader. Before we give the end of the proof of proposition 2.7.

Taking R sufficiently large and using (2.44), we obtain

$$\int_{\mathbb{R}}\|\phi_R(h_k^2\Delta)\psi_A(h_k^2D_t)1_{[0,T]}\chi_1w_k(t)\|^2_{L^2(\mathbb{R}^d)}dt \ge \frac{1}{4}.$$

Return to the proof of (2.45). We have $|1-\phi_R(t)| \le C\dfrac{h|\xi|}{\sqrt{R}}$ then we obtain,

$$\begin{aligned}\int_{\mathbb{R}}\|\mathrm{II}\|^2_{L^2(\mathbb{R}^d)}dt &\le C\frac{h_k^2}{R}\int_{\mathbb{R}}\sum_j\|\partial_j\psi_A(h_k^2D_t)1_{[0,T]}\chi_1w_k\|^2_{L^2(\mathbb{R}^d)}dt\\ &\le C\frac{h_k^2}{R}\int_{\mathbb{R}}\sum_j\|\partial_j\psi_A(h_k^2D_t)1_{[0,T]}\chi_1u_k\|^2_{L^2(\Omega)}dt\\ &\le \frac{h_k^2}{R}\sum_j\Big(\int_{\mathbb{R}}\|\partial_j\widetilde{\theta}(h_k^2P)\psi_A(h_k^2D_t)1_{[0,T]}\chi_1u_k\|^2_{L^2(\Omega)}dt\\ &\quad+\int_{\mathbb{R}}\|\partial_j(1-\widetilde{\theta}(h_k^2P))\psi_A(h_k^2D_t)1_{[0,T]}\chi_1u_k\|^2_{L^2(\Omega)}dt\Big)\\ &:= \frac{h_k^2}{R}(C_k^1+C_k^2),\end{aligned} \tag{2.46}$$

where $\widetilde{\theta} \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfying $\widetilde{\theta}(t) = 1$ if $t \in \mathrm{supp}(\theta_1)$ and $\widetilde{\theta}\theta_1 = \theta_1$.

We have by Lemma 6.3 [30]

$$C_k^1 \leq C h_k^{-2} \int_0^T \|\chi_1 u_k\|^2_{L^2(\Omega)} dt \leq c h_k^{-2}, \tag{2.47}$$

and

$$\begin{aligned} C_k^2 &\leq \int_{\mathbb{R}} \|\partial_j[\widetilde{\theta}(h_k^2 P), \chi_1]\psi_A(h_k^2 D_t) 1_{[0,T]} \widetilde{\chi}_1 u_k\|^2_{L^2(\Omega)} dt \\ &\leq \int_{\mathbb{R}} \|\psi_A(h_k^2 D_t) 1_{[0,T]} \widetilde{\chi}_1 u_k\|^2_{L^2(\Omega)} dt \\ &\leq C \int_0^T \|\widetilde{\chi}_1 u_k\|^2_{L^2(\Omega)} dt \leq C_{R_1} \int_0^T \|\langle x\rangle^{-s} u_k\|^2_{L^2(\Omega)} dt, \end{aligned} \tag{2.48}$$

where $\widetilde{\chi}_1 \in \mathscr{C}_0^\infty(\overline{\Omega})$, $\widetilde{\chi}_1 = 1$ on $\mathrm{supp}(\chi_1)$.
Combining (2.46), (2.47) and (2.48), we obtain (2.45). ∎

2.4 The microlocal defect measure vanishes in the incoming set

In this section we prove that the microlocal defect measure μ vanishes in the incoming set.

First remind some notation introduced in [30] section 7. We keep the same notation when it is possible.

We denote by

$$b(x,\xi) = \sum_{j,k=1}^{d} b^{jk}(x) x_j \xi_k.$$

Proposition 2.9. *Let $m_0 = (x_0, t_0, \xi_0, \tau_0) \in T^\star(\mathbb{R}^{d+1})$ be such $\xi_0 \neq 0$, $\tau_0 + p(x_0, \xi_0) = 0$, $|x_0| \geq 3R_0$, $b(x_0, \xi_0) \leq -3\delta|x_0||\xi_0|$ for some $\delta > 0$ small enough. Then $m_0 \notin \mathrm{supp}\,\mu$.*

We remind the results proved in [30] in section 7, Lemma 7.5 and Corollary 7.6. A part of the proof is in Doi [15]. We use the Weyl quantification of symbol which is denoted by Op^w.

There exist a symbol $\Phi \in S(1,g)$ such that $0 \leq \Phi \leq 1$ and a symbol $\lambda_1 \in S(1,g)$ such that,

$$\mathrm{supp}\,\lambda_1 \subset \mathrm{supp}\,\Phi \subset \{(x,\xi) \in T^*(\mathbb{R}^d),\ |x| \geq 2R_0,\ b(x,\xi) \leq -\frac{\delta}{2}|x||\xi|,\ |\xi| \geq \frac{|\xi_0|}{4}\}, \tag{2.49}$$

$$\{(x,\xi) \in T^*(\mathbb{R}^d),\ |x| \geq \frac{5}{2}R_0,\ b(x,\xi) \leq -\delta|x||\xi|,\ |\xi| \geq \frac{|\xi_0|}{2}\} \subset \{(x,\xi) \in T^*(\mathbb{R}^d),\ \Phi(x,\xi) = 1\},$$

$$\Phi(x, h\xi) = \Phi(x,\xi) \text{ when } |h\xi| \geq \frac{|\xi_0|}{2}, \text{ and } 0 < h \leq 1,$$

$$H_p\Phi(x,\xi) \leq 0 \text{ on the support of } \lambda_1,$$

$$\lambda_1 \geq 0,$$

$$[\tilde{P}, Op^w(\lambda_1)] - \frac{1}{i} Op^w(H_p\lambda_1) \in Op^w(S(1,g)), \tag{2.50}$$

$$\text{there exist two positive constants } C,\ C' \text{ such that,}$$

$$-H_p\lambda_1 \geq C\langle x\rangle^{-2s}\Phi^2(x,\xi)(|x| + |\xi|) - C'\Phi^2(x,\xi). \tag{2.51}$$

Proof. Let $\varphi_1 \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ such that

$$\varphi_1(x) = 1 \text{ if } |x| \leq \frac{4}{3}R_0,\ \mathrm{supp}\,\varphi_1 \subset \{x,\ |x| \leq \frac{3}{2}R_0\}. \tag{2.52}$$

Let M large enough such that,

$$|((1-\varphi_1)Op^w(\lambda_1)(1-\varphi_1)u|u)| \leq \frac{M}{2}\|u\|^2.$$

Here and in the sequel $(\cdot|\cdot)$ and $\|\cdot\|$ denote the $L^2(\Omega)$ inner product and norm respectively. The cutoff make sense with this L^2 product. We set,

$$N(t) = ((M - (1-\varphi_1)\mathcal{O}p(\lambda_1)(1-\varphi_1))u_k(t)|u_k(t)),$$

and we have

$$\frac{M}{2}\|u_k(t)\|^2 \leq N(t) \leq 2M\|u_k(t)\|^2. \tag{2.53}$$

Setting $\Lambda = M - (1-\varphi_1)\mathcal{O}p(\lambda_1)(1-\varphi_1)$, we have,

$$\frac{d}{dt}N(t) = (\Lambda\frac{d}{dt}u_k(t)|u_k(t)) + (\Lambda u_k(t)|\frac{d}{dt}u_k(t)).$$

From (2.12) we have

$$\frac{d}{dt}u_k = -iPu_k - h_k^{-1}a(h_k^2P)^{1/2}(au_k) + ih_k^{-1}g_k.$$

We obtain,

$$\begin{aligned}\frac{d}{dt}N =&(i[P,\Lambda]u_k|u_k)\\ &- h_k^{-1}(\Lambda a(h^2P)^{1/2}au_k|u_k) - h_k^{-1}(\Lambda u_k|a(h_k^2P)^{1/2}au_k)\\ &+ ih_k^{-1}(\Lambda g_k|u_k) - ih_k^{-1}(\Lambda u_k|g_k)\\ =&A_1 + A_2 + A_3.\end{aligned} \tag{2.54}$$

For support reasons, we have $a(1-\varphi_1) = 0$ thus we deduce,

$$\begin{aligned}A_2 =& -\frac{M}{h_k}[(a(h_k^2P)^{1/2}(au_k)|u_k) + (u_k|a(h_k^2P)^{1/2}(au_k))]\\ =& -\frac{2M}{h_k}\|(h_k^2P)^{1/4}(au_k)\|^2 \leq 0.\end{aligned} \tag{2.55}$$

We have, for a constant $C_1 > 0$

$$|A_3| \leq \frac{C_1}{h_k}\|\langle x\rangle^s g_k\|\|\langle x\rangle^{-s}u_k\|. \tag{2.56}$$

To estimate A_1 we remark that $[P,\Lambda] = [\tilde{P},\Lambda]$ and

$$[P,\Lambda] = [\tilde{P},\varphi_1]Op^w(\lambda_1)(1-\varphi_1) - (1-\varphi_1)[\tilde{P},Op^w(\lambda_1)](1-\varphi_1) + (1-\varphi_1)Op^w(\lambda_1)[\tilde{P},\varphi_1]. \tag{2.57}$$

Following (2.49) and (2.52), the support of λ_1 and φ_1 are disjoint, thus, taking account of (2.53), we have

$$|(\big[[\tilde{P},\varphi_1]Op^w(\lambda_1)(1-\varphi_1) + (1-\varphi_1)Op^w(\lambda_1)[\tilde{P},\varphi_1]\big]u_k|u_k)| \leq C_2N(t). \tag{2.58}$$

