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# Nonlocal damage model: boundary and evolving boundary effects

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*ABSTRACT. The present contribution aims at providing a closer insight on boundary effects in non local damage modelling. From micromechanics, we show that on a boundary interaction stress components normal to the surface should vanish. These interaction stresses are at the origin of nonlocality and therefore the material response of points located on the boundary should be local. Then we discuss a tentative modification of the classical non local damage model aimed at accounting for this effect due to existing boundaries and also boundaries that arise from crack propagation. One-dimensional computations show that the profiles of damage are quite different compared to those obtained with the original formulation. The region in which damage is equal to 1 is small. The modified model performs better at complete failure, with a consistent description of discontinuity of the displacement field after failure.*

*RÉSUMÉ. Cette contribution a pour objet principal d'étudier les effets de surface dans une formulation non locale de l'endommagement. A partir d'arguments micromécaniques, nous montrons que sur la surface d'un solide et selon la normale à cette surface, les effets non locaux disparaissent. Une modification du modèle d'endommagement non local scalaire est proposée pour tenir compte des effets de surface et des effets liés à l'apparition de nouvelles surfaces (fissures). Les résultats sur un calcul unidimensionnel montrent que le modèle modifié permet de mieux représenter la rupture complète de la barre et l'apparition d'une discontinuité dans le champs de déplacement calculé.*

*KEYWORDS: damage; non locality; boundary effects; interactions*

*MOTS-CLÉS : endommagement; endommagement non local; effets de surface; interactions*

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## 1. Introduction

Most failure models for strain softening materials involve non locality. Whether non locality is introduced in an integral or in a gradient format, an internal length is added to the material description. Such constitutive relations provide consistent continuum models for progressive cracking in quasibrittle materials (see e.g. (Bažant *et al.*, 2002)). In those materials at least, non locality finds its origin in the interaction between growing defects in the course of failure. These interactions may be approximated following the superposition scheme due to Kachanov (Kachanov, 1987) for instance and folded into micromechanical damage based models (see e.g. Refs. (Bažant, 1994, Pijaudier-Cabot *et al.*, 1990)).

Nearby the boundary of the solid, interactions between defects are expected to be different compared to those observed in the bulk material. It has been stated, from qualitative arguments, that on the boundary of a solid the material response should be local (Krayani *et al.*, 2009). In the first part of this paper, we provide a derivation of this result. Krayani and co-workers have restricted their discussion to existing boundaries. Surfaces generated during cracking are evolving boundaries for which the same effect is expected. Cracks should modify the nonlocal interaction and induce some shielding effect: the interaction between two points located apart from a crack should not exist. This means also that non locality, to some extent, should depend on the state of damage in the material and, for instance, that the internal length entering in nonlocal models should evolve in the course of failure.

Boundary effects on fracture properties have been investigated, e.g. by (Duan *et al.*, 2003, Duan *et al.*, 2006). Grassl and Jirasek (Grassl *et al.*, 2006) investigated boundary effects on the fracture energy within a nonlocal continuum description. More precisely, they looked at the notch effect and the way it was treated in the averaging process. Surprisingly, the best results they could obtain was in the case where the notch is represented by a material which is totally damaged. For material points close to the notch, the strain in this damage zone is so large that other contributions in the averaging process become negligible. Boundary effects cannot, however, be treated by removing boundaries and placing fictitious damaged material outside the boundaries of the solid. From a practical and computational point of view, adding material where there is none is cumbersome. Furthermore, as we will see, this is not consistent with results from micromechanics.

Capturing existing and evolving boundaries in nonlocal model is still a pending issue. In integral models, the weight function involved in the non local average is chopped off and normalized (Pijaudier-Cabot *et al.*, 1987). In gradient enhanced models, the free boundary condition on the non local variable is the same as the condition that would be induced by an axis of symmetry (no normal flux of the regularised variable). Therefore, the non local interactions nearby a boundary of the solid are the same as the non local interactions that would be observed nearby an axis of symmetry. In damage models with nonlocal displacements, the local displacements are taken equal to the nonlocal displacements at the boundary (Rodriguez-Ferran *et al.*, 2005). This

condition is rather arbitrary but it is the only case where it is consistent with the assertion of locality of the material response on the boundary. Krayani and co-workers (Krayani *et al.*, 2009) proposed some modifications of the integral non local damage model aimed at accounting for the locality of the material response on the boundary. In order to evaluate the pertinence of the modification of the non local model numerical simulations of size effect were considered. The issue was to analyse size effect in specimen geometries where the crack initiates from a notch or from a flat surface (unnotched specimens). It was demonstrated that the original non local model was not capable to describe size effect in both cases with a single set of material parameters. On the contrary, the modified model which accounts for a local behaviour at the surface of the specimen (where the crack initiates in unnotched specimens) provided a more consistent description of size effect in both geometries.

The purpose of this paper is twofold: first we provide some justification based on micromechanics of interacting cracks to the fact that the constitutive model should be local at the boundary of the solid. Second we consider how existing and emerging boundary effects (crack surfaces) can be folded into a new nonlocal damage formulation. A crack is composed of two free surfaces facing each other. In a continuum damage setting, a crack is represented by a band of intense damage. When damage is almost maximum, close to 1 at material failure reproducing macro-crack formation, a local response should be recovered and non local interactions should vanish. A one dimensional example illustrates the capabilities of this new model to describe progressive failure and also complete failure with a view to the estimate of crack opening.

## 2. Interactions in an elastic solid with micro-cracks

In this section, we are going to look at the material at a scale where the microstructure can be described in details. We will discuss the case of a damageable material, viewed as an elastic material which contains a distribution of defects. The purpose is to exhibit the various interactions which develops at this level, between cracks and between cracks and the boundary of the solid. We are going to discuss the case where the distance between the micro-cracks is not very small compared to their length (of the same order of magnitude). Under this assumption, interaction is the same as if the cracks were two voids. The spatial distribution of the interaction forces is the same, except that the prefactor may differ between cracks and voids (Bažant, 1994, Pijaudier-Cabot *et al.*, 1991). Thus we will speak about cracks or voids indifferently.

