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Noise prevents collapse of Vlasov–Poisson point charges

F. Delarue, F. Flandoli, D. Vincenzi

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Abstract

We elucidate the effect of noise on the dynamics of N point charges in a Vlasov-Poisson model driven by a singular bounded interaction force. We show that a too simple noise does not impact the structure inherited from the deterministic case and, in particular, cannot prevent the emergence of coalescence. Inspired by the theory of random transport in passive scalars, we identify a class of random fields which generate random pulses that are chaotic enough to disorganize the deterministic structure and prevent any collapse of the particles. We thus obtain strong unique solvability of the stochastic model for any initial configuration of different point charges. Moreover, in the case where there are exactly two particles, we implement the "vanishing noise method" for determining the continuation of the deterministic model after collapse.

1 Introduction

It is a well-known fact that white noise perturbations improve the well-posedness properties of Ordinary Differential Equations (ODEs): for instance, strong existence and uniqueness hold for equations of the form $dX_t = b(X_t) dt + \varepsilon dW_t$ with $\varepsilon \neq 0$, even if b is of L^p class only, for p greater than the dimension, see Krylov and Röckner [20]. In such a case, the noise has a twofold effect: not only Peano-type phenomena are excluded, but also singularities of b are somewhat avoided. Unfortunately, the influence of noise on pathologies of Partial Differential Equations (PDEs) is not as well understood, since the underlying deterministic structure is much more complex

and is subject to a wider variety of phenomena. A review of examples and recent results can be found in Flandoli [13]. Most of them are actually concerned with uniqueness or, more specifically, with uniqueness gained by noise. On the contrary, whether noise can prevent the emergence of singularities in PDEs is still quite obscure. A further challenging question is whether noise can select a natural candidate for the continuation of solutions after the singularities.

A well-known system in which singularities may develop explicitly is the Vlasov–Poisson equation on the line. We refer the reader to Majda and Bertozzi [27] for several examples and for an extensive discussion of the underlying stakes, including the connection with the 2D Euler equations. (See also Carmack and Majda [5], Majda, Majda and Zheng [28] and Zheng and Majda [35].) The motivation for the present study is to understand the influence of noise on such singularities.

1.1 Vlasov–Poisson equation on the line

Consider the following system in the unknown $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \ni (t, x, v) \mapsto f(t, x, v) \in \mathbb{R}$:

$$\begin{aligned} \frac{\partial f}{\partial t}(t, x, v) + v \frac{\partial f}{\partial x}(t, x, v) + E(t, x) \frac{\partial f}{\partial v}(t, x, v) &= 0, \\ \text{with } f(0, x, v) &= f_0(x, v), \\ \text{and } \rho(t, x) &= \int_{\mathbb{R}} f(t, x, v) dv, \quad E(t, x) = \int_{\mathbb{R}} F(x - y) \rho(t, y) dy, \end{aligned} \tag{1}$$

where $F(x)$ is a bounded function that is continuous everywhere except at $x = 0$ and has side limits in 0^+ and 0^- . If $F(x) = \text{sign}(x)$, Eq. (1) is the one-dimensional Vlasov–Poisson model describing the evolution of the phase-space density f of a system of electrons (in natural units, where the elementary charge and the mass of the electron are set equal to one). Following [27] and the works discussed therein, it is possible to design examples of so-called *electron-sheet* structures that persist in positive time, but collapse to one point at a certain time. Precisely, the initial density f_0 is said to be an electron sheet if it is concentrated on lines, namely if it is written as $f_0(x, v) = f_0(x) \cdot \delta(v - v_0(x))$. Examples given in [27] show that the electron-sheet structure may hold on some interval $[0, t_0)$ and then degenerate into $f(t_0, x, v) = \delta(x - x_0) \delta(v - v_0)$ at time t_0 . In such examples, the

charge is initially distributed in space with a deterministic velocity at each point, but becomes a single point charge at time t_0 . By time inversion, these solutions of (1) also provide examples of non-uniqueness, referred to as *fission*. Moreover, not only is the emergence of singularities for electron sheets an interesting phenomenon in its own, but it also gives some indication on singularities of vortex lines in 2D Euler fluids, these ones being not so explicit as in the case of charges.

A simplified version of this phenomenon is the coalescence of N point charges in finite time: in a similar manner as above, it is possible to design examples of initial conditions of the form $f_0(x, v) = \sum_{i=1}^N a_i \delta(z - z_i)$, with different $z_1, \dots, z_N \in \mathbb{R}^2$, that remain of the same form on some interval $[0, t_0)$, but for which $f(t_0, x, v) = \delta(z - z_0)$ with $z_0 \in \mathbb{R}^2$. An explicit example is given in the next section.

The main question motivating our work is the following: does the presence of noise modify these facts? A subsequent question is: what does the word “noise” stand for in this context? The simplest conceivable noisy structure consists of an additive noise plugged into the right-hand-side of the PDE. This structure, however, is both difficult to investigate and questionable from a physical point of view. A natural way to conceive the form of the random perturbation may instead be the following: when the electric charge is not totally isolated, but lives in a medium which mildly acts on it (a sort of electric bath), a random external force adds to the force given by the electric field (self-produced by the electric charge). To a first approximation, it is then reasonable to assume that the electric charge does not affect the external random field. The simplest structure modelling this situation is a Stochastic Partial Differential Equation (SPDE) of the form

$$\frac{\partial f}{\partial t}(t, x, v) + v \frac{\partial f}{\partial x}(t, x, v) + \left(E(t, x) + \varepsilon \circ \frac{dW_t}{dt} \right) \frac{\partial f}{\partial v}(t, x, v) = 0, \quad (2)$$

where W is a standard Brownian motion and where the Stratonovich integral is used; this is indeed the natural choice when passing to the limit along regular noises. Unfortunately, the noisy structure in Eq. (2) is too simple to avoid the emergence of singularities such as those described above. This is a consequence of the fact that the random field $\tilde{f}(t, x, v) = f(t, x, v + \varepsilon W_t)$ formally satisfies $[\partial \tilde{f} / \partial t] + v[\partial \tilde{f} / \partial x] + E[\partial \tilde{f} / \partial v] = 0$, and therefore if \tilde{f} is a solution of the deterministic equation that concentrates in finite time, the same holds for the solution $f(t, x, v) = \tilde{f}(t, x, v - \varepsilon W_t)$ of the stochastic

equation. The concentration point $z_0 = (x_0, v_0)$ of \tilde{f} at some time t_0 translates into the random concentration point $(x_0, v_0 + \varepsilon W_{t_0})$ of f at the same time t_0 .

To expect a positive effect of the noise, we must consider noises of a refined structure. This is exactly the strategy followed in the present paper. Here we prove that a space-dependent noise with a sufficiently complex space structure does affect the dynamics of the system and prevents charge concentration. Specifically, we consider a noisy equation of the form

$$df(t, x, v) + v \frac{\partial f}{\partial x}(t, x, v) dt + \left(E(t, x) dt + \varepsilon \sum_{k=1}^{\infty} \sigma_k(x) \circ dW_t^k \right) \frac{\partial f}{\partial v}(t, x, v) = 0, \quad (3)$$

where $((W_t^k)_{t \geq 0})_{k \geq 1}$ is a family of independent Brownian motions, and we prove that, under very general conditions on the covariance function $Q(x, y) = \sum_{k=1}^{\infty} \sigma_k(x) \sigma_k(y)$, the following result holds:

Theorem 1 *Given $f_0(x, v) = \sum_{i=1}^N a_i \delta(z - z_i)$ as initial condition, with different initial points $z_i \in \mathbb{R}^2$, $i = 1, \dots, N$, there exists a unique global solution to system (3) of the form $f(t, x, v) = \sum_{i=1}^N a_i \delta(z - z_i(t))$, where $((z_i(t))_{t \geq 0})_{1 \leq i \leq N}$ is a continuous adapted stochastic process with values in \mathbb{R}^{2N} without coalescence in \mathbb{R}^2 , i.e., with probability one, $z_i(t) \neq z_j(t)$ for all $t \geq 0$ and $1 \leq i, j \leq N$, $i \neq j$.*

The precise assumptions of Theorem 1 and the definitions used therein will be clarified in the remainder of the paper. Here, it is worth remarking that our study does not cover the case of an electron sheet, i.e. the case in which f_0 is not an atomic measure, but has an absolutely continuous marginal distribution in the space variable x . The key simplification of point charges is the reducibility to a finite-dimensional stochastic differential equation (the Lagrangian dynamics). A direct approach to electron sheets (of sufficiently general form to incorporate perturbations produced by noise) is to our knowledge not available in the literature and, furthermore, it seems to us really difficult to preserve the non-coalescent structure of point charges in the continuum limit. We nonetheless expect our result to be a first step forward in this direction. Our study indeed identifies a potentially interesting class of random perturbations, whose structure is more complex than that of the

simple noises of which one might think for such a problem. The assumption that we shall impose on $Q(x, y)$ actually guarantees that any N -tuple, with arbitrary N , of different points (z_1, \dots, z_N) in the (x, v) -space feels the original noise as random impulses that are not too coordinated one with each other. Such a propagation may be seen as a sort of mild spatial chaos produced by the noise. We emphasize that the negative example discussed above does not enjoy a similar property since the noise $(\varepsilon W_t)_{t \geq 0}$ plugged therein produces the same impulse at every space point, thus acting as a random Galilean transformation.

1.2 Non-Markovian continuation after a singularity

As mentioned above, the random perturbation introduced in Eq. (3) may provide some indications about the natural continuation of the solutions after the coalescence of two point charges. (More difficult or generic cases are not clear at this stage of our understanding of the problem.) Consider the simple example in which $F(x) = \text{sign}(x)$ and

$$f_0(x, v) = \delta(z - z_1) + \delta(z - z_2), \quad z_1 = (-1, v_0), \quad z_2 = (1, -v_0)$$

with $v_0 > 0$. As we shall discuss below, the Lagrangian dynamics corresponding to (1) consists of the system

$$\begin{aligned} \frac{dx_1}{dt}(t) &= v_1(t), & \frac{dx_2}{dt}(t) &= v_2(t) \\ \frac{dv_1}{dt}(t) &= F(x_1(t) - x_2(t)), & \frac{dv_2}{dt}(t) &= F(x_2(t) - x_1(t)), \quad t \geq 0, \end{aligned} \quad (4)$$

with $(x_1(0), v_1(0)) = (-1, v_0)$, $(x_2(0), v_2(0)) = (1, -v_0)$ as initial conditions. The functions

$$\begin{aligned} v_1^*(t) &= v_0 - t, & v_2^*(t) &= -v_0 + t, \\ x_1^*(t) &= -1 + v_0 t - \frac{t^2}{2}, & x_2^*(t) &= 1 - v_0 t + \frac{t^2}{2} \end{aligned} \quad (5)$$

solve the system for $t \in [0, v_0)$, and the limits of $x_1^*(t)$ and $x_2^*(t)$ as $t \rightarrow v_0^-$ coincide when $v_0^2 = 2$. This means that, with the choice $v_0 = \sqrt{2}$, the solutions $(x_i^*(t), v_i^*(t))$, $i = 1, 2$, converge to the same point $(0, 0)$ as $t \rightarrow v_0^-$. The origin $(0, 0)$ is thus the singular point of the Lagrangian dynamics.

On the contrary, Theorem 1 states that, for any positive level of noise ε in the noisy formulation (3), the random solutions $((x_i^\varepsilon(t), v_i^\varepsilon(t))_{t \geq 0})_{i=1,2}$ never meet, with probability one. It is then natural to investigate the behavior of the stochastic solution as $\varepsilon \rightarrow 0$. In Section 2, we shall prove, under general conditions on the covariance function Q :

Theorem 2 *Assume that $v_0 = \sqrt{2}$. Then, as $\varepsilon \rightarrow 0$, the pair process $((z_i^\varepsilon(t))_{t \geq 0})_{i=1,2}$ converges in distribution on the space $\mathcal{C}([0, \infty), \mathbb{R}^2 \times \mathbb{R}^2)$ toward*

$$\frac{1}{2} \delta((z_1^*(t), z_2^*(t))_{t \geq 0}) + \frac{1}{2} \delta((z_1^{**}(t), z_2^{**}(t))_{t \geq 0}), \quad (6)$$

where $(z_i^*(t) = (x_i^*(t), v_i^*(t)))$, for $t \geq 0$ and $i = 1, 2$, and

$$\begin{aligned} \forall t \in [0, \sqrt{2}], \quad (z_1^{**}(t), z_2^{**}(t)) &= (z_1^*(t), z_2^*(t)) \\ \forall t \in (\sqrt{2}, +\infty), (z_1^{**}(t), z_2^{**}(t)) &= (z_2^*(t), z_1^*(t)). \end{aligned}$$

Denoting by $f^\varepsilon(t, x, v) = \sum_{i=1}^2 \delta(z - z_i^\varepsilon(t))$, $t \geq 0$, the stochastic solution to (1), we can recast Theorem 2 as follows: the process $(f^\varepsilon(t, \cdot, \cdot))_{t \geq 0}$, seen as a continuous measure-valued process, converges in distribution on the space $\mathcal{C}([0, +\infty), \mathcal{M}_f(\mathbb{R}^4))$ toward the measure-valued process

$$f^0(t, x, v) = \sum_{i=1,2} \delta(z - z_i^0(t)), \quad t \geq 0,$$

where $(z_1^0(t), z_2^0(t))_{t \geq 0}$ is a stochastic continuous process admitting (6) as distribution and where $\mathcal{M}_f(\mathbb{R}^4)$ stands for the set of finite Borel measures on \mathbb{R}^4 endowed with the weak topology. (See Section 3 in [7] for more details on the convergence in distribution of measure-valued processes.)

Theorem 2 must be seen as a rule for the continuation of the solutions to the deterministic system (1) after singularity. When the particles meet, they split instantaneously, but they can do it in two different ways: (i) with probability 1/2, the trajectories meet at coalescence time and then split without crossing each other, namely each of the two trajectories keeps of constant sign before and after coalescence; (ii) with probability 1/2, the trajectories meet, cross each other and then split for ever, namely the sign of each of them changes at coalescence time exactly. This rule for continuation provides a very precise mathematical description of the repulsive effect of the interaction force F : there is no way for the particles to merge and then form a single particle with a double charge; from a physical point of view,

one might think that, when the two particles are extremely close to each other in phase space, but do not have equal position and velocity yet, any minor imperfection in the system makes them move far apart from each other because of the repulsive effect of F .

This situation sounds as a physical loss of the Markov property: just after coalescence, splitting occurs because the system keeps memory of what its state was before. Since the differential equation (4) has several solutions after coalescence time, there is indeed no mathematical way to prove that splitting occurs after coalescence by considering the state of the system at coalescence time only. More precisely, if we model the dynamics of a single particle with a double charge by a pair $(z_1^{00}(t), z_2^{00}(t))_{t \geq 0}$ in phase space, submitted to $z_1^{00}(t) = z_2^{00}(t)$ and

$$\dot{v}_1^{00}(t) = \dot{v}_2^{00}(t) = 0, \quad \dot{x}_1^{00}(t) = \dot{x}_2^{00}(t) = v_1^{00}(t) = v_2^{00}(t), \quad t \geq 0,$$

we get a non-Markovian family of solutions to (4). Indeed, when the trajectories z_1^0 and z_2^0 meet, they do not restart with the same dynamics as z_1^{00} and z_2^{00} do. We refer the reader to [10] for other mathematical examples of non-Markovian continuations.

1.3 Vlasov–Poisson type system of N particles in \mathbb{R}^d

The problem described in Section 1.1 will be treated as a particular case of the following problem in \mathbb{R}^d . Namely, we seek f from $[0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla_x f(t, x, v) + E(t, x) \cdot \nabla_v f(t, x, v) = 0,$$

$$\text{with } f(0, x, v) = f_0(x, v),$$

$$\text{and } \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv, \quad E(t, x) = \int_{\mathbb{R}^d} F(x - y) \rho(t, y) dy.$$

As detailed below, F will be assumed to be bounded everywhere on \mathbb{R}^d and locally Lipschitz continuous on any compact subset of $\mathbb{R}^d \setminus \{0\}$, the Lipschitz constant on any ring of the form $\{x \in \mathbb{R}^d : r \leq |x| \leq 1\}$ growing at most as $1/r$ as r tends to 0. In particular, F may be discontinuous at 0. A relevant case is when $F = \nabla U$ where U is a potential with a *Lipschitz point at zero*, namely U is Lipschitz continuous on \mathbb{R}^d and smooth on $\mathbb{R}^d \setminus \{0\}$.

This framework includes the example $F(x) = x/|x|$, $x \in \mathbb{R}^d$, and, as a particular case, the one-dimensional model discussed above, i.e. $F(x) = \text{sign}(x)$, $x \in \mathbb{R}$. On the contrary, the d -dimensional Poisson case, where

$$F(x) = \pm \frac{x}{|x|^d}, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad (7)$$

does not satisfy the aforementioned assumptions, and therefore falls outside this study. The case in which the sign above is “+” describes repulsive interactions and is referred to as the *electrostatic* model; the case in which the sign is “−” describes attractive interactions and is referred to as the *gravitational* model. For the electrostatic potential, developing a coalescence-free stochastic version of the Vlasov–Poisson equation is not relevant in dimension $d \geq 2$. Indeed, the deterministic system itself is free of coalescence: the system conserves energy and energy blows up when particles get close. In other words, only the gravitational model is challenging.

Assuming F to be bounded may seem rather restrictive in comparison with the existing literature on the deterministic Vlasov–Poisson equation. For example, unique solvability of the three-dimensional deterministic Vlasov–Poisson equation, with F as in (7), was investigated in Lions and Perthame [25] for a sufficiently regular initial condition f_0 . A key argument in the method consists in proving boundedness of the kinetic energy and of the force field. Nevertheless, these energy estimates fail in the setting considered here since the initial condition is written as a combination of Dirac masses. As already mentioned, the Lagrangian description of atomic-measure-valued solutions consists of a system of N interacting particles, whereas function-valued solutions are expected to model the density of the system when the number of particles is infinite. To our knowledge, the best bound for the kinetic energy and for the force field of finite interacting particle systems of Vlasov–Poisson type was established by Hauray and Jabin [17]. The main assumptions of Ref. [17] are that time is sufficiently small or the number of particles is sufficiently large and the rate of explosion of $F(x)$ in 0 is at most $|x|^\alpha$, for $\alpha < 1$. Our assumptions on the function F are more stringent; however, the extension of our study to the case studied by Hauray and Jabin [17] seems non-trivial, and is therefore left for future investigation.

The Lagrangian motion associated with the PDE is

$$\begin{aligned}\frac{dX_t}{dt} &= V_t, \\ \frac{dV_t}{dt} &= E(t, X_t) = \int_{\mathbb{R}^d} F(X_t - x') \rho(t, x') dx', \quad t \geq 0,\end{aligned}$$

where $\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$. The formal relation between (X_t, V_t) and f is

$$f(t, X_t, V_t) = \text{constant},$$

as one can readily verify by a formal differentiation in time. A particular class of distributional solutions is given by

$$f(t, x, v) = \sum_{i=1}^N a_i \delta(x - X_t^i) \delta(v - V_t^i) \quad (8)$$

where

$$\begin{aligned}\frac{dX_t^i}{dt} &= V_t^i \\ \frac{dV_t^i}{dt} &= \sum_{j \in N_i} a_j F(X_t^i - X_t^j), \quad t \geq 0, \quad 1 \leq i \leq N\end{aligned} \quad (9)$$

with N_i denoting the set of integers $N_i = \{j = 1, \dots, N : j \neq i\}$.

According to the discussion above, we shall consider the following stochastic perturbation of the Lagrangian dynamics:

$$\begin{aligned}\frac{dX_t^i}{dt} &= V_t^i \\ dV_t^i &= \sum_{j \in N_i} a_j F(X_t^i - X_t^j) dt + \sum_{k=1}^{\infty} \sigma_k(X_t^i) \circ dW_t^k, \quad t \geq 0,\end{aligned} \quad (10)$$

for $1 \leq i \leq N$, where $\sigma_k : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are Lipschitz-continuous fields subject to additional assumptions, which will be specified later (see (A.2–3) in Section 4), and $((W_t^k)_{t \geq 0})_{k \in \mathbb{N} \setminus \{0\}}$ are independent one-dimensional Brownian motions.

Formally, this system gives a solution of the form (8) to the following SPDE:

$$df(t, x, v) + (v \cdot \nabla_x f(t, x, v) + E(t, x) \cdot \nabla_v f(t, x, v)) dt + \sum_{k=1}^{\infty} \sigma_k(x) \cdot \nabla_v f(t, x, v) \circ dW_t^k = 0,$$

$$\text{with } f(0, x, v) = f_0(x, v),$$

$$\text{and } \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv, \quad E(t, x) = \int_{\mathbb{R}^d} F(x - y) \rho(t, y) dy.$$

(Apply Itô's formula in Stratonovich form to $(\sum_{1 \leq i \leq N} a_i \varphi(X_t^i, V_t^i))_{t \geq 0}$, for a smooth test function $\varphi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, and deduce that the atomic measure in (8) is a weak solution to the above SPDE.)

The paper is organized as follows. In Section 2, we start with the proof of Theorem 2, as we feel it to have a striking interpretation from the physical point of view. The vanishing noise method for selecting solutions to singular differential equations goes back to the earlier paper by Bafico and Baldi [3], but the examples investigated therein are one-dimensional only. (See also Flandoli [13], together with Gradinaru et al. [16] and Attanasio and Flandoli [2].) Here Theorem 2 applies to a four-dimensional system. In [3], the vanishing noise behavior of the stochastic differential equation at hand is derived from the vanishing viscosity behavior of the associated generator, so that the core of the proof is mainly of analytical essence. Below, the proof of Theorem 2 relies on a new approach, which might be promising for more general systems: the asymptotic dynamics of $(z_i^\varepsilon)_{i=1,2}$ (with $(z_i^\varepsilon)_{i=1,2}$ as in the statement of the theorem) are investigated pathwise, the paths being expanded with respect to the parameter ε till coalescence occurs; the possible limit regimes, as $\varepsilon \rightarrow 0$, then read on the (random) coefficients of the expansion. The result is obtained under general conditions on the structure of the covariance matrix Q : specific examples when these conditions are satisfied are given in Section 4.

The remainder of the paper is devoted to the proof of Theorem 1. We examine first the deterministic case and prove that the Lagrangian dynamics is well posed (namely it has existence and uniqueness of solutions without coalescence of particles in the phase space) for Lebesgue a.e. initial configuration of different particles, see Section 3. This result is not strictly necessary, but since the proof of the same result in the stochastic case is particularly cumbersome, it is useful to see first the main details in the deterministic case.

We emphasize that the result and the approach are inspired by the analysis of point vortex dynamic reported in [25], although there are many technical differences.

Then we turn to the stochastic case. Specifically, we first discuss the structure of the noise that will prevent coalescence from emerging. It should be emphasized that the effect of the noise on the $2N$ -system (10) is highly non-trivial. Indeed, although it is doubly singular, the noise makes the system fluctuate enough to avoid pathological phenomena such as those observed in the deterministic case. The first singularity is due to the fact that the same Brownian motions act on all the particles. If the $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$ were constant, the particles would feel the same impulses and the noise would not have any real effect; basically, it would just act as a random translation of the system, as in (2). Thus the point is to design a noise with a specific spatial structure allowing displacements of different particles in different directions. To reach the desired effect, we require the covariance matrix $(Q(x^i, x^j))_{1 \leq i, j \leq N}$ to be strictly positive for any vector $(x^1, \dots, x^N) \in (\mathbb{R}^d)^N$ with pairwise different entries. We give examples and sufficient conditions to ensure strict positivity in Section 4.2. These conditions are inspired from the Kraichnan noise used in the theory of random transport of passive scalars. (See for example Falkovich, Gawedzki and Vergassola [12], Le Jan [23] and Monin and Yaglom [30].) However, the model considered here does not have the same interpretation as the Kraichnan model, since the noise appears in the equations for the velocities and not in the equations for the positions. A possible way to relate Eq. (10) to turbulence theory would consist in penalizing the dynamics of the velocity of the i th particle by $-V_t^i$. This model would describe the motion of interacting heavy particles in a fluid whose velocity is a white-in-time random field with non-trivial spatial correlations, see [4].

The second singularity of the model is inherited from the kinetic structure of the deterministic counterpart: the noise acts as an additional random force only, namely it is plugged into the equation of the velocity only. In other words, there is no additional noise in the basic relation $dX_t^i = V_t^i dt$, and consequently the coupled system for $(X_t^i, V_t^i)_{1 \leq i \leq N}$ is degenerate. We shall show in Subsection 4.3 that the ellipticity properties of the noise in \mathbb{R}^{Nd} actually lift-up to hypoellipticity properties in \mathbb{R}^{2Nd} .