Let $d(x,\xi) \in \mathscr{C}_0^\infty(\mathbb{R}^{2d})$ supported in $\{|x-x_0| \leq 1,\ |\xi-\xi_0| \leq 1\}$, and $d(x_0,\xi_0) = 1$. According to (2.50), (2.51) and Gårding inequality, we get,

$$(-i(1-\varphi_1)[\tilde{P},Op^w(\lambda_1)](1-\varphi_1)u_k|u_k) \geq C_3h_k^{-1}\|\langle x\rangle^{-s}d(x,h_kD_x)u_k\|^2 - C_4N(t). \tag{2.59}$$

From (2.57), (2.58) and (2.59) we obtain,

$$A_1 \geq C_3h_k^{-1}\|\langle x\rangle^{-s}d(x,h_kD_x)u_k\|^2 - C_5N(t). \tag{2.60}$$

Following (2.54), (2.55), (2.56) and (2.60), we have

$$N'(t) + C_3h_k^{-1}\|\langle x\rangle^{-s}d(x,h_kD_x)u_k\|^2 \leq \beta(t) + C_6N(t), \tag{2.61}$$

where we have set

$$\beta(t) = \frac{C_1}{h_k}\|\langle x\rangle^s g_k(t)\|.\|\langle x\rangle^{-s} u_k(t)\|.$$

Integrating (2.61) between 0 and t for $t \in [0,T]$ we obtain,

$$N(t) + C_3 h_k^{-1}\|\langle x\rangle^{-s} d(x,h_k D_x)u_k\|^2_{L^2([0,T]\times\Omega)} \leq \int_0^T \beta(t)dt + N(0) + C_8\int_0^t N(s)ds. \tag{2.62}$$

By Gronwall's inequality we have for $t \in [0,T]$,

$$N(t) \leq C_7\int_0^T \beta(t)dt + C_8 N(0). \tag{2.63}$$

Using (2.63) in (2.62), we get

$$\begin{aligned}\|\langle x\rangle^{-s} d(x,h_k D_x)u_k\|^2_{L^2([0,T]\times\Omega)}&\\ \leq C_8\|\langle x\rangle^s g_k\|_{L^2([0,T]\times\Omega)}&\|\langle x\rangle^{-s} u_k\|_{L^2([0,T]\times\Omega)} + C_9 h_k\|u_k(0)\|^2_{L^2(\Omega)}.\end{aligned}$$

Following (2.14) and (2.16) we obtain

$$\|\langle x\rangle^{-s} d(x,h_k D_x)u_k\|^2_{L^2([0,T]\times\Omega)} \to 0 \text{ when } k \to +\infty.$$

Let $\chi(t,\tau) \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ supported in a neighborhood sufficiently small around (t_0,τ_0) and taking account that d is supported in a neighborhood of (x_0,ξ_0), we have

$$\|\chi(t,h_k^2) d(x,h_k D_x)u_k\|_{L^2([0,T]\times\Omega)} \to 0 \text{ when } k \to +\infty,$$

then $\langle \mu, \chi^2 d^2\rangle = 0$ thus $(x_0,t_0,\xi_O,\tau_0) \notin \operatorname{supp}\mu$. ∎

2.5 The microlocal defect measure vanishes on $\{a^2 > 0\}$

The goal of this section is to prove that the microlocal defect measure vanishes on $\{a^2 > 0\}$. More precisely we have the following proposition.

Proposition 2.10. *Let $u_k = \psi(h_k^2 P)u$ satisfying*

$$h_k^2(D_t + P)u_k - ih_k a(h_k^2 P)^{1/2}(au_k) = h_k g_k, \tag{2.64}$$

$$\|\langle x\rangle^s g_k\|^2_{L^2([0,T]\times\Omega)} + h_k \sup_{t\in[0,T]}\|u_k(t)\|^2_{L^2(\Omega)} + h_k \underset{k\to+\infty}{\to} 0, \tag{2.65}$$

and

$$\left\|\langle x\rangle^{-s} u_k\right\|^2_{L^2([0,T]\times\Omega)} \underset{k\to+\infty}{\to} 1. \tag{2.66}$$

We assume that the sequence $(W_k) = (1_{[0,T]}1_\Omega u_k)$ admits a microlocal defect measure μ then $a^2\mu = 0$.

Proof. Taking the imaginary part of the $L^2([0,T]\times\Omega)$ inner product of (2.64) with u_k/h_k, we obtain,

$$\Im m[(h_k(D_t+P)u_k|u_k) - i(a(h_k^2P)^{1/2}(au_k)|u_k) = \Im m(g_k|u_k). \tag{2.67}$$

Using that P is self-adjoint, we get

$$\Im m(h_k\int_0^T\int_\Omega \frac{1}{2}D_t|u_k|^2 dxdt) - ((h_k^2P)^{1/2}(au_k)|au_k) = \Im m(\langle x\rangle^s g_k|\langle x\rangle^{-s}u_k). \tag{2.68}$$

From (2.65) and (2.66), we have

$$h_k \int_0^T \int_\Omega D_t |u_k|^2 dxdt = ih_k \|u_k(0)\|^2_{L^2(\Omega)} - ih_k \|u_k(T)\|^2_{L^2(\Omega)} \underset{k\to+\infty}{\to} 0,$$

and

$$|(\langle x\rangle^s g_k | \langle x\rangle^{-s} u_k)| \leq \|\langle x\rangle^s g_k\|_{L^2(\Omega)} \|\langle x\rangle^{-s} u_k\|_{L^2(\Omega)} \underset{k\to+\infty}{\to} 0.$$

Following (2.68), we deduce

$$((h_k^2 P)^{1/2}(au_k)|au_k) \underset{k\to+\infty}{\to} 0. \tag{2.69}$$

Let $\theta \in \mathscr{C}_0^\infty((0,+\infty))$ with $\theta = 1$ on the support of ψ. Thus we have $\theta(h_k^2 P)u_k = u_k$. Let $\tilde{\theta}(t) = t^{-1/4}\theta(t)$, we have $\tilde{\theta} \in \mathscr{C}_0^\infty((0,+\infty))$ and,

$$\begin{aligned}(au_k|au_k) &= (a\theta^2(h_k^2P)u_k|au_k) = (a(h_k^2P)^{1/2}\tilde{\theta}^2(h_k^2P)u_k|au_k)\\ &= ((h_k^2P)^{1/2}\tilde{\theta}^2(h_k^2P)au_k|au_k) + ([a,(h_k^2P)^{1/2}\tilde{\theta}^2(h_k^2P)]u_k|au_k).\end{aligned} \tag{2.70}$$

From Lemma 6.3 [30], we have

$$\|[a,(h_k^2P)^{1/2}\tilde{\theta}^2(h_k^2P)]u_k\|_{L^2(\Omega)} \leq Ch_k\|u_k\|_{L^2(\Omega)}. \tag{2.71}$$

We have also,

$$\begin{aligned}((h_k^2P)^{1/2}\tilde{\theta}^2(h_k^2P)au_k|au_k) &= \|(h_k^2P)^{1/4}\tilde{\theta}(h_k^2P)au_k\|^2_{L^2([0,T]\times\Omega)}\\ &\leq \|(h_k^2P)^{1/4}au_k\|^2_{L^2([0,T]\times\Omega)} = ((h_k^2P)^{1/2}au_k|au_k) \underset{k\to+\infty}{\to} 0,\end{aligned} \tag{2.72}$$

from (2.69). Following (2.70), (2.71) and (2.72), we obtain,

$$(au_k|au_k) \underset{k\to+\infty}{\to} 0. \tag{2.73}$$

Let $b(x,t,\xi,\tau) \in \mathscr{C}_0^\infty(\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R})$, we have by standard symbolic semi-classical calculus

$$\begin{aligned}(a^2(x)b(x,t,h_kD_x,h_k^2D_t)W_k|W_k) =&(b(x,t,h_kD_x,h_k^2D_t)(aW_k)|aW_k)\\ &+ h_k(r(x,t,h_kD_x,h_k^2D_t)W_k|W_k),\end{aligned} \tag{2.74}$$

where $r(x,t,h_kD_x,h_k^2D_t)$ is bounded on $L^2([0,T]\times\mathbb{R}^d)$. Thus from (2.65), we have,

$$h_k|(r(x,t,h_kD_x,h_k^2D_t)W_k|W_k)| \leq Ch_k\|W_k\|^2_{L^2([0,T]\times\mathbb{R}^d)} \underset{k\to+\infty}{\to} 0. \tag{2.75}$$

From (2.73) and using $\|aW_k\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d)} = \|au_k\|^2_{L^2([0,T]\times\Omega)}$ we obtain,

$$|(b(x,t,hD_x,h^2D_t)(aW_k)|aW_k)_{L^2(\mathbb{R}\times\mathbb{R}^d)}| \leq C\|aW_k\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d)} \underset{k\to+\infty}{\to} 0. \tag{2.76}$$

According to the definition of the microlocal defect measure μ, (2.74), (2.75) and (2.76) imply the Proposition 2.10 ∎

2.6 Propagation properties of microlocal defect measure and end of proof

The statement of our results requires some geometric notions which are classical in the microlocal study of boundary problems (cf. [18] p. 424 and 430-432).
Let $M = \Omega\times\mathbb{R}_t$. We set

$$T_b^*M = T^*M\backslash\{0\} \cup T^*\partial M\backslash\{0\}.$$

We have the natural restriction map

$$\pi : T^*\mathbb{R}^{d+1}_M \to T_b^*M,$$

which is the identity on $T^*\mathbb{R}^{d+1}_M\backslash\{0\}$ (see [30] for details). Consider, near a point of the boundary $z=(x_1,x',t)\in\partial M$ a geodesic system of coordinates given by the diffeomorphism F in (2.29), for which $z=(0,0,t)$, $M=\{(x_1,x',t),x_1>0)\}$ and the operator D_t+P has the form (near z)

$$P=D_t+D_{x_1}^2+R(x_1,x',D_{x'})+S(x,D_x),$$

with R a second order tangential operator and S a first order operator. Denoting $r(x_1,x',\xi')$ the principal symbol of R and $r_0=r|_{x_1=0}$, the cotangent bundle to the boundary $T^\star\partial M\backslash\{0\}$ can be decomposed (in this coordinate system) as the disjoint union of the following regions:

- the elliptic region $\mathcal{E}=\{(x',t,\xi',\tau)\in T^\star\partial M\backslash\{0\};\quad r_0(x',\xi')+\tau>0\}$,
- the hyperbolic region $\mathcal{H}=\{(x',t,\xi',\tau)\in T^\star\partial M\backslash\{0\};\quad r_0(x',\xi')+\tau<0\}$,
- and the glancing region $\mathcal{G}=\{(x',t,\xi',\tau)\in T^\star\partial M\backslash\{0\};\quad r_0(x',\xi')+\tau=0\}$.