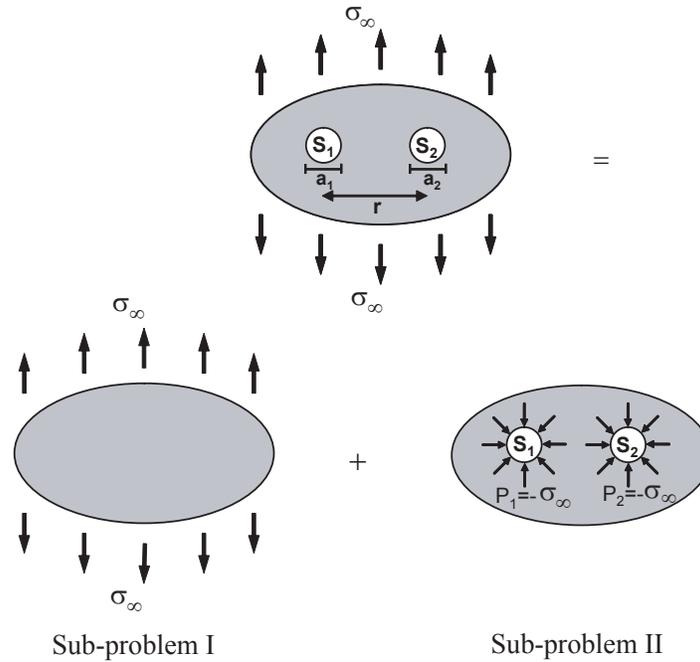
### 2.1. Infinite body

Let us consider for the sake of simplicity the case of an elastic material containing two voids and subjected to a remote stress field  $\sigma_\infty$  (Figure 1). The two voids denoted as  $S_1$  and  $S_2$  are of diameter  $a_1$  and  $a_2$  respectively and the distance between their center is denoted as  $r$ . We may compute the state of stress in the solid, accounting for the presence of the voids and their interaction, according to a superposition scheme

that is similar to the technique devised by Kachanov for crack interaction (Kachanov, 1987, Fond *et al.*, 1995). We decompose this problem into two sub-problems:

– Sub-problem I: the solid is considered without any void and loaded by the remote boundary condition corresponding to  $\sigma_\infty$ .

– Sub-problem II: the remote traction  $\sigma_\infty$  is transformed into distributed forces  $\vec{P}_1 = \vec{P}_2 = -\sigma_\infty \cdot \vec{n}$  acting inside each void whose inner surface is defined by the normal vector  $\vec{n}$ .

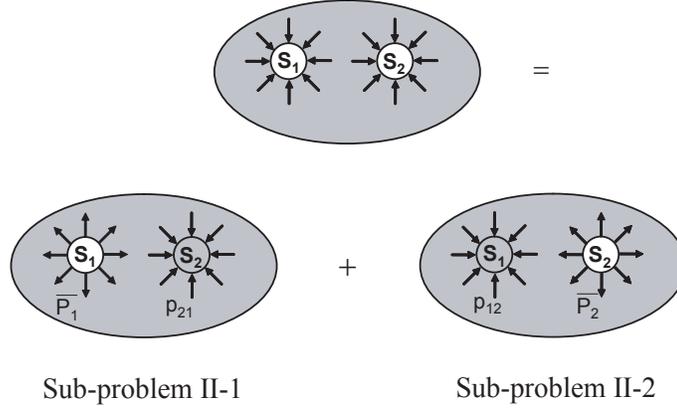


**Figure 1.** Infinite elastic body containing two voids.

Consider now sub-problem II. Again, we can apply the principle of superposition (Figure 2), in order to compute the interaction stress field due to the presence of the voids.

In sub-problem II-1, the void  $S_1$  is considered alone loaded by a force distribution  $\vec{p}_1$  on its inner surface. These surface forces generate on the imaginary contour of void  $S_2$  stress vectors denoted as  $\vec{p}_{21}$ . Similar considerations are performed for void  $S_2$  in sub-problem II-2. Superposition of these sub-problems II yields:

$$\begin{aligned}\vec{P}_1 &= \vec{p}_1 + \vec{p}_{12} \\ \vec{P}_2 &= \vec{p}_{21} + \vec{p}_1\end{aligned}\quad [1]$$



**Figure 2.** Superposition scheme in sub-problem II.

The difficulty is now that the distributions of the forces  $\vec{p}_i$  acting inside void  $S_i$  and of the influence pressure  $\vec{p}_{ji}$  on the imaginary contours of void  $S_j$  are not known. Usually, these distributions are approximated. A possibility is to consider the distribution of each component to be piecewise linear on the contour of the voids (Pijaudier-Cabot *et al.*, 1991). The distribution of each component can be also expanded into Fourier series with constant and periodic terms (Kachanov, 1987). The level of approximation required for a proper representation depends on the complexity of the interactions.

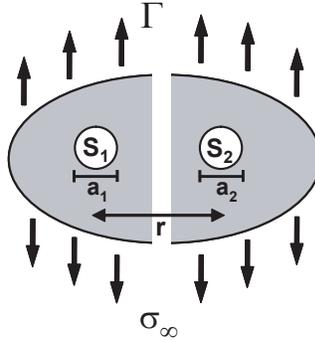
The resulting state of stress (stress tensor) in the body at point  $x$  in the infinite solid is the sum of the remote stress, and the interaction stresses due to the force distributions  $\vec{p}_i$  inside void  $S_i$  denoted as  $\sigma(x)_{xi}$ :

$$\sigma(x) = \sigma_\infty + \sum_{i=1}^N \sigma(x)_{xi} \quad [2]$$

where we have extended the results to the case of  $N$  voids, each void  $i$  being located at point  $x_i$ . In the very simplistic case of an elastic medium subjected to a remote isotropic stress containing two weakly interacting voids, the states of stress  $\sigma(x)_{xi}$  are proportional to the remote stress field and depend on the ratio of the diameter of the void to the distance to the void  $a_i/r(x, x_i)$  (Pijaudier-Cabot *et al.*, 2004).