Once the set-up for the noise is defined, we are ready to tackle the problem of no-coalescence in the stochastic case. We first establish the analog of the deterministic solvability result, namely the Lagrangian dynamics are well posed for Lebesgue a.e. initial configuration of different particles, see

Subsection 5.2. This does not require any special features of the noise. Then we specify the form of the noise according to the requirements discussed in Section 4 and prove well-posedness and no-collapse for *all* initial conditions of the particle system with pairwise different entries by taking advantage of the hypoellipticity property of the whole system, see Subsection 5.3.

Actually, we must say that the main lines for establishing Theorem 1 are connected with the strategy already developed by Flandoli, Gubinelli and Priola [14] in order to prove that noise may prevent N point vertices, when driven by 2D Euler equations, from collapsing. Anyhow, we emphasize that both the framework and the results are here quite different. First, the reader must keep in mind that the noise is here of infinite dimension, whereas it is finite-dimensional only in [14]. The motivation for it is not a matter of technicality only –although it induces additional technical difficulties–, but it is also of practical essence: taking benefit of the infinite-dimensional framework, we are here able to design explicit examples when the noise prevents the system from collapsing. Second, the reader must also remember that, whatever the structure of the noise is, the stochastic Vlasov-Poisson system is intrinsically degenerate as the velocity only is randomly forced. As noticeable in the core of the proof, the degeneracy of the system requires an additional significant effort in comparison with the 2D Euler framework, which appears as non-degenerate. Third, we point out that we here discuss the vanishing noise method for selecting solutions to the deterministic system in the case when there are two particles exactly.

Finally, we stress that the problem would be much easier to handle if each particle were to be forced with its own Brownian motion, independently of the other ones. This choice of the noise, however, would break the relation between the Lagrangian dynamics and the SPDE introduced above. With the random forcing considered here, the same noise realization acts on all the particles simultaneously, but the noise is assumed to be of a great degree of space complexity so that each single realization makes particles being at different space positions move in different directions in a rather chaotic way, thus simulating the effect of independent pulses. These irregular perturbations are sufficiently different at any small distance to break down the coherence that is required for coalescence to occur.

1.4 Assumption

In what follows, we assume that $a_i = 1/N$ for all $1 \leq i \leq N$ for the sake of simplicity. We also assume that F satisfies Assumption (A.1):

(A.1) F is bounded everywhere on \mathbb{R}^d and locally Lipschitz-continuous on any compact subset of $\mathbb{R}^d \setminus \{0\}$. Moreover,

$$\sup_{0 < r \leq 1} \sup_{r \leq |x|, |y| \leq 1, x \neq y} \left[r \frac{|F(x) - F(y)|}{|x - y|} \right] < +\infty.$$

(In other words, the Lipschitz constant of F on the ring $\{r \leq |x| \leq 1\}$ explodes at most as $1/r$ as r tends to zero.)

The examples we have in mind are:

1. For $d = 1$, $F(x) = \text{sgn}(x)$, $x \in \mathbb{R} \setminus \{0\}$.
2. For $d \geq 2$, $F(x) = x/|x|$, $x \in \mathbb{R}^d \setminus \{0\}$.

Remark 3 *For simplicity's sake, the assumptions on the noise in (10) are detailed in Subsection 4.1.*

1.5 Useful notation

For any $n \in \mathbb{N} \setminus \{0\}$, $z \in \mathbb{R}^n$ and $r > 0$, $B_n(z, r)$ stands for the closed ball of dimension n , of center z and of radius r ; Leb_n stands for the Lebesgue measure on \mathbb{R}^n . The volume of $B_n(z, r)$ is denoted by $V_n(r)$, i.e. $V_n(r) = \text{Leb}_n(B_n(z, r))$.

The configurations of the N -particle system in the phase space are generally denoted by the letters z or Z . Positions are denoted by the letters x or X and velocities by the letters v or V . Similarly, the typical notation for a single particle in the phase space is $\tilde{z} = (\tilde{x}, \tilde{v})$, \tilde{x} standing for its position and \tilde{v} for its velocity. We then define the projection mappings:

$$\begin{aligned} \Pi_x : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} = ((x^j, v^j))_{1 \leq j \leq N} &\mapsto \pi_x(z) = (x^j)_{1 \leq j \leq N} \in \mathbb{R}^{Nd}, \\ \Pi_v : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} = ((x^j, v^j))_{1 \leq j \leq N} &\mapsto \pi_v(z) = (v^j)_{1 \leq j \leq N} \in \mathbb{R}^{Nd}, \\ \tilde{\pi}_x : \mathbb{R}^{2d} \ni \tilde{z} = (\tilde{x}, \tilde{v}) &\mapsto \tilde{x} \in \mathbb{R}^d, \\ \tilde{\pi}_v : \mathbb{R}^{2d} \ni \tilde{z} = (\tilde{x}, \tilde{v}) &\mapsto \tilde{v} \in \mathbb{R}^d, \end{aligned}$$

and

$$\begin{aligned}\pi_i : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} &\mapsto \pi_i(z) = z^i \in \mathbb{R}^{2d}, \\ \pi_{i,x} : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} = ((x^j, v^j))_{1 \leq j \leq N} &\mapsto \pi_{i,x}(z) = x^i \in \mathbb{R}^d, \\ \pi_{i,v} : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} = ((x^j, v^j))_{1 \leq j \leq N} &\mapsto \pi_{i,v}(z) = v^i \in \mathbb{R}^d.\end{aligned}$$

The set of pairs of different indices in the particle system is denoted by Δ_N , that is $\Delta_N = \{(i, j) \in \{1, \dots, N\}^2 : i \neq j\}$. We also denote $\Gamma_N = \{(z^1, \dots, z^N) \in \mathbb{R}^{2Nd} : \forall (i, j) \in \Delta_N, z_i \neq z_j\}$ and $\Gamma_{x,N} = \{(x^1, \dots, x^N) \in \mathbb{R}^{Nd} : \forall (i, j) \in \Delta_N, x_i \neq x_j\}$.

In the whole paper, we work with a fixed number N of particles, so that we do not have to take into account the dependence on N of constants, functions, and so on.

2 Continuation: Proof of Theorem 2

In this section, we identify general conditions on the structure of the noise in (3) under which Theorem 2 holds. Typical examples that satisfy these general conditions are given in Proposition 37 in Section 4.

In the whole section, we thus consider the 4D system of Vlasov-Poisson type:

$$\begin{aligned}dX_t^{i,\varepsilon} &= V_t^{i,\varepsilon} dt \\ dV_t^{i,\varepsilon} &= \text{sign}(X_t^{i,\varepsilon} - X_t^{\bar{i},\varepsilon}) + \varepsilon \sum_{k=1}^{\infty} \sigma_k(X_t^{i,\varepsilon}) dW_t^k, \quad t \geq 0,\end{aligned}\tag{11}$$

for $i = 1, 2$ and $\bar{i} = 3 - i$ (i.e. $\bar{i} = 2$ if $i = 1$ and vice versa). Below, we assume that $(X_0^{1,\varepsilon}, V_0^{1,\varepsilon}) = (1, -\sqrt{2})$ and $(X_0^{2,\varepsilon}, V_0^{2,\varepsilon}) = -(X_0^{1,\varepsilon}, V_0^{1,\varepsilon}) = (-1, \sqrt{2})$.

As a first general condition, we set:

Condition 4 *For any $\varepsilon > 0$, Theorem 1 applies and thus (11) has a unique strong solution. Moreover,*

$$\sup_{x \in \mathbb{R}} \left[\sum_{k \geq 1} |\sigma_k(x)|^2 \right] < +\infty.\tag{12}$$

Again, we refer to Proposition 37 in Section 4 for typical examples for which Condition 4 holds.

For notational convenience, we will set $Z_t^{i,\varepsilon} = (X_t^{i,\varepsilon}, V_t^{i,\varepsilon})$, $\varepsilon > 0$, $i = 1, 2$. When $\varepsilon = 0$, the curves

$$\begin{aligned} X_t^{1,0} &= \left(1 - \frac{t}{\sqrt{2}}\right)^2, & V_t^{1,0} &= -\sqrt{2} + t, & t &\geq 0, \\ X_t^{2,0} &= -\left(1 - \frac{t}{\sqrt{2}}\right)^2, & V_t^{2,0} &= \sqrt{2} - t, & t &\geq 0, \end{aligned}$$

solve the system (11) on the time interval $[0, t_0)$, with $t_0 = \sqrt{2}$, but the particles merge at time t_0 , making the system singular after t_0 . Again, we will set $Z_t^{i,0} = (X_t^{i,0}, V_t^{i,0})$, $i = 1, 2$. Note that

$$\forall t \geq 0, \quad Z_t^{2,0} = -Z_t^{1,0}.$$

We are to prove that $(Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})_{t \geq 0}$ converges in distribution on the space $\mathcal{C}([0, +\infty), \mathbb{R}^4)$ towards $(1/2)\delta_{(Z_t^{1,0}, Z_t^{2,0})_{t \geq 0}} + (1/2)\delta_{(Z_t^{2,0}, Z_t^{1,0})_{t \geq 0}}$.

The whole point is to investigate the differences:

$$\mathbb{X}_t^\varepsilon = \frac{X_t^{1,\varepsilon} - X_t^{2,\varepsilon}}{2}, \quad \mathbb{V}_t^\varepsilon = \frac{V_t^{1,\varepsilon} - V_t^{2,\varepsilon}}{2}, \quad t \geq 0, \quad \varepsilon > 0. \quad (13)$$

We will use the second general condition (see as an example Proposition 37):

Condition 5 *Assume that $(Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})_{t \geq 0}$ satisfies (11), but with an arbitrary random initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$, independent of the noise $((W_t^k)_{t \geq 0})_{k \geq 1}$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the augmented filtration generated by the initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$ and by the noise $((W_t^k)_{t \geq 0})_{k \geq 1}$. Then, there exists an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $(B_t^\varepsilon)_{t \geq 0}$ such that, for all $t \geq 0$,*

$$\begin{aligned} d\mathbb{X}_t^\varepsilon &= \mathbb{V}_t^\varepsilon dt \\ d\mathbb{V}_t^\varepsilon &= \text{sign}(\mathbb{X}_t^\varepsilon) dt + \varepsilon \sigma(\mathbb{X}_t^\varepsilon) dB_t^\varepsilon, \end{aligned} \quad (14)$$

where σ is \mathcal{C}^2 function from \mathbb{R} to \mathbb{R} , depending on the $(\sigma_k)_{k \geq 1}$ only (in particular, σ is independent of the initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$ and of ε), with bounded derivatives of order 1 and 2, such that $\sigma(0) = 0$ and $\sigma(1) > 0$.

Setting $\mathbb{Z}_t^\varepsilon = (\mathbb{X}_t^\varepsilon, \mathbb{V}_t^\varepsilon)$, for any $t \geq 0$, we first investigate the solutions of (14) when $\varepsilon = 0$. We have the obvious

Lemma 6 For $\varepsilon = 0$, all the solutions of

$$\begin{aligned}\dot{\mathbb{X}}_t^0 &= \mathbb{V}_t^0 \\ \dot{\mathbb{V}}_t^0 &= \text{sign}(\mathbb{X}_t^0),\end{aligned}\tag{15}$$

with $(\mathbb{X}_0^0, \mathbb{V}_0^0) = (1, -\sqrt{2})$, have the form

$$\mathbb{Z}_t^0 = (\mathbb{X}_t^0, \mathbb{V}_t^0) = \left(\frac{(t_0 - t)^2}{2}, t - t_0 \right) \quad \text{for } 0 \leq t \leq t_0 = \sqrt{2}.\tag{16}$$

Remark 7 We emphasize that uniqueness fails after coalescence time t_0 . Indeed, any $(\mathbb{Z}_t^0)_{t \geq 0}$, with $(\mathbb{Z}_t^0)_{0 \leq t \leq t_0}$ as in (16) and

$$\mathbb{Z}_t^0 = \begin{cases} (0, 0) & \text{for } t_0 \leq t \leq t_1, \\ \pm \left(\frac{(t - t_1)^2}{2}, t - t_1 \right) & \text{for } t \geq t_1, \end{cases}$$

where $t_1 \geq t_0$ may be real or infinite, is a solution to (15). (We could prove that these are the only possible solutions, but this is useless for our purpose.)

Then, we can prove

Proposition 8 Define $\tau^\varepsilon = \inf\{t \geq 0 : \mathbb{X}_t^\varepsilon \leq 0\}$. Then, for any $\delta > 0$ and $M > t_0$,

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon \in (t_0 - \delta, t_0 + \delta)\} &= \frac{1}{2}, \\ \lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon \geq M\} &= \frac{1}{2}.\end{aligned}\tag{17}$$

Moreover, defining $\tau_2^\varepsilon = \inf\{t > \tau^\varepsilon : \mathbb{X}_t^\varepsilon \geq 0\}$, we have, for any $M > 0$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau_2^\varepsilon \geq M\} = 1.\tag{18}$$

Proposition 8 suggests that, in the limit regime $\varepsilon \rightarrow 0$, the trajectories of the two particles cross with probability 1/2 exactly, and, if so, they cross at coalescence time exactly. As well understood, this is one step forward in the proof of Theorem 2. Precisely, we prove below that Proposition 8 implies Theorem 2.

Proof. (Proposition 8 \Rightarrow Theorem 2.) By (12) in Condition 4, it is plain to see that the family $((Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})_{t \geq 0})_{0 < \varepsilon \leq 1}$ is tight. We denote by μ a weak limit, on the space of continuous functions $\mathcal{C}([0, +\infty), \mathbb{R}^4)$, of the family of measures $(\mathbb{P}_{(Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})})_{0 < \varepsilon \leq 1}$ as $\varepsilon \rightarrow 0$, the canonical process on $\mathcal{C}([0, +\infty), \mathbb{R}^4)$ being denoted by $(\xi^1 = (\chi^1, \nu^1), \xi^2 = (\chi^2, \nu^2))$. We will also denote $\chi_t = (\chi_t^1 - \chi_t^2)/2$ and $\nu_t = (\nu_t^1 - \nu_t^2)/2, t \geq 0$.

Under the measure μ , $\xi^i = (\chi^i, \nu^i), i = 1, 2$, satisfies

$$\dot{\chi}_t^i = \nu_t^i, \quad |\dot{\nu}_t^1| \leq 1, \quad t \geq 0, \quad i = 1, 2.$$

We now make use of Proposition 8. Given $M > 0$, we have, on the set $\{\tau^\varepsilon \geq M\}$,

$$V_t^{i,\varepsilon} = V_0^{i,\varepsilon} + (3 - 2i)t + \varepsilon \sum_{k \geq 1} \int_0^t \sigma_k(X_s^{i,\varepsilon}) dW_s^k, \quad 0 \leq t \leq M,$$

so that, for any $\eta > 0$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{P} \left\{ \sup_{0 \leq t \leq M} |V_t^{i,\varepsilon} - V_0^{i,\varepsilon} - (3 - 2i)t| \leq \eta, \quad i = 1, 2 \right\} \\ & \geq \lim_{\varepsilon \rightarrow 0} \mathbb{P} \{ \tau^\varepsilon \geq M \} = \begin{cases} 1 & \text{if } M < t_0, \\ 1/2 & \text{if } M > t_0. \end{cases} \end{aligned}$$

We deduce that,

$$\mu \left\{ \nu_t^i = \nu_0^i + (3 - 2i)t, \quad 0 \leq t \leq M, \quad i = 1, 2 \right\} \begin{cases} = 1 & \text{if } M < t_0, \\ \geq 1/2 & \text{if } M > t_0. \end{cases} \quad (19)$$

We deduce from (19) that, for any $\delta \in (0, t_0)$, under μ ,

$$\nu_t^i = \nu_0^i + (3 - 2i)t, \quad 0 \leq t \leq t_0 - \delta, \quad i = 1, 2.$$

Letting δ tend to 0, we deduce that, under μ , $(\xi_t^1)_{0 \leq t \leq t_0}$ coincides with $(Z_t^{1,0})_{0 \leq t \leq t_0}$ and $(\xi_t^2)_{0 \leq t \leq t_0}$ coincides with $(Z_t^{2,0})_{0 \leq t \leq t_0}$. In particular, under μ , $\xi_{t_0}^1 = \xi_{t_0}^2 = (0, 0)$. Similarly, we also deduce from (19) that, with probability greater than 1/2 under μ , $(\xi_t^1, \xi_t^2) = (Z_t^{1,+}, Z_t^{2,+})$ for any $t \geq 0$.

Now, we know that, on the event $\{\tau^\varepsilon \leq t_0 + \delta, \tau_2^\varepsilon \geq M\}$, for $\delta > 0$ small and $M > t_0 + \delta$, we have

$$V_t^{i,\varepsilon} = V_{t_0+\delta}^{i,\varepsilon} - (3 - 2i)[t - (t_0 + \delta)] + \varepsilon \sum_{k \geq 1} \int_{t_0+\delta}^t \sigma_k(X_s^{i,\varepsilon}) dW_s^k,$$

for $t_0 + \delta \leq t \leq M$, so that, by Proposition 8, for any $\eta > 0$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{P} \left\{ \sup_{t_0 + \delta \leq t \leq M} |V_t^{i,\varepsilon} - V_{t_0 + \delta}^{i,\varepsilon} + (3 - 2i)[t - (t_0 + \delta)]| \leq \eta, i = 1, 2 \right\} \\ & \geq \lim_{\varepsilon \rightarrow 0} \mathbb{P} \{ \tau^\varepsilon \leq t_0 + \delta, \tau_2^\varepsilon \geq M \} = \frac{1}{2}. \end{aligned}$$

We deduce that, under μ ,

$$v_t^i = v_{t_0 + \delta}^i - (3 - 2i)[t - (t_0 + \delta)], \quad t_0 + \delta \leq t \leq M, \quad i = 1, 2,$$

at least with probability $1/2$. Letting δ tend toward 0 and M toward $+\infty$, we deduce that, with probability greater than $1/2$ under μ , $(\xi_t^1, \xi_t^2) = (Z_t^{1,-}, Z_t^{2,-})$ for any $t \geq 0$. ■

2.1 Key Lemmas by Integration by Parts

The proof of Proposition 8 relies on two key lemmas, each of them being proven by integration by parts. The first one is

Lemma 9 *Denote by N^+ the set $N^+ = [0, +\infty)^2 \setminus \{(0, 0)\}$ and, similarly, by N^- the set $N^- = (-\infty, 0]^2 \setminus \{(0, 0)\}$. Consider also the sets of initial conditions for the 4D system $\Gamma^+ = \{(z^1 = (x^1, v^1), z^2 = (x^2, v^2)) : (x^1 - x^2, v^1 - v^2) \in N^+\}$ and $\Gamma^- = \{(z^1 = (x^1, v^1), z^2 = (x^2, v^2)) : (x^1 - x^2, v^1 - v^2) \in N^-\}$. Then, there exists a constant $c > 0$ such that, for any $M > 0$ and any compact subset $K \subset \mathbb{R}^4$,*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \inf_{(z^1, z^2) \in K \cap \Gamma^+} \mathbb{P} \{ \forall t \in [0, M], \mathbb{X}_t^\varepsilon \geq ct^2 | (Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) = (z^1, z^2) \} = 1, \\ & \lim_{\varepsilon \rightarrow 0} \inf_{(z^1, z^2) \in K \cap \Gamma^-} \mathbb{P} \{ \forall t \in [0, M], \mathbb{X}_t^\varepsilon \leq -ct^2 | (Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) = (z^1, z^2) \} = 1. \end{aligned}$$

Proof. In the whole proof, the initial condition $(z^1, z^2) \in K \cap \Gamma^+$ is given, i.e. $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) = (z^1, z^2) \in K \cap \Gamma^+$. Writing $z^i = (x^i, v^i)$, $i = 1, 2$, we set $x = (x^1 - x^2)/2$ and $v = (v^1 - v^2)/2$. Without loss of generality, we can assume that $x > 0$. Indeed, in the case when $x = 0$, v must be positive, so that, in very short time, both \mathbb{X}^ε and \mathbb{V}^ε are positive. By Markov property (for the 4D system), we are then led back to the case when x and v are positive.

By Condition 5, we can write

$$d\mathbb{V}_t^\varepsilon = \text{sign}(\mathbb{X}_t^\varepsilon) dt + \varepsilon \sigma(\mathbb{X}_t^\varepsilon) dB_t^\varepsilon, \quad t \geq 0,$$

where $(B_t^\varepsilon)_{t \geq 0}$ is a 1D Brownian motion. Taking advantage of the smoothness of σ , we perform the following integration by parts:

$$d(\mathbb{V}_t^\varepsilon - \varepsilon\sigma(\mathbb{X}_t^\varepsilon)B_t^\varepsilon) = (\text{sign}(\mathbb{X}_t^\varepsilon) - \varepsilon\sigma'(\mathbb{X}_t^\varepsilon)\mathbb{V}_t^\varepsilon B_t^\varepsilon)dt.$$

Keeping in mind that $\tau^\varepsilon = \inf\{t \geq 0 : \mathbb{X}_t^\varepsilon \leq 0\}$, we have

$$\mathbb{V}_t^\varepsilon - \varepsilon\sigma(\mathbb{X}_t^\varepsilon)B_t^\varepsilon \geq t - \varepsilon \int_0^t \sigma'(\mathbb{X}_s^\varepsilon)\mathbb{V}_s^\varepsilon B_s^\varepsilon ds, \quad 0 \leq t \leq \tau^\varepsilon.$$

Introducing the event

$$A_1^\varepsilon = \left\{ \sup_{0 \leq t \leq M} |\sigma'(\mathbb{X}_t^\varepsilon)\mathbb{V}_t^\varepsilon B_t^\varepsilon| \leq \frac{1}{2\varepsilon} \right\},$$

we deduce that, on A_1^ε ,

$$d\mathbb{X}_t^\varepsilon \geq \left(\frac{t}{2} + \varepsilon\sigma(\mathbb{X}_t^\varepsilon)B_t^\varepsilon \right) dt \geq \left(\frac{t}{2} - C\varepsilon\mathbb{X}_t^\varepsilon|B_t^\varepsilon| \right) dt, \quad 0 \leq t \leq \tau^\varepsilon \wedge M,$$

where C here stands for the Lipschitz constant of σ . Setting

$$\bar{\mathbb{X}}_t^\varepsilon = \mathbb{X}_t^\varepsilon \exp\left(C\varepsilon \int_0^t |B_s^\varepsilon| ds \right),$$

we deduce

$$d\bar{\mathbb{X}}_t^\varepsilon \geq \frac{t}{2} \exp\left(C\varepsilon \int_0^t |B_s^\varepsilon| ds \right) dt, \quad 0 \leq t \leq \tau^\varepsilon \wedge M.$$

Therefore, on A_1^ε , τ^ε must be greater than M , so that the above expression holds up to time M (at least). We deduce

$$d\bar{\mathbb{X}}_t^\varepsilon \geq \frac{t}{2} dt, \quad 0 \leq t \leq M,$$

so that

$$\bar{\mathbb{X}}_t^\varepsilon \geq \frac{t^2}{4}, \quad 0 \leq t \leq M.$$

Intersect now A_1^ε with

$$A_2^\varepsilon = \left\{ \sup_{0 \leq t \leq M} |B_t^\varepsilon| \leq \frac{1}{\varepsilon M} \right\}.$$

Then, on $A_1^\varepsilon \cap A_2^\varepsilon$,

$$\mathbb{X}_t^\varepsilon \geq \frac{t^2}{4} \exp(-C), \quad 0 \leq t \leq M.$$

To complete the proof, it remains to notice (from a standard tightness argument) that $\mathbb{P}(A_1^\varepsilon \cap A_2^\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$, uniformly in $(z^1, z^2) \in K$. (The proof when (z^1, z^2) is in Γ^- is similar.) ■

We now come back to the case when the initial condition of the 4D system is $((1, -\sqrt{2}), (-1, \sqrt{2}))$. The second key lemma consists in expanding the difference process $(\mathbb{X}^\varepsilon, \mathbb{V}^\varepsilon)$ w.r.t. ε , up to the hitting time $\tau^\varepsilon = \inf\{t \geq 0 : \mathbb{X}_t^\varepsilon \leq 0\}$.