For the purpose of the proofs, it is important to consider the following subsets of the glancing region:

- the diffractive region $\mathcal{G}_d=\{\zeta\in\mathcal{G},\partial_{x_1}r|_{x_1=0}(\zeta)<0\}$,
- the gliding region $\mathcal{G}_g=\{\zeta\in\mathcal{G},\partial_{x_n}r|_{x_n=0}(\zeta)>0\}$; we set $\mathcal{G}^2=\mathcal{G}_d\cup\mathcal{G}_g$,
- and $\mathcal{G}^k=\{\zeta\in\mathcal{G},H^j_{r_0}(\partial_{x_1}r|_{x_1=0})(\zeta)=0,\ 0\le j<k-2,\ H^{k-2}_{r_0}(\partial_{x_1}r|_{x_1=0})(\zeta)\neq 0\}\quad k\ge 3$, where
$$H_{r_0}=\frac{\partial r_0}{\partial\xi'}\frac{\partial}{\partial x'}-\frac{\partial r_0}{\partial x'}\frac{\partial}{\partial\xi'}$$
.

Definition 2.11. We say that the bicaracteristics have no contact of infinite order with the boundary if $\mathcal{G}=\bigcup_{k=2}^{+\infty}\mathcal{G}^k$.

Now, we recall the definition of ν the measure on the boundary. By the Lemma 2.4, we see that the sequence $(1_{[0,T]}h_k(\frac{\partial w_k}{\partial n}))$ is bounded in $L^2(\mathbb{R}_t\times L^2(\partial\Omega))$. Therefore with the notations in (2.18) and Proposition 2.2, we have the following Lemma.

Lemma 2.12. *There exists a subsequence $(W_{\sigma_1(k)})$ of $(W_{\sigma(k)})$ and a Radon measure ν on $T^*(\partial\Omega\times\mathbb{R}_t)$ such that for every $b\in\mathscr{C}_0^\infty(T^*(\partial\Omega\times\mathbb{R}_t))$ we have*

$$\lim_{k\to+\infty}\left(\mathcal{O}p(b)\left(x,t,h_{\sigma_1(k)}D_x,h^2_{\sigma_1(k)}D_t\right)h_{\sigma_1(k)}\frac{1}{i}\frac{\partial W_{\sigma_1(k)}}{\partial n},h_{\sigma_1(k)}\frac{1}{i}\frac{\partial W_{\sigma_1(k)}}{\partial n}\right)_{L^2(\partial\Omega\times\mathbb{R}_t)}=\langle\nu,b\rangle\,.$$

We give now two results on propagation of support of microlocal defect measure. The first, Proposition 2.13 for point inside $T^\star M$ and the second, Proposition 2.15 at the boundary of M.

Proposition 2.13. *Let $m_0=(x_0,\xi_0,t_0,\tau_0)\in T^\star M$ and U_{m_0} be a neighborhood of this point in $T^\star M$. Then for every $b\in\mathscr{C}_0^\infty(U_{m_0})$, we have*

$$\langle\mu,H_pb\rangle=0. \tag{2.77}$$

Proof. It is enough to prove (2.77) when $b(x,t,\xi,\tau)=\Phi(x,\xi)\chi(t,\tau)$ with $\pi_x\operatorname{supp}\Phi\subset V_{x_0}\subset\Omega$. Let $\varphi\in\mathscr{C}_0^\infty(\Omega)$ be such that $\varphi=1$ on V_{x_0}. We introduce

$$\begin{aligned}A_k=\frac{i}{h_k}[&(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi h_k^2(D_t+P)1_{[0,T]}w_k,1_{[0,T]}w_k)_{L^2(\Omega\times\mathbb{R})}\\&-(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi 1_{[0,T]}w_k,h_k^2(D_t+P)1_{[0,T]}w_k)_{L^2(\Omega\times\mathbb{R})}].\end{aligned}$$

We claim that we have

$$\lim_{k\to+\infty} A_k = 0. \tag{2.78}$$

We have

$$\begin{aligned} A_k = \frac{i}{h_k}[&(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi h_k^2[D_t,1_{[0,T]}]w_k,1_{[0,T]}w_k)_{L^2(\Omega\times\mathbb{R})}\\ &-(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi 1_{[0,T]}w_k,h_k^2[D_t,1_{[0,T]}]w_k)_{L^2(\Omega\times\mathbb{R})}]\\ &-2\Im(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi g_k,1_{[0,T]}w_k)_{L^2(\Omega\times\mathbb{R})}\\ &-2\Re(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi 1_{[0,T]}a(h_k^2P)^{1/2}aw_k,1_{[0,T]}w_k)_{L^2(\Omega\times\mathbb{R})}+o(1), \end{aligned}$$

where we used that $(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi)-(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi)^*=o(1)$ by pseudo-differential calculus. It was proved in [30, proof of Proposition A.9] that the first and the second terms tend to zero when $k\to+\infty$. Since $g_k\to 0$ in L^2_{loc}, the third term tends also to zero when $k\to+\infty$. For the fourth term, according to (2.74) and (2.76), it is easy to see that it tends to zero. Thus (2.78) is proved.

In another side, it was shown in the Proposition A.9 [30] that

$$\lim_{k\to+\infty} A_k = -\langle\mu, H_p(\Phi\chi)\rangle.$$

It follows from (2.78), (2.77) that $\langle\mu,H_pb\rangle=0$ if $b=\Phi\chi$, which implies our proposition. ∎

We consider now the case of point $m_0=(x_0,\xi_0,t_0,\tau_0)\in T^\star\mathbb{R}^{d+1}$ with $x_0\in\partial\Omega$. We take, as in [30], a neighborhood U_{x_0} so small that we can perform the diffeomorphism F described in (2.29).

Let μ and ν be the measures on $T^\star\mathbb{R}^{d+1}$ and $T^\star(\partial\Omega\times\mathbb{R}_t)$ defined in Proposition 2.2 and Lemma 2.12. We denote by $\tilde\mu$ and $\tilde\nu$ the measures on $T^\star(U_{x_0}\times\mathbb{R}_t)$ and $T^\star(U_{x_0}\cap\{y_1=0\}\times\mathbb{R}_t)$ which are the pullback of μ and ν by the diffeomorphism $\tilde F:(x,t)\mapsto(F(x),t)$.

We first recall the Lemma A.10 established in [30].

Lemma 2.14. *Let $b\in\mathscr{C}_0^\infty(T^\star(U_{x_0}\times\mathbb{R}_t))$. We can find $b_j\in\mathscr{C}_0^\infty(U_{x_0}\times\mathbb{R}_t\times\mathbb{R}^{d-1}_{\eta'}\times\mathbb{R}_\tau)$, $j=0,1$ and $b_2\in\mathscr{C}_0^\infty(T^\star(U_{x_0}\times\mathbb{R}_t))$ with compact support in (y,t,η',τ) such that with the notations of (2.29),*

$$b(y,t,\eta,\tau)=b_0(y,t,\eta',\tau)+b_1(y,t,\eta',\tau)\eta_1+b_2(y,t,\eta,\tau)(\tau+\eta_1^2+r(y,\eta')),$$

where r is the principal symbol of $R(y,D')$.

Proposition 2.15. *With the notations of Lemma 2.14 for every $b\in\mathscr{C}_0^\infty(T^\star(U_0\times\mathbb{R}_t))$, we have*

$$\langle\tilde\mu,H_pb\rangle=-\langle\tilde\nu,b_{1|Y_1=0}\rangle.$$

Proof. This proof is similar to the one of Proposition A.12 [30]. We recall some results from [30] used to prove Proposition A.12. ∎

Lemma 2.16 (Lemma A.13 [30])**.** *Let for $j=0,1$, $b_j=b_j(Y,t,\eta',\tau)\in\mathscr{C}_0^\infty(U_0\times\mathbb{R}^{d+1})$ and $\varphi\in\mathscr{C}_0^\infty(U_0)$, $\varphi=1$ on $\pi_Y\operatorname{supp}a_j$. Then,*

$$\begin{aligned} &\frac{i}{h_k}[((b_0(\Lambda_k)+b_1(\Lambda_k)h_kD_1)\varphi h_k^2(D_t+P)1_{[0,T]}v_k|1_{[0,T]}v_k)_{L^2_+}\\ &\quad-\int_{U_0^+}\langle(b_0(\Lambda_k)+b_1(\Lambda_k)h_kD_1)\varphi 1_{[0,T]}v_k,h_k^2(D_t+P)\overline{1_{[0,T]}v_k}\rangle dY]\\ &=-\frac{i}{h_k}([h_k^2(D_t+P),(b_0(\Lambda_k)+b_1(\Lambda_k)h_kD_1)\varphi 1_{[0,T]}]v_k|1_{[0,T]}v_k)_{L^2_+}\\ &\quad-(a_1(0,Y',t,h_kD_{Y'},h_k^2D_t)\varphi_{|Y_1=0}1_{[0,T]}(h_kD_1v_{k|Y_1=0})|1_{[0,T]}(h_kD_1v_{k|Y_1=0}))_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}. \end{aligned}\tag{2.79}$$

Here $\langle.,.\rangle$ denotes the bracket in $\mathcal{D}'(\mathbb{R}_t)$.