## 2.2. Semi-infinite body

The same infinite elastic body is now cut into two pieces (Figure 3). The two voids have the same radius and are symmetrically located with respect to this new boundary. We shall be interested in the body containing void  $S_1$  only. The objective is the same as in the previous problem - we want to compute the elastic strain in the body - but now we have to deal with a free boundary  $\Gamma$ .



**Figure 3.** *Semi-infinite body with a single inclusion.*

In sub-problem I, the solid without void  $S_1$  is subjected to the remote stress field. The corresponding distribution of stress satisfies the boundary conditions on the free surface  $\Gamma$ :

$$\sigma_\infty \cdot \vec{n} = \vec{0} \quad [3]$$

In the new sub-problem II, the interaction between the void and the free surface is computed. A formulation that is similar to the previous one in section 2.1 may be followed. The surface forces  $\vec{p}_1$  acting on the inner surface of void  $S_1$  generate on  $\Gamma$  surface forces denoted as  $\vec{p}_{\Gamma,1}$ . Surface forces acting on  $\Gamma$  are introduced and they generate surface forces  $\vec{p}_{1,\Gamma}$  on the inner contour of  $S_1$ . Superposition yields:

$$\begin{aligned} \vec{P}_1 &= \vec{p}_1 + \vec{p}_{1,\Gamma} \\ \vec{0} &= \vec{p}_{\Gamma,1} + \vec{p}_\Gamma \end{aligned} \quad [4]$$

Compared to Eq. (1-b), the left handside term in Eq. (4-b) is zero because of the condition of free surface on  $\Gamma$ . The resulting interaction stress vectors on this surface should vanish.

Within the context of an approximate interaction scheme, the introduction of the surface forces on  $\Gamma$  may not be necessary and a new sub-problem II may be considered: we imagine that the two semi-infinite pieces in Fig. (3) containing voids  $S_1$  and

$S_2$  are glued again. Inside void  $S_2$  the pressure distribution is known, it is  $-\vec{p}_1$ , the opposite of the pressure inside void  $S_1$ . applied. Equation (1-a) becomes:

$$\vec{P}_1 = \vec{p}_1 + \vec{p}_{12} \quad [5]$$

$\vec{p}_{12}$  is proportional to  $-\vec{p}_1$  because  $-\vec{p}_1$  is applied inside void  $S_2$ . Since  $\vec{P}_1$  is known from sub-problem I, the distribution of forces  $\vec{p}_1$  inside void  $S_1$  is the solution of this equation. The stress distribution at any point  $x$  due to the distribution of forces inside the two voids is

$$\sigma(x) = \sigma(x)_{x1} + \sigma(x)_{x2} \quad [6]$$

Because of the symmetry of the problem and because void  $S_2$  is loaded by a distribution of forces that is the opposite of that in void  $S_1$ , we have on the free surface  $\Gamma$ :

$$\vec{n} \cdot (\sigma(x)_{x1} + \sigma(x)_{x2}) \cdot \vec{n} = 0 \quad [7]$$

The normal component of the resulting interaction stress vector vanishes. The sum of the tangential component over  $\Gamma$ , assumed to be infinitely long, vanishes too (this is for the stress fields due to the voids taken separately in fact). Therefore, the conditions of free boundary are verified pointwise on  $\Gamma$  for the normal component, and in an average form for the tangential component.

In this new sub-problem II, an infinite body is considered although the aim is the stress distribution in a semi-infinite body. The stress field solution of this sub-problem is such that surface forces cancel on the imaginary location of the free surface (for the shear component, the average over the surface cancels). In return, it generates a specific state of stress, that is due to the interaction between voids  $S_1$  and  $S_2$ , or equivalently between the boundary  $\Gamma$  and the defect  $S_1$ .

In the case of a semi-infinite body containing  $N$  voids, the resulting state of stress (stress tensor) in the body at point  $x$  is the sum of the remote stress, the interaction stresses due to the force distributions  $\vec{p}_{i1}$  inside void  $S_i$  denoted as  $\sigma(x)_{xi1}$ , and the interaction stresses due to the force distributions  $\vec{p}_{i2}$  inside the symmetric of void  $S_i$  with respect to the boundary of the solid denoted as  $\sigma(x)_{xi2}$ :

$$\sigma(x) = \sigma_\infty + \sum_{i=1}^N \sigma(x)_{xi1} + \sum_{i=1}^N \sigma(x)_{xi2} \quad [8]$$

Obviously, the difficulties that are involved in the practical calculation of the distribution of surface forces inside voids  $S_i$  (and on the free surface) remains the same as for the infinite body and the same approximation techniques may be devised. It is important to notice, however, that whatever the approximation of the distributions of forces  $\vec{p}_i$ , Eq. (7) still holds and that the condition of free surface on  $\Gamma$  is still satisfied (in an average format for the tangential component).

The same type of reasoning can be extended without difficulties to any type of boundary condition at the limit of the solid. Whatever the approximation of the surface forces in the interaction schemes, the interaction stress vectors should vanish on the free boundary of a solid. Far from this boundary, the state of stress induced by the surface forces  $\vec{p}_\Gamma$  vanishes and the influence of the free boundary disappears. *If non-locality of the constitutive response is due to the presence of these interaction terms, nonlocality vanishes on the free surface, in the normal direction to this surface.*

### 3. Upscaling and consequences in continuum modelling

The equivalent homogeneous material which is the macroscopic counterpart of the above solid with micro-cracks is a continuum with an overall stiffness which is a function of the density, size and shape of the micro-cracks (see e.g. (Budiansky *et al.*, 1976)). The present issue is not to define accurately what should be the function relating the distribution of micro-cracks to the overall stiffness but rather *how the variable which defines the growth of damage in the continuum model can be obtained from upscaling techniques*. As opposed to standard homogenisation theories, we are not looking at a frozen state of micro-cracking and at the corresponding continuum state-related quantities but at the evolution problem and therefore at evolution-related quantities. Still, it is important to see how state-related quantities are defined because most probably it is a combination of those variables which will control damage growth in the end.