Lemma 10 *There exist a family of 1D Brownian motions $(B_t^\varepsilon)_{\varepsilon>0}$ and a family of random continuous processes $(g^\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R})_{\varepsilon>0}$, such that*

$$\forall T > 0, \quad \lim_{R \rightarrow +\infty} \sup_{0 < \varepsilon \leq 1} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} |g_t^\varepsilon| > R \right\} = 0, \quad (20)$$

and the processes

$$\begin{aligned} d\mathbb{V}_t^{(0)} &= dt, & d\mathbb{X}_t^{(0)} &= \mathbb{V}_t^{(0)} dt, & (\mathbb{X}_0^{(0)}, \mathbb{V}_0^{(0)}) &= (1, -\sqrt{2}), \\ d\mathbb{V}_t^{(1,\varepsilon)} &= \sigma(\mathbb{X}_t^{(0)}) dB_t^\varepsilon, & d\mathbb{X}_t^{(1,\varepsilon)} &= \mathbb{V}_t^{(1,\varepsilon)} dt, & (\mathbb{X}_0^{(1,\varepsilon)}, \mathbb{V}_0^{(1,\varepsilon)}) &= (0, 0), \end{aligned}$$

satisfy

$$|\mathbb{X}_t^\varepsilon - (\mathbb{X}_t^{(0)} + \varepsilon \mathbb{X}_t^{(1,\varepsilon)})| + |\mathbb{V}_t^\varepsilon - (\mathbb{V}_t^{(0)} + \varepsilon \mathbb{V}_t^{(1,\varepsilon)})| \leq \varepsilon^2 |g_t^\varepsilon|, \quad 0 \leq t \leq \tau^\varepsilon. \quad (21)$$

Proof. By Condition 5, we can write

$$d\mathbb{V}_t^\varepsilon = dt + \varepsilon \sigma(\mathbb{X}_t^\varepsilon) dB_t^\varepsilon, \quad 0 \leq t \leq \tau^\varepsilon,$$

for some 1D Brownian motion $(B_t^\varepsilon)_{t \geq 0}$, so that

$$\begin{aligned} d[\delta \mathbb{X}_t^\varepsilon] &= \delta \mathbb{V}_t^\varepsilon dt, \\ d[\delta \mathbb{V}_t^\varepsilon] &= \varepsilon [\sigma(\mathbb{X}_t^\varepsilon) - \sigma(\mathbb{X}_t^{(0)})] dB_t^\varepsilon, \quad 0 \leq t \leq \tau^\varepsilon, \end{aligned}$$

with

$$\delta \mathbb{X}_t^\varepsilon = \mathbb{X}_t^\varepsilon - (\mathbb{X}_t^{(0)} + \varepsilon \mathbb{X}_t^{(1,\varepsilon)}), \quad \delta \mathbb{V}_t^\varepsilon = \mathbb{V}_t^\varepsilon - (\mathbb{V}_t^{(0)} + \varepsilon \mathbb{V}_t^{(1,\varepsilon)}).$$

We perform the same integration by parts as above, with

$$\delta\bar{\mathbb{V}}_t^\varepsilon = \delta\mathbb{V}_t^\varepsilon - \varepsilon(\sigma(\mathbb{X}_t^\varepsilon) - \sigma(\mathbb{X}_t^{(0)}))B_t^\varepsilon.$$

Then, we can find a family of random continuous function $((v_t^{0,\varepsilon})_{t \geq 0})_{\varepsilon \geq 0}$, satisfying (20), such that

$$\begin{aligned} d[\delta\bar{\mathbb{V}}_t^\varepsilon] &= -\varepsilon(\sigma'(\mathbb{X}_t^\varepsilon)\mathbb{V}_t^\varepsilon - \sigma'(\mathbb{X}_t^{(0)})\mathbb{V}_t^{(0)})B_t^\varepsilon dt \\ &= -\varepsilon(\sigma'(\mathbb{X}_t^\varepsilon)\mathbb{V}_t^\varepsilon - \sigma'(\mathbb{X}_t^{(0)} + \varepsilon\mathbb{X}_t^{(1,\varepsilon)})(\mathbb{V}_t^{(0)} + \varepsilon\mathbb{V}_t^{(1,\varepsilon)}))B_t^\varepsilon dt + \varepsilon^2 v_t^{0,\varepsilon} dt \\ &= -\varepsilon\sigma'(\mathbb{X}_t^\varepsilon)B_t^\varepsilon \delta\mathbb{V}_t^\varepsilon dt - \varepsilon(\sigma'(\mathbb{X}_t^\varepsilon) - \sigma'(\mathbb{X}_t^{(0)} + \varepsilon\mathbb{X}_t^{(1,\varepsilon)}))(\mathbb{V}_t^{(0)} + \varepsilon\mathbb{V}_t^{(1,\varepsilon)})B_t^\varepsilon dt \\ &\quad + \varepsilon^2 v_t^{0,\varepsilon} dt. \end{aligned}$$

Modifying the definition of $v^{0,\varepsilon}$, the above expression reads

$$\begin{aligned} d[\delta\bar{\mathbb{V}}_t^\varepsilon] &= -\varepsilon\sigma'(\mathbb{X}_t^\varepsilon)B_t^\varepsilon \delta\bar{\mathbb{V}}_t^\varepsilon dt \\ &\quad - \varepsilon[(\mathbb{V}_t^{(0)} + \varepsilon\mathbb{V}_t^{(1,\varepsilon)})B_t^\varepsilon (\sigma'(\mathbb{X}_t^\varepsilon) - \sigma'(\mathbb{X}_t^{(0)} + \varepsilon\mathbb{X}_t^{(1,\varepsilon)}))(\delta\mathbb{X}_t^\varepsilon)^{-1} \mathbf{1}_{\{\delta\mathbb{X}_t^\varepsilon \neq 0\}}] \delta\mathbb{X}_t^\varepsilon dt \\ &\quad + \varepsilon^2 v_t^{0,\varepsilon} dt. \end{aligned}$$

Put it differently, we can find two families of random functions $((v_t^{1,\varepsilon})_{t \geq 0})_{\varepsilon \geq 0}$ and $((v_t^{2,\varepsilon})_{t \geq 0})_{\varepsilon \geq 0}$, satisfying (20), such that

$$d[\delta\bar{\mathbb{V}}_t^\varepsilon] = \varepsilon v_t^{1,\varepsilon} \delta\mathbb{X}_t^\varepsilon dt + \varepsilon v_t^{2,\varepsilon} \delta\bar{\mathbb{V}}_t^\varepsilon dt + \varepsilon^2 v_t^{0,\varepsilon} dt.$$

In a similar way, we can find two families of random functions $((x_t^{0,\varepsilon})_{t \geq 0})_{\varepsilon \geq 0}$ and $((x_t^{1,\varepsilon})_{t \geq 0})_{\varepsilon \geq 0}$, satisfying (20), such that

$$d[\delta\mathbb{X}_t^\varepsilon] = \varepsilon x_t^{1,\varepsilon} \delta\mathbb{X}_t^\varepsilon dt + \delta\bar{\mathbb{V}}_t^\varepsilon dt + \varepsilon^2 x_t^{0,\varepsilon} dt.$$

In a vectorial form, we write

$$d \begin{pmatrix} \delta\mathbb{X}_t^\varepsilon \\ \delta\bar{\mathbb{V}}_t^\varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon x_t^{1,\varepsilon} & 1 \\ \varepsilon v_t^{1,\varepsilon} & \varepsilon v_t^{2,\varepsilon} \end{pmatrix} \begin{pmatrix} \delta\mathbb{X}_t^\varepsilon \\ \delta\bar{\mathbb{V}}_t^\varepsilon \end{pmatrix} dt + \begin{pmatrix} \varepsilon^2 x_t^{0,\varepsilon} \\ \varepsilon^2 v_t^{0,\varepsilon} \end{pmatrix} dt.$$

Introduce the resolvent

$$\frac{d}{dt}(R_{s,t}^\varepsilon) = \begin{pmatrix} \varepsilon x_t^{1,\varepsilon} & 1 \\ \varepsilon v_t^{1,\varepsilon} & \varepsilon v_t^{2,\varepsilon} \end{pmatrix} R_{s,t}^\varepsilon, \quad R_{s,s}^\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$\begin{pmatrix} \delta \mathbb{X}_t^\varepsilon \\ \delta \mathbb{V}_t^\varepsilon \end{pmatrix} = \int_0^t R_{s,t}^\varepsilon \begin{pmatrix} \varepsilon^2 x_s^{0,\varepsilon} \\ \varepsilon^2 v_s^{0,\varepsilon} \end{pmatrix} ds, \quad 0 \leq t \leq \tau^\varepsilon.$$

It is plain to bound the resolvent in terms of the bounds for $x^{1,\varepsilon}$, $v^{1,\varepsilon}$ and $v^{2,\varepsilon}$. The result easily follows. ■

2.2 Proof of Proposition 8

We start with

Remark 11 *We emphasize that*

$$\mathbb{V}_t^{(0)} = \mathbb{V}_0 + t = -\sqrt{2} + t, \quad \mathbb{X}_t^{(0)} = 1 - \sqrt{2}t + \frac{t^2}{2} = \left(1 - \frac{t}{\sqrt{2}}\right)^2, \quad (22)$$

and

$$\begin{aligned} \mathbb{V}_t^{(1,\varepsilon)} &= \int_0^t \sigma(\mathbb{X}_s^{(0)}) dB_s^\varepsilon = \int_0^t \sigma\left[\left(1 - \frac{s}{\sqrt{2}}\right)^2\right] dB_s^\varepsilon, \\ \mathbb{X}_t^{(1,\varepsilon)} &= \int_0^t \int_0^s \sigma\left[\left(1 - \frac{r}{\sqrt{2}}\right)^2\right] dB_r^\varepsilon = \int_0^t (t-r) \sigma\left[\left(1 - \frac{r}{\sqrt{2}}\right)^2\right] dB_r^\varepsilon. \end{aligned} \quad (23)$$

Setting $t_0 = \sqrt{2}$ (t_0 is the critical point of the deterministic Vlasov-Poisson system), it holds

$$\begin{aligned} \mathbb{X}_{t_0}^{(1,\varepsilon)} &= \int_0^{\sqrt{2}} (\sqrt{2} - r) \sigma\left[\left(1 - \frac{r}{\sqrt{2}}\right)^2\right] dB_r^\varepsilon \\ &= \sqrt{2} \int_0^{\sqrt{2}} \left(1 - \frac{r}{\sqrt{2}}\right) \sigma\left[\left(1 - \frac{r}{\sqrt{2}}\right)^2\right] dB_r^\varepsilon. \end{aligned}$$

Since $\sigma(1) \neq 0$, we deduce

Lemma 12 *The r.v.'s $(\mathbb{X}_{t_0}^{(1,\varepsilon)})_{\varepsilon > 0}$ have a common Gaussian distribution with a zero mean and a non-zero variance. In particular,*

$$\mathbb{P}\{\mathbb{X}_{t_0}^{(1,\varepsilon)} > 0\} = \mathbb{P}\{\mathbb{X}_{t_0}^{(1,\varepsilon)} \geq 0\} = \frac{1}{2}.$$

We now claim

Lemma 13 For any real $M > t_0 = \sqrt{2}$ and any $\delta > 0$,

- (i) $\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\{\tau^\varepsilon \leq M\} \cap \{\mathbb{X}_{t_0}^{(1,\varepsilon)} > \delta\}) = 0$,
- (ii) $\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\{\tau^\varepsilon > M\} \cap \{\mathbb{X}_{t_0}^{(1,\varepsilon)} < -\delta\}) = 0$,
- (iii) $\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\{\tau^\varepsilon \leq M\} \cap \{\tau^\varepsilon \notin (t_0 - \delta, t_0 + \delta)\}) = 0$.

Proof. Given $M > \sqrt{2}$, we set:

$$E_M^\varepsilon = \{\tau^\varepsilon \leq M\}.$$

We know from Lemma 10, that there exists a tight family of random variables $(\zeta_M^\varepsilon)_{0 < \varepsilon \leq 1}$ such that

$$|\mathbb{X}_t^\varepsilon - (\mathbb{X}_t^{(0)} + \varepsilon \mathbb{X}_t^{(1,\varepsilon)})| \leq \varepsilon^2 \zeta_M^\varepsilon, \quad 0 \leq t \leq \tau^\varepsilon \wedge M. \quad (24)$$

Therefore, on E_M^ε , we can choose $t = \tau^\varepsilon$ above. From (22), we deduce that

$$\left| \left(1 - \frac{\tau^\varepsilon}{\sqrt{2}}\right)^2 + \varepsilon \mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} \right| \leq \varepsilon^2 \zeta_M^\varepsilon, \quad (25)$$

that is

$$\left(1 - \frac{\tau^\varepsilon}{\sqrt{2}}\right)^2 \leq \varepsilon \sup_{0 \leq t \leq M} |\mathbb{X}_t^{(1,\varepsilon)}| + \varepsilon^2 \zeta_M^\varepsilon.$$

Up to a modification of ζ_M^ε , we deduce that

$$|\tau^\varepsilon - t_0|^2 \leq \varepsilon \zeta_M^\varepsilon. \quad (26)$$

This proves (iii).

We now prove (i). From (25), we deduce that

$$\mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} \leq \varepsilon \zeta_M^\varepsilon,$$

on E_M^ε . Since $\mathbb{X}^{(1,\varepsilon)}$ is Lipschitz continuous on the interval $[0, M]$, we deduce from (26) that there exists a tight family of random variables $(C_M^\varepsilon)_{0 < \varepsilon \leq 1}$ such that

$$\begin{aligned} \mathbb{X}_{t_0}^{(1,\varepsilon)} &= \mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} + \mathbb{X}_{t_0}^{(1,\varepsilon)} - \mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} \\ &\leq \varepsilon \zeta_M^\varepsilon + C_M^\varepsilon |\tau^\varepsilon - t_0| \leq \varepsilon \zeta_M^\varepsilon + C_M^\varepsilon \varepsilon^{1/2} (\zeta_M^\varepsilon)^{1/2}. \end{aligned}$$

That is, for every $\delta > 0$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(E_M^\varepsilon \cap \{\mathbb{X}_{t_0}^{(1,\varepsilon)} > \delta\}) = 0.$$

We finally prove (ii). From (24), we know that

$$\varepsilon \mathbb{X}_{t_0}^{(1,\varepsilon)} \geq \mathbb{X}_{t_0}^\varepsilon - \varepsilon^2 \zeta_{t_0}^\varepsilon.$$

Therefore, on $(E_M^\varepsilon)^c$,

$$\varepsilon \mathbb{X}_{t_0}^{(1,\varepsilon)} \geq -\varepsilon^2 \zeta_{t_0}^\varepsilon.$$

This proves that, for every $\delta > 0$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}((E_M^\varepsilon)^c \cap \{\mathbb{X}_{t_0}^{(1,\varepsilon)} < -\delta\}) = 0.$$

This completes the proof. ■

We now deduce

Lemma 14 *It holds*

- (i) $\forall M > \sqrt{2}, \quad \lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon > M\} = 1/2,$
- (ii) $\forall \delta > 0, \quad \lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon \in (t_0 - \delta, t_0 + \delta)\} = 1/2.$

In particular, τ^ε converges in law towards $\frac{1}{2}\delta_{t_0} + \frac{1}{2}\delta_{+\infty}$.

Proof. From Lemmas 12 and 13, we know that, for any $M > \sqrt{2}$ and any $\delta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon > M\} \leq \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\{\mathbb{X}_{t_0}^{(1,\varepsilon)} \geq -\delta\} = \mathbb{P}\{\mathbb{X}_{t_0}^{(1,1)} \geq -\delta\}.$$

Letting δ tend toward 0, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon > M\} \leq \frac{1}{2}.$$

Similarly,

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon \leq M\} \leq \mathbb{P}\{\mathbb{X}_{t_0}^{(1,1)} \leq 0\} = \frac{1}{2}.$$

We deduce (i). Then, (ii) follows from (iii) in Lemma 13. ■

We finally claim

Lemma 15 Set $\sigma^\varepsilon = \inf\{t \geq 0 : \mathbb{V}_t^\varepsilon \geq 0\}$. Then, for all $M > 0$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon \leq M, \sigma^\varepsilon < \tau^\varepsilon\} = 0, \quad (27)$$

and

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau_2^\varepsilon \geq M\} = 1. \quad (28)$$

Proof. We start with the proof of (27). By Markov property, we notice that

$$\begin{aligned} & \mathbb{P}\{\tau^\varepsilon \leq M, \sigma^\varepsilon < \tau^\varepsilon\} \\ & \leq \int_{\mathbb{R}^4} \mathbf{1}_{\{\tilde{\pi}_x(z^1 - z^2) > 0, \tilde{\pi}_v(z^1 - z^2) = 0\}} \mathbb{P}\{\tau^\varepsilon \leq M \mid (Z_0^{1,\varepsilon}, Z_0^\varepsilon) = (z^1, z^2)\} d\eta^\varepsilon(z^1, z^2), \end{aligned}$$

where η^ε is the conditional law of $(Z_{\rho^\varepsilon}^{1,\varepsilon}, Z_{\rho^\varepsilon}^{2,\varepsilon})$ given $\rho^\varepsilon \leq M$, with $\rho^\varepsilon = \inf(\sigma^\varepsilon, \tau^\varepsilon)$, under the initial condition $((1, -\sqrt{2}), (-1, \sqrt{2}))$. By Lemma 9, this shows that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau^\varepsilon \leq M, \sigma^\varepsilon < \tau^\varepsilon\} = 0.$$

We deduce that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tau_2^\varepsilon \leq M, \sigma^\varepsilon < \tau^\varepsilon\} = 0. \quad (29)$$

Moreover, following the proof of (27), we have

$$\begin{aligned} & \mathbb{P}\{\tau_2^\varepsilon \leq M, \tau^\varepsilon < \sigma^\varepsilon\} \\ & \leq \int_{\mathbb{R}^4} \mathbf{1}_{\{\tilde{\pi}_x(z^1 - z^2) = 0, \tilde{\pi}_v(z^1 - z^2) < 0\}} \mathbb{P}\{\tau^\varepsilon \leq M \mid (Z_0^{1,\varepsilon}, Z_0^\varepsilon) = (z^1, z^2)\} d\eta^\varepsilon(z^1, z^2), \end{aligned}$$

which tends to 0 by the same argument as above. Together with (29), we deduce (28). (Keep in mind that $\mathbb{P}\{\tau^\varepsilon = \sigma^\varepsilon\} = 0$.) ■

3 Deterministic case: no coalescence for Lebesgue a.e. initial configuration

The result of this section is known but not easy to find in the literature and we prefer to report it as a guideline to the next section, where it will be generalized to the stochastic model. (To be complete, we also refer the reader to the seminal works by DiPerna and Lions [9] and Ambrosio [1] for more general results about almost-everywhere solvability of ordinary differential equations, but we emphasize that a different notion of uniqueness is investigated therein, namely uniqueness of a regular Lagrangian flow.)

Theorem 16 *For Lebesgue a.e. $z \in \mathbb{R}^{2Nd}$, there exists a unique solution $(Z_t)_{t \geq 0}$ to system (9) with initial condition z . It is free of coalescence in the phase space, that is $Z_t \in \Gamma_N$ for any $t > 0$.*

Moreover, the set of instants at which coalescence occur in the space of positions is of zero Lebesgue measure, that is $\Pi_x(Z_t) \in \Gamma_{x,N}$ for a.e. $t \geq 0$.

The proof of Theorem 16 is divided into three parts: in Subsection 3.1, we investigate a mollified version of (9); in Subsection 3.2, we establish the existence of a solution to (9) (for a.e. initial condition) by a compactness argument along the mollification introduced in Subsection 3.1; uniqueness is discussed in Subsection 3.3.

3.1 Smoothed system of equations

For every $\varepsilon > 0$, let F_ε be equal to F outside $B_d(0, \varepsilon)$, but smooth inside $B_d(0, \varepsilon)$, with $\sup_{x \in \mathbb{R}^d} |F_\varepsilon(x)| \leq \sup_{x \in \mathbb{R}^d} |F(x)| + 1$. Given such an F_ε , consider the smoothed system

$$\begin{aligned} \frac{dX_t^{i,\varepsilon}}{dt} &= V_t^{i,\varepsilon}, \\ \frac{dV_t^{i,\varepsilon}}{dt} &= \frac{1}{N} \sum_{j \in N_i} F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}), \quad t \geq 0, \quad i \in \{1, \dots, N\}. \end{aligned} \tag{30}$$

Since F_ε is bounded and Lipschitz-continuous on compact subsets of \mathbb{R}^d , we have global existence and uniqueness, for every initial condition $z \in \mathbb{R}^{2Nd}$.

Following the notations defined in Subsection 1.5, we set $Z_t^{i,\varepsilon} = (X_t^{i,\varepsilon}, V_t^{i,\varepsilon})$ for any $i \in \{1, \dots, N\}$ and $t \geq 0$, that is $Z_t^{i,\varepsilon}$ stands for the coordinate of the i th particle in the phase space, $X_t^{i,\varepsilon}$ for its position and $V_t^{i,\varepsilon}$ for its velocity. We also set $Z_t^\varepsilon = (Z_t^{i,\varepsilon})_{1 \leq i \leq N}$, $X_t^\varepsilon = (X_t^{i,\varepsilon})_{1 \leq i \leq N}$ and $V_t^\varepsilon = (V_t^{i,\varepsilon})_{1 \leq i \leq N}$.

With these notations, we can write Eq. (30) in a global way:

$$\frac{dZ_t^\varepsilon}{dt} = \mathbb{F}_\varepsilon(Z_t^\varepsilon), \quad t \geq 0,$$

for a suitably defined vector field $\mathbb{F}_\varepsilon : \mathbb{R}^{2Nd} \rightarrow \mathbb{R}^{2Nd}$. Specifically, \mathbb{F}_ε may be written as $\mathbb{F}_\varepsilon = (\mathbb{F}_\varepsilon^1, \dots, \mathbb{F}_\varepsilon^N)$, $\mathbb{F}_\varepsilon^i : \mathbb{R}^{2Nd} \rightarrow \mathbb{R}^{2d}$, $1 \leq i \leq N$, denoting its $2d$ -dimensional components. For any $1 \leq i \leq N$,

$$\frac{dZ_t^{i,\varepsilon}}{dt} = \mathbb{F}_\varepsilon^i(Z_t^\varepsilon), \quad t \geq 0,$$

\mathbb{F}_ε^i having the form $\mathbb{F}_\varepsilon^i = (\mathbb{F}_\varepsilon^{i,x}, \mathbb{F}_\varepsilon^{i,v})$, with

$$\begin{aligned} \mathbb{F}_\varepsilon^{i,x}(z) &= v^i, \\ \mathbb{F}_\varepsilon^{i,v}(z) &= \frac{1}{N} \sum_{j \in N_i} F_\varepsilon(x^i - x^j), \quad z = ((x^1, v^1), \dots, (x^N, v^N)) \in \mathbb{R}^{2Nd}. \end{aligned} \quad (31)$$

By existence and uniqueness, we can define the (uniquely and well defined) maps

$$T_t^\varepsilon : \mathbb{R}^{2Nd} \ni z \mapsto T_t^\varepsilon(z) = Z_t^\varepsilon \text{ whenever } Z_0^\varepsilon = z, \quad t \geq 0.$$

In Lemmas 17, 18, 19 and 20 below, we establish several crucial properties of the dynamical system $(T_t^\varepsilon)_{t \geq 0}$. Lemma 17 is a conservation lemma. Lemmas 18, 19 and 20 investigate the coalescence phenomenon in the phase space and in the position space.

Lemma 17 *For any $t \geq 0$, T_t^ε preserves the Lebesgue measure on \mathbb{R}^{2Nd} :*

$$\int_{\mathbb{R}^{2Nd}} \varphi(T_t^\varepsilon(z)) dz = \int_{\mathbb{R}^{2Nd}} \varphi(z') dz'$$

for all φ that are in $C_0^\infty(\mathbb{R}^{2Nd})$ and all φ that are measurable and non-negative.