Lemma 2.17 (Lemma A.15 [30]). *Let for $j = 0,1,2$, $b_j = b_j(Y,t,\eta',\tau) \in \mathscr{C}_0^\infty(U_0 \times \mathbb{R}^d)$ and $\varphi \in \mathscr{C}_0^\infty(U_0)$, $\varphi = 1$ on $\pi_Y \operatorname{supp} b_j$. Let us set*

$$L_k^j = (b_j(\Lambda_k)\varphi(h_k D_1)^j 1_{[0,T]} v_k, 1_{[0,T]} v_k)_{L^2_+}.$$

Then we have for $j = 0,1,2$

$$\lim_{k \to +\infty} L^j_{\sigma(k)} = \langle \widetilde{\mu}, b_j \eta_1^j \rangle.$$

The previous Lemmas still hold in our case, since they are independent of the equation.

Lemma 2.18. *Let $b = b(Y,t,\eta',\tau) \in \mathscr{C}_0^\infty(U_0 \times \mathbb{R}^{d+1})$ and $\varphi \in \mathscr{C}_0^\infty(U_0)$, $\varphi = 1$ on $\pi_Y \operatorname{supp} b_j$. For $j = 0,1$ we set,*

$$I_k^j = (h_k^{-1} b(\Lambda_k)\varphi(h_k D_1)^j h_k^2 (D_t + P) 1_{[0,T]} v_k | 1_{[0,T]} v_k)_{L^2_+},$$
$$J_k^j = \int_{U_0^+} \langle h_k^{-1} b(\Lambda_k)\varphi(h_k D_1)^j 1_{[0,T]} v_k | h_k^2 (D_t + P) 1_{[0,T]} v_k \rangle dY.$$

Then $\lim_{k \to +\infty} I_k^j = \lim_{k \to +\infty} J_k^j = 0.$

Proof. The proof is similar to the one of Lemma A.14 [30]. We have,

$$I_k^j = \frac{1}{i}[(h_k b(\Lambda_k)\delta_{t=0}\varphi(h_k D_1)^j v_k(0,.) | 1_{[0,T]} v_k)_{L^2_+} - (h_k b(\Lambda_k)\delta_{t=T}\varphi(h_k D_1)^j v_k(0,.) | 1_{[0,T]} v_k)_{L^2_+}]$$
$$+ (b(\Lambda_k)\varphi(h_k D_1)^j 1_{[0,T]} g_k | 1_{[0,T]} v_k)_{L^2_+} + (b(\Lambda_k)\varphi(h_k D_1)^j 1_{[0,T]} a(h_k^2 P)^{1/2} a v_k | 1_{[0,T]} v_k)_{L^2_+}.$$

From Lemma A.14 [30], the first and the second terms of the RHS in the previous identity tend to zero.
Using that $\|g_k\|_{L^2} \to 0$, we can prove that the third term tends also to zero.

Following Lemma A.6 and (2.73) the forth term tends to zero. We conclude that I_k^j tends to zero. For J_k^j we argue as for I_k^j. ∎

Proof of Proposition 2.15. From Proposition 2.3 $(\tau + p)\mu = 0$, so we have

$$\langle \widetilde{\mu}, H_p b \rangle = \langle \widetilde{\mu}, H_p(b_0 + b_1 \eta_1) \rangle.$$

Let consider the identity (2.79), by Lemma 2.18, the LHS tends to zero when $k \to +\infty$. By the semiclassical symbolic calculus, we have

$$\frac{i}{h_k}[k^2(D_t + P), (b_0(\Lambda_k) + b_1(\Lambda_k) h_k D_1)\varphi] = \sum_{j=0}^{2} c_j(\Lambda_k)\varphi(h_k D_1)^j,$$

where $c_j \in \mathscr{C}_0^\infty(U_0 \times \mathbb{R}^{d+1})$, $\varphi_1 = 1$ on $\operatorname{supp}\varphi$, and $\{p, b_0 + b_1\eta_1\} = \sum_{j=0}^{2} c_j \eta_1^j$. Hence, using Lemma 2.17 and Lemma 2.12, the RHS of (2.79) tends to

$$-\langle \widetilde{\mu}, H_p(b_0 + b_1\eta_1) \rangle - \langle \widetilde{\nu}, b_{1\,|Y_1=0} \rangle,$$

when $k \to +\infty$.

We conclude that

$$\langle \widetilde{\mu}, H_p b \rangle = \langle \widetilde{\mu}, H_p(b_0 + b_1\eta_1) \rangle = -\langle \widetilde{\nu}, b_{1\,|Y_1=0} \rangle,$$

which proves the Proposition 2.15. ∎

Proposition 2.19. *With the notations of [30], we have*

$$\widetilde{\nu}(\mathcal{G}_d \cup (\bigcup_{k=3}^{+\infty} \mathcal{G}^k)) = 0.$$

Proof. The proof is the same as of Lemma A.17 in [30]. ∎

By measure theory methods (see [8], [9] and [30]), the propagation of the measure μ along the generalized bicharacteristic flow is equivalent to Propositions 2.13, 2.15 and 2.19.

A Appendix

In this appendix, we prove some Lemmas used above.
We recall the Helffer-Sjöstrand formula (see [14]) used extensively in this section. To introduce it we recall some notations.
Let $\theta \in \mathscr{C}_0^\infty(\mathbb{R})$ and let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ if $|t| \leq 1$ and $\varphi(t) = 0$ if $|t| \geq 2$. Let $N \geq 2$, we set

$$\tilde{\theta}(t,\sigma) = \sum_{q=1}^{N} \frac{\theta^{(q)}(t)}{q!}(i\sigma)^q \varphi(\sigma).$$

then $\tilde{\theta} \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ and satisfies

$$|\bar{\partial}\tilde{\theta}(t,\sigma)| \leq C|\sigma|^N \text{ where } \bar{\partial}\tilde{\theta}(t,\sigma) = \frac{1}{2}(\partial_t\tilde{\theta} + i\partial_\sigma\tilde{\theta})(t,\sigma). \tag{A.1}$$

We call $\tilde{\theta}$ an almost analytic extension of θ. Let P a self adjoint operator. We have the following Helffer-Sjöstrand formula

$$\theta(h^2P) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \bar{\partial}\tilde{\theta}(t,\sigma)(z - h^2P)^{-1} dt d\sigma \text{ where } z = t + i\sigma. \tag{A.2}$$

The formula does not depend of N and φ. We recall the estimates proved in [30], Lemma A.22, we have for $f = (z - h^2P)^{-1}u$ and $\Im mz \neq 0$,

$$\|h^2Pf\|^2_{L^2(\Omega)} + \|hD_jf\|^2_{L^2(\Omega)} + \|hV^{1/2}f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \leq C\frac{\langle |z| \rangle^2}{|\Im mz|^2}\|u\|^2_{L^2(\Omega)}. \tag{A.3}$$

Let h_n a sequence such that $h_n > 0$ and $h_n \to 0$ when $n \to +\infty$. In the sequel, for simplicity we denote such a sequence by h. We say $h \to 0$ instead of $h_n \to 0$ when $n \to +\infty$.

Lemma A.1. *Let u_h and g_h satisfying*

$$\begin{cases} h^2(D_t + P)u_h - iha(h^2P)^{1/2}(au_h) = hg_h \text{ in } [0,T] \times \Omega \\ u_h = 0 \text{ on } [0,T] \times \partial\Omega \end{cases}$$

and we assume that $\|\langle x \rangle^{-s} u_h\|^2_{L^2([0,T]\times\Omega)} \leq 1$, $h\|u_h(0)\|^2_{L^2(\Omega)} \to 0$ and $\|\langle x \rangle^s g_h\|^2_{L^2([0,T]\times\Omega)} \to 0$ when $h \to 0$. Then $\sup_{t\in[0,T]} h\|u_h(t)\|^2_{L^2(\Omega)} \to 0$.

Proof. Let $k(t) = h\|u_h(t)\|^2_{L^2(\Omega)}$, using $h\partial_t u_h = -ihPu_h - a(h^2P)^{1/2}(au_h) + ig_h$, we have

$$\begin{aligned} k'(t) &= 2\Re e(h\partial_t u_h(t)|u_h(t)) \\ &= 2\Re e(-ihPu_h(t)|u_h(t)) - 2\Re e(a(h^2P)^{1/2}(au_h)(t)|u_h(t)) + 2\Re e(ig_h|u_h). \end{aligned}$$

Using

$$\Re e(iPu_h(t)|u_h(t)) = 0,$$

and

$$\Re e(a(h^2P)^{1/2}(au_h)(t)|u_h(t)) = \Re e((h^2P)^{1/2}(au_h)(t)|au_h(t)) \geq 0,$$

we obtain

$$k'(t) \leq 2\|\langle x \rangle^s g_h(t)\|_{L^2(\Omega)}\|\langle x \rangle^{-s} u_h(t)\|_{L^2(\Omega)}.$$

Thus

$$k(t) \leq k(0) + 2\|\langle x \rangle^s g_h\|_{L^2([0,T]\times\Omega)}\|\langle x \rangle^{-s} u_h\|_{L^2([0,T]\times\Omega)}.$$

The assumptions and the definition of k imply the Lemma. ∎

Let $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(t) = 0$ if $t \leq \alpha$ or $t \geq \beta$ where $0 < \alpha < \beta$.