As suggested in the constitutive relationship proposed by Bazant, (Bažant, 1994), the interaction stresses computed in the previous section are at the origin of non locality in a continuum description. Another framework based on a generalisation of Hashin-Shtrikman variational principles (Hashin *et al.*, 1962) was discussed by Drugan and co-workers (Drugan *et al.*, 1996, Monetto *et al.*, 1962) and it is quite instructive as it discusses RVE sizes and nonlocal effects.

In the case of a two phase composite with isotropic and statistically uniform distribution of phases, Drugan and Willis arrived to a nonlocal constitutive relation in the elastic regime relating the average stress  $\langle \sigma \rangle$  to the average strain  $\langle \epsilon \rangle$ :

$$\langle \sigma \rangle = \hat{L} : \langle \epsilon \rangle - \hat{T} : \nabla^2 (\langle \epsilon \rangle) \quad [9]$$

where the average strain  $\langle \epsilon \rangle$  is defined as:

$$\langle \epsilon \rangle = \epsilon_0 + \int_{\Omega} \Gamma(\mathbf{x}, \xi) d\xi \quad [10]$$

Function  $\Gamma(\mathbf{x}, \xi)$  is a measure of the local interactions derived with the help of Green's functions and  $\epsilon_0$  is the strain in the homogeneous comparison solid (equal to the strain in the matrix if inclusions are very dilute). The above equation is very similar to Eq. (2), except that it is expressed in term of strain rather than stress. Since the material is elastic, the difference is not really fundamental.

In the approach by Drugan (Drugan *et al.*, 1996), there is an evaluation of the size of the representative volume element (RVE) in the homogenised material. The size of the RVE is such that the second order gradient appearing in Eq. (9) becomes negligible. In other words, the constitutive equations become local, relating the average stress to the average strain. This definition of the RVE was also used by Delaplace and co-workers (Delaplace *et al.*, 1996) in order to compute a correlation length in a lattice model. According to Drugan and Willis, the size of the RVE for a material containing voids with a volume fraction less than 0.33 is about two times the diameter of the void. As pointed out in the same paper, this size of RVE is much less than what is commonly expected in heterogeneous materials. A ratio of five to ten times the size of the voids or cracks is usually used for defining the size of the RVE needed in order to average in a statistical sense the fluctuations of the stress or strain. Hence, the stress-strain relation may be defined locally as nonlocal second order gradient terms are negligible but at the same size of RVE, averages of strains and stresses may not be statistically representative.

We expect that the variable which controls the growth of damage be a function of the average stress or the average strain computed according to Eq. (10). The issue is now whether averaging results into a nonlocal quantity or not at the scale defined by the size of the RVE where constitutive relations become local.

There is at least one instance where the average strain is local whatever the interaction terms, it is the case of a uniform distribution of voids. When the distribution of micro-cracks remains homogeneous, the integral of the interaction terms in Eq. (10) becomes independent of  $\mathbf{x}$ :

$$\langle \epsilon(\mathbf{x}) \rangle = \epsilon_0(\mathbf{x}) + \int_{\Omega} \Gamma(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi} = \epsilon_{\infty}(\mathbf{x}) + \int_{\Omega} \Gamma(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad [11]$$

Interactions induce a corrective term which is homogeneous over the solid containing micro-cracks. Obviously, the average of the strain becomes local in this case, whatever the magnitude of the interactions. We might expect also that when the second order gradient of the average variable remains small, nonlocal effect remain negligible. For this, variation of the strain and of the interactions outside the RVE should be negligible:

$$\langle \epsilon(\mathbf{x}) \rangle = \epsilon_0(\mathbf{x}) + \int_{\Omega} \Gamma(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi} = \epsilon_{\infty}(\mathbf{x}) + \int_{RVE} \Gamma(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi} \quad [12]$$

or

$$\int_{\Omega} \Gamma(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi} = \int_{RVE} \Gamma(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi} \quad [13]$$

Equation (13) is a sort of insulation condition: at the center  $\mathbf{x}$  of a RVE, a variation of the interactions due to micro-crack growth outside the RVE may not be felt or be negligible.

The size of the RVE is about two times the size of the voids or cracks. If the spacing between the micro-cracks/voids is of the same order as the micro-crack length or the

void radii, interactions cannot be neglected. According to Xueli (Xueli *et al.*, 1999), interactions between two voids become sensitive when their relative distance is less than twice their diameter. This corresponds to volume fractions of voids higher than 0.18 approximately. For volume fraction close to 0.33, interactions induce a change of the hoop stress on the voids surface by 10 percent. Interactions are even more sensitive when cracks are considered instead of voids. According to Kachanov (Kachanov *et al.*, 1989, Gorbatikh *et al.*, 2000) the stress distribution may change by as much as 20 percent in the case of colinear cracks or stacked cracks. Interactions are much more sensitive for stacked cracks with crack densities from 0.2 to 0.3. Therefore, if a (micro-)crack next to a considered RVE grows, the interactions between micro-cracks inside and outside the RVE will change, more specifically the influence pressure quantities  $\vec{p}_{ij}$  computed in section 2 will change. Hence, the average strain in RVE centered at point  $x$  should change although the configuration of micro-cracks in the RVE does not change. This is clearly non local effect which is getting stronger as micro-cracking develops. To summarise, as stated in (Kachanov, 1994), interactions may vanish when considering the deterioration of the elastic stiffness but field variables such as stresses or strain, and local stress intensity factors which control the process of failure are still sensitive to these interactions. This provides a justification for introducing a non local variable in the equation governing the growth of damage.