Proof. This is a consequence of $\operatorname{div} \mathbb{F}_\varepsilon = 0$, which holds because

$$\operatorname{div} \mathbb{F}_\varepsilon = \sum_{i=1}^N (\operatorname{div}_{x^i} \mathbb{F}_\varepsilon^{i,x} + \operatorname{div}_{v^i} \mathbb{F}_\varepsilon^{i,v}) = 0$$

almost everywhere for the Lebesgue measure since $\mathbb{F}_\varepsilon^{i,x}$ is a locally Lipschitz function that depends only on v^i and $\mathbb{F}_\varepsilon^{i,v}$ is a locally Lipschitz function that depends only on (x^1, \dots, x^N) . ■

Lemma 18 *Let $\log^+ : (0, +\infty) \ni r \mapsto \log^+(r)$ be the function equal to 0 for $r \geq 1$ and to $-\log r$ for $r \in (0, 1)$. For every $R \geq 1$, set*

$$h_R(z) = 1_{\{|z| \leq R\}} \sum_{(i,j) \in \Delta_N} \log^+ |z^i - z^j|, \quad z \in \mathbb{R}^{2Nd}.$$

Then, for any $T > 0$, there exists a constant $C_{R,T}$, depending on R , T and the parameters in (\mathbf{A}) only, such that for any $\varepsilon > 0$

$$\int_{\mathbb{R}^{2Nd}} \sup_{t \in [0, T]} h_R(T_t^\varepsilon(z)) dz \leq C_{R,T}.$$

Proof. Step 1. Let $\theta_R : \mathbb{R}^{2Nd} \rightarrow [0, 1]$ be a smooth function equal to 1 on $B_{2Nd}(0, R)$, equal to 0 outside $B_{2Nd}(0, R + 2)$, with values in $[0, 1]$ and with $\sup_{z \in \mathbb{R}^{2Nd}} |\nabla \theta_R(z)| \leq 1$. For a given smooth function $\phi : \mathbb{R} \rightarrow [0, 1]$, with support included in $(0, 1)$, let $\log_\phi^+ : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the smooth function defined as

$$\log_\phi^+(r) = \int_r^1 \frac{\phi(s)}{s} ds \quad \text{for } r \geq 0, \quad (32)$$

so that

$$\left| \frac{d}{dr} \log_\phi^+(r) \right| \leq \frac{1}{r} \quad \text{for } r > 0. \quad (33)$$

As ϕ increases towards the indicator function $\mathbf{1}_{(0,1)}$ of the interval $(0, 1)$, $\log_\phi^+(r)$ increases towards $\log^+(r)$. Define now

$$h_\phi^{\theta_R}(z) = \theta_R(z) \sum_{(i,j) \in \Delta_N} \log_\phi^+(|z^i - z^j|), \quad z \in \mathbb{R}^{2Nd}. \quad (34)$$

As ϕ increases towards the indicator function $\mathbf{1}_{(0,1)}$, $h_\phi^{\theta_R}(z)$ increases towards $h_R(z)$.

In the next step we prove that there exists a constant C , independent of ε , ϕ and the details of θ_R in $B_{2Nd}(0, R + 2) \setminus B_{2Nd}(0, R)$, such that

$$\int_{\mathbb{R}^{2Nd}} \sup_{t \in [0, T]} h_\phi^{\theta_R}(T_t^\varepsilon(z)) dz \leq C. \quad (35)$$

By monotonous convergence of $h_\phi^{\theta_R}$ (restricted to $B_{2Nd}(0, R)$) towards h_R as ϕ increases towards $\mathbf{1}_{(0,1)}$, the claim of the proposition easily follows.

Step 2. Since \log_ϕ^+ is constant in the neighborhood of zero, we can differentiate $\log_\phi^+(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|)$ w.r.t. t . Given an initial condition $z \in B_{2Nd}(0, R)$, we deduce from (33)

$$\left| \frac{d}{dt} \log_\phi^+(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|) \right| \leq \frac{|\mathbb{F}_\varepsilon^i(Z_t^\varepsilon) - \mathbb{F}_\varepsilon^j(Z_t^\varepsilon)|}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|}.$$

(Note that the ratio in the above right-hand side is always finite since \mathbb{F}_ε is Lipschitz-continuous on compact subsets.) Thus,

$$\begin{aligned} \frac{d}{dt} h_\phi^{\theta_R}(Z_t^\varepsilon) &\leq \theta_R(Z_t^\varepsilon) \sum_{(i,j) \in \Delta_N} \frac{|\mathbb{F}_\varepsilon^i(Z_t^\varepsilon) - \mathbb{F}_\varepsilon^j(Z_t^\varepsilon)|}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|} \\ &\quad + |\nabla \theta_R(Z_t^\varepsilon)| \sum_{(i,j) \in \Delta_N} \log_\phi^+(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|). \end{aligned}$$

Since $\theta_R(z) \leq \mathbf{1}_{\{|z| \leq R+2\}}$ and $|\nabla \theta_R(z)| \leq \mathbf{1}_{\{|z| \leq R+2\}}$, we can also write

$$\begin{aligned} \frac{d}{dt} h_\phi^{\theta_R}(Z_t^\varepsilon) &\leq \mathbf{1}_{\{|Z_t^\varepsilon| \leq R+2\}} \sum_{(i,j) \in \Delta_N} \frac{|\mathbb{F}_\varepsilon^i(Z_t^\varepsilon) - \mathbb{F}_\varepsilon^j(Z_t^\varepsilon)|}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|} \\ &\quad + \mathbf{1}_{\{|Z_t^\varepsilon| \leq R+2\}} \sum_{(i,j) \in \Delta_N} \log_\phi^+(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|). \end{aligned}$$

Now we apply very crude estimates to simplify the expressions:

$$\begin{aligned} |\mathbb{F}_\varepsilon^i(Z_t^\varepsilon) - \mathbb{F}_\varepsilon^j(Z_t^\varepsilon)| &\leq 2 |\mathbb{F}_\varepsilon(Z_t^\varepsilon)|, \\ \log_\phi^+(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|) &\leq \frac{1}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|}, \end{aligned}$$

and thus

$$\frac{d}{dt} h_\phi^{\theta_R}(Z_t^\varepsilon) \leq C \mathbf{1}_{\{|Z_t^\varepsilon| \leq R+2\}} (1 + |\mathbb{F}_\varepsilon(Z_t^\varepsilon)|) \sum_{(i,j) \in \Delta_N} \frac{1}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|}.$$

Therefore,

$$\begin{aligned} &\sup_{t \in [0, T]} h_\phi^{\theta_R}(Z_t^\varepsilon) \\ &\leq h_\phi^{\theta_R}(Z_0^\varepsilon) + C \int_0^T \mathbf{1}_{\{|Z_s^\varepsilon| \leq R+2\}} (1 + |\mathbb{F}_\varepsilon(Z_s^\varepsilon)|) \sum_{(i,j) \in \Delta_N} \frac{1}{|Z_s^{i,\varepsilon} - Z_s^{j,\varepsilon}|} ds. \end{aligned} \quad (36)$$

(Pay attention that the above right-hand side may be infinite.)

Step 3. We now integrate (36) on \mathbb{R}^{2Nd} with respect to the Lebesgue measure and apply Lemma 17. We get

$$\begin{aligned} &\int_{\mathbb{R}^{2Nd}} \sup_{t \in [0, T]} h_\phi^{\theta_R}(T_t^\varepsilon(z)) dz \\ &\leq \int_{\mathbb{R}^{2Nd}} h_\phi^{\theta_R}(z) dz + C \int_{\mathbb{R}^{2Nd}} \mathbf{1}_{\{|z| \leq R+2\}} (1 + |\mathbb{F}_\varepsilon(z)|) \sum_{(i,j) \in \Delta_N} \frac{1}{|z^i - z^j|} dz. \end{aligned}$$

By boundedness of F_ε , we have $|\mathbb{F}_\varepsilon(z)| \leq C(1 + |v|)$, with $v = \Pi_v(z)$, so that

$$\int_{\mathbb{R}^{2Nd}} \sup_{t \in [0, T]} h_\phi^{\theta_R}(T_t^\varepsilon(z)) dz \leq C(1 + R) \left(1 + \int_{B_{2Nd}(0, R+2)} \sum_{(i,j) \in \Delta_N} \frac{1}{|z^i - z^j|} dz \right).$$

The above integral is finite since the variables “ z^j ”, $1 \leq j \leq N$, are $2d$ dimensional. The proof is complete. ■

Lemma 19 *Given $R_0, T > 0$, there exists a constant $C_{R_0, T}$, depending on R_0, T and the parameters in (A.1) only, such that, for any $\epsilon \in (0, 1)$ and $\varepsilon > 0$,*

$$\begin{aligned} & \text{Leb}_{2Nd} \left\{ z \in B_{2Nd}(0, R_0) : \inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |\pi_i(T_t^\varepsilon(z)) - \pi_j(T_t^\varepsilon(z))| < \epsilon \right\} \\ & \leq \frac{C_{R_0, T}}{|\log(\epsilon)|}. \end{aligned}$$

Proof. By boundedness of F , given $R_0 > 0$ (the maximum size of the initial conditions) and $T > 0$, there exists $R_1 = R_1(R_0, T) > 0$ such that

$$\sup_{\varepsilon > 0} \sup_{|z| \leq R_0} \sup_{t \in [0, T]} |T_t^\varepsilon(z)| \leq R_1. \quad (37)$$

From Lemma 18 and Chebyshev inequality we get, for $K > 0$,

$$\begin{aligned} & \text{Leb}_{2Nd} \left\{ z \in B_{2Nd}(0, R_0) : \right. \\ & \left. \sup_{t \in [0, T]} \left(\mathbb{1}_{\{|T_t^\varepsilon(z)| \leq R_1\}} \sum_{(i, j) \in \Delta_N} \log^+ |\pi_i(T_t^\varepsilon(z)) - \pi_j(T_t^\varepsilon(z))| \right) > K \right\} \leq \frac{C_{R_1, T}}{K}. \end{aligned}$$

By (37), we can get rid of the indicator function $\mathbb{1}_{\{|T_t^\varepsilon(z)| \leq R_1\}}$ above. Setting

$$A_K = \left\{ z \in B_{2Nd}(0, R_0) : \sup_{t \in [0, T]} \sum_{(i, j) \in \Delta_N} \log^+ |\pi_i(T_t^\varepsilon(z)) - \pi_j(T_t^\varepsilon(z))| \leq K \right\},$$

we obtain $\text{Leb}_{2Nd}(B_{2Nd}(0, R_0) \setminus A_K) \leq C_{R_1, T}/K$. For $\epsilon = \exp(-K)$, we have

$$\inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |Z_t^{i, \varepsilon} - Z_t^{j, \varepsilon}| \geq \epsilon$$

whenever $Z_0^\varepsilon \in A_K$. ■

Lemma 20 *Given $R_0, T > 0$, there exists a constant $C_{R_0, T}$, depending on R_0, T and the parameters in (A.1) only, such that, for any $\varepsilon > 0$, $\epsilon, \delta_0 \in (0, 1)$,*

$$\begin{aligned} & \text{Leb}_{2Nd} \left\{ z \in B_{2Nd}(0, R_0) : \right. \\ & \left. \text{Leb}_1 \left(t \in [0, T] : \inf_{(i, j) \in \Delta_N} |\pi_{i, x}(T_t^\varepsilon(z)) - \pi_{j, x}(T_t^\varepsilon(z))| \leq \frac{\delta_0 \epsilon^2}{C^2} \right) \leq C \delta_0 \right\} \\ & \geq 1 - \frac{C}{|\log(\epsilon)|}. \end{aligned}$$

Proof. Choose $z \in B_{2Nd}(0, R_0)$ such that

$$\inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |Z_t^{i, \epsilon} - Z_t^{j, \epsilon}| \geq \epsilon, \quad (38)$$

with $Z_t^\epsilon = T_t^\epsilon(z)$. Then, the proof is easily completed by Lemma 19, (38) and Proposition 21 below. ■

Proposition 21 *Given $L, R_0, T > 0$, let $(\zeta_t = (\chi_t, \nu_t))_{0 \leq t \leq T}$ be a continuous path with values in \mathbb{R}^{2Nd} such that $\zeta_0 = z \in B_{2Nd}(0, R_0)$, $(\nu_t^i)_{t \geq 0}$ is a L -Lipschitz continuous path with values in \mathbb{R}^d , for $1 \leq i \leq N$, and*

$$\frac{d\chi_t^i}{dt} = \nu_t^i, \quad t \geq 0, \quad i \in \{1, \dots, N\}.$$

Then, there exists a constant C , depending on d, L, N, R_0 and T only, such that, for any $\epsilon, \delta_0 \in (0, 1)$,

$$\begin{aligned} & \inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |\zeta_t^i - \zeta_t^j| \geq \epsilon \\ \Rightarrow & \text{Leb}_1 \left(t \in [0, T] : \inf_{(i, j) \in \Delta_N} |\chi_t^i - \chi_t^j| \leq \delta_0 \frac{\epsilon^2}{C^2} \right) \leq C\delta_0. \end{aligned}$$

Proof. Assume that there exist $\delta \in (0, \epsilon)$, $t_0 \in [0, T)$ and $(i, j) \in \Delta_N$ such that $|\chi_{t_0}^i - \chi_{t_0}^j| \leq \delta$. Since $\inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |\zeta_t^i - \zeta_t^j| \geq \epsilon$, we deduce

$$|\nu_{t_0}^i - \nu_{t_0}^j| \geq \sqrt{\epsilon^2 - \delta^2}.$$

By Lipschitz property of $(\nu_t)_{0 \leq t \leq T}$, there exists a constant C , independent of ϵ, t_0 and δ , such that

$$|\nu_t^i - \nu_t^j| \geq \sqrt{\epsilon^2 - \delta^2} - C(t - t_0), \quad t_0 \leq t \leq T.$$

Therefore, there exists one coordinate $\ell \in \{1, \dots, d\}$ such that

$$|(\nu_t^i - \nu_t^j)_\ell| \geq \frac{\sqrt{\epsilon^2 - \delta^2} - C(t - t_0)}{\sqrt{d}}, \quad t_0 \leq t \leq T.$$

For $C(t - t_0) < \sqrt{\epsilon^2 - \delta^2}$, the right-hand side is always positive so that $(\nu_t^i - \nu_t^j)_\ell$ cannot vanish. By continuity, it is of constant sign. Therefore,

$$|\chi_t^i - \chi_t^j| \geq |(\chi_t^i - \chi_t^j)_\ell| \geq (t - t_0) \frac{\sqrt{\epsilon^2 - \delta^2} - C(t - t_0)}{\sqrt{d}} - \delta,$$

for $C(t - t_0) \leq \sqrt{\epsilon^2 - \delta^2}$. For $\delta \leq \epsilon/2$, we deduce

$$|\chi_t^i - \chi_t^j| \geq (t - t_0) \frac{\epsilon}{4\sqrt{d}} - \delta,$$

for $C(t - t_0) \leq \epsilon/4$. Finally, for $8\sqrt{d}\delta/\epsilon \leq t - t_0 \leq \epsilon/(4C)$, $|\chi_t^i - \chi_t^j| \geq \delta$. Modifying C if necessary, we deduce that

$$|\chi_t^i - \chi_t^j| \geq \delta, \quad (39)$$

for $C\delta/\epsilon \leq t - t_0 \leq \epsilon/C$ and $\delta \leq \epsilon/2$. Assume now w.l.o.g. that $C \geq 2$ and choose δ of the form $\delta_0\epsilon^2/C^2$ with $\delta_0 \leq 1$. Define the set

$$\mathcal{I}_x(\delta_0, \epsilon) = \{t \in [0, T] : |\chi_t^i - \chi_t^j| \leq \delta_0\epsilon^2/C^2\}.$$

By (39), $t_0 \in \mathcal{I}_x(\delta_0, \epsilon) \Rightarrow [t_0 + \delta_0\epsilon/C, t_0 + \epsilon/C] \cap \mathcal{I}_x(\delta_0, \epsilon) = \emptyset$. Therefore,

$$\text{Leb}_1(\mathcal{I}_x(\delta_0, \epsilon)) \leq \delta_0\epsilon/C \lceil TC/\epsilon \rceil \leq \delta_0(T + 1).$$

This completes the proof. ■

3.2 Existence via relative compactness of distributions

We now establish the existence of a solution without coalescence in the phase space by establishing the tightness of the paths $((T_t^\epsilon : z \mapsto T_t^\epsilon(z))_{t \geq 0})_{\epsilon > 0}$, seen as processes from $B_{2Nd}(0, R_0)$ endowed with the Lebesgue measure Leb_{2Nd} into the space $\mathcal{C}([0, +\infty), \mathbb{R}^{2Nd})$ of continuous paths from $[0, +\infty)$ into \mathbb{R}^{2Nd} .

Below, we set $\Xi = \mathcal{C}([0, +\infty), \mathbb{R}^{2Nd})$ and we denote by \mathcal{X} the Borel σ -field generated by the topology of uniform convergence on compact subsets.

Lemma 22 *Given $R_0 > 0$ and $\epsilon > 0$, define the probability measure*

$$\mathbb{Q}^\epsilon(A) = \frac{\text{Leb}_{2Nd}(z \in B_{2Nd}(0, R_0) : (T_t^\epsilon(z))_{t \geq 0} \in A)}{\text{Leb}_{2Nd}(B_{2Nd}(0, R_0))}, \quad A \in \mathcal{X},$$

on the measurable space (Ξ, \mathcal{X}) . Then, the family of measures $(\mathbb{Q}^\epsilon)_{\epsilon > 0}$ is tight.

Proof. The proof is a straightforward consequence of the uniform boundedness of the family $(F_\epsilon)_{\epsilon > 0}$. ■

Lemma 23 *Let $(\varepsilon_n)_{n \geq 0}$ be a decreasing sequence of positive reals with 0 as limit such that the family $(\mathbb{Q}^{\varepsilon_n})_{n \geq 0}$ converges towards some probability \mathbb{Q} on (Ξ, \mathcal{X}) . Then,*

i) The random variable $\Xi \ni \xi \mapsto \xi_0$ has a uniform distribution on the ball $B_{2Nd}(0, R_0)$ under the probability measure \mathbb{Q} .

ii) Define \mathbb{F} as \mathbb{F}_ε in (31) but replacing F_ε therein by F . Then, for \mathbb{Q} almost-every $\xi \in \Xi$,

$$\xi_t = \xi_0 + \int_0^t \mathbb{F}(\xi_s) ds, \quad t \geq 0.$$

iii) For any $T > 0$,

$$\mathbb{Q} \left(\xi \in \Xi : \inf_{(i,j) \in \Delta_N} \inf_{t \in [0, T]} |\pi_i(\xi_t) - \pi_j(\xi_t)| = 0 \right) = 0,$$

$$\mathbb{Q} \left(\xi \in \Xi : \text{Leb}_1 \left(t \in [0, T] : \inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| = 0 \right) > 0 \right) = 0.$$

Proof. The proof of (i) is straightforward. By definition of \mathbb{Q}^ε , the random variable $\Xi \ni \xi \mapsto \xi_0$ has a uniform distribution on the ball $B_{2Nd}(0, R_0)$ under the probability \mathbb{Q}^ε , for $\varepsilon > 0$. Passing to the limit, the same holds under \mathbb{Q} .

We now establish (iii). By Lemma 19, for any $T, \varepsilon > 0$ and $\epsilon \in (0, 1)$,

$$\begin{aligned} & \text{Leb}_{2Nd} \left(z \in B_{2Nd}(0, R_0) : \inf_{t \in [0, T]} \inf_{(i,j) \in \Delta_N} |\pi_i(T_t^\varepsilon(z)) - \pi_j(T_t^\varepsilon(z))| < \epsilon \right) \\ & \leq \frac{C}{|\log(\epsilon)|}, \end{aligned}$$

for some constant C independent of ε and ϵ . That is

$$\mathbb{Q}^\varepsilon \left(\xi \in \Xi : \inf_{t \in [0, T]} \inf_{(i,j) \in \Delta_N} |\pi_i(\xi_t) - \pi_j(\xi_t)| < \epsilon \right) \leq \frac{C}{|\log(\epsilon)|}.$$

The above event is open in Ξ . Passing to the limit we deduce that the same holds under \mathbb{Q} . Letting ϵ tend to zero, we obtain the first point in (iii).

By Lemma 20, we know that, for any $(i, j) \in \Delta_N$ and $0 < \delta_0 < 1$,

$$\begin{aligned} & \mathbb{Q}^\varepsilon \left(\xi \in \Xi : \text{Leb}_1 \left(t \in [0, T] : |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| < \delta_0 \epsilon^2 / C^2 \right) \leq C \delta_0 \right) \\ & \geq 1 - C / |\log(\epsilon)|. \end{aligned}$$

The above event is closed in Ξ . Passing to the limit, we deduce that the same holds under \mathbb{Q} . We deduce that, for any $\delta_0 \in (0, 1)$,

$$\mathbb{Q}(\xi \in \Xi : \text{Leb}_1(t \in [0, T] : |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| = 0) \leq C\delta_0) = 1,$$

that is

$$\mathbb{Q}(\xi \in \Xi : \text{Leb}_1(t \in [0, T] : |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| = 0) = 0) = 1.$$

We now establish (ii). We know that, for any $T, \varepsilon > 0$,

$$\mathbb{Q}^\varepsilon \left(\xi \in \Xi : \xi_t = \xi_0 + \int_0^t \mathbb{F}_\varepsilon(\xi_s) ds, 0 \leq t \leq T \right) = 1. \quad (40)$$

To prove (ii), it is sufficient to prove that

$$\left(\xi_t - \xi_0 - \int_0^t \mathbb{F}_\varepsilon(\xi_s) ds \right)_{0 \leq t \leq T} \cdot \mathbb{Q}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \left(\xi_t - \xi_0 - \int_0^t \mathbb{F}(\xi_s) ds \right)_{0 \leq t \leq T} \cdot \mathbb{Q}, \quad (41)$$

where the left- and the right-hand sides indicate the distributions of the indicated processes under the indicated measures on $\mathcal{C}([0, T], \mathbb{R}^{2Nd})$ and \Rightarrow stands for the convergence in distribution. By boundedness of F_ε , we emphasize that there exists a constant $C > 0$, independent of ε such that, for any $a > 0$ and any $\varepsilon' > \varepsilon$,

$$\begin{aligned} & \mathbb{Q}^\varepsilon \left(\sup_{0 \leq t \leq T} \left| \int_0^t \mathbb{F}_\varepsilon(\xi_s) ds - \int_0^t \mathbb{F}(\xi_s) ds \right| > a \right) \\ & \leq \mathbb{Q}^\varepsilon \left(C \text{Leb}_1 \left(t \in [0, T] : \inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| \leq \varepsilon' \right) \geq a \right). \end{aligned}$$

The event in the right-hand side is closed in Ξ , so that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \mathbb{Q}^\varepsilon \left(\sup_{0 \leq t \leq T} \left| \int_0^t \mathbb{F}_\varepsilon(\xi_s) ds - \int_0^t \mathbb{F}(\xi_s) ds \right| > a \right) \\ & \leq \mathbb{Q} \left(C \text{Leb}_1 \left(t \in [0, T] : \inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| \leq \varepsilon' \right) \geq a \right). \end{aligned} \quad (42)$$

By (iii), $\text{Leb}_1(t \in [0, T] : \inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| = 0) = 0$ with probability 1 under \mathbb{Q} . Letting ε' tend to 0 in (42), we deduce that the left-hand side is 0. Therefore, to prove (41), it is sufficient to prove

$$\left(\xi_t - \xi_0 - \int_0^t \mathbb{F}(\xi_s) ds \right)_{0 \leq t \leq T} \cdot \mathbb{Q}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \left(\xi_t - \xi_0 - \int_0^t \mathbb{F}(\xi_s) ds \right)_{0 \leq t \leq T} \cdot \mathbb{Q}.$$

By dominated convergence Theorem, the map $\mathcal{C}([0, T], \mathbb{R}^{2Nd}) \ni (\xi_t)_{0 \leq t \leq T} \mapsto \left(\int_0^t \mathbb{F}(\xi_s) ds \right)_{0 \leq t \leq T} \in \mathcal{C}([0, T], \mathbb{R}^{2Nd})$ is continuous at any path ξ for which $\text{Leb}_1 \left(t \in [0, T] : \inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\xi_t) - \pi_{j,x}(\xi_t)| = 0 \right) = 0$. By (iii) again, this is true a.s. under \mathbb{Q} : by continuous mapping Theorem, we complete the proof of (41). We deduce that (40) holds true with $\varepsilon = 0$. Letting T tend to $+\infty$, we complete the proof of (ii). ■

Corollary 24 *For Lebesgue a.e. $z \in \mathbb{R}^{2Nd}$, there exists a solution $(Z_t)_{t \geq 0}$ to system (9) with initial condition z , on $[0, \infty)$, without coalescence in the phase space, that is, for any $t \geq 0$, $Z_t \in \Gamma_N$. Moreover, the set of instants at which coalescence occur in the space of positions is of zero Lebesgue measure, that is, $\Pi_x(Z_t) \in \Gamma_{x,N}$ for a.e. $t \geq 0$.*

Proof. The proof is a consequence of Lemmas 22 and 23. Given R_0 as in Lemmas 22 and 23, there exists a probability measure \mathbb{Q} on (Ξ, \mathcal{X}) such that (i), (ii) and (iii) in Lemma 23 hold. We then denote by $(\Omega(\cdot, z))_{z \in B_{2Nd}(0, R_0)}$ a family of regular conditional probabilities of \mathbb{Q} given the random variable $\Xi \ni \xi \mapsto \xi_0$. We get

$$\begin{aligned} 1 &= \mathbb{Q} \left\{ \xi \in \Xi : \forall t \geq 0, \xi_t = \xi_0 + \int_0^t \mathbb{F}(\xi_s) ds \right\} \\ &= \int_{B_{2Nd}(0, R_0)} \Omega \left(\left\{ \xi \in \Xi : \forall t \geq 0, \xi_t = \xi_0 + \int_0^t \mathbb{F}(\xi_s) ds \right\}, z \right) \frac{dz}{V_{2Nd}(R_0)}. \end{aligned}$$

We deduce that, for a.e. $z \in B_{2Nd}(0, R_0)$, the set of solutions to (9) with z as initial condition is not empty. Applying the same argument and using (iii) in Lemma 23, we deduce that, for a.e. $z \in B_{2Nd}(0, R_0)$, the set of solutions to (9) with z as initial condition, without coalescence in the phase space and with Lebesgue null set of instants of coalescence in the space of positions, is not empty. ■

3.3 Uniqueness

We now complete the proof of Theorem 16 by proving uniqueness.