Lemma A.2. *Let $a \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and $s \leq 1$, there exist $C > 0$, h_0 such that, if $0 < h < h_0$ we have, for all $u \in L^2(\Omega)$,*

$$\|\langle x\rangle^s [a, \psi(h^2P)](h^2P)^{1/2} u\|^2_{L^2(\Omega)} \leq Ch^2 \|u\|^2_{L^2(\Omega)}. \tag{A.4}$$

Proof. We prove (A.4) for $u \in \mathscr{C}_0^\infty(\Omega)$.
Taking the adjoint, (A.4) is equivalent to

$$\|(h^2P)^{1/2}[a, \psi(h^2P)]\langle x\rangle^s u\|^2_{L^2(\Omega)} \leq Ch^2 \|u\|^2_{L^2(\Omega)},$$

which is equivalent to

$$|(\langle x\rangle^s [a, \psi(h^2P)](h^2 \sum \partial_{x_j} a_{jk}(x) \partial_{x_k} + h^2 V)[a, \psi(h^2P)]\langle x\rangle^s u | u) \leq Ch^2 \|u\|^2_{L^2(\Omega)}.$$

Thus it is enough to prove

$$\|h\partial_{x_j}[a, \psi(h^2P)]\langle x\rangle^s u\| \leq Ch\|u\|_{L^2(\Omega)}, \tag{A.5}$$

and

$$\|hV^{1/2}[a, \psi(h^2P)]\langle x\rangle^s u\| \leq Ch\|u\|_{L^2(\Omega)}. \tag{A.6}$$

Now we prove (A.5). Following the Helffer-Sjöstrand formula, where $\tilde{\psi}$ is an almost analytic extension of ψ, we have

$$\begin{aligned} h\partial_{x_j}[a, \psi(h^2P)]\langle x\rangle^s &= -\frac{1}{\pi}\int \bar{\partial}\tilde{\psi}(z) h\partial_{x_j}[a, (z - h^2P)^{-1}]\langle x\rangle^s dt d\sigma \\ &= \frac{1}{\pi}\int \bar{\partial}\tilde{\psi}(z) h\partial_{x_j}(z - h^2P)^{-1}[a, z - h^2P](z - h^2P)^{-1}\langle x\rangle^s dt d\sigma \\ &= \frac{1}{\pi}\int \bar{\partial}\tilde{\psi}(z) h\partial_{x_j}(z - h^2P)^{-1}[a, z - h^2P]\langle x\rangle^s (z - h^2P)^{-1} dt d\sigma + A, \end{aligned} \tag{A.7}$$

where $A = \dfrac{1}{\pi}\displaystyle\int \bar{\partial}\tilde{\psi}(z) h\partial_{x_j}(z - h^2P)^{-1}[a, z - h^2P](z - h^2P)^{-1}[\langle x\rangle^s, z - h^2P](z - h^2P)^{-1} ds d\sigma.$

We have

$$[a, z - h^2P] = h^2 \sum_{j=1}^d \alpha_j(x)\partial_{x_j} + h^2 c(x), \tag{A.8}$$

where α_j and c are compact supported. Following (A.7), we have two types of terms to control. First we remark that

$$(h^2 \sum_{j=1}^d \alpha_j(x)\partial_{x_j} + h^2 c(x))\langle x\rangle^s = h^2 \beta_j \partial_{x_j} + h^2 d(x),$$

where β_j and d are compact supported, following (A.7) and estimates (A.3) (with $N = 3$) we obtain

$$\|h\partial_{x_j}(z - h^2P)^{-1}(h^2\beta_j\partial_{x_j} + h^2 d(x))(z - h^2P)^{-1} u\|_{L^2(\Omega)} \leq Ch \frac{\langle |z|\rangle^2}{|\Im m z|^2}\|u\|_{L^2(\Omega)}. \tag{A.9}$$

Thus following (A.1), we have

$$\|\int \bar{\partial}\tilde{\psi}(z) h\partial_{x_j}(z - h^2P)^{-1}(h^2\beta_j\partial_{x_j} + h^2 d(x))(z - h^2P)^{-1} u dt d\sigma\|_{L^2(\Omega)} \leq Ch\|u\|_{L^2(\Omega)}. \tag{A.10}$$

Second, we have

$$[\langle x\rangle^s, z - h^2P] = h^2 \sum_{k=1}^d \gamma_k(x)\partial_{x_k} + h^2\gamma(x),$$

where $|\gamma_k(x)| + |\gamma(x)| \leq C\langle x\rangle^{s-1} \leq C'$, with the above notations, we have following (A.3),

$$\begin{aligned} \|h\partial_{x_j}(z - h^2P)^{-1}(h^2\alpha_j(x)\partial_{x_j} + h^2 c(x))(z - h^2P)^{-1}(h^2\gamma_k(x)\partial_{x_k} + h^2\gamma(x))(z - h^2P)^{-1} u\| \\ \leq Ch^2 \frac{\langle |z|\rangle^3}{|\Im m z|^3}\|u\|_{L^2(\Omega)}, \end{aligned} \tag{A.11}$$

thus, following the proof of (A.10), we prove (A.5).

To prove (A.6), following the Helffer-Sjöstrand formula we have,

$$hV^{1/2}[a,\psi(h^2P)]\langle x\rangle^s = \frac{1}{\pi}\int \bar\partial\tilde\psi(z)hV^{1/2}(z-h^2P)^{-1}[a,z-h^2P](z-h^2P)^{-1}\langle x\rangle^s dtd\sigma.$$

With the notation above, it is enough to prove

$$\|hV^{1/2}(z-h^2P)^{-1}(h^2\sum_{j=1}^d \alpha_j(x)\partial_{x_j}+h^2c(x))(z-h^2P)^{-1}\langle x\rangle^s u\|_{L^2(\Omega)} \le Ch\frac{\langle|z|\rangle^3}{|\Im mz|^3}\|u\|_{L^2(\Omega)}. \quad (A.12)$$

Writing $(z-h^2P)^{-1}\langle x\rangle^s = \langle x\rangle^s(z-h^2P)^{-1}+[(z-h^2P)^{-1},\langle x\rangle^s]$, the first term is estimated following the proof of (A.9). To estimate the second term, we follow the proof of (A.11). Thus we obtain (A.12) which achieve the proof of Lemma. ∎

Lemma A.3. *Let $s\in[0,1]$ and χ a smooth function such that $\chi=1$ for $|x|\ge 1$. We set $\chi_R(x)=\chi(x/R)$. There exists $C>0$ such that for all $u\in L^2(\Omega)$,*

$$\|(h^2P)^{1/2}\langle x\rangle^s[\psi(h^2P),\chi_R]u\| \le Ch\|u\|.$$

Proof. The proof is very close to the one of Lemma A.2. By the same argument it is sufficient to prove

$$\|h\partial_{x_j}\langle x\rangle^s[\psi(h^2P),\chi_R]u\| \le Ch\|u\|, \quad (A.13)$$

$$\|hV^{1/2}\langle x\rangle^s[\psi(h^2P),\chi_R]u\| \le Ch\|u\|. \quad (A.14)$$

From the Helffer-Sjöstrand formula, we obtain (as in (A.7))

$$h\partial_{x_j}\langle x\rangle^s[\psi(h^2P),\chi_R] = \frac{1}{\pi}\int \bar\partial\tilde\psi(z)h\partial_{x_j}(z-h^2P)^{-1}\langle x\rangle^s[(z-h^2P),\chi_R](z-h^2P)^{-1}dtd\sigma \quad (A.15)$$
$$+\frac{1}{\pi}\int \bar\partial\tilde\psi(z)h\partial_{x_j}[\langle x\rangle^s,(z-h^2P)^{-1}][(z-h^2P),\chi_R](z-h^2P)^{-1}dtd\sigma.$$

Modulo negative power of $\Im mz$, in the first term of (A.15) $h\partial_{x_j}(z-h^2P)^{-1}$ is bounded on $L^2(\Omega)$ and, because $\langle x\rangle^s/R$ is bounded on the support of $\chi'(x/R)$, we can write $\langle x\rangle^s[(z-h^2P),\chi_R]$ as a sum of term $\alpha(x)h^2\partial_{x_j}$. This yields that $\langle x\rangle^s[(z-h^2P),\chi_R](z-h^2P)^{-1}$ is bounded on $L^2(\Omega)$ by Ch modulo negative power of $\Im mz$. This gives the result for the first term in (A.15).

Writing

$$[\langle x\rangle^s,(z-h^2P)^{-1}] = -(z-h^2P)^{-1}[\langle x\rangle^s,z-h^2P](z-h^2P)^{-1}$$

and arguing as for the first term, we obtain (A.13). By the same arguments and using that $hV^{1/2}(z-h^2P)^{-1}$ is bounded on $L^2(\Omega)$ modulo negative power of $\Im mz$ (see [30, Lemma A.22]), we obtain (A.14). ∎

Lemma A.4. *Let s such that $|s|\le 1$, let $b\in\mathscr{C}^\infty(\overline{\Omega})$ such that $|b(x)|\le C\langle x\rangle^s$ and $|\partial_{x_j}b(x)|+|\partial^2_{x_jx_k}b|\le C\langle x\rangle^{s-1}$, there exist $C>0$, $h_0>0$ such that, if $0<h<h_0$ we have, for all $u\in L^2(\Omega)$,*

$$\|\langle x\rangle^{-s}[\psi(h^2P),b]u\|_{L^2(\Omega)} \le Ch\|u\|_{L^2(\Omega)}.$$