Another point of view to the same problem is that the interaction between voids or cracks may not be necessarily symmetric. This is important when strong gradients of the crack/void density is encountered. The influence pressure quantities  $\vec{p}_{ij}$  computed in the previous section are not symmetric.  $\vec{p}_{ij} \neq \vec{p}_{ji}$  if the micro-crack  $i$  is not the same as micro-crack  $j$  or if the remote state of stress in the solid without cracks or voids is non homogeneous. A small micro-crack sitting near the crack tip of a large crack is influenced by this one, whereas the large crack is not influenced by the small one a lot. In the equivalent homogeneous material, the conditions of growth of damage of a material point sitting nearby the tip of a damage zone should feel the damage zone but the damage zone may not feel the state of damage of a material point outside it as much. This is typical of a non local effect. The condition of damage growth at a material point sitting in a homogeneous distribution of damage should not be the same as the conditions of damage growth of the same material point sitting nearby a zone that is completely damaged. This is also the case of the interactions between a free boundary and a micro-crack. A free boundary may be viewed as a large crack face and results into a strong gradient in the distribution of micro-cracking.

Crack interaction at the microstructural scale results into non locality at the macro scale in the presence of strong gradients which does not vanish upon upscaling. This non locality vanishes only in three cases: (i) when the distribution of micro-cracking is uniform and the solid is subjected to a homogeneous remote state of stress, local quantities are equal to non local ones; (ii) at the boundary of a solid, non locality should vanish in the direction normal to the boundary; (iii) upon a macro-crack formation, non locality should also vanish on the macro-crack faces (in their normal direction).

#### 4. Non local damage model

We are going to examine now how such results can be incorporated in a continuum formulation and extended to the case of evolving boundaries, macro-crack surfaces described by a zone in which damage is equal to 1 according to the continuum model (Mazars *et al.*, 1996). Although the constitutive relations are discussed in a full 3D format, we will restrict applications of the modified non local model to the one-dimensional case which is simple but not entirely representative of full boundary effects (no tangential effects on the boundary). 2D and 3D generalisation are left for further developments.

##### 4.1. Isotropic (scalar) damage model

The classical stress-strain relation for this type of model reads:

$$\sigma_{ij} = (1 - D)C_{ijkl}\varepsilon_{kl} \quad [14]$$

where  $\sigma_{ij}$  and  $\varepsilon_{kl}$  are the components of the stress and strain tensors, respectively ( $i, j, k, l \in [1, 3]$ ) and  $C_{ijkl}$  are the components of the fourth-order elastic stiffness tensor. The damage variable  $D$  represents a measure of material degradation which grows from zero (undamaged material with the virgin stiffness) to one (at complete loss of integrity). The material is isotropic, with  $E$  and  $\nu$  the initial Young's modulus and Poisson's ratio respectively.

For the purpose of defining damage growth, a scalar equivalent strain  $\varepsilon_{eq}$  is introduced, which quantifies the local deformation state in the material in terms of its effect on damage. In this contribution, Mazars' definition of the equivalent strain is used (Mazars, 1984):

$$\varepsilon_{eq} = \sqrt{\sum_{i=1}^3 (\langle \varepsilon_i \rangle_+)^2} \quad [15]$$

where  $\langle \varepsilon_i \rangle_+$  are the positive principal strains. Damage growth is governed by the loading function:

$$g(\varepsilon, k) = \varepsilon_{eq}(\varepsilon) - k \quad [16]$$

$k$  equals the damage threshold  $\varepsilon_{D_0}$  initially, and during the damage process it is the largest ever reached value of  $\varepsilon_{eq}$ . The evolution of damage is governed by the Kuhn-Tucker loading-unloading condition:

$$g(\varepsilon, k) \leq 0, \quad \dot{k} \geq 0, \quad \dot{k}g(\varepsilon, k) = 0 \quad [17]$$

The damage variable  $D$  is determined as a linear combination of two damage variables  $D_t$  and  $D_c$ , that represent tensile damage and compressive damage respectively, by the

help of two coefficients  $\alpha_t$  and  $\alpha_c$  which depend on the type of stress state (Mazars, 1984):

$$D = \alpha_t D_t + \alpha_c D_c \quad [18]$$

$$D_{t,c} = 1 - \frac{1 - A_{t,c}}{\varepsilon_{eq}} + \frac{A_{t,c}}{\exp(B_{t,c}(\varepsilon_{eq} - \varepsilon_{D_0}))} \quad [19]$$

Standard values of the model parameters in the damage have been given in Ref. (Mazars, 1984).

#### 4.2. Original non local formulation

In the integral-type non local damage models, the local equivalent strain is replaced by its weighted average:

$$\bar{\varepsilon}_{eq}(\mathbf{x}) = \int_{\Omega} \Psi(\mathbf{x}, \boldsymbol{\xi}) \varepsilon_{eq}(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad [20]$$

with  $\Omega$  the volume of the structure and  $\Psi(\mathbf{x}, \boldsymbol{\xi})$  the weight function. It is required that the non local operator does not alter the uniform field, which means that the weight function must satisfy the condition:

$$\int_{\Omega} \Psi(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi} = 1 \quad \forall \mathbf{x} \in \Omega \quad [21]$$

For this reason, the weight function is recast in the following form (Pijaudier-Cabot *et al.*, 1987):

$$\Psi(\mathbf{x}, \boldsymbol{\xi}) = \frac{\Psi_0(\mathbf{x} - \boldsymbol{\xi})}{\Omega_r(\mathbf{x})} \quad \text{with} \quad \Omega_r(\mathbf{x}) = \int_{\Omega} \Psi_0(\mathbf{x} - \boldsymbol{\xi}) d\boldsymbol{\xi} \quad [22]$$

where  $\Omega_r(\mathbf{x})$  is a representative volume and  $\Psi_0(\mathbf{x} - \boldsymbol{\xi})$  is the basic non local weight function which is often taken as the polynomial bell-shaped function (Bažant *et al.*, 2002), or here as the Gauss distribution function:

$$\Psi_0(\mathbf{x} - \boldsymbol{\xi}) = \exp\left(-\frac{4\|\mathbf{x} - \boldsymbol{\xi}\|^2}{l_c^2}\right) \quad [23]$$

$l_c$  is the internal length of the non local continuum. Preserving the uniform field in the vicinity of the boundary makes the averaging in Eq. (22) not symmetric with respect to its arguments  $\mathbf{x}$  and  $\boldsymbol{\xi}$ . This lack of symmetry leads to the non-symmetry of the tangent operator (Bažant *et al.*, 1988, Pijaudier-Cabot, 1995, Jirasek *et al.*, 2002). A symmetric non local formulation exists also (Borino *et al.*, 2003).