Lemma 25 *Given an initial condition $z \in \Gamma_N$, there exists at most one solution to (9).*

Proof. For a given z as in the statement, consider two solutions $(\zeta_t)_{t \geq 0}$ and $(\zeta'_t)_{t \geq 0}$ to (9) with z as initial condition, $(\zeta_t)_{t \geq 0}$ being free of coalescence in the phase space. As soon as $(\zeta_t)_{t \geq 0}$ and $(\zeta'_t)_{t \geq 0}$ coincide, $(\zeta'_t)_{t \geq 0}$ is also free of coalescence in the phase space. Therefore, if the first splitting time $\tau = \inf\{t \geq 0 : \zeta_t \neq \zeta'_t\}$ is finite, the splitting location $\zeta_\tau = \zeta'_\tau$ is in Γ_N , exactly as $z \in \Gamma_N$. By time-shifting, it is thus sufficient to prove that there exists $\eta > 0$ such that ζ and ζ' coincide on the interval $[0, \eta]$. Below, we write $z = ((x^i)_{1 \leq i \leq N}, (v^i)_{1 \leq i \leq N})$ with $\pi_{i,x}(z) = x^i$ and $\pi_{i,v}(z) = v^i$.

If $\inf_{(i,j) \in \Delta_N} |x^i - x^j| > 0$, there exist small reals $\varepsilon, \eta > 0$ such that both $(\zeta_t)_{0 \leq t \leq \eta}$ and $(\zeta'_t)_{0 \leq t \leq \eta}$ solve the regularized system (30) driven by F_ε on the interval $[0, \eta]$. Uniqueness follows in short time. The existence of such a pair (ε, η) follows from the boundedness of F and the assumption $\inf_{(i,j) \in \Delta_N} |x^i - x^j| > 0$. Indeed, by boundedness of F , both ζ and ζ' stay in a compact subset of \mathbb{R}^{2Nd} in finite time. Therefore, the x -components of $\zeta = (\chi, \nu)$ and $\zeta' = (\chi', \nu')$ are Lipschitz continuous on finite intervals. In particular, in small time, the minimal distance between the positions of the particles $(\chi^i)_{1 \leq i \leq N}$, that is $\inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\zeta) - \pi_{j,x}(\zeta)| = \inf_{(i,j) \in \Delta_N} |\chi^i - \chi^j|$, is bounded from below by a positive real. Similarly, $\inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\zeta') - \pi_{j,x}(\zeta')| = \inf_{(i,j) \in \Delta_N} |(\chi')^i - (\chi')^j|$ is bounded from below by a positive real in small time.

If $\inf_{(i,j) \in \Delta_N} |x^i - x^j| = 0$, we denote by $\mathcal{M} = \{(i, j) \in \Delta_N : |\pi_{i,x}(z) - \pi_{j,x}(z)| = 0\}$. For $(i, j) \in \mathcal{M}$, we know that $|v^i - v^j| > 0$. Therefore, in short time,

$$\nu_t^i - \nu_t^j = v^i - v^j + O(t), \quad (43)$$

where $O(\cdot)$ stands for the Landau notation. Therefore,

$$\chi_t^i - \chi_t^j = t(v^i - v^j) + t^2 \gamma_t, \quad (44)$$

γ being bounded on compact subsets of $[0, +\infty)$. Thus, we can find $\eta > 0$ such that $|\chi_t^i - \chi_t^j| \geq t|v^i - v^j|/2$ for $0 \leq t \leq \eta$. Similarly, $|(\chi'_t)^i - (\chi'_t)^j| \geq t|v^i - v^j|/2$ in small time. By (A.1), we deduce that

$$|F(\chi_t^i - \chi_t^j) - F((\chi'_t)^i - (\chi'_t)^j)| \leq Ct^{-1} |\chi_t - \chi'_t| \leq C \sup_{0 \leq s \leq t} |\nu_s - \nu'_s|, \quad (45)$$

for $t \in (0, \eta]$, η being small enough. (The constant C above depends on $|v^i - v^j|$.) Concerning the pairs $(i, j) \in \Delta_N \setminus \mathcal{M}$, we notice that $\inf_{(i,j) \in \Delta_N \setminus \mathcal{M}} |x^i - x^j| > 0$. Applying the same argument as above and modifying η if necessary, there exists $\varepsilon > 0$ such that $F(\chi_t^i - \chi_t^j)$ and $F((\chi'_t)^i - (\chi'_t)^j)$ coincide with

$F_\varepsilon(\chi_t^i - \chi_t^j)$ and $F_\varepsilon((\chi_t')^i - (\chi_t')^j)$ for $t \in [0, \eta]$ and $(i, j) \in \Delta_N \setminus \mathcal{M}$. Therefore, (45) holds as well, provided η is small enough. Finally, we deduce that, for any $t \in [0, \eta]$,

$$|\nu_t - \nu'_t| \leq C \int_0^t \sup_{0 \leq r \leq s} |\nu_r - \nu'_r| ds.$$

Uniqueness easily follows on $[0, \eta]$. ■

4 Stochastic dynamics: properties of the noise

We now specify the structure of the noise we plug into (9).

Stochastic differential equations considered in Subsection 1.3, see (10), are written in Stratonovich form since Stratonovich integral is the most natural one for approximations and for conservation laws. Anyhow, in this particular case, Stratonovich integrals are (formally) equal to Itô integrals, so that (10) will be interpreted in the usual Itô form

$$\begin{aligned} \frac{dX_t^i}{dt} &= V_t^i \\ dV_t^i &= \frac{1}{N} \sum_{j \in N_i} F(X_t^i - X_t^j) dt + \sum_{k=1}^{\infty} \sigma_k(X_t^i) dW_t^k, \end{aligned} \quad (46)$$

$t \geq 0, i \in \{1, \dots, N\}$. The (formal) equivalence is based on the fact that

$$\int_0^t \sigma_k(X_s^i) \circ dW_s^k = \int_0^t \sigma_k(X_s^i) dW_s^k + \frac{1}{2} [\sigma_k(X^i), W^k]_t, \quad t \geq 0,$$

where $[\cdot, \cdot]_t$ denotes the mutual covariation of two semimartingales. Here the local martingale part of $(\sigma_k(X_t^i))_{t \geq 0}$ is zero, since $(\sigma_k(X_t^i))_{t \geq 0}$ is of bounded variation as the composition of a Lipschitz function and of a process of bounded variation. We shall not treat more rigorously this equivalence, which is of secondary importance here, and, from now, we shall adopt the Itô formulation.

4.1 Structure of the noise

We here discuss several crucial properties of the functions $((\sigma_k^i)_{1 \leq i \leq d})_{k \in \mathbb{N} \setminus \{0\}}$.

Assuming that the series

$$\sum_{k=1}^{\infty} \sigma_k^\alpha(\tilde{x}) \sigma_k^\beta(\tilde{y}) \quad (47)$$

converges uniformly on compact sets in (\tilde{x}, \tilde{y}) , for each $\alpha, \beta = 1, \dots, d$, we can denote by $Q(\tilde{x}, \tilde{y})$ the limit $d \times d$ matrix-valued function

$$Q(\tilde{x}, \tilde{y}) = \sum_{k=1}^{\infty} \sigma_k(\tilde{x}) \otimes \sigma_k(\tilde{y}). \quad (48)$$

This is the covariance function of the random field $\mathbb{R}^d \ni \tilde{x} \mapsto \sum_{k=1}^{\infty} \sigma_k(\tilde{x}) W_1^k$. The covariance function $Q(\tilde{x}, \tilde{y})$ plays a very important role in the next discussion and we shall impose specific assumptions on it. Recall that it satisfies

$$\begin{aligned} \sum_{i,j=1}^N \langle Q(x^j, x^i) v^i, v^j \rangle_{\mathbb{R}^d} &= \sum_{i,j=1}^N \sum_{k=1}^{\infty} \langle \sigma_k(x^i), v^i \rangle_{\mathbb{R}^d} \langle \sigma_k(x^j), v^j \rangle_{\mathbb{R}^d} \\ &= \sum_{k=1}^{\infty} \left(\sum_{i=1}^N \langle \sigma_k(x^i), v^i \rangle_{\mathbb{R}^d} \right)^2 \geq 0 \end{aligned}$$

for all $(x^1, \dots, x^N) \in \mathbb{R}^{Nd}$ and $(v^1, \dots, v^N) \in \mathbb{R}^{Nd}$.

Definition 26 Let us call $Q(\tilde{x}, \tilde{y})$ strictly positive on $\Gamma_{x,N}$ if

$$\sum_{i,j=1}^N \langle Q(x^j, x^i) v^i, v^j \rangle_{\mathbb{R}^d} > 0$$

for all $(x^1, \dots, x^N) \in \Gamma_{x,N}$ and $v = (v^1, \dots, v^N) \in \mathbb{R}^{Nd} \setminus \{0\}$.

In the following, we shall assume (in addition to (A.1)) that:

(A.2) For each $k \in \mathbb{N} \setminus \{0\}$, σ_k is Lipschitz-continuous on \mathbb{R}^d , and the series in (47) converges uniformly on every compact subset of $\mathbb{R}^d \times \mathbb{R}^d$.

(A.3) The mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (\tilde{x}, \tilde{y}) \mapsto Q(\tilde{x}, \tilde{y})$ is bounded on the diagonal, that is $\sup_{\tilde{x} \in \mathbb{R}^d} |Q(\tilde{x}, \tilde{x})| < +\infty$. Moreover, it satisfies the Lipschitz type regularity property

$$\sup_{\tilde{x}, \tilde{y} \in \mathbb{R}^d, \tilde{x} \neq \tilde{y}} \frac{|Q(\tilde{x}, \tilde{x}) + Q(\tilde{y}, \tilde{y}) - 2Q(\tilde{x}, \tilde{y})|}{|\tilde{x} - \tilde{y}|^2} < +\infty. \quad (49)$$

Finally, $Q(\tilde{x}, \tilde{y})$ is strictly positive on $\Gamma_{x,N}$ as in Definition 26.

Remark 27 *The regularity assumptions on σ_k and Q in (A.2) and (A.3) respectively are strongly related one with each other. Specifically, the Lipschitz condition (49) implies a strong Lipschitz property of the fields $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$:*

$$\begin{aligned} & \sum_{k=1}^{\infty} (\sigma_k^\alpha(\tilde{x}) - \sigma_k^\alpha(\tilde{y})) (\sigma_k^\beta(\tilde{x}) - \sigma_k^\beta(\tilde{y})) \\ &= Q_{\alpha\beta}(\tilde{x}, \tilde{x}) - Q_{\alpha\beta}(\tilde{x}, \tilde{y}) - Q_{\alpha\beta}(\tilde{y}, \tilde{x}) + Q_{\alpha\beta}(\tilde{y}, \tilde{y}) \leq C |\tilde{x} - \tilde{y}|^2. \end{aligned} \quad (50)$$

Conversely, Eq. (49) holds if the Lipschitz constant of the $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$ are assume to be square summable.

This connection must be understood in the following way: in practice, we are given the covariance function Q first. Precisely, given a function $Q : \mathbb{R}^{2d} \ni (\tilde{x}, \tilde{y}) \mapsto Q(\tilde{x}, \tilde{y})$ with values in the set of symmetric matrices of size $d \times d$, satisfying (A.3) and of positive type, that is

$$\forall n \geq 1, \forall \tilde{x}^1, \dots, \tilde{x}^n, \tilde{v}^1, \dots, \tilde{v}^n \in \mathbb{R}^d, \quad \sum_{i,j=1}^n \langle Q(\tilde{x}^j, \tilde{x}^i) \tilde{v}^i, \tilde{v}^j \rangle_{\mathbb{R}^d} \geq 0,$$

Q may be expressed as a covariance function as in (48), for some fields $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$ satisfying (A.2). We refer to Theorem 4.2.5 in Kunita [21].

Remark 28 *When Q is given first, a typical way to guarantee (49) consists in assuming Q to be of class \mathcal{C}^2 with bounded mixed derivatives, that is $\sup_{(\tilde{x}, \tilde{y}) \in \mathbb{R}^d \times \mathbb{R}^d} |\partial_{\tilde{x}, \tilde{y}}^2 Q(\tilde{x}, \tilde{y})| < +\infty$. Indeed, in such a case,*

$$\begin{aligned} & Q_{\alpha\beta}(\tilde{x}, \tilde{y}) - Q_{\alpha\beta}(\tilde{x}, \tilde{y}) - Q_{\alpha\beta}(\tilde{y}, \tilde{x}) + Q_{\alpha\beta}(\tilde{y}, \tilde{y}) \\ &= \sum_{i,j=1}^d \left[(\tilde{x} - \tilde{y})_i (\tilde{x} - \tilde{y})_j \right. \\ & \quad \left. \times \int_0^1 \int_0^1 [\partial_{1,j} \partial_{2,i} Q_{\alpha\beta}(s\tilde{x} + (1-s)\tilde{y}, r\tilde{x} + (1-r)\tilde{y})] ds dr \right], \end{aligned}$$

where $\partial_{1,j}$ and $\partial_{2,i}$ denote the partial derivatives of the function $\tilde{x} \mapsto Q(\tilde{x}, \tilde{y})$ and $\tilde{y} \mapsto Q(\tilde{x}, \tilde{y})$ respectively. (Basically, $\partial_{1,j}$ and $\partial_{2,i}$ also read $\partial_{\tilde{x}_j}$ and $\partial_{\tilde{y}_i}$ respectively, but we feel here more understandable to use the notations $\partial_{1,j}$ and $\partial_{2,i}$.)

Remark 29 By Eq. (50), one can apply classical Lipschitz arguments to prove existence and uniqueness of a strong solution to an equation of the form

$$dY_t = b(Y_t) dt + \sum_{k=1}^{\infty} \sigma_k(Y_t) dW_t^k, \quad t \geq 0,$$

(or to systems of similar type) when b is also Lipschitz continuous. Obviously, we shall use Eq. (50) as well to investigate (46).

Remark 30 In all the space homogeneous examples of the next subsection, $Q(\tilde{x}, \tilde{x})$ is constant, so that it is bounded on \mathbb{R}^d .

4.2 Ellipticity of N -point motion and examples

For every $k \in \mathbb{N} \setminus \{0\}$, introduce the vector field $A_k : \mathbb{R}^{Nd} \rightarrow \mathbb{R}^{Nd}$ defined as

$$A_k(x^1, \dots, x^N) = (\sigma_k(x^1), \dots, \sigma_k(x^N)).$$

These fields govern the N -point motion of the simple equation

$$dY_t = \sum_{k=1}^{\infty} A_k(Y_t) dW_t^k, \quad t \geq 0, \quad (51)$$

which may be also expressed as

$$dY_t^i = \sum_{k=1}^{\infty} \sigma_k(Y_t^i) dW_t^k, \quad t \geq 0, \quad i \in \{1, \dots, N\},$$

the unknown process $(Y_t)_{t \geq 0}$ being \mathbb{R}^{Nd} -valued. In this subsection, we investigate the ellipticity properties of the fields $(A_k)_{k \in \mathbb{N} \setminus \{0\}}$ or, equivalently, of Eq. (51). On $\Gamma_{x,N}^{\mathbb{C}}$, $\text{Span} \{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}}$ cannot be \mathbb{R}^{Nd} so that the N -point motion there moves along a restricted number of directions. On $\Gamma_{x,N}$, the non-degeneracy is equivalent to the *strict positivity* of $Q(x, y)$ on $\Gamma_{x,N}$:

Lemma 31 Q satisfies Definition 26, that is $Q(\tilde{x}, \tilde{y})$ is strictly positive on $\Gamma_{x,N}$, if and only if

$$\text{Span} \{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}} = \mathbb{R}^{Nd}, \quad \forall x = (x^1, \dots, x^N) \in \Gamma_{x,N}. \quad (52)$$

Proof. The set $\text{Span} \{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}}$ is a vector subspace of \mathbb{R}^{Nd} , hence (52) is equivalent to the fact that there is no vector $v = (v^1, \dots, v^N) \in \mathbb{R}^{Nd} \setminus \{0\}$ orthogonal to $\text{Span} \{A_k(x)\}_{k \in \mathbb{N}}$, that is, given $v = (v^1, \dots, v^N) \in \mathbb{R}^{Nd} \setminus \{0\}$, we have

$$\sum_{k=1}^{\infty} \langle A_k(x), v \rangle_{\mathbb{R}^{Nd}}^2 > 0.$$

Now,

$$\sum_{k=1}^{\infty} \langle A_k(x), v \rangle_{\mathbb{R}^{Nd}}^2 = \sum_{k=1}^{\infty} \left(\sum_{i=1}^N \langle \sigma_k(x^i), v^i \rangle_{\mathbb{R}^d} \right)^2 = \sum_{i,j=1}^N \langle Q(x^j, x^i) v^i, v^j \rangle_{\mathbb{R}^d}.$$

The proof is complete. ■

We want to exhibit interesting examples of strictly positive covariance functions $Q(\tilde{x}, \tilde{y})$. Let us restrict ourselves to the space-homogeneous case, namely we assume that there is a $d \times d$ matrix-valued function $Q(\tilde{x})$ such that $Q(\tilde{x}, \tilde{y}) = Q(\tilde{x} - \tilde{y})$ and we assume that the following spectral representation holds:

$$Q(\tilde{x}) = \int_{\mathbb{R}^d} e^{ik \cdot \tilde{x}} \mathcal{Q}(k) dk, \quad x \in \mathbb{R}^d, \quad (53)$$

where the matrix-valued spectral density \mathcal{Q} , with coordinates in $L^1(\mathbb{R}^d)$, takes values in the space of non-negative real symmetric $d \times d$ matrices and satisfies $\mathcal{Q}(-k) = \mathcal{Q}(k)$, $k \in \mathbb{R}^d$. (Above, $k \cdot \tilde{x}$ is a shorten notation for $\langle k, \tilde{x} \rangle_{\mathbb{R}^d}$.)

Lemma 32 *Assume that \mathcal{Q} has the following property: for any \mathbb{R}^d -valued trigonometric polynomial $v(k)$ of the form $v(k) = \sum_{j=1}^N v^j e^{ik \cdot x^j}$, for some $(x^1, \dots, x^N), (v^1, \dots, v^N) \in \mathbb{R}^{Nd}$ and for $i^2 = -1$, the almost-everywhere equality*

$$\langle \mathcal{Q}(k) v(k), v(k) \rangle_{\mathbb{C}^d} = 0 \quad \text{for a.e. } k \in \mathbb{R}^d$$

implies $v(k) = 0$ for any $k \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}^d}$ denotes the Hermitian product in \mathbb{C}^d . Then $Q(\tilde{x}, \tilde{y})$ is strictly positive on $\Gamma_{x,N}$.

(Keep in mind that, for any $u, u' \in \mathbb{C}^d$, $\langle u, u' \rangle_{\mathbb{C}^d} = \sum_{j=1}^d \bar{u}^j (u')^j$, \bar{u} denoting the conjugate of u . We will also write $\langle u, u' \rangle_{\mathbb{C}^d} = \langle \bar{u}, u' \rangle_{\mathbb{R}^d}$ with an abuse of notation.)

Proof. We have

$$\begin{aligned} \sum_{j,\ell=1}^N \langle Q(x^j, x^\ell) v^\ell, v^j \rangle_{\mathbb{R}^d} &= \sum_{j,\ell=1}^N \int_{\mathbb{R}^d} e^{ik \cdot (x^j - x^\ell)} \langle Q(k) v^\ell, v^j \rangle_{\mathbb{R}^d} dk \\ &= \int_{\mathbb{R}^d} \langle Q(k) v(k), v(k) \rangle_{\mathbb{C}^d} dk, \end{aligned} \quad (54)$$

where $v(k) = \sum_{j=1}^N v^j e^{ik \cdot x^j}$. Since $Q(k)$ is a non-negative symmetric real matrix, $\langle Q(k) v(k), v(k) \rangle_{\mathbb{C}^d}$ is real and non-negative.

Choose $x \in \Gamma_{x,N}$ and $v \in \mathbb{R}^{Nd}$ such that the left-hand side in (54) is zero. Since $\langle Q(k) v(k), v(k) \rangle_{\mathbb{C}^d} \geq 0$ for every $k \in \mathbb{R}^d$, we deduce that $\langle Q(k) v(k), v(k) \rangle_{\mathbb{C}^d} = 0$ for a.e. $k \in \mathbb{R}^d$. By assumption, this implies $v(k) = 0$ for any $k \in \mathbb{R}^d$ and thus $(v^1, \dots, v^N) = 0$ since $v(k)$ is a (vector valued) trigonometric polynomial driven by pairwise different vectors x^1, \dots, x^N . (See Remark 38 below.) ■

It is now easy to produce examples of covariances $Q(\tilde{x}, \tilde{y})$ that are *strictly positive* on $\Gamma_{x,N}$: for instance, it is sufficient to take Q such that $Q(k)$ is (strictly) positive definite for a.e. $k \in \mathbb{R}^d$. In fact, it is sufficient that $Q(k)$ is (strictly) positive definite on a set of positive Lebesgue measure:

Proposition 33 *If the matrix $Q(k)$ is strictly positive definite on a Borel subset of \mathbb{R}^d of positive Lebesgue measure, then $Q(\tilde{x}, \tilde{y})$ is strictly positive on $\Gamma_{x,N}$.*

Proof. If A denotes the Borel subset of \mathbb{R}^d of positive Lebesgue measure in the assumption and $\langle Q(k) v(k), v(k) \rangle_{\mathbb{C}^d} = 0$ for a.e. k , then $v(k) \equiv 0$ a.s. on a Borel subset $\tilde{A} \subset A \subset \mathbb{R}^d$, still of the same (positive) Lebesgue measure. This implies $v(k) \equiv 0$ because v is a trigonometric polynomial, see remark 38 below. ■

The previous proposition does not cover important examples intensively used in the literature on diffusion of passive scalars. We now prove that the matrix $Q(\tilde{x}, \tilde{y})$ is *strictly positive* on $\Gamma_{x,N}$ in these examples as well, provided very general conditions on the fields $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$. Precisely, the description we make below corresponds to an isotropic random field $\sum_{k=1}^{\infty} \sigma_k(\tilde{x}) W_1^k$ on \mathbb{R}^d .

Example 34 We here call isotropic case the case when

$$Q(U\tilde{x}) = UQ(\tilde{x})U^\top, \quad \tilde{x} \in \mathbb{R}^d, \quad U \in \mathcal{O}(\mathbb{R}^d),$$

where $\mathcal{O}(\mathbb{R}^d)$ is the set of orthogonal matrices of dimension d (U^\top denotes the transpose of U). This is the case when $\mathcal{Q}(k)$ in (53) has the form

$$\mathcal{Q}(k) = \pi_k f(|k|), \quad \text{that is} \quad Q(x) = \int_{\mathbb{R}^d} e^{ik \cdot x} \pi_k f(|k|) dk,$$

where

$$\pi_k = \begin{cases} (1 - \mathfrak{p}) I_d + (\mathfrak{p}d - 1) \frac{k \otimes k}{|k|^2} & \text{for } d \geq 2 \\ 1 & \text{for } d = 1 \end{cases}$$

with $\mathfrak{p} \in [0, 1]$ and $f : [0, +\infty) \rightarrow \mathbb{R}$ is in $L^1([0, +\infty))$ and satisfies

$$f(r) \geq 0 \quad \text{for a.e. } r > 0. \quad (55)$$

The matrix $\mathcal{Q}(k)$ is symmetric, it satisfies $\mathcal{Q}(-k) = \mathcal{Q}(k)$, and it is almost-everywhere non-negative because (we restrict the proof to $d \geq 2$ for obvious reasons)

$$|k|^2 \langle \pi_k w, w \rangle_{\mathbb{C}^d} = (1 - \mathfrak{p}) |k|^2 |w|^2 + (\mathfrak{p}d - 1) |\langle k, w \rangle_{\mathbb{C}^d}|^2, \quad w \in \mathbb{C}^d,$$

and, by Schwartz inequality,

$$\begin{aligned} |k|^2 \langle \pi_k w, w \rangle_{\mathbb{C}^d} &\geq (1 - \mathfrak{p}) |\langle k, w \rangle_{\mathbb{C}^d}|^2 + (\mathfrak{p}d - 1) |\langle k, w \rangle_{\mathbb{C}^d}|^2 \\ &= \mathfrak{p}(d - 1) |\langle k, w \rangle_{\mathbb{C}^d}|^2 \geq 0 \end{aligned} \quad (56)$$

(the dimension d is a positive integer and $\mathfrak{p} \in [0, 1]$). (Inequality is here given in \mathbb{C}^d but only the \mathbb{R}^d part is useful to prove non-negativity of $\mathcal{Q}(k)$. The full inequality in \mathbb{C}^d will be used next.)