Proof. By Helffer-Sjöstrand formula, we have, with the notation of Lemma A.2,

$$\langle x\rangle^{-s}[\psi(h^2P),b] = \frac{1}{\pi}\int \bar\partial\tilde\psi(z)\langle x\rangle^{-s}(z-h^2P)^{-1}[z-h^2P,b](z-h^2P)^{-1}dtd\sigma \quad (A.16)$$
$$= \frac{1}{\pi}\int \bar\partial\tilde\psi(z)\langle x\rangle^{-s}(z-h^2P)^{-1}(h^2\sum_{k=1}^d\gamma_k(x)\partial_{x_k}+h^2\gamma(x))(z-h^2P)^{-1}dtd\sigma,$$

where $|\gamma_k(x)| + |\gamma(x)| \leq C\langle x\rangle^{s-1}$.
If $s \geq 0$, following (A.3), we have

$$\|\langle x\rangle^{-s}(z-h^2P)^{-1}(h^2\sum_{k=1}^{d}\gamma_k(x)\partial_{x_k}+h^2\gamma(x))(z-h^2P)^{-1}u\|_{L^2(\Omega)} \leq Ch\frac{\langle|z|\rangle}{|\Im mz|}\|u\|_{L^2(\Omega)}, \quad (A.17)$$

thus, following the proof of (A.10), we achieve the proof of Lemma in this case.
If $s < 0$, we write

$$\langle x\rangle^{-s}(z-h^2P)^{-1} = (z-h^2P)^{-1}\langle x\rangle^{-s} - (z-h^2P)^{-1}[\langle x\rangle^{-s},(z-h^2P)](z-h^2P)^{-1}.$$

Putting this in (A.16), we obtain two terms. The first gives

$$\|(z-h^2P)^{-1}\langle x\rangle^{-s}(h^2\sum_{k=1}^{d}\gamma_k(x)\partial_{x_k}+h^2\gamma(x))(z-h^2P)^{-1}u\|_{L^2(\Omega)} \leq Ch\frac{\langle|z|\rangle}{|\Im mz|}\|u\|_{L^2(\Omega)}. \quad (A.18)$$

The second gives

$$\begin{aligned}\|(z-h^2P)^{-1}(h^2\sum_{k=1}^{d}\tilde{\gamma}_k(x)\partial_{x_k}+h^2\tilde{f}(x))(z-h^2P)^{-1}(h^2\sum_{k=1}^{d}\gamma_k(x)\partial_{x_k}+h^2\gamma(x))(z-h^2P)^{-1}u\| \\ \leq Ch^2\frac{\langle|z|\rangle^2}{|\Im mz|^2}\|u\|,\end{aligned} \quad (A.19)$$

because $|\tilde{\gamma}_k(x)| + |\tilde{\gamma}(x)| \leq C\langle x\rangle^{-s-1}$. Following (A.18), (A.19) and the Helffer-Sjöstrand formula, we obtain the Lemma. ∎

Remarks A.5. In the Lemma A.4, we can remove the assumption $|s| \leq 1$, by commuting $\langle x\rangle^s$ with $(z-h^2P)^{-1}$ several times, but Lemma A.4 is sufficient for us in the sequel.

Lemma A.6. *Let $a \in \mathscr{C}_0^\infty(\mathbb{R}^d)$, there exist $C > 0$, h_0 such that, if $0 < h < h_0$ we have, for all $u \in L^2(\Omega)$,*

$$\|(h^2P)^{1/2}a\psi(h^2P)u\|^2_{L^2(\Omega)} \leq Ch^2\|u\|^2_{L^2(\Omega)} + C\|au\|^2_{L^2(\Omega)}.$$

Proof. Writing

$$(h^2P)^{1/2}a\psi(h^2P)u = (h^2P)^{1/2}[a,\psi(h^2P)]u + (h^2P)^{1/2}\psi(h^2P)au,$$

then using the Lemma A.2 with $s = 0$,

$$\begin{aligned}\|(h^2P)^{1/2}a\psi(h^2P)u\|^2_{L^2(\Omega)} &\leq \|(h^2P)^{1/2}[a,\psi(h^2P)]u\|^2_{L^2(\Omega)} + \|(h^2P)^{1/2}\psi(h^2P)au\|^2_{L^2(\Omega)} \\ &\leq Ch^2\|u\|^2_{L^2(\Omega)} + C\|au\|^2_{L^2(\Omega)},\end{aligned}$$

which proves the Lemma. ∎

Lemma A.7. *For all $s \in [-1,1]$, there exists $C > 0$ such that for all $u \in \mathscr{C}_0^\infty(\Omega)$and all $h \in (0,1]$, we have*

$$\|\langle x\rangle^s\psi(h^2P)\langle x\rangle^{-s}u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

Proof. We have by Lemma A.4

$$\begin{aligned}\|\langle x\rangle^s\psi(h^2P)\langle x\rangle^{-s}u\|_{L^2(\Omega)} &\leq \|\psi(h^2P)u\|_{L^2(\Omega)} + \|\langle x\rangle^s[\psi(h^2P),\langle x\rangle^{-s}]u\|_{L^2(\Omega)} \\ &\leq \|\psi(h^2P)u\|_{L^2(\Omega)} + Ch\|u\|_{L^2(\Omega)},\end{aligned}$$

which proves the Lemma. ∎

Lemma A.8. *Let $\alpha \in (-1,1)$ and $s \in [-1,1]$, then there exist $C_1 >$ and $C_2 > 0$ such that for all $u \in \mathscr{C}_0^\infty(\Omega)$, we have*

$$C_1 \sum_{n=0}^{+\infty} h_n^{-2\alpha} \|\langle x\rangle^s \psi(h_n^2 P)u\|^2_{L^2(\Omega)} \leq \sum_{n=0}^{+\infty} h_n^{-2\alpha} \|\psi(h_n^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \leq C_2 \sum_{n=0}^{+\infty} h_n^{-2\alpha} \|\langle x\rangle^s \psi(h_n^2 P)u\|^2_{L^2(\Omega)},$$

where ψ was defined in Section 2.1 and $h_n = 2^{-n}$.

Proof. We have

$$\begin{aligned}
\|\psi(h_n^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} &= \|\psi(h_n^2 P)\langle x\rangle^s \sum_{k=0}^{+\infty} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} \\
&\leq 2\|\psi(h_n^2 P)\langle x\rangle^s \sum_{k=0}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} \\
&\quad + 2\|\psi(h_n^2 P)\langle x\rangle^s \sum_{k=n+2}^{+\infty} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} = 2A + 2B.
\end{aligned}$$

To estimate A, we can write

$$A \leq 2\|\langle x\rangle^s \psi(h_n^2 P) \sum_{k=0}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} + 2\|[\psi(h_n^2 P), \langle x\rangle^s] \sum_{k=0}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} = 2A_1 + 2A_2.$$

By support properties of ψ and by the Lemma A.7, we have

$$A_1 = \|\langle x\rangle^s \psi(h_n^2 P) \sum_{k=n-1}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} \leq \|\langle x\rangle^s \sum_{k=n-1}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)}. \tag{A.20}$$

By Lemma A.4 we see easily that

$$A_2 \leq C h_n^2 \|\langle x\rangle^s \sum_{k=0}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)}.$$

Summing with respect n, we obtain

$$\sum_{n=0}^{+\infty} h_n^{-2\alpha} h_n^2 \|\langle x\rangle^s \sum_{k=0}^{n+1} \psi(h_k^2 P)u\|^2_{L^2(\Omega)} \leq \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n+1} h_n^{-\alpha+1} h_k^\alpha \left(h_k^{-\alpha} \|\langle x\rangle^s \psi(h_k^2 P)u\|_{L^2(\Omega)} \right) \right)^2. \tag{A.21}$$

We have $h_n^{-\alpha+1} h_k^\alpha = 2^{-(1-\alpha)(n-k)} 2^{-k} \leq 2^{-(1-\alpha)(n-k)}$ and $(2^{-(1-\alpha)j})_{j\geq 0} \in \ell^1$ because $1-\alpha > 0$. We can consider the right hand side of (A.21) as a convolution $\ell^1 * \ell^2$ and we obtain the estimation of this term by $C \sum\limits_{n=0}^{+\infty} h_n^{-2\alpha} \|\langle x\rangle^s \psi(h_n^2 P)u\|^2_{L^2(\Omega)}$ which estimates, with (A.20), the term A.

Now we estimate B. By support properties of ψ and Lemma A.4 it follows that

$$\begin{aligned}
B &= \|\psi(h_n^2 P)\langle x\rangle^s \sum_{k=n+2}^{+\infty} \psi(h_k^2 P) \sum_{j=k-1}^{k+1} \psi(h_j^2 P)u\|^2_{L^2(\Omega)} \\
&= \|\psi(h_n^2 P) \sum_{k=n+2}^{+\infty} [\langle x\rangle^s, \psi(h_k^2 P)] \sum_{j=k-1}^{k+1} \psi(h_j^2 P)u\|^2_{L^2(\Omega)} \\
&\leq C \left(\sum_{k=n+2}^{+\infty} h_k \|\langle x\rangle^s \sum_{j=k-1}^{k+1} \psi(h_j^2 P)u\|_{L^2(\Omega)} \right)^2.
\end{aligned}$$

Summing with respect n, we obtain

$$\sum_{n=0}^{+\infty} h_n^{-2\alpha} \left(\sum_{k=n+2}^{+\infty} h_k \|\langle x\rangle^s \sum_{j=k-1}^{k+1} \psi(h_j^2 P)u\|_{L^2(\Omega)} \right)^2 \leq \sum_{n=0}^{+\infty} \left(\sum_{k=n+2}^{+\infty} h_n^{-\alpha} h_k^{1+\alpha} \left(h_k^{-\alpha} \|\langle x\rangle^s \sum_{j=k-1}^{k+1} \psi(h_j^2 P)u\|_{L^2(\Omega)} \right) \right)^2.$$

We have $h_n^{-\alpha} h_k^{1+\alpha} = 2^{-(1+\alpha)(k-n)} 2^{-n} \leq 2^{-(1+\alpha)(k-n)}$ and $(2^{-(1+\alpha)j}) \in \ell^1$ since $1+\alpha > 0$. We can conclude as for the term A above. We have proved the right inequality of the Lemma.

We prove the other inequality.