### 4.3. Modified non local formulation

The goals of the modified non local formulation are twofold: (1) at the boundary of the solid, non locality should vanish; (2) when a crack has formed - i.e. when a band of damage close to 1 has been created, the constitutive relations should again be local in this band. By comparison to the modifications discussed in (Krayani *et al.*, 2009) which dealt with a boundary effect only, here we try to account also for boundary effects occurring when a new free surface has formed in the process of cracking. We are going to rely on a simple argument: in the process of cracking, the quantity of information that can be transmitted between two neighbors depends on the state of damage. The argument is rather similar to that of Desmorat and Gatuingt (Desmorat *et al.*, 2007). Some internal time is introduced in the constitutive relation which induces a decrease of the internal length upon damage growth. Here, the analogy with the propagation of wave is used but the weight function is regarded as an attenuation function in the transmission of information from one point to its neighbor.

Let us denote as  $s$  the distance appearing in Eq. (23), defined as  $\|\mathbf{x} - \boldsymbol{\xi}\|$ . This distance is now modified according to the following incremental relation:

$$du = \frac{ds}{\gamma(\boldsymbol{\xi})} = \frac{\|\mathbf{x} - d\boldsymbol{\xi}\|}{\gamma(\boldsymbol{\xi})} \quad [24]$$

where  $\gamma(\boldsymbol{\xi})$  is a function of the state of damage at point of coordinate  $\boldsymbol{\xi}$ . In the computation of the average centered at point  $\mathbf{x}$ , we use now the distance  $u$  instead of the distance  $s$ :

$$u = \int_{\Omega} \frac{ds}{\gamma(\boldsymbol{\xi})} \quad [25]$$

In this remapping of the neighborhood around point of coordinate  $\mathbf{x}$ , each point of coordinate  $\boldsymbol{\xi}$  is now defined by the new coordinate  $u$ . In a spherical coordinate system centered at point  $\mathbf{x}$ , the distance is defined by the above equation and the two angles are kept the same, they are invariant through the mapping function. The weighted average in Eq. (20) becomes:

$$\bar{\varepsilon}_{eq}(\mathbf{x}) = \int_{\Omega} \Psi(u) \varepsilon_{eq}(\mathbf{x} + \mathbf{u}) du \quad [26]$$

$\gamma(\boldsymbol{\xi})$  should be equal to 1 when the material is not damaged at point  $\boldsymbol{\xi}$ , and it decreases when damage grows. As a consequence, the distance between two close neighbors will be increased as damage grows and in the weighted average process, the nonlocal effect of point  $\boldsymbol{\xi}$  will decrease.

Function  $\gamma(\boldsymbol{\xi})$  remains to be defined completely. We use for this an analogy with the attenuation during the propagation of waves. In the non local average defined in Eq. (20) the weight function  $\Psi(\mathbf{x} - \boldsymbol{\xi})$  is seen as the attenuation applied to a wave of amplitude  $\varepsilon_{eq}$  generated at point  $\boldsymbol{\xi}$  and propagating toward point  $\mathbf{x}$ .  $\Psi(\mathbf{x} - \boldsymbol{\xi}) \cdot \varepsilon_{eq}$  is the interaction of point  $\boldsymbol{\xi}$  on point  $\mathbf{x}$ . It is attenuated as the information is propagated in the solid. We may now consider that the wave speed depends on the state

of damage: the time needed for the interaction effect to cover distance  $s$  is going to increase if the material on the path of propagation of the wave is getting damaged. In a one-dimensionnal setting, the wave speed  $c = \sqrt{E/\rho}$  becomes  $c_d = \sqrt{(1-D) \cdot E/\rho}$  where  $c_d$  is the velocity of the damaged material. This increase of time for the information to be transmitted in the damaged medium between the two points  $\xi$  and  $x$  is converted into a fictitious increase of distance between these two points. The time needed for the interaction to propagate between the two points is kept the same, but since we consider now that the interaction is propagated in an undamaged medium the distance between the points is increased. When this distance is increased, the attenuation is increased accordingly. Let us consider Eq. (24) and divide both terms by the wave speed  $c$ :

$$\frac{du}{c} = \frac{ds}{\gamma(\xi) * c} = \frac{ds}{c_d} \quad [27]$$

This equation states that the time needed for an information to propagate over a distance  $du$  in an undamaged medium is the same as the time needed for an information to propagate over the length  $ds$  in the damaged medium. Upon damage, the distance between these two points is increased, which is equivalent to a slower wave propagation and consequently, it is more attenuated than in the case where damage is not observed. From this equation we have:

$$\gamma(\xi) = \frac{c_d}{c} \quad [28]$$

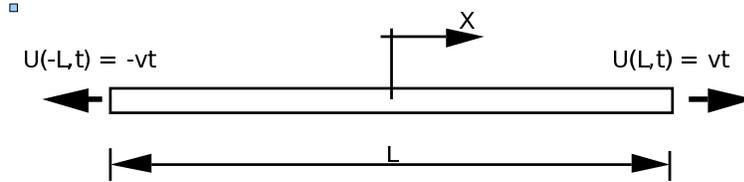
This qualitative reasoning implies that damage enters in the function  $\gamma$ . The difficulty is that the formulation becomes implicit. In order to keep the simplicity of the approach, we substitute to the non local value of damage the local expression and we take:

$$\gamma(\xi) = (1 - F(\varepsilon_{eq}(\xi)))^{\frac{1}{2}} \quad [29]$$

where  $F(\varepsilon_{eq})$  is the function defined in Eq. (19).  $F(\varepsilon_{eq})$  is the local value of damage computed at the considered point. This definition of function  $\gamma$  is rather arbitrary. In fact, it fulfills the requirements stated above: when the material is not damaged, this function is equal to 1 and it becomes equal to zero if the material is totally damaged locally. At this stage, the incremental value of the modified distance between two points becomes infinite and the nonlocal interaction in between them vanishes. The exponent  $1/2$  has been chosen by analogy to wave propagation. This new definition of the nonlocal averaging needs also to fulfill the conditions on the free boundary demonstrated in the previous section. This can be performed by considering that on a free boundary local damage is set equal to 1. According to Eq. (24), a close neighbor to a point located on the boundary of the solid is located at a distance  $du$  which becomes infinite. Consequently, the nonlocal effect of this neighbor to the nonlocal average centered at the boundary is equal to zero.