We refer the reader to Le Jan [23], Monin and Yaglom [30] and Yaglom [34] for references where this form (for particular choices of f) is used or investigated. This class of covariances is related to the Batchelor regime of the Kraichnan model, where $f(r) = (r_0^2 + r^2)^{-(d+\varpi)/2}$ with $\varpi = 2$, see Gawedzki, Falkovich and Vergassola [12]. In the limit $r_0 \rightarrow 0$, the covariance of the increments of the noise is scale invariant with scaling exponent equal to 2. The ‘‘turbulent regime’’ of the Kraichnan model ($0 \leq \varpi < 2$) is in contrast not included in our main final result because of the regularity properties we require on Q .

Proposition 35 *If there exists a Borel set $A \subset [0, \infty)$ such that*

$$\text{Leb}_1(A) > 0, \quad \text{and} \quad f(r) > 0 \quad \text{for } r \in A, \quad (57)$$

then $Q(\tilde{x}, \tilde{y})$ is strictly positive on $\Gamma_{x,N}$.

Proof. Step 1. From Lemma 32 it is sufficient to prove that the condition

$$f(|k|) \langle \pi_k v(k), v(k) \rangle_{\mathbb{C}^d} = 0 \quad \text{for a.e. } k \in \mathbb{R}^d$$

implies $v(k) = 0$ for any $k \in \mathbb{R}^d$, when $v(k)$ has the form $v(k) = \sum_{j=1}^N v^j e^{ik \cdot x^j}$ for some $(v^1, \dots, v^N) \in \mathbb{R}^{Nd}$ and $(x^1, \dots, x^N) \in \Gamma_{x,N}$. Let A^* be the set of all $k \in \mathbb{R}^d$ such that $|k| \in A$. It holds $\text{Leb}_d(A^*) > 0$. If $f(|k|) \langle \pi_k v(k), v(k) \rangle_{\mathbb{C}^d} = 0$ for a.e. $k \in \mathbb{R}^d$, then $\langle \pi_k v(k), v(k) \rangle_{\mathbb{C}^d} = 0$ for a.e. $k \in A^*$, namely for all k in a positive measure set $\widetilde{A^*}$. We have to prove that this implies $v(k) \equiv 0$.

Let us understand what the condition $\langle \pi_k w, w \rangle_{\mathbb{C}^d} = 0$ means for some $w \in \mathbb{C}^d$. In the case $d = 1$ it simply implies $w = 0$. For $\mathfrak{p} \in (0, 1]$, $d > 1$, inequality (56) implies $\mathfrak{p}(d-1) \langle k, w \rangle_{\mathbb{C}^d}^2 = 0$, and thus $\langle k, w \rangle_{\mathbb{C}^d}^2 = 0$. Finally, in the case $\mathfrak{p} = 0$, $d > 1$, for all $w \in \mathbb{C}^d$ we have

$$|k|^2 \langle \pi_k w, w \rangle_{\mathbb{C}^d} = (|k| |w| - \langle k, w \rangle_{\mathbb{C}^d})^2$$

and thus $\langle \pi_k w, w \rangle_{\mathbb{C}^d} = 0$ implies that w is parallel to k , if $k \neq 0$. (That is $w = \lambda k$ for some $\lambda \in \mathbb{R}$.)

Let us go back to the main line of the proof. We have $\langle \pi_k v(k), v(k) \rangle_{\mathbb{C}^d} = 0$ for all $k \in \widetilde{A^*}$. Depending on the values of \mathfrak{p} and d , this implies at least one of the three following conditions: $v(k) = 0$ for all $k \in \widetilde{A^*}$, or $\langle k, v(k) \rangle_{\mathbb{C}^d} = 0$ for all $k \in \widetilde{A^*}$, or $v(k)$ is parallel to k for all $k \in \widetilde{A^*}$ (except maybe at $k = 0$). From Remark 38 below, all these three inequalities can be extended to the whole \mathbb{R}^d , see¹. Therefore, depending on the values of \mathfrak{p} and d , at least one of the three following conditions is satisfied: $v(k) = 0$ for a.e. $k \in \mathbb{R}^d$, or $\langle k, v(k) \rangle_{\mathbb{C}^d} = 0$ for a.e. $k \in \mathbb{R}^d$, or $v(k)$ is parallel to k for a.e. $k \in \mathbb{R}^d$. In the first case, we have $v(k) = 0$ for any $k \in \mathbb{R}^d$. The third case is treated in Theorem 4.7 of [18], see also [11] and also implies $v(k) = 0$ for any $k \in \mathbb{R}^d$. Inspired by those works, we prove below that the same holds in the second case.

¹Note that $v(k) \parallel k$ may be written as $\sum_{j=1}^N \langle v_\ell^j e_{\ell'} - v_{\ell'}^j e_\ell, k \rangle_{\mathbb{C}^d} e^{ix^j \cdot k} = 0$, for $1 \leq \ell, \ell' \leq d$, where $(e_\ell)_{1 \leq \ell \leq d}$ is the canonical basis of \mathbb{C}^d .

Step 2. We have to prove that

$$\sum_{j=1}^N \langle k, v^j \rangle_{\mathbb{C}^d} e^{ik \cdot x^j} = 0$$

for a.e. $k \in \mathbb{R}^d$ implies $v = 0$ and thus $v(k) = 0$ for any $k \in \mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function with compact support. Denoting by $\hat{\varphi}$ the Fourier transform of φ , we deduce

$$\sum_{j=1}^N \langle k, v^j \rangle_{\mathbb{C}^d} e^{ik \cdot x^j} \hat{\varphi}(k) = 0$$

for a.e. $k \in \mathbb{R}^d$, hence the integral in k is equal to zero and thus

$$\sum_{j=1}^N \langle v^j, \nabla \varphi(x^j) \rangle_{\mathbb{R}^d} = 0.$$

Since the points x^i are all different, we may construct a function φ such that $\nabla \varphi(x^i)$ is equal to v^i , and thus we get $v = 0$. The proof is complete. ■

Remark 36 *Proposition 35 can be called ellipticity of the of N -point motion of the isotropic noise introduced in Example 34. Consider indeed Eq. (51) with N initial conditions $x^1, \dots, x^N \in \mathbb{R}^d$ and denote the corresponding solution by $(Y_t^1, \dots, Y_t^N)_{t \geq 0}$.*

The noise in (53) may be quite degenerate a priori: for instance, if the fields $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$ are constant, then the fields $(A_k)_{k \geq 1}$ act along the “diagonal” of \mathbb{R}^{Nd} only, so that the law of Y_t , for $t \geq 0$, is concentrated on a subset of \mathbb{R}^{Nd} of dimension smaller than Nd . On the contrary, Proposition 35 states that the family of fields $\{A_k(Y_t)\}_{k \in \mathbb{N} \setminus \{0\}}$ point in all directions as soon as Y_t is in $\Gamma_{x,N}$. By Malliavin calculus, this implies that the law of Y_t , for $t > 0$, is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^{Nd} when $Y_0 = (x^1, \dots, x^N) \in \Gamma_{x,N}$. The proof consists in applying Bouleau and Hirsch criterion for absolute continuity of the distribution of a Malliavin differentiable random vector, on the same model as in Theorem 2.3.1 in Nualart [32] for SDEs driven by a finite-dimensional noise; we refer to Nguyen [31] for Malliavin differentiability of the solution to (53) under (A.1–3) and to Proposition 2.3 therein for a typical application of the Bouleau

and Hirsch criterion. In particular, there is no need to compute some Lie brackets of the fields $(A_k)_{k \geq 1}$ with themselves to establish absolute continuity of the marginal laws. The regime is thus said to be elliptic. Note however that absolute continuity fails when the starting point is not in $\Gamma_{x,N}$: if $x^j = x^\ell$ for some $j \neq \ell$, then $Y_t^j = Y_t^\ell$ for any $t \geq 0$, so that $(Y_t)_{t \geq 0}$ stays in a hyperplane of \mathbb{R}^{Nd} .

We are now able to give examples for which Theorem 2 applies:

Proposition 37 *In the case when $d = 1$, consider Q as in Example 34, with $f \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying (57) and*

$$\int_0^{+\infty} k^4 f(k) dk < \infty,$$

then Conditions 4 and 5 in Section 2 are satisfied with

$$\sigma(x) = \text{sign}(x) \sqrt{\frac{Q(0) - Q(x)}{2}}, \quad x \in \mathbb{R}.$$

Proof. Go back to the framework of Section 2 and recall (11) and (13).

The existence and uniqueness part in Condition 4 follows from Theorem 41 below. The boundedness condition (12) is straightforward. Actually, it coincides with (A.3).

We now prove (14). Considering an arbitrary random initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$, independent of the noise $((W_t^k)_{t \geq 0})_{k \geq 1}$ as it is in Condition 5, we have:

$$d\mathbb{X}_t^\varepsilon = \text{sign}(\mathbb{X}_t^\varepsilon) dt + \varepsilon \sum_{k \geq 1} \frac{\sigma_k(X_t^{1,\varepsilon}) - \sigma_k(X_t^{2,\varepsilon})}{2} dW_t^k, \quad t \geq 0.$$

We set, for $t \geq 0$,

$$\begin{aligned} \rho_t^\varepsilon &= Q(0) - Q(\mathbb{X}_t^\varepsilon) \geq 0, \\ B_t^\varepsilon &= \sum_{k \geq 1} \int_0^t \text{sign}(\mathbb{X}_s^\varepsilon) \mathbf{1}_{\{\rho_s^\varepsilon > 0\}} (\rho_s^\varepsilon)^{-1/2} \frac{\sigma_k(X_s^{1,\varepsilon}) - \sigma_k(X_s^{2,\varepsilon})}{\sqrt{2}} dW_s^k \\ &\quad + \int_0^t \text{sign}(\mathbb{X}_s^\varepsilon) \mathbf{1}_{\{\rho_s^\varepsilon = 0\}} dW_s^1. \end{aligned}$$

The bracket of B^ε is given by

$$d\langle B^\varepsilon \rangle_t = \left(\mathbf{1}_{\{\rho_t^\varepsilon > 0\}} (\rho_t^\varepsilon)^{-1} \sum_{k \geq 1} \frac{|\sigma_k(X_t^{1,\varepsilon}) - \sigma_k(X_t^{2,\varepsilon})|^2}{2} + \mathbf{1}_{\{\rho_t^\varepsilon = 0\}} \right) dt = dt.$$

By Lévy's Theorem, $(B_t^\varepsilon)_{t \geq 0}$ is a Brownian motion w.r.t. the augmented filtration generated by the initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$ and by the noise $((W_t^k)_{t \geq 0})_{k \geq 1}$.

We now investigate the properties of σ . We first notice that $\sigma(0) = 0$. Below, we prove that σ is \mathcal{C}^2 with bounded derivatives and that $\sigma(1) > 0$. We have

$$Q(0) - Q(x) = x^2 \int_{\mathbb{R}} \frac{1 - \cos(kx)}{x^2} f(|k|) dk = x^2 \int_{\mathbb{R}} k^2 \varphi(kx) f(|k|) dk,$$

with

$$\varphi(u) = \frac{1 - \cos(u)}{u^2}.$$

It is well checked that φ is infinitely differentiable with bounded derivatives. Therefore, the function

$$\Phi : \mathbb{R} \ni x \mapsto \int_{\mathbb{R}} k^2 \varphi(kx) f(|k|) dk$$

is twice continuously differentiable with bounded derivatives. At $x = 0$, $\Phi(0) > 0$, so that $\sqrt{\Phi}$ is twice continuously differentiable in the neighborhood of 0. Therefore, the function σ , which reads

$$\sigma(x) = x \sqrt{\Phi(x)}, \quad x \in \mathbb{R},$$

is twice continuously differentiable in the neighborhood of 0. Away from 0, the function $\mathbb{R} \ni x \mapsto Q(0) - Q(x)$ has positive values so that its square root is also twice continuously differentiable and σ is twice continuously differentiable as well. The derivatives of order one and two of σ are bounded since the derivatives of order one and two of Q are bounded and $Q(0) - Q(x) \rightarrow Q(0) > 0$ as $|x| \rightarrow +\infty$. Moreover, $\sigma(1)$ is clearly positive. ■

Remark 38 Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function of the form

$$f(k) = \sum_{j=1}^N (a_j + \langle k, v^j \rangle_{\mathbb{C}^d}) e^{ix^j \cdot k}, \quad k \in \mathbb{R}^d,$$

where $a_j \in \mathbb{C}$, $v^j \in \mathbb{C}^d$ and $(x^1, \dots, x^N) \in \Gamma_{x,N}$. If there is a Borel set $A \subset \mathbb{R}^d$ of positive Lebesgue measure such that $f = 0$ on A , then $f \equiv 0$. This is presumably well known fact of complex analysis (true for much more general functions) but we prefer to give a proof. Since $(x^1, \dots, x^N) \in \Gamma_{x,N}$, we can find linearly independent vectors $u_1, \dots, u_d \in \mathbb{R}^d$ such that, for any $1 \leq n \leq d$, the scalar products $\langle u_n, x^j \rangle_{\mathbb{R}^d}$ are pairwise different². It is then sufficient to prove that $a_j = 0$ and $\langle u_n, v^j \rangle_{\mathbb{C}^d} = 0$ for any $1 \leq j \leq N$ and $1 \leq n \leq d$. Below, we prove that $a_j = 0$ and $\langle u_d, v^j \rangle_{\mathbb{C}^d} = 0$ for any $1 \leq j \leq N$. Rotating the space if necessary, we can assume without loss of generality that $u_d = e_d = (0, \dots, 0, 1)$. Since $\int_{\mathbb{R}^d} \mathbf{1}_A(k) dk > 0$, it follows from Fubini's Theorem that there exists $(k_2^*, \dots, k_d^*) \in \mathbb{R}^{d-1}$ such that $\int_{\mathbb{R}} \mathbf{1}_A(k_1, k_2^*, \dots, k_d^*) dk_1 > 0$, namely the line $t \mapsto k^* + te_1$ intersects A infinitely many times with at least one accumulation point, where $k^* = (0, k_2^*, \dots, k_d^*)$ and $e_1 = (1, 0, \dots, 0)$. Therefore, the function $t \mapsto f(k^* + te_1)$ is an analytic function with such set of zeros, thus it is identically equal to zero. Choosing $t = 0$, we deduce that $f(k^*) = 0$ for $k^* \in \{0\} \times A^*$ with $A^* = \{(k_2^*, \dots, k_d^*) \in \mathbb{R}^{d-1} : \int_{\mathbb{R}} \mathbf{1}_A(k_1, k_2^*, \dots, k_d^*) dk_1 > 0\}$. By Fubini's Theorem, $\text{Leb}_{d-1}(A^*) > 0$. Iterating the process, we deduce that $f(0, \dots, 0, k_d) = 0$ for k_d in a non-zero Lebesgue measure subset of \mathbb{R} , that is $f(0, \dots, 0, k_d) = 0$ for any $k_d \in \mathbb{R}$. Following Step 2 in the proof of Proposition 35, we deduce that $a_j = 0$ and $v_d^j = 0$ for any $j \in \{1, \dots, N\}$.

²The proof is as follows. Clearly, we can find a non-zero vector u_1 such that the set $\{\langle u_1, x^\ell \rangle_{\mathbb{R}^d}, 1 \leq \ell \leq N\}$ has at least two different elements. We then construct u_2, \dots, u_d by induction. Assume that there exist m disjoint subsets I_1, \dots, I_m , $1 \leq m \leq N-1$, such that $\{1, \dots, N\} = I_1 \cup \dots \cup I_m$ and n linearly independent vectors u_1, \dots, u_n , $1 \leq n \leq d-1$, such that the mapping $\{1, \dots, N\} \ni \ell \mapsto \langle u_j, x^\ell \rangle_{\mathbb{R}^d}$, $1 \leq j \leq n$, takes the same values on each I_p but different values on different I_p 's. Choose one of the I_p 's at least of cardinal two. Denote it by I_{p^*} . There exists u_{n+1}^* in the orthogonal of $\{u_1, \dots, u_n\}$ such that the mapping $\ell \mapsto \langle u_{n+1}^*, x^\ell \rangle_{\mathbb{R}^d}$ takes at least two values on I_{p^*} . For any $\eta > 0$, the mapping $\ell \mapsto \langle u_j + \eta u_{n+1}^*, x^\ell \rangle_{\mathbb{R}^d}$ also takes at least two values on I_{p^*} . Clearly, we can choose η small enough such that the mapping $\ell \mapsto \langle u_j + \eta u_{n+1}^*, x^\ell \rangle_{\mathbb{R}^d}$ takes different values on different I_p 's. Setting $u_{n+1} = u_n + \eta u_{n+1}^*$, we obtain at least $m+1$ classes. Induction ends when $m = N$. If $m = N$ and $n < d$, we can complete the family (u_1, \dots, u_n) by the same argument as above: setting $u_{n+1} = u_n + \eta u_{n+1}^*$ for $\eta > 0$ and u_{n+1}^* in the orthogonal of $\{u_1, \dots, u_n\}$, we can choose η small enough such that all the $\langle u_{n+1}, x^\ell \rangle_{\mathbb{R}^d}$, $\ell = 1 \dots N$, are different. Note that we cannot have $m < N$ and $n \geq d$ as otherwise there would be two identical x^ℓ and $x^{\ell'}$.

4.3 Hypocoellipticity of the N -point motion for the second order system

We consider now the second order system

$$\frac{dX_t}{dt} = V_t, \quad dV_t = \sum_{k=1}^{\infty} \sigma_k(X_t) dW_t^k, \quad t \geq 0,$$

when the noise satisfies the assumptions of the previous section. Given N initial conditions $(x^1, v^1), \dots, (x^N, v^N) \in \mathbb{R}^{2d}$, the corresponding solutions are denoted by $(X_t^1, V_t^1), \dots, (X_t^N, V_t^N)$. We then write $X_t = (X_t^1, \dots, X_t^N)$, $V_t = (V_t^1, \dots, V_t^N)$, so that

$$\frac{dX_t}{dt} = V_t, \quad dV_t = \sum_{k=1}^{\infty} A_k(X_t) dW_t^k, \quad t \geq 0.$$

Rewrite this as

$$dZ_t = \mathbb{F}_0(Z_t) dt + \sum_{k=1}^{\infty} \mathbb{A}_k(Z_t) dW_t^k, \quad t \geq 0,$$

where $Z_t = (X_t, V_t)$, $\mathbb{A}_k : \mathbb{R}^{2Nd} \ni (x, v) \mapsto (0, A_k(x)) \in \mathbb{R}^{2Nd}$ and $\mathbb{F}_0 : \mathbb{R}^{2Nd} \ni (x, v) \mapsto (v, 0) \in \mathbb{R}^{2Nd}$. The action of the noise on such equation in \mathbb{R}^{2Nd} is certainly not elliptic. Anyhow it is hypoelliptic in the following relatively simple sense:

Proposition 39 *Assume that, for any $k \in \mathbb{N} \setminus \{0\}$, σ_k is of class \mathcal{C}^1 on \mathbb{R}^d and that, for every $x = (x^1, \dots, x^N) \in \Gamma_{x,N}$,*

$$\text{Span} \{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}} = \mathbb{R}^{Nd}.$$

Then, for every $z \in \mathbb{R}^{2Nd}$ of the form $z = (x, v)$ with $x = (x^1, \dots, x^N) \in \Gamma_{x,N}$, $v \in \mathbb{R}^{Nd}$, we have

$$\text{Span} \{\mathbb{A}_k(z), [\mathbb{A}_k, \mathbb{F}_0](z)\}_{k \in \mathbb{N} \setminus \{0\}} = \mathbb{R}^{2Nd}.$$

(Keep in mind that, given two differentiable vector fields V_1 and V_2 in \mathbb{R}^n , their commutator $[V_1, V_2]$ is the vector field in \mathbb{R}^n with components

$$[V_1, V_2]_j = V_1 \cdot \nabla V_2^j - V_2 \cdot \nabla V_1^j.)$$

Proof. Step 1. We compute the commutators of the vector fields $\mathbb{A}_k, \mathbb{F}_0$, $k \in \mathbb{N} \setminus \{0\}$. The point is to be careful with the notation. A vector $z \in \mathbb{R}^{2Nd}$ is a pair $z = (x, v) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, where $x \in \mathbb{R}^{Nd}$ may be written in blocks as $x = (x^1, \dots, x^N) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$ (N times), each block $x^i \in \mathbb{R}^d$ being a vector of the form $x^i = (x_1^i, \dots, x_d^i)$, similar notations being used for the second component $v \in \mathbb{R}^{Nd}$ of z , that is $v = (v^1, \dots, v^N)$, $v^i = (v_1^i, \dots, v_d^i)$. We use similar decompositions for \mathbb{F}_0 and \mathbb{A}_k and we write the decomposition in blocks as

$$\mathbb{F}_0 = \left(\mathbb{F}_0^{(1)}, \mathbb{F}_0^{(2)} \right); \quad \mathbb{F}_0^{(\alpha)} = \left(\mathbb{F}_0^{(\alpha,1)}, \dots, \mathbb{F}_0^{(\alpha,N)} \right) \quad \text{for } \alpha = 1, 2,$$

with

$$\mathbb{F}_0^{(\alpha,i)} = \left(\mathbb{F}_0^{(\alpha,i,1)}, \dots, \mathbb{F}_0^{(\alpha,i,d)} \right) \quad \text{for } \alpha = 1, 2 \text{ and } i = 1, \dots, N,$$

and similarly for \mathbb{A}_k and $[\mathbb{A}_k, \mathbb{F}_0]$.

With these notations, we have

$$\begin{aligned} \mathbb{A}_k^{(1,i)} &\equiv 0, & \mathbb{F}_0^{(2,i)} &\equiv 0, \\ \mathbb{A}_k^{(2,i)}(x, v) &= \sigma_k(x^i), & \mathbb{F}_0^{(1,i)}(x, v) &= v^i, \end{aligned}$$

for $(x, v) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ and $i = 1, \dots, N$.

and thus

$$\begin{aligned} \nabla \mathbb{A}_k^{(1,i,j)} &\equiv 0, & \partial_{x_q^p} \mathbb{A}_k^{(2,i,j)}(x, v) &= \delta_{ip} \partial_q \sigma_k^j(x^i), & \partial_{v_q^p} \mathbb{A}_k^{(2,i,j)}(x, v) &= 0, \\ \nabla \mathbb{F}_0^{(2,i,j)} &\equiv 0, & \partial_{x_q^p} \mathbb{F}_0^{(1,i,j)}(x, v) &= 0, & \partial_{v_q^p} \mathbb{F}_0^{(1,i,j)}(x, v) &= \delta_{ip} \delta_{jq}, \end{aligned}$$

for $(x, v) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, $i \in \{1, \dots, N\}$, $j \in \{1, \dots, d\}$, $k \in \mathbb{N} \setminus \{0\}$. Therefore,

$$\begin{aligned} [\mathbb{A}_k, \mathbb{F}_0]^{(1,i,j)} &= \mathbb{A}_k \cdot \nabla \mathbb{F}_0^{(1,i,j)} - \mathbb{F}_0 \cdot \nabla \mathbb{A}_k^{(1,i,j)} = \mathbb{A}_k \cdot \nabla \mathbb{F}_0^{(1,i,j)} \\ &= \sum_{p=1}^N \sum_{q=1}^d \mathbb{A}_k^{(2,p,q)} \partial_{v_q^p} \mathbb{F}_0^{(1,i,j)} = \sigma_k^j \circ \pi_{i,x}, \end{aligned}$$

that is $[\mathbb{A}_k, \mathbb{F}_0]^{(1,i,j)}(x, v) = \sigma_k^j(x^i)$. Similarly,

$$\begin{aligned} [\mathbb{A}_k, \mathbb{F}_0]^{(2,i,j)} &= \mathbb{A}_k \cdot \nabla \mathbb{F}_0^{(2,i,j)} - \mathbb{F}_0 \cdot \nabla \mathbb{A}_k^{(2,i,j)} = -\mathbb{F}_0 \cdot \nabla \mathbb{A}_k^{(2,i,j)} \\ &= -\sum_{p=1}^N \sum_{q=1}^d \mathbb{F}_0^{(1,p,q)} \partial_{x_q^p} \mathbb{A}_k^{(2,i,j)} = -\pi_{i,v} \cdot \nabla \sigma_k^j \circ \pi_{i,x}, \end{aligned}$$

that is $[\mathbb{A}_k, \mathbb{F}_0]^{(2,i,j)}(x, v) = -v^i \cdot \nabla \sigma_k^j(x^i)$.