We have,

$$\begin{aligned}
\|\langle x\rangle^s \psi(h_n^2 P)u\|^2_{L^2(\Omega)} &= \|\langle x\rangle^s \psi(h_n^2 P)\langle x\rangle^{-s} \sum_{k=0}^{+\infty} \psi(h_k^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \\
&\leq 2\|\langle x\rangle^s \psi(h_n^2 P)\langle x\rangle^{-s} \sum_{k=0}^{n+1} \psi(h_k^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \\
&\quad + 2\|\langle x\rangle^s \psi(h_n^2 P)\langle x\rangle^{-s} \sum_{k=n+2}^{+\infty} \psi(h_k^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} = 2D + 2E.
\end{aligned}$$

We have by properties of support of ψ,

$$D \leq 2\|\psi(h_n^2 P) \sum_{k=n-1}^{n+1} \psi(h_k^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} + 2\|[\langle x\rangle^s, \psi(h_n^2 P)]\langle x\rangle^{-s} \sum_{k=0}^{n+1} \psi(h_k^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)}.$$

The estimate of the first term is clear, for the second using Lemma A.4, we get

$$\sum_{n=0}^{+\infty} h_n^{-2\alpha} \|[\langle x\rangle^s, \psi(h_n^2 P)]\langle x\rangle^{-s} \sum_{k=0}^{n+1} \psi(h_k^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \leq \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n+1} h_n^{-\alpha+1} h_k^{\alpha} \left(h_k^{-\alpha} \|\psi(h_k^2 P)\langle x\rangle^s u\|_{L^2(\Omega)} \right) \right)^2.$$

We have $h_n^{-\alpha+1} h_k^{\alpha} \leq 2^{-(1-\alpha)(n-k)}$ and we can conclude as above by convolution argument.

For E, it follows from the support properties of ψ, Lemma A.7 and Lemma A.4,

$$\begin{aligned}
E &= \|\langle x\rangle^s \psi(h_n^2 P)\langle x\rangle^{-s} \sum_{k=n+2}^{+\infty} \psi(h_k^2 P) \sum_{j=k-1}^{k+1} \psi(h_j^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \\
&\leq \|\langle x\rangle^s \psi(h_n^2 P) \sum_{k=n+2}^{+\infty} [\langle x\rangle^{-s}, \psi(h_k^2 P)] \sum_{j=k-1}^{k+1} \psi(h_j^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \\
&\leq C\|\langle x\rangle^s \sum_{k=n+2}^{+\infty} [\langle x\rangle^{-s}, \psi(h_k^2 P)] \sum_{j=k-1}^{k+1} \psi(h_j^2 P)\langle x\rangle^s u\|^2_{L^2(\Omega)} \\
&\leq C \left(\sum_{k=n+2}^{+\infty} h_k \| \sum_{j=k-1}^{k+1} \psi(h_j^2 P)\langle x\rangle^s u\|_{L^2(\Omega)} \right)^2.
\end{aligned}$$

Summing with respect n, we obtain,

$$\sum_{n=0}^{+\infty} h_n^{-2\alpha} \|\langle x\rangle^s \psi(h_n^2 P)\langle x\rangle^{-s} \sum_{k=n+2}^{+\infty} \psi(h_k^2 P)\langle x\rangle^s u\|_{L^2(\Omega)}^2$$
$$\leq \sum_{n=0}^{+\infty} \left(\sum_{k=n+2}^{+\infty} h_n^{-\alpha} h_k^{1+\alpha} \left(h_k^{-\alpha} \| \sum_{j=k-1}^{k+1} \psi(h_j^2 P)\langle x\rangle^s u\|_{L^2(\Omega)} \right) \right)^2.$$

We have $h_n^{-\alpha} h_k^{1+\alpha} \leq 2^{-(n-k)(1+\alpha)}$ and we can conclude by convolution argument. ∎

Lemma A.9. *Let $s \in [-1,1]$, $\alpha \in (-1, 3/2)$ there exists $C > 0$ such that for all $u \in L^2(\Omega)$, we have*

$$\sum_{k=0}^{+\infty} h_k^{-1} \|\langle x\rangle^s [\psi(h_k^2 P), a](h_k^2 P)^{1/2} a (h_k^2 P)^{-\alpha/2} u\|_{L^2(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2.$$

Proof. Following the properties of ψ, we have

$$(h_k^2 P)^{1/2} = \sum_{j=0}^{+\infty} h_k h_j^{-1} \psi_0(h_j^2 P)$$

where $\psi_0(\sigma) = \sigma^{1/2}\psi(\sigma)$ and

$$(h_k^2 P)^{-\alpha/2} = \sum_{n=0}^{+\infty} h_k^{-\alpha} h_n^{\alpha} \psi_1(h_n^2 P)$$

where $\psi_1(\sigma) = \sigma^{-\alpha/2}\psi(\sigma)$. Thus we must prove,

$$\sum_{k=1}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in\mathbb{N}^{*2}} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_0(h_j^2 P) a \psi_1(h_n^2 P) u\|_{L^2(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2. \tag{A.22}$$

Let us introduce for each k the following partition of $\mathbb{N}^2$.

$$A_k^1 = \{(j,n) \in \mathbb{N}^2,\ k \geq j-2 \text{ or } k \geq n-2, \text{ and } j \geq n-2\},$$
$$A_k^2 = \{(j,n) \in \mathbb{N}^2,\ k \geq j-2 \text{ or } k \geq n-2,\ textandj \leq n-3\},$$
$$A_k^3 = \{(j,n) \in \mathbb{N}^2,\ k \leq j-3 \text{ and } k \leq n-3\}.$$

In the sequel, for each set A_k^p we will prove (A.22).

Let $\psi_2 \in \mathscr{C}_0^\infty(0,+\infty)$ such that $\psi_2 = 1$ on the support of ψ. We have,

$$\sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^1} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_2(h_j^2 P)\psi_0(h_j^2 P) a \psi_1(h_n^2 P) u\|_{L^2(\Omega)}^2 \leq 2A + 2B,$$

where

$$A = \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^1} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_2(h_j^2 P) a \psi_0(h_j^2 P)\psi_1(h_n^2 P) u\|_{L^2(\Omega)}^2$$
$$\leq C \sum_{k=0}^{+\infty} h_k^{-1} \left(\sum_{\substack{(j,n)\in A_k^1 \\ |j-n|\leq 1}} h_k^{1-\alpha} h_n^{-1+\alpha} \|\langle x\rangle^s [\psi(h_k^2 P), a]\psi_2(h_j^2 P) a \psi_0(h_j^2 P)\psi_1(h_n^2 P) u\|_{L^2(\Omega)} \right)^2$$
$$\leq C \sum_{k=0}^{+\infty} \left(\sum_{n\leq k+4} h_k^{3/2-\alpha} h_n^{-1+\alpha} \|\psi_1(h_n^2 P) u\|_{L^2(\Omega)} \right)^2 \text{ (by Lemma A.4).} \tag{A.23}$$

We have $h_k^{3/2-\alpha}h_n^{-1+\alpha} = 2^{-(k-n)(3/2-\alpha)}2^{-n/2} \leq 2^{-(k-n)(3/2-\alpha)}$ and we can see (A.23) as a convolution $\ell^1 * \ell^2$ if $\alpha < 3/2$ which prove (A.22) for this term.

For B, we can see that

$$\begin{aligned} B &= \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^1} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_2(h_j^2 P)[\psi_0(h_j^2 P), a]\psi_1(h_n^2 P)u\|^2_{L^2(\Omega)} \\ &\leq 2C + 2D, \end{aligned}$$

where

$$C = \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^1} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a][\psi_0(h_j^2 P), a]\psi_2(h_j^2 P)\psi_1(h_n^2 P)u\|^2_{L^2(\Omega)}.$$

In the last sum $|j-n| \leq 1$, then we can estimate this term as the term A.

We have

$$\begin{aligned} D &= \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^1} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a][\psi_2(h_j^2 P), [\psi_0(h_j^2 P), a]]\psi_1(h_n^2 P)u\|^2_{L^2(\Omega)} \\ &\leq \sum_{k=0}^{+\infty} \left(\sum_{(j,n)\in A_k^1} h_j h_k^{3/2-\alpha} h_n^{\alpha} \|\psi_1(h_n^2 P)u\|_{L^2(\Omega)} \right)^2 \text{ (by Lemma A.4 and Lemma A.10).} \end{aligned}$$

In A_k^1, we have $j \geq n-2$ then the sum over j gives a constant time h_n. Then,

$$\begin{aligned} D &\leq C \sum_{k=0}^{+\infty} \left(\sum_{n\leq k+4} h_k^{3/2-\alpha} h_n^{1+\alpha} \|\psi_1(h_n^2 P)u\|_{L^2(\Omega)} \right)^2 \\ &\leq C \sum_{k=0}^{+\infty} h_k^{3-2\alpha} \left(\sum_{n\leq k+4} h_n^{2+2\alpha} \right) \left(\sum_{n\leq k+4} \|\psi_1(h_n^2 P)u\|^2_{L^2(\Omega)} \right), \end{aligned}$$

by Cauchy-Schwarz inequality and as all the sums converge if $\alpha \in (-1, 3/2)$, we obtain (A.22).

Now we will estimate the sum over A_k^2. We have with the function ψ_2 defined above, as $\psi_0(h_j^2 P)\psi_2(h_n^2 P) = 0$, because $j \leq n-2$,

$$\begin{aligned} &\sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^2} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_0(h_j^2 P) a \psi_2^2(h_n^2 P)\psi_1(h_n^2 P)u\|^2_{L^2(\Omega)} \\ &= \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^2} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_0(h_j^2 P)[[a, \psi_2(h_n^2 P)], \psi_2(h_n^2 P)]\psi_1(h_n^2 P)u\|^2_{L^2(\Omega)} \\ &\leq C \sum_{k=0}^{+\infty} \left(\sum_{(j,n)\in A_k^2} h_k^{3/2-\alpha} h_j^{-1} h_n^{2+\alpha} \|\psi_1(h_n^2 P)u\|_{L^2(\Omega)} \right)^2 \text{ (by Lemma A.4 and the Lemma A.10).} \end{aligned}$$

As $\sum_{j\leq n-3} h_j^{-1} \leq C h_n^{-1}$, we can end the proof as for the term D above.