## 5. One-dimensional example

In order to compare the original and the modified nonlocal formulation, we are going to use the simple one-dimensional example devised in (Pijaudier-Cabot *et al.*, 1987). A bar of length  $2L$ , in which two constant strain waves converge toward its center is considered. The amplitude of the wave is 0.7 times the deformation at the peak load in tension. When the two waves meet at the center, the strain amplitude is doubled, the material enters the softening regime suddenly and failure occurs.

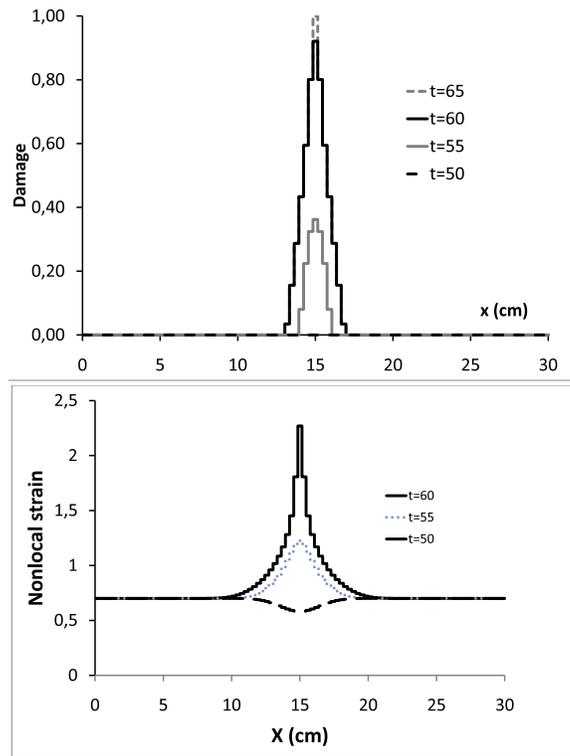


**Figure 4.** Principle of the one dimensional computation.

The bar length is taken equal to 30 cm. The parameters used in this example are: the volumic mass  $\rho = 1 \text{ kg/m}^3$ , the Young's modulus  $E = 1 \text{ MPa}$  and the velocity boundary condition  $v = 0.7 \text{ cm/s}$  applied at the bar ends. The other model parameters are  $A_t = 1$ ,  $B_t = 2$ ,  $\varepsilon_{D_0} = 1$  and the internal length  $l_c$  is 2 cm (there is no damage in compression). A fixed mesh of 99 constant strain elements is used. Time integration is performed according to an explicit, central difference scheme. The time step is  $\Delta t = 1 \text{ s}$ .

Figures 5 present the evolution with time of the profiles of damage and non local strain. The two waves meet at the center of the bar at time  $t = 50 \text{ s}$ . Note that damage develops over a band of finite – non zero – width. Compared to computations with the original damage model shown in Fig. 6, the profile of damage is almost triangular instead of being almost rectangular, forming a band of damage equal to 1 according to the original formulation. This difference is due to the modification of the weight function as damage develops. This is illustrated in Figs. 7. The weight function centered in the middle element where complete failure is expected to occur (e.g. where damage is equal to 1) shrinks progressively as damage develops. When damage is equal to 1 in this element, it is a Dirac delta function and the material response becomes local. When damage is equal to 1, it is as if the bar would be cut into two pieces. At this point, the material response is local and this is in agreement with micromechanics of crack interaction as demonstrated in this paper.

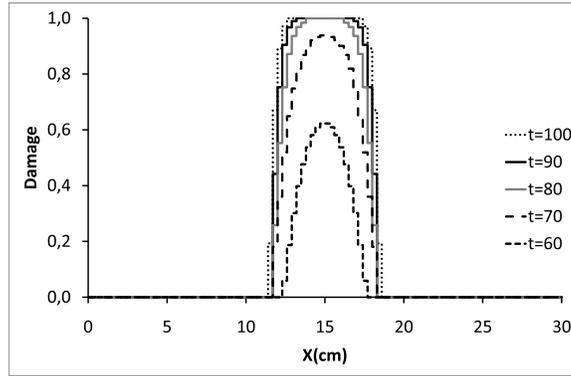
In neighboring elements, however, the weight function evolves differently. We have computed the weight function nearby the element in which failure occurs, two elements farther on the left. Figure (7) shows that the weight becomes cut at the center of the bar. Information coming from material points located behind the point at which failure occurs is screened by damage. This is again a difference with the original non-local damage formulation in which nonlocal interactions (weights) is transmitted even



**Figure 5.** Evolution of the damage (top) and nonlocal strain (bottom) profiles with time (in s) over the bar.

across a macro-crack. We have also checked in Fig. (8) that the distribution of damage is not subject to spurious mesh dependency. For the finite element meshes used which are already quite refined, the largest element size is smaller than the internal length used in the computations, the profiles of damage are almost the same. Convergence of the damage profiles with respect to mesh refinement means also that the energy dissipated at failure is a constant. It is the sum of the energy dissipated due to damage at each material point in the damage band.

The damage profiles are not dependent on the finite element size. The issue is now whether the strain distribution should be also independent of the finite element size or not. Before complete failure, this result is expected. Without this property, the damage band which is a function of the maximum nonlocal strain recorded in each finite element over the history of loading would not be expected to be mesh independent. When damage is equal to one in the middle element, convergence of the strain with respect to the size of this element may be questioned.