We thus have $[\mathbb{A}_k, \mathbb{F}_0]^{(1,i)}(x, v) = \sigma_k(x^i)$, namely

$$[\mathbb{A}_k, \mathbb{F}_0]^{(1)}(x, v) = A_k(x). \quad (58)$$

Similarly, we can also write

$$[\mathbb{A}_k, \mathbb{F}_0]^{(2,i)}(x, v) = - (v^i \cdot \nabla) \sigma_k(x^i). \quad (59)$$

Step 2. Let us now show that the family of vectors $\mathbb{A}_k(z)$, $[\mathbb{A}_k, \mathbb{F}_0](z)$ spans the entire \mathbb{R}^{2Nd} , at any point z of the form specified in the assumptions. Given $w = (w^{(1)}, w^{(2)}) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, we want to express it as

$$w = \sum_{k=1}^{\infty} \lambda_k \mathbb{A}_k(z) + \sum_{k=1}^{\infty} \mu_k [\mathbb{A}_k, \mathbb{F}_0](z) \quad (60)$$

with only a finite number of coefficients λ_k and μ_k different from zero. From (58) and (59), a necessary condition for (60) is

$$w^{(1)} = \sum_{k=1}^{\infty} \mu_k A_k(x). \quad (61)$$

By assumption, $\text{Span}\{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}} = \mathbb{R}^{Nd}$, so that (61) is solvable, that is we can find a family $(\mu_k)_{k \in \mathbb{N} \setminus \{0\}}$ with a finite number of non-zero elements satisfying (61). A second necessary condition for (60) is

$$w^{(2)} = \sum_{k=1}^{\infty} \lambda_k A_k(x) + \sum_{k=1}^{\infty} \mu_k [\mathbb{A}_k, \mathbb{F}_0]^{(2)}(x, v). \quad (62)$$

Given a family $(\mu_k)_{k \in \mathbb{N} \setminus \{0\}}$ with a finite number of non-zero elements, Eq. (62) is solvable, that is we can find a family $(\lambda_k)_{k \in \mathbb{N} \setminus \{0\}}$ with a finite number of non-zero elements such that

$$\sum_{k=1}^{\infty} \lambda_k A_k(x) = w^{(2)} - \sum_{k=1}^{\infty} \mu_k [\mathbb{A}_k, \mathbb{F}_0]^{(2)}(x, v)$$

Again, solvability relies on the assumption $\text{Span}\{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}} = \mathbb{R}^{Nd}$. ■

Here is the precise formulation of hypoellipticity in our framework:

Proposition 40 *Under (A.1–3) and under the previous notation, the stochastic approximate system*

$$\begin{aligned} \frac{dX_t^{i,\varepsilon}}{dt} &= V_t^{i,\varepsilon}, \\ dV_t^{i,\varepsilon} &= \frac{1}{N} \sum_{j \in N_i} F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}) dt + \sum_{k=1}^{\infty} \sigma_k(X_t^{i,\varepsilon}) dW_t^k, \quad t \geq 0, \end{aligned} \tag{63}$$

admits a unique strong solution for every initial condition $((X_0^i, V_0^i))_{1 \leq i \leq N} = ((x^i, v^i))_{1 \leq i \leq N} \in \mathbb{R}^{2Nd}$, where F_ε is as in (30). Moreover, the mappings $\varphi_t^\varepsilon : \mathbb{R}^{2Nd} \ni ((x^i, v^i))_{1 \leq i \leq N} \in \mathbb{R}^{2Nd} \mapsto ((X_t^i, V_t^i))_{1 \leq i \leq N}$ with $((X_0^i, V_0^i))_{1 \leq i \leq N} = ((x^i, v^i))_{1 \leq i \leq N}$, $t \geq 0$, form a stochastic flow of homeomorphisms on \mathbb{R}^{2Nd} . Finally, for any $t > 0$, the marginal law of the $2Nd$ -dimensional vector

$$Z_t^\varepsilon = \left(X_t^{1,\varepsilon}, V_t^{1,\varepsilon}, \dots, X_t^{N,\varepsilon}, V_t^{N,\varepsilon} \right)$$

is absolutely continuous with respect to Lebesgue measure when $(x^1, \dots, x^N) \in \Gamma_{x,N}$.

Proof. Eq. (63) may be written as

$$dZ_t^\varepsilon = \mathbb{F}_\varepsilon(Z_t^\varepsilon) dt + \sum_{k=1}^{+\infty} \mathbb{A}_k(Z_t^\varepsilon) dW_t^k, \quad t \geq 0, \tag{64}$$

with \mathbb{F}_ε as in (31). Unique strong solvability and homeomorphism property of the flow may be found in Nguyen, Nualart and Sanz [31] and Kunita [21, Chapter 4, Section 5]. We then notice that Proposition 39 remains true with \mathbb{F}_0 replaced by \mathbb{F}_ε , provided \mathbb{F}_ε is continuously differentiable. The proof is the same: on the one hand, (58) remains true with \mathbb{F}_0 replaced by \mathbb{F}_ε , so that (61) remains true as well; on the other hand, (59) is modified but this has no real consequence on (62). When the coefficients $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$ are smooth, with derivatives of any order in $\ell^2(\mathbb{N} \setminus \{0\})$, and \mathbb{F}_ε is smooth as well, absolute continuity then follows from a suitable version of Hörmander's Theorem, namely Hörmander's Theorem for systems driven by an infinite-dimensional noise. See for example Theorem 4.3 in [31], which extends the classical results of Malliavin calculus to infinite-dimensional noises.

Here the coefficients are not smooth. Anyhow, absolute continuity follows from the Bouleau and Hirsch criterion directly. By Proposition 2.2 in [31],

we know that $(Z_t^\varepsilon)_{t \geq 0}$ is differentiable in the sense of Malliavin with

$$\sum_{k=1}^{+\infty} \mathbb{E} \int_0^t |D_s^k Z_t^\varepsilon|^2 ds < +\infty,$$

for any $t \geq 0$. We also know that

$$D_r^k Z_t^\varepsilon = Y_t^\varepsilon (Y_r^\varepsilon)^{-1} \mathbb{A}_k(Z_r^\varepsilon), \quad 0 \leq r \leq t,$$

the equality holding true in \mathbb{R}^{2Nd} , where $(Y_t^\varepsilon)_{t \geq 0}$ is an $\mathbb{R}^{2Nd \times 2Nd}$ -valued process, solution a linear SDE of the form

$$Y_t^\varepsilon = I_{2Nd} + \sum_{k=1}^{+\infty} \int_0^t \alpha_k(s) Y_s^\varepsilon dW_s^k + \int_0^t \alpha_0(s) Y_s^\varepsilon ds, \quad t \geq 0, \quad (65)$$

the processes $(\alpha_k(s))_{s \geq 0}$, $k \in \mathbb{N}$, being progressively-measurable and the infinite-dimensional process $((|\alpha_k(s)|)_{s \geq 0})_{k \in \mathbb{N} \setminus \{0\}}$ being bounded in $\ell^2(\mathbb{N} \setminus \{0\})$. When the coefficients \mathbb{F}_ε and \mathbb{A}_k , $k \in \mathbb{N} \setminus \{0\}$, are smooth, $\alpha_0(s) = \nabla \mathbb{F}_\varepsilon(Z_s^\varepsilon)$ and $\alpha_k(s) = \nabla \mathbb{A}_k(Z_s^\varepsilon)$, $k \in \mathbb{N} \setminus \{0\}$, the general case being treated by a mollification argument. In particular, it is always true that $[\alpha_k(s)]_{1,1}$, $[\alpha_k(s)]_{1,2}$ and $[\alpha_k(s)]_{2,2}$ are zero for $k \in \mathbb{N} \setminus \{0\}$, where $[\alpha_k(s)]_{1,1}$, $[\alpha_k(s)]_{1,2}$ and $[\alpha_k(s)]_{2,2}$ stand for the $(1,1)$, $(1,2)$ and $(2,2)$ -blocks of dimension Nd of $\alpha_k(s)$, that is $[\alpha_k(s)]_{1,1} = (\partial_{x_j} \mathbb{A}_k^{(1,i)})_{1 \leq i, j \leq N}$, $[\alpha_k(s)]_{1,2} = (\partial_{v_j} \mathbb{A}_k^{(1,i)})_{1 \leq i, j \leq N}$ and $[\alpha_k(s)]_{2,2} = (\partial_{v_j} \mathbb{A}_k^{(2,i)})_{1 \leq i, j \leq N}$ are zero. Similarly, $[\alpha_0(s)]_{1,1} = [\alpha_0(s)]_{2,2} = 0$ and $[\alpha_0(s)]_{1,2} = I_{Nd}$. As in the finite-dimensional framework, we can check that, a.s., for any $t > 0$, Y_t^ε is invertible, the inverse being of finite polynomial moments of any order.

For r small, $Y_r^\varepsilon = I_{2Nd} + o_r(1)$, $o_r(1)$ standing for the Landau notation and almost-surely converging to 0 with r . Therefore, by the equalities $[\alpha_0(s)]_{1,1} = [\alpha_k(s)]_{1,1} = [\alpha_k(s)]_{1,2} = [\alpha_0(s)]_{2,2} = [\alpha_k(s)]_{2,2} = 0$ and $[\alpha_0(s)]_{1,2} = I_{Nd}$, we deduce that $[Y_r^\varepsilon]_{1,1} = I_{Nd} + r o_r(1)$ and $[Y_r^\varepsilon]_{2,2} = I_{Nd} + o_r(1)$ and that $[Y_r^\varepsilon]_{1,2}$ may expanded as

$$[Y_r^\varepsilon]_{1,2} = r I_{Nd} + r o_r(1).$$

Setting $\mathcal{Z}_r = (Y_r^\varepsilon)^{-1} \mathbb{A}_k(Z_r^\varepsilon) \in \mathbb{R}^{2Nd}$ and writing \mathcal{Z}_r under the form $\mathcal{Z}_r = ((\mathcal{X}_r^i, \mathcal{V}_r^i))_{1 \leq i \leq N}$, we have

$$\begin{aligned} [Y_r^\varepsilon]_{1,1} \mathcal{X}_r + [Y_r^\varepsilon]_{1,2} \mathcal{V}_r &= 0, \\ [Y_r^\varepsilon]_{2,1} \mathcal{X}_r + [Y_r^\varepsilon]_{2,2} \mathcal{V}_r &= \mathbb{A}_k(Z_r^\varepsilon), \end{aligned}$$

that is $\mathcal{X}_r + r\mathcal{V}_r = ro_r(1)$, and $o_r(1)\mathcal{X}_r + \mathcal{V}_r = A_k(X_r^\varepsilon) + o_r(1)$. We deduce $\mathcal{V}_r = A_k(X_r^\varepsilon) + o_r(1)$ and $\mathcal{X}_r = rA_k(X_r^\varepsilon) + ro_r(1)$, so that

$$(Y_t^\varepsilon)^{-1} D_r^k Z_t^\varepsilon = (rA_k(x) + ro_r(1), A_k(x) + o_r(1)). \quad (66)$$

(The above equality holds almost-surely, $o_r(1)$ being random itself.) For a given $\omega \in \Omega$ for which (66) holds true, consider $\zeta = ((\chi^i, \nu^i))_{1 \leq i \leq N} \in \mathbb{R}^{2Nd}$ such that $\langle D_r^k Z_t^\varepsilon, \zeta \rangle_{\mathbb{R}^d} = 0$ for any $0 \leq r \leq t$ and $k \in \mathbb{N} \setminus \{0\}$. Changing ζ into $((Y_t^\varepsilon)^{-1})^\top \zeta$, we deduce from (66) that

$$r \sum_{i=1}^N \langle \sigma_k(x^i), \chi^i \rangle_{\mathbb{R}^d} + \sum_{i=1}^N \langle \sigma_k(x^i), \nu^i \rangle_{\mathbb{R}^d} = ro_r(1)|\chi| + o_r(1)|\nu|,$$

Letting $r \rightarrow 0$, we deduce that $\nu \perp A_k(x)$ for any $k \in \mathbb{N} \setminus \{0\}$. Since $\text{Span}\{A_k(x)\}_{k \in \mathbb{N} \setminus \{0\}} = \mathbb{R}^{Nd}$ for $x \in \Gamma_{x,N}$, we deduce that $\nu = 0$. Dividing the above equality by r and letting $r \rightarrow 0$ again, we deduce that $\chi = 0$. We complete the proof by Bouleau and Hirsch criterion, see Theorem 2.1.2 in Nualart [32]. ■

5 Stochastic dynamics of N particles: no coalescence

We now prove the main result of the paper:

Theorem 41 *Assume that (A.1–3) are in force. Then, for any $z \in \Gamma_N$, there exists a unique solution $(Z_t(z))_{t \geq 0}$ to (46) with z as initial condition. It satisfies $\mathbb{P}\{\forall t \geq 0, Z_t(z) \in \Gamma_N\} = 1$ and $\mathbb{P}\{\text{Leb}_1\{t \geq 0 : \Pi_x(Z_t(z)) \in \Gamma_{x,N}^c\} = 0\} = 1$.*

The proof is split into three parts: we first establish a priori estimates for a regularized version of (46); using a compactness argument as in the deterministic case, we deduce that strong unique solvability holds for Lebesgue almost-every starting point; taking advantage of the absolute continuity of the marginal laws of the regularized system, we finally prove that strong unique solvability holds for any $z \in \Gamma_N$.

5.1 Preliminary estimates

Given the smoothed system (63)–(64) with $(\varphi_t^\varepsilon)_{t \geq 0}$ as associated stochastic flow of diffeomorphisms on \mathbb{R}^{2Nd} , we aim at extending the results in Subsection 3.1 to the stochastic framework.

Lemma 42 *For any $t \geq 0$, $\varphi_t^\varepsilon(\cdot)$ preserves Lebesgue measure:*

$$\mathbb{E} \int_{\mathbb{R}^{2Nd}} g(\varphi_t^\varepsilon(z)) dz = \int_{\mathbb{R}^{2Nd}} g(z') dz',$$

for all g that belong to $C_0^\infty(\mathbb{R}^{2Nd})$ and all g that are measurable and non negative.

Proof. This is a consequence of the almost everywhere equality $\operatorname{div} \mathbb{F}_\varepsilon = 0$ and $\operatorname{div} \mathbb{A}_k = 0$ for all $k \in \mathbb{N}$. ■

Proposition 43 *With the same notation as in Lemma 18, for any $R_0, R > 0$, there exists a constant $C_{R_0, R}$ independent of ε such that*

$$\int_{B_{2Nd}(0, R_0)} \mathbb{E} \left[\sup_{t \in [0, T]} h_R(\varphi_t^\varepsilon(z)) \right] dz \leq C_{R_0, R}.$$

Proof. Step 1. Let $\log_\phi^+(r)$ be defined as in (32). Let us write \tilde{z} for the generic point of \mathbb{R}^{2d} and consider the function $\mathbb{R}^{2d} \ni \tilde{z} \mapsto \log_\phi^+(|\tilde{z}|)$. We have

$$\nabla [\log_\phi^+(|\tilde{z}|)] = -\frac{\phi(|\tilde{z}|)}{|\tilde{z}|^2} \tilde{z}$$

and thus

$$|\nabla [\log_\phi^+(|\tilde{z}|)]| \leq \frac{\mathbf{1}_{\{|\tilde{z}| \leq 1\}}}{|\tilde{z}|} \quad \text{for all } \tilde{z} \in \mathbb{R}^{2d}. \quad (67)$$

Moreover, denoting by \tilde{z}_α the generic component of \tilde{z} , for $\alpha = 1, \dots, 2d$, we have

$$\frac{\partial^2}{\partial \tilde{z}_\alpha \partial \tilde{z}_\beta} [\log_\phi^+(|\tilde{z}|)] = 2 \frac{\phi(|\tilde{z}|)}{|\tilde{z}|^4} \tilde{z}_\alpha \tilde{z}_\beta - \frac{\phi'(|\tilde{z}|)}{|\tilde{z}|^3} \tilde{z}_\alpha \tilde{z}_\beta - \frac{\phi(|\tilde{z}|)}{|\tilde{z}|^2} \delta_{\alpha\beta}.$$

We easily deduce that

$$\left| \frac{\partial^2}{\partial \tilde{z}_\alpha \partial \tilde{z}_\beta} \log_\phi^+ |\tilde{z}| \right| \leq \frac{C}{|\tilde{z}|^2} (1 + \phi'(|\tilde{z}|)) \cdot \mathbf{1}_{\{|\tilde{z}| \leq 1\}} \quad (68)$$

for a simple constant C independent of the details of ϕ .

Given $R > 0$, let $\theta_R : \mathbb{R}^{2Nd} \rightarrow \mathbb{R}$ be defined as in the proof of Lemma 18. As in (34), define also

$$h_\phi^{\theta_R}(z) = \theta_R(z) \sum_{(i,j) \in \Delta_N} \log_\phi^+(|z^i - z^j|), \quad z \in \mathbb{R}^{2Nd}.$$

In the next step we prove that, given $R_0, R > 0$, there exists a constant $C_{R_0, R}$, independent of ε, ϕ and the details of θ_R in $B_{2Nd}(0, R+2) \setminus B_{2Nd}(0, R)$, such that

$$\mathbb{E} \int_{B(0, R_0)} \sup_{t \in [0, T]} h_\phi^{\theta_R}(T_t^\varepsilon(z)) dz \leq C_{R_0, R}.$$

By monotonous convergence, this easily implies the claim of the proposition.

Step 2. Denote mutual variation between two continuous semimartingales X and Y by $[X, Y]$. Then, with the notation $g(\tilde{z}) = \log_\phi^+(|\tilde{z}|)$,

$$d(\theta_R(Z_t^\varepsilon)g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})) = dI_t^1 + dI_t^2 + dI_t^3, \quad \begin{cases} dI_t^1 = dI_t^{11} + dI_t^{12} \\ dI_t^2 = dI_t^{21} + dI_t^{22} \end{cases} \quad (69)$$

where

$$\begin{aligned} dI_t^{11} &= \theta_R(Z_t^\varepsilon) \langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), d(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) \rangle_{\mathbb{R}^{2d}}, \\ dI_t^{12} &= \frac{\theta_R(Z_t^\varepsilon)}{2} \sum_{\alpha, \beta=1}^{2d} \frac{\partial^2 g}{\partial \tilde{z}_\alpha \partial \tilde{z}_\beta}(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) d[(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})_\alpha, (Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})_\beta]_t, \end{aligned}$$

and,

$$\begin{aligned} dI_t^{21} &= g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) \langle \nabla \theta_R(Z_t^\varepsilon), dZ_t^\varepsilon \rangle_{\mathbb{R}^{2Nd}}, \\ dI_t^{22} &= \frac{g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})}{2} \sum_{i,j=1}^N \sum_{\alpha, \beta=1}^{2d} \frac{\partial^2 \theta_R}{\partial (z^i)_\alpha \partial (z^j)_\beta}(Z_t^\varepsilon) d[(Z_t^{i,\varepsilon})_\alpha, (Z_t^{j,\varepsilon})_\beta]_t, \end{aligned}$$

and

$$\begin{aligned} dI_t^3 &= d \left[\sum_{k=1}^{\infty} \int_0^\cdot \langle \nabla g(Z_s^{i,\varepsilon} - Z_s^{j,\varepsilon}), \mathbb{A}_k^i(Z_s^\varepsilon) - \mathbb{A}_k^j(Z_s^\varepsilon) \rangle_{\mathbb{R}^{2d}} dW_s^k, \right. \\ &\quad \left. \sum_{k=1}^{\infty} \int_0^\cdot \langle \nabla \theta_R(Z_s^\varepsilon), \mathbb{A}_k(Z_s^\varepsilon) \rangle_{\mathbb{R}^{2Nd}} dW_s^k \right]_t. \end{aligned}$$

In this step let us handle the three terms with the mutual variations. We have

$$\begin{aligned}
dI_t^3 &= \sum_{k=1}^{\infty} \langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), \mathbb{A}_k^i(Z_t^\varepsilon) - \mathbb{A}_k^j(Z_t^\varepsilon) \rangle_{\mathbb{R}^{2d}} \langle \nabla \theta_R(Z_t^\varepsilon), \mathbb{A}_k(Z_t^\varepsilon) \rangle_{\mathbb{R}^{2Nd}} dt \\
&= \sum_{k=1}^{\infty} \left[\langle \tilde{\pi}_v(\nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})), \sigma_k(X_t^{i,\varepsilon}) - \sigma_k(X_t^{j,\varepsilon}) \rangle_{\mathbb{R}^d} \right. \\
&\quad \left. \times \left\langle \Pi_v(\nabla \theta_R(Z_t^\varepsilon)), \left(\sigma_k(X_t^{1,\varepsilon}), \dots, \sigma_k(X_t^{N,\varepsilon}) \right) \right\rangle_{\mathbb{R}^{Nd}} \right] dt.
\end{aligned}$$

For all $\tilde{x}, \tilde{y}, \tilde{v} \in \mathbb{R}^d$, we deduce from Cauchy-Schwarz inequality and from Remark 27,

$$\sum_{k=1}^{\infty} \langle v, (\sigma_k(\tilde{x}) - \sigma_k(\tilde{y})) \rangle_{\mathbb{R}^d}^2 \leq C |\tilde{x} - \tilde{y}|^2 |\tilde{v}|^2. \quad (70)$$

Similarly

$$\sum_{k=1}^{\infty} \langle \tilde{v}, \sigma_k(\tilde{x}) \rangle_{\mathbb{R}^d}^2 \leq C |\tilde{v}|^2 \quad (71)$$

using the assumption of boundedness of $|Q(\tilde{x}, \tilde{x})|$ on \mathbb{R}^d . By (67) and Cauchy-Schwarz inequality, this implies

$$|dI_t^3| \leq C |\nabla \theta_R(Z_t^\varepsilon)| dt. \quad (72)$$

In order to deal with the term I_t^{12} we need to analyze the mutual variation $[(Z^{i,\varepsilon} - Z^{j,\varepsilon})_\alpha, (Z^{i,\varepsilon} - Z^{j,\varepsilon})_\beta]_t$. Obviously, it holds

$$\left[(X^{i,\varepsilon} - X^{j,\varepsilon})_p, (Z^{i,\varepsilon} - Z^{j,\varepsilon})_\beta \right]_t = 0$$

for all $p = 1, \dots, d$ and $\beta = 1, \dots, 2d$, since $X^{i,\varepsilon} - X^{j,\varepsilon}$ is of bounded variation. On the contrary, we have, for $p, q = 1, \dots, d$,

$$\begin{aligned}
&\left| d \left[(V_t^{i,\varepsilon} - V_t^{j,\varepsilon})_p, (V_t^{i,\varepsilon} - V_t^{j,\varepsilon})_q \right]_t \right| \\
&= \left| \sum_{k=1}^{\infty} (\sigma_k^p(X_t^{i,\varepsilon}) - \sigma_k^p(X_t^{j,\varepsilon})) (\sigma_k^q(X_t^{i,\varepsilon}) - \sigma_k^q(X_t^{j,\varepsilon})) \right| dt \\
&\leq C |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|^2 dt,
\end{aligned}$$

by Remark 27 again. Combining the above bound with inequality (68), we get (renaming the constant C)

$$\begin{aligned} |dI_t^{12}| &\leq \frac{C\theta_R(Z_t^\varepsilon) (1 + \phi'(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|))}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|^2} |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|^2 dt \\ &\leq C\theta_R(Z_t^\varepsilon) (1 + \phi'(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|)) dt. \end{aligned} \quad (73)$$