Finally we treat the sum over A_k^3. We have, as $\psi(h_k^2 P)\psi_0(h_j^2 P) = 0$.

$$\sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^3} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s [\psi(h_k^2 P), a]\psi_0(h_j^2 P) a \psi_1(h_n^2 P) u\|^2_{L^2(\Omega)}$$
$$= \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^3} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s \psi(h_k^2 P) a \psi_0(h_j^2 P)\psi_2(h_j^2 P) a \psi_2(h_n^2 P)\psi_1(h_n^2 P) u\|^2_{L^2(\Omega)}$$
$$\leq 2E + 2F,$$

where,

$$E = \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in A_k^3} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s \psi(h_k^2 P)[[a,\psi_0(h_j^2 P)],\psi_2(h_j^2 P)][a,\psi_2(h_n^2 P)]\psi_1(h_n^2 P) u\|^2_{L^2(\Omega)}$$
$$\leq \sum_{k=0}^{+\infty} \left(\sum_{(j,n)\in A_k^3} h_k^{1/2-\alpha} h_j h_n^{1+\alpha} \|\psi_1(h_n^2 P) u\|_{L^2(\Omega)} \right)^2 .$$

If $(j,n) \in A_k^3$, we have $j \geq k+3$ then the sum over j is less than Ch_k. We obtain,

$$E \leq C \sum_{k=0}^{+\infty} \left(\sum_{n\geq k+3} h_k^{3/2-\alpha} h_n^{1+\alpha} \|\psi_1(h_n^2 P) u\|_{L^2(\Omega)} \right)^2$$
$$\leq C \sum_{k=0}^{+\infty} h_k^{3-2\alpha} \left(\sum_{n\geq k+3} h_n^{2+2\alpha} \right) \left(\sum_{n\geq k+3} \|\psi_1(h_n^2 P) u\|^2_{L^2(\Omega)} \right)$$
$$\leq C \sum_{k=0}^{+\infty} h_k^5 \|u\|^2_{L^2(\Omega)} \leq C\|u\|^2_{L^2(\Omega)}.$$

And we have

$$F = \sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{\substack{(j,n)\in A_k^3 \\ |j-n|\leq 1}} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s \psi(h_k^2 P)[[a,\psi_0(h_j^2 P)],\psi_2(h_j^2 P)]\psi_2(h_n^2 P) a \psi_1(h_n^2 P) u\|^2_{L^2(\Omega)}$$
$$\leq C \sum_{k=0}^{+\infty} h_k^{1-2\alpha} \left(\sum_{n\geq k+3} h_n^{1+\alpha} \|\psi_1(h_n^2 P) u\|_{L^2(\Omega)} \right)^2 \text{ (by Lemma A.10)}$$
$$\leq C \sum_{k=0}^{+\infty} h_k^{1-2\alpha} \left(\sum_{n\geq k+3} h_n^{2+2\alpha} \right) \left(\sum_{n\geq k+3} \|\psi_1(h_n^2 P) u\|^2_{L^2(\Omega)} \right) \leq C \sum_{k=0}^{+\infty} h_k^3 \|u\|^2_{L^2(\Omega)}.$$

Which achieve the proof of Lemma. ∎

Lemma A.10. *Let $b \in \mathscr{C}^\infty(\Omega)$ with support in $\{|x| \leq R\}$, let $\theta_1,\ \theta_2 \in \mathscr{C}_0^\infty(\mathbb{R})$, let $s \in [0,1]$ there exist $h_0 > 0$ and $C > 0$ such that for all $u \in L^2(\Omega)$ and $h \in (0, h_0)$ we have,*

$$\|\langle x\rangle^s [[\theta_1(h^2 P), b], \theta_2(h^2 P)] u\|_{L^2(\Omega)} \leq Ch^2 \|u\|_{L^2(\Omega)}.$$

Proof. We give only a sketch of proof, we use the same technic than before. By the Helffer-Sjöstrand formula, we have

$$[[\theta_1(h^2 P), b]\theta_2(h^2 P)] u = \frac{1}{\pi^2} \int_{\mathbb{R}^4} \bar\partial \tilde\theta_1(t_1,\sigma_1) \bar\partial \tilde\theta_2(t_2,\sigma_2) [[(z_1 - h^2 P)^{-1}, b], (z_2 - h^2 P)^{-1}] dt d\sigma,$$

where $z = (z_1, z_2)$ and $z_j = t_j + i\sigma_j$.

First, we can write

$$\begin{aligned}&[[(z_1 - h^2P)^{-1}, b], (z_2 - h^2P)^{-1}]\\ &\quad = (z_1 - h^2P)^{-1}(z_2 - h^2P)^{-1}[[z_1 - h^2P, b], z_2 - h^2P](z_1 - h^2P)^{-1}(z_2 - h^2P)^{-1},\end{aligned}$$

and

$$[[z_1 - h^2P, b], z_2 - h^2P] = h^4 \sum_{j,k} \gamma_{jk}(x)\partial_{jk}^2 + h^4 \sum_j \gamma_j(x)\partial_j + h^4\gamma_0(x),$$

where the γ's are compactly supported. Second, as

$$\begin{aligned}\langle x\rangle^s (z_1 - h^2P)^{-1}(z_2 - h^2P)^{-1} =&(z_1 - h^2P)^{-1}(z_2 - h^2P)^{-1}\langle x\rangle^s\\ &+ [\langle x\rangle^s, (z_1 - h^2P)^{-1}](z_2 - h^2P)^{-1}\\ &+ (z_1 - h^2P)^{-1}[\langle x\rangle^s, (z_2 - h^2P)^{-1}],\end{aligned}$$

and $[\langle x\rangle^s, (z - h^2P)^{-1}] = -(z - h^2P)^{-1}[\langle x\rangle^s, (z - h^2P)](z - h^2P)^{-1}$, then we can obtain the Lemma by using the estimate (A.3) and writing the commutator $[\langle x\rangle^s, (z - h^2P)]$ as in the Formula (A.16). ■

Lemma A.11. *Let $s \in [-1, 1]$, $\alpha < 3/2$, there exists $C > 0$ such that for all $u \in L^2(\Omega)$, we have*

$$\sum_{k=0}^{+\infty} h_k^{-1} \|\langle x\rangle^s a(h_k^2P)^{1/2}[\psi(h_k^2P), a](h_k^2P)^{-\alpha/2}u\|_{L^2(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2.$$

Proof. We follow the same strategy than the one for the proof of Lemma A.9. We have to prove,

$$\sum_{k=0}^{+\infty} h_k^{-1} \| \sum_{(j,n)\in\mathbb{N}^2} h_k^{1-\alpha} h_j^{-1} h_n^{\alpha} \langle x\rangle^s a\psi_0(h_j^2P)[\psi(h_k^2P), a]\psi_1(h_n^2P)u\|_{L^2(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2. \tag{A.24}$$

If $[j - k| \geq 2$ and $|n - k| \geq 2$, the corresponding term in the sum is null. If $|j - k| \leq 1$ (the case $|n - k| \leq 1$ is symmetric and let to the reader). We consider two cases, the first if $n \geq k + 2$, term A in the sequel, and the second if $k \geq n + 2$ term B in the sequel.

$$\begin{aligned}A &\leq C \sum_{k=0}^{+\infty} \left(\sum_{\substack{|j-k|\leq 1\\ n\geq k+2}} h_k^{-1/2-\alpha} h_n^{\alpha} \|\langle x\rangle^s a\psi_0(h_j^2P)\psi(h_k^2P)a\psi_2(h_n^2P)\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2\\ &\leq C \sum_{k=0}^{+\infty} \left(\sum_{n\geq k+2} h_k^{-1/2-\alpha} h_n^{\alpha} \|\psi(h_k^2P)[a, \psi_2(h_n^2P)]\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2\\ &\leq C \sum_{k=0}^{+\infty} \left(\sum_{n\geq k+2} h_k^{-1/2-\alpha} h_n^{1+\alpha} \|\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2\\ &\leq C \sum_{k=0}^{+\infty} h_k^{-1-2\alpha} \left(\sum_{n\geq k+2} h_n^{2+2\alpha} \right) \left(\sum_{n\geq k+2} \|\psi_1(h_n^2P)u\|_{L^2(\Omega)}^2 \right)\\ &\leq C \sum_{k=0}^{+\infty} h_k \|u\|_{L^2(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2.\end{aligned}$$

$$
\begin{aligned}
B &\leq C \sum_{k=0}^{+\infty} \left(\sum_{\substack{|j-k|\leq 1 \\ k\geq n+2}} h_k^{-1/2-\alpha} h_n^{\alpha} \|\langle x\rangle^s a\psi_0(h_j^2P)\psi_2(h_k^2P)\psi(h_k^2P)a\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2 \\
&\leq C \sum_{k=0}^{+\infty} \left(\sum_{k\geq n+2} h_k^{-1/2-\alpha} h_n^{\alpha} \|[\psi_2(h_k^2P),[\psi(h_k^2P),a]]\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2 \\
&\leq C \sum_{k=0}^{+\infty} \left(\sum_{k\geq n+2} h_k^{3/2-\alpha} h_n^{\alpha} \|\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2 \\
&\leq C \sum_{k=0}^{+\infty} \left(\sum_{k\geq n+2} 2^{-(k-n)(3/2-\alpha)} \|\psi_1(h_n^2P)u\|_{L^2(\Omega)} \right)^2 \leq C\|u\|^2_{L^2(\Omega)},
\end{aligned}
$$

because the last term can be seen as a convolution $\ell^1 * \ell^2$ if $\alpha < 3/2$. The estimations on A and B prove (A.24). ∎

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