**Figure 6.** Evolution of the damage profile with time (in s) over the bar according to the original damage model.

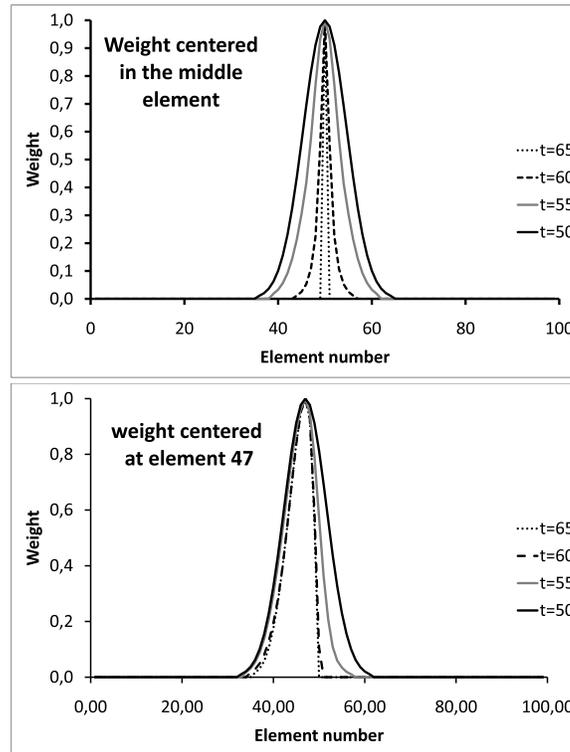
Indeed, in the finite element located in the middle of the mesh, where complete failure occurs, the stress is vanishing and the relative displacement at the extremity of the finite element should be equal to the crack opening. According to the finite element discretisation however, there is no displacement and strain discontinuity and the crack opening is smeared over the element. Let us denote as  $h$  the length of this finite element, the crack opening  $[u]$  is:

$$[u] = \epsilon * h \quad [30]$$

If the finite element calculation has converged with respect to mesh refinement, upon complete failure the crack opening should be independent of the element size and therefore that the strain in the element located in the middle of the mesh evolves as a function of  $h^{-1}$ . We have checked this in Fig. (9). According to the modified nonlocal model, the maximum strain is indeed a power law of the finite element size. Exponent  $-1$  is recovered. At the same time, the crack opening displacement is measured. We obtain  $[u] = 4,81cm$ . Interestingly, the same crack opening displacement, computed according to the technique proposed by Dufour and co-workers (Dufour *et al.*, 2008) is rather close  $[u] = 5,07cm$ . For this calculation, we have chosen a constant weight function for the computation of the estimate of the crack opening. It is important to remark that the original nonlocal formulation is far from exhibiting the same property. The maximum strain is constant and therefore, in this simple application it may not provide the displacement discontinuity properly.

## 6. Conclusions

Boundary effects are difficult issues in nonlocal models. In either integral or gradient formulations, boundary conditions are rather arbitrary. In this paper, we have

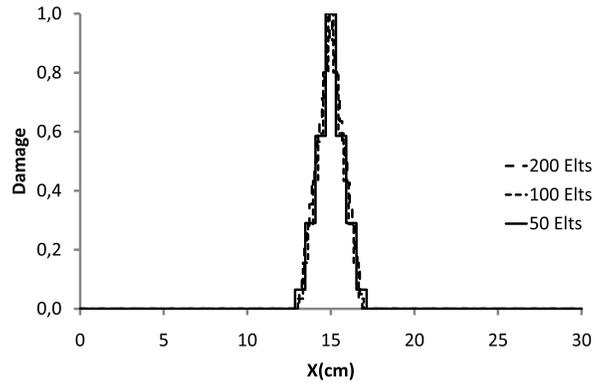


**Figure 7.** Evolution of weight function with damage: weight centered in the middle element (top), weight centered near the middle element (bottom).

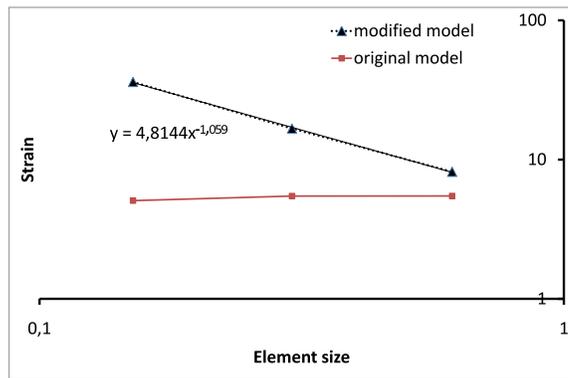
demonstrated first that on a free boundary non local interactions should vanish. Some qualitative argument had already been discussed in the past (Krayani *et al.*, 2009), here we provide a more rigorous mathematical derivation.

In the course of failure, when a macro-crack is formed, new boundary surfaces are appearing. This should also be taken into account in non local formulation, with the same requirements as for an initial boundary in the solid. Grassl and Jirasek (Grassl *et al.*, 2006) showed that these boundaries had an influence on the energy of fracture. We have proposed here a prototype damage model that accounts for the progressive shielding effect induced by a crack appearing in the material. This is achieved by a remapping of the non local averaging, in the same spirit as in (Krayani *et al.*, 2009), but with a different mathematical formulation.

One dimensionnal finite element calculations show that the modified non local model still describes failure with a finite non zero fracture energy, that the damage profiles are triangular and not rectangular as observed in the original formulation,



**Figure 8.** Evolution of the damage profiles for different meshes.



**Figure 9.** Evolution of the maximum strain in the middle element with the finite element size.

and that at complete failure, the model is capable of approaching a discontinuous formulation, whereas the original formulation is not capable of reaching this limit case.

This modified non local model should be seen as a first step toward a comprehensive modelling of boundaries and interface effect in nonlocal damage models. A two dimensional implementation is required at least in order to check the capabilities of this new model. For instance, a description of size effect that is consistent with Bazant's universal size effect law for notched and unnotched specimens needs to be checked (Krayani *et al.*, 2009).

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