Finally, let us deal with I_t^{22} . As above, the only terms to be non-zero in the variation $d[(Z^{i,\varepsilon})_\alpha, (Z^{j,\varepsilon})_\beta]_t$ are the terms $d[(V_t^{i,\varepsilon})_p, (V_t^{j,\varepsilon})_q]_t$. We claim

$$\left| d \left[(V_t^{i,\varepsilon})_p, (V_t^{j,\varepsilon})_q \right]_t \right| = \left| \sum_{k=1}^{\infty} \sigma_k^p(X_s^{i,\varepsilon}) \sigma_k^q(X_s^{j,\varepsilon}) \right| dt \leq C dt$$

and thus

$$|dI_t^{22}| \leq Cg(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) |\nabla^2 \theta_R(Z_t^\varepsilon)| dt. \quad (74)$$

Step 3. We collect the previous results and get from (69), (72), (73) and (74),

$$d[\theta_R(Z_t^\varepsilon)g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})] \leq dI_t^{111} + dI_t^{112} + dI_t^{211} + dI_t^{212} + dI_t^4$$

where

$$\begin{aligned} dI_t^{111} &= \theta_R(Z_t^\varepsilon) \langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), \mathbb{F}_\varepsilon^i(Z_t^\varepsilon) - \mathbb{F}_\varepsilon^j(Z_t^\varepsilon) \rangle_{\mathbb{R}^{2d}} dt \\ dI_t^{112} &= \theta_R(Z_t^\varepsilon) \sum_{k=1}^{\infty} \langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), \mathbb{A}_k^i(Z_t^\varepsilon) - \mathbb{A}_k^j(Z_t^\varepsilon) \rangle_{\mathbb{R}^{2d}} dW_t^k \end{aligned}$$

and

$$\begin{aligned} dI_t^{211} &= g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) \langle \nabla \theta_R(Z_t^\varepsilon), \mathbb{F}_\varepsilon(Z_t^\varepsilon) \rangle_{\mathbb{R}^{2Nd}} dt \\ dI_t^{212} &= g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) \sum_{k=1}^{\infty} \langle \nabla \theta_R(Z_t^\varepsilon), \mathbb{A}_k(Z_t^\varepsilon) \rangle_{\mathbb{R}^{2Nd}} dW_t^k \end{aligned}$$

and

$$\begin{aligned} dI_4 &= C \left[\theta_R(Z_t^\varepsilon) (1 + \phi'(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|)) \right. \\ &\quad \left. + |\nabla \theta_R(Z_t^\varepsilon)| + g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) |\nabla^2 \theta_R(Z_t^\varepsilon)| \right] dt. \end{aligned}$$

By (67) and by boundedness of \mathbb{F}_ε on $B_{2Nd}(0, R)$, we have, as in the deterministic case,

$$dI_t^{111} \leq C \frac{\theta_R(Z_t^\varepsilon)}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|} dt,$$

and similarly

$$dI_t^{211} \leq Cg(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) |\nabla\theta_R(Z_t^\varepsilon)| dt.$$

Therefore,

$$d[\theta_R(Z_t^\varepsilon)g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})] \leq dI_t^{112} + dI_t^{212} + dI_t^5, \quad (75)$$

where (up to new value of C)

$$\begin{aligned} dI_t^5 = C & \left[\theta_R(Z_t^\varepsilon) \left(1 + |Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|^{-1} + \phi'(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|) \right) \right. \\ & + (1 + g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})) |\nabla\theta_R(Z_t^\varepsilon)| \\ & \left. + g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) |\nabla^2\theta_R(Z_t^\varepsilon)| \right] dt. \end{aligned} \quad (76)$$

Step 4. We now deal with the martingale terms I^{112} and I^{212} . By (67), (70) and Doob's inequality

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |I_T^{112}|^2 \right] \\ & \leq C\mathbb{E} \left[\int_0^T \theta_R^2(Z_s^\varepsilon) \sum_{k=1}^{\infty} \langle \nabla g(Z_s^{i,\varepsilon} - Z_s^{j,\varepsilon}), \sigma_k(Z_s^{i,\varepsilon}) - \sigma_k(Z_s^{j,\varepsilon}) \rangle_{\mathbb{R}^d}^2 ds \right] \\ & \leq C\mathbb{E} \left[\int_0^T \theta_R^2(Z_s^\varepsilon) ds \right]. \end{aligned}$$

Similarly, from (71),

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |I_T^{212}|^2 \right] \\ & \leq C\mathbb{E} \left[\int_0^T g^2(Z_s^{i,\varepsilon} - Z_s^{j,\varepsilon}) \sum_{k=1}^{\infty} \langle \nabla\theta_R(Z_s^\varepsilon), \mathbb{A}_k(Z_s^\varepsilon) \rangle_{\mathbb{R}^{2Nd}}^2 ds \right] \\ & \leq C\mathbb{E} \left[\int_0^T g^2(Z_s^{i,\varepsilon} - Z_s^{j,\varepsilon}) |\nabla\theta_R(Z_s^\varepsilon)|^2 ds \right]. \end{aligned}$$

As in the deterministic case, we apply the following estimates

$$\begin{aligned} \max(\theta_R(z), |\nabla_R \theta(z)|, |\nabla_R^2 \theta(z)|) &\leq \mathbf{1}_{\{|z| \leq R+2\}}, \quad z \in \mathbb{R}^{2Nd}, \\ \max(g(\tilde{z}), g^2(\tilde{z})) &\leq \frac{C}{|\tilde{z}|}, \quad \tilde{z} \in \mathbb{R}^{2d}. \end{aligned}$$

From (75) and (76), we deduce (with $Z_0^\varepsilon = z$)

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} (\theta_R(Z_t^\varepsilon) \log_\phi^+(|Z_t^{i, \varepsilon} - Z_t^{j, \varepsilon}|)) \right] &\leq \theta_R(z) \log_\phi^+(|z^i - z^j|) \\ &+ C \left(1 + \mathbb{E} \int_0^T \mathbf{1}_{\{|Z_s^\varepsilon| \leq R+2\}} \left(\frac{1}{|Z_s^{i, \varepsilon} - Z_s^{j, \varepsilon}|} + \phi'(|Z_s^{i, \varepsilon} - Z_s^{j, \varepsilon}|) \right) ds \right). \end{aligned}$$

Step 5. We now integrate on a ball $B_{2Nd}(0, R_0)$ of \mathbb{R}^{2Nd} with respect to the initial conditions. Applying Lemma 42 and following the deterministic case, we get

$$\begin{aligned} &\int_{B_{2Nd}(0, R_0)} \mathbb{E} \left[\sup_{t \in [0, T]} (\theta_R(\varphi_t^\varepsilon(z)) \log_\phi^+(|\varphi_t^{i, \varepsilon}(z) - \varphi_t^{j, \varepsilon}(z)|)) \right] dz \\ &\leq \int_{B_{2Nd}(0, R_0)} \theta_R(z) \log_\phi^+(|z^i - z^j|) dz \\ &+ C \left[R_0^{2Nd} + \mathbb{E} \int_0^T \int_{\{|z| \leq R+2\}} \left(\frac{1}{|z^i - z^j|} + \phi'(|z^i - z^j|) \right) dz ds \right]. \end{aligned}$$

By a spherical change of variable, the integral of $\phi'(|z^i - z^j|)$ can be bounded independently of ϕ . This completes the proof. ■

Lemma 44 *Given $R_0, T > 0$, we have*

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \sup_{\varepsilon > 0} (\text{Leb}_{2Nd} \otimes \mathbb{P}) \left\{ (z, \omega) \in B_{2Nd}(0, R_0) \times \Omega : \right. \\ \left. \inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |\varphi_t^{i, \varepsilon}(z, \omega) - \varphi_t^{j, \varepsilon}(z, \omega)| \leq \varepsilon \right\} = 0. \end{aligned}$$

Proof. By boundedness of F_ε and $Q(\tilde{x}, \tilde{x})$ and by Markov inequality, it is well-seen that, for any $R_0 > 0$, there exists a constant C_{R_0} such that, for any $R > 0$,

$$(\text{Leb}_{2Nd} \otimes \mathbb{P}) \left\{ (z, \omega) \in B_{2Nd}(0, R_0) \times \Omega : \sup_{t \in [0, T]} |\varphi_t^\varepsilon(z, \omega)| > R \right\} \leq \frac{C_{R_0}}{R}.$$

Moreover, by Proposition 43,

$$(\text{Leb}_{2Nd} \otimes \mathbb{P}) \left\{ (z, \omega) \in B_{2Nd}(0, R_0) \times \Omega : \right. \\ \left. \sup_{t \in [0, T]} \mathbf{1}_{\{|\varphi_t^\varepsilon(z, \omega)| \leq R\}} \sum_{(i, j) \in \Delta_N} \log^+ |\varphi_t^{i, \varepsilon}(z, \omega) - \varphi_t^{j, \varepsilon}(z, \omega)| > K \right\} \leq \frac{C_{R_0, R}}{K}.$$

The proof is easily completed on the same model as Lemma 19. ■

Lemma 45 *Given $R_0, T > 0$, we have*

$$\lim_{(\varepsilon, A) \rightarrow (0, +\infty)} \sup_{\varepsilon > 0} \sup_{0 < \delta_0 < 1} (\text{Leb}_{2Nd} \otimes \mathbb{P}) \left\{ (z, \omega) \in B_{2Nd}(0, R_0) \times \Omega : \right. \\ \left. \text{Leb}_1 \left(t \in [0, T] : \inf_{(i, j) \in \Delta_N} |\tilde{\pi}_x [\varphi_t^{i, \varepsilon}(z, \omega) - \varphi_t^{j, \varepsilon}(z, \omega)]| \leq \frac{\delta_0 \varepsilon}{A} \right) > A \delta_0 \right\} = 0.$$

Proof. The proof is performed on the same model as the proof of Lemma 20, but with an appropriate version of Proposition 21. Indeed, the paths of the process $(\varphi_t^\varepsilon(z))_{0 \leq t \leq T}$ cannot be Lipschitz continuous, but they are 1/4-Hölder continuous. Specifically, from the boundedness of F_ε and $Q(\tilde{x}, \tilde{x})$ and by a standard tightness argument, we can prove that the probability that $(\varphi_t^\varepsilon(z))_{0 \leq t \leq T}$ is 1/4-Hölder continuous with A as Hölder constant converges towards 1 as A tends to $+\infty$, uniformly in $\varepsilon > 0$ and in $z \in B_{2Nd}(0, R_0)$.

The statement then follows from the following version of Proposition 21, the proof of which is left to the reader:

Proposition 46 *Given $A, R_0, T > 0$, let $(\zeta_t = (\chi_t, \nu_t))_{0 \leq t \leq T}$ be a continuous path with values in \mathbb{R}^{2Nd} such that $\zeta_0 = z \in B_{2Nd}(0, R_0)$, $(\nu_t^i)_{t \geq 0}$ is a 1/4-Hölder continuous \mathbb{R}^d -valued path with A as Hölder constant, for $1 \leq i \leq N$, and*

$$\frac{d\chi_t^i}{dt} = \nu_t^i, \quad t \geq 0, \quad i \in \{1, \dots, N\}.$$

Then, there exists a constant C , depending on d, A, N, R_0 and T only, such that, for any $\varepsilon, \delta_0 \in (0, 1)$,

$$\inf_{t \in [0, T]} \inf_{(i, j) \in \Delta_N} |\zeta_t^i - \zeta_t^j| \geq \varepsilon \\ \Rightarrow \text{Leb}_1 \left(t \in [0, T] : \inf_{(i, j) \in \Delta_N} |\chi_t^i - \chi_t^j| \leq \delta_0 \frac{\varepsilon^5}{C} \right) \leq C \delta_0.$$

■

5.2 No coalescence for a.e. initial configuration

As a first consequence of the previous estimates, we prove the same result of the deterministic case.

Theorem 47 *Under Assumptions (A.1–3), for Lebesgue almost every z , equation (46) has one and only one global strong solution.*

Proof. Step 1. We here consider $\Xi = \mathcal{C}([0, +\infty), \mathbb{R}^{2Nd}) \otimes \mathcal{C}([0, +\infty), \mathbb{R})^{\otimes \mathbb{N} \setminus \{0\}}$ endowed with the product σ -field \mathcal{X} of the Borel σ -fields. In comparison with the deterministic case, we here enlarge Ξ to support the paths of all the Brownian motions involved in the dynamics of the particle system.

Given $R_0 > 0$ and $\varepsilon > 0$, we endow the pair (Ξ, \mathcal{X}) with the probability \mathbb{Q}^ε defined on the cylinders as

$$\begin{aligned} \mathbb{Q}^\varepsilon & (A_0 \times A_1 \times \cdots \times A_k \times \mathcal{C}([0, +\infty), \mathbb{R}) \times \cdots) \\ & = V_{2Nd}^{-1}(R_0) \cdot (\text{Leb}_{2Nd} \otimes \mathbb{P}) \left\{ (z, \omega) \in B_{2Nd}(0, R_0) \times \Omega : \right. \\ & \quad \left. (\varphi_t^\varepsilon(z))_{t \geq 0} \in A_0, (W_t^1, \dots, W_t^k) \in A_1 \times \cdots \times A_k \right\}, \end{aligned}$$

where A_0 is a Borel subset of $\mathcal{C}([0, +\infty), \mathbb{R}^{2Nd})$ and A_1, \dots, A_N are Borel subsets of $\mathcal{C}([0, +\infty), \mathbb{R})$.

We emphasize that the σ -field \mathcal{X} coincides with the Borel σ -field generated by the standard product metric on the product space Ξ . In particular, the notion of tightness is relevant for probability measures on the pair (Ξ, \mathcal{X}) : it is well-checked that the family $(\mathbb{Q}^\varepsilon)_{\varepsilon > 0}$ is tight.

Denoting by \mathbb{Q} the limit of some convergent sequence $(\mathbb{Q}^{\varepsilon_n})_{n \in \mathbb{N}}$ for a decreasing sequence of positive reals $(\varepsilon_n)_{n \in \mathbb{N}}$ converging towards 0, the point is to discuss the properties of the canonical process under \mathbb{Q} , as done in Lemma 23 for the deterministic case. Below, we will denote by $\xi_t = (\xi_t^k)_{k \in \mathbb{N}}$ the canonical process on Ξ , $(\xi_t^0)_{t \geq 0}$ being \mathbb{R}^{2Nd} -valued and the $(\xi_t^k)_{t \geq 0}$, $k \geq 1$, being \mathbb{R} -valued.

We first notice that the family $((\xi_t^k)_{t \geq 0})_{k \in \mathbb{N} \setminus \{0\}}$ is a family of independent Brownian motions under \mathbb{Q} . Following (i) in Lemma 23, the marginal law of $\Xi \ni \xi \mapsto \xi_0^0$ is the uniform distribution on the ball $B_{2Nd}(0, R_0)$.

Following the proof of (iii) in Lemma 23 and applying Lemmas 44 and 45, we deduce that, for any $T > 0$,

$$\mathbb{Q} \left\{ \xi \in \Xi : \inf_{(i,j) \in \Delta_N} \inf_{t \in [0, T]} |\pi_i(\xi_t^0) - \pi_j(\xi_t^0)| = 0 \right\} = 0. \quad (77)$$

and

$$\mathbb{Q} \left\{ \xi \in \Xi : \text{Leb}_1 \left(t \in [0, T] : \inf_{(i,j) \in \Delta_N} |\pi_{i,x}(\xi_t^0) - \pi_{j,x}(\xi_t^0)| = 0 \right) > 0 \right\} = 0. \quad (78)$$

Set now $\xi_t^0 = (\chi_t^0, \nu_t^0)$, with $\chi_t^0 = \Pi_x(\xi_t^0)$ and $\nu_t^0 = \Pi_v(\xi_t^0)$, $t \geq 0$. Set also

$$\tilde{\nu}_t^0 = \nu_t^0 - \int_0^t \Pi_v(\mathbb{F}(\chi_s^0)) ds, \quad t \geq 0.$$

We claim that $(\tilde{\nu}_t^0)_{t \geq 0}$ is a square-integrable continuous martingale under \mathbb{Q} w.r.t. the filtration $(\mathcal{G}_t^0 = \sigma(\xi_s^k, s \leq t, k \in \mathbb{N}))_{t \geq 0}$ with

$$\left[(\tilde{\nu}^0)^i, (\tilde{\nu}^0)^j \right]_t = \int_0^t Q((\chi_s^0)^i, (\chi_s^0)^j) ds, \quad i, j \in \{1, \dots, d\}, \quad (79)$$

as quadratic variation. We also claim that

$$\left[(\tilde{\nu}^0)^i, \xi^k \right]_t = \int_0^t \sigma_k((\chi_s^0)^i) ds, \quad i \in \{1, \dots, d\}, k \in \mathbb{N} \setminus \{0\}. \quad (80)$$

The proof is quite standard and consists in passing to the limit in the martingale properties characterizing the dynamics of $(Z_t^\varepsilon)_{t \geq 0}$. The only difficulty is to pass to the limit along the mollified drifts, but this can be done as in the proof of Lemma 23 for the deterministic case.

Step 2. Denote by $(\mathcal{G}_t)_{t \geq 0}$ the right-continuous version of $(\mathcal{G}_t)_{t \geq 0}$ augmented with \mathbb{Q} -null sets. Clearly, $(\tilde{\nu}_t^0)_{t \geq 0}$ is a square-integrable continuous martingale under \mathbb{Q} w.r.t. $(\mathcal{G}_t)_{t \geq 0}$ and both (79) and (80) remain true. In particular, we can compute

$$\left[\tilde{\nu}^0 - \sum_{k \in \mathbb{N} \setminus \{0\}} \int_0^\cdot A_k(\xi_s^0) d\xi_s^k \right]_t = 0, \quad t \geq 0,$$

so that, \mathbb{Q} -a.s.,

$$\xi_t^0 = \xi_0^0 + \int_0^t \mathbb{F}(\xi_s^0) ds + \sum_{k \in \mathbb{N} \setminus \{0\}} \int_0^t \mathbb{A}_k(\xi_s^0) d\xi_s^k, \quad t \geq 0. \quad (81)$$

We now denote by $(\mathfrak{Q}(\cdot, z))_{z \in B_{2Nd}(0, R_0)}$ a family of regular conditional probabilities of \mathbb{Q} given the random variable $\Xi \ni \xi \mapsto \xi_0^0$. It is then plain to

see that, for a.e. $z \in B_{2Nd}(0, R_0)$, $((\xi_t^k)_{t \geq 0})_{k \in \mathbb{N} \setminus \{0\}}$ are independent Brownian motions under $\mathfrak{Q}(\cdot, z)$, $(\tilde{\nu}_t^0)_{t \geq 0}$ is a square-integrable continuous martingale under $\mathfrak{Q}(\cdot, z)$ w.r.t. $(\mathcal{G}_t)_{t \geq 0}$ and both (79) and (80) remain true under $\mathfrak{Q}(\cdot, z)$. We then deduce that, for a.e. $z \in B_{2Nd}(0, R_0)$, (81) holds true $\mathfrak{Q}(\cdot, z)$ -a.s. with $\xi_0^0 = z$ therein. (Notice that, almost everywhere on $B_{2Nd}(0, R_0)$, the version of the stochastic integral may be chosen independently of z . Of course, its distribution under $\mathfrak{Q}(\cdot, z)$ depends on z .) As in the deterministic case, we deduce from (77) and (78) that, for a.e. $z \in B_{2Nd}(0, R_0)$, there is no coalescence in the phase space with probability 1 under $\mathfrak{Q}(\cdot, z)$ and that the set of instants where coalescence occurs in the space of positions is of zero Lebesgue measure with probability 1 under $\mathfrak{Q}(\cdot, z)$.

Step 3 We now prove that pathwise uniqueness holds for solutions starting from Γ_N . Following the proof of Lemma 25, we are given two solutions $(\zeta_t)_{t \geq 0}$ and $(\zeta'_t)_{t \geq 0}$ to (46) with $z \in \Gamma_N$, $(\zeta_t)_{t \geq 0}$ being almost-surely free of coalescence in the phase space. Denoting by $\tau = \inf\{t \geq 0 : \zeta_t \neq \zeta'_t\}$, the point is thus to prove that $\mathbb{P}\{\tau = +\infty\} = 1$.

On the set $\{\tau < +\infty\}$ (if not empty), we have $\zeta_\tau = \zeta'_\tau \in \Gamma_N$ since $(\zeta_t)_{t \geq 0}$ is free of coalescence in the phase space. Put it differently, we have, \mathbb{P} -a.s., $\zeta_{\tau \wedge n} = \zeta'_{\tau \wedge n} \in \Gamma_N$ for all $n \in \mathbb{N}$. Therefore, pathwise uniqueness holds if we can prove:

Lemma 48 *Let Z_0 be a random variable with values in Γ_N and let $(\zeta_t)_{t \geq 0}$ and $(\zeta'_t)_{t \geq 0}$ stand for two solutions to (46) with Z_0 as initial condition, $(\zeta_t)_{t \geq 0}$ being free of coalescence in the phase space. Then, there exists a stopping time ρ , $\mathbb{P}\{\rho > 0\} = 1$, such that $(\zeta_t)_{t \geq 0}$ and $(\zeta'_t)_{t \geq 0}$ match almost-surely on $[0, \rho]$.*

Applying Lemma 48 with $Z_0 = \zeta_{\tau \wedge n}$ as initial conditions, we deduce that $\mathbb{P}\{\tau \geq n\} = 1$ for any $n \in \mathbb{N}$, that is $\mathbb{P}\{\tau = +\infty\} = 1$, as announced.

The proof of Lemma 48 is based on the same argument as the proof of Lemma 25. We write $Z_0 = (X_0, V_0)$, with $X_0 = \Pi_x(Z_0)$ and $V_0 = \Pi_v(Z_0)$, and $\zeta_t = (\chi_t, \nu_t)$, with $\chi_t = \Pi_x(\zeta_t)$ and $\nu_t = \Pi_v(\zeta_t)$, for $t \geq 0$. On the same model, we write $\zeta'_t = (\chi'_t, \nu'_t)$ for $t \geq 0$. Setting $\rho^{1,i,j} = \inf(\rho_\zeta^{1,i,j}, \rho_{\zeta'}^{1,i,j})$, with

$$\rho_\zeta^{1,i,j} = \inf \{t \geq 0 : |\chi_t^i - \chi_t^j| \leq |X_0^i - X_0^j|/2\}$$

(with a similar definition for $\rho_{\zeta'}^{1,i,j}$) and $\rho^{2,i,j} = \inf(\rho_\zeta^{2,i,j}, \rho_{\zeta'}^{2,i,j})$, with

$$\rho_\zeta^{2,i,j} = \inf \{t > 0 : |\chi_t^i - \chi_t^j| \leq t|V_0^i - V_0^j|/2\}$$

(with the convention that $\rho_\zeta^{2,i,j} = 0$ if $V_0^i - V_0^j = 0$, and with a similar definition for $\rho_{\zeta'}^{2,i,j}$), we put $\rho^{i,j} = \max(\rho^{1,i,j}, \rho^{2,i,j})$.

We first prove that $\rho^{i,j}$ is almost-surely positive. If $\rho^{1,i,j}(\omega)$ is zero for a given $\omega \in \Omega$, it holds $\rho_\zeta^{1,i,j}(\omega) = 0$ or $\rho_{\zeta'}^{1,i,j}(\omega) = 0$, so that $|X_0^i(\omega) - X_0^j(\omega)| = 0$. Since Z_0 has values in Γ_N , we deduce that $|V_0^i(\omega) - V_0^j(\omega)| > 0$. Since the paths of $(\zeta_t)_{t \geq 0}$ and $(\zeta'_t)_{t \geq 0}$ are 1/4-Hölder continuous (up to a null event), we have the following analogue of (43):

$$|\nu_t^i(\omega) - \nu_t^j(\omega) - (V_0^i(\omega) - V_0^j(\omega))| \leq C(\omega)t^{1/4}, \quad (82)$$

for a finite constant $C(\omega)$ depending on ω . We deduce that, for any $t \in (0, 1]$,

$$|\chi_t^i(\omega) - \chi_t^j(\omega) - t(V_0^i(\omega) - V_0^j(\omega))| \leq C(\omega)t^{5/4}.$$

Therefore,

$$|\chi_t^i(\omega) - \chi_t^j(\omega)| \geq \frac{t}{2} |V_0^i(\omega) - V_0^j(\omega)|, \quad (83)$$

for $C(\omega)t^{1/4} \leq (1/2)|V_0^i(\omega) - V_0^j(\omega)|$. Therefore, $\rho_\zeta^{2,i,j}(\omega) > 0$. Similarly, $\rho_{\zeta'}^{2,i,j}(\omega) > 0$.

We now follow the end of the proof of Lemma 25. On $[0, \rho]$, the drift $F(\zeta_t^i - \zeta_t^j)$ in (46) satisfies (45): it thus coincides with some functional $G((\nu_s^i - \nu_s^j)_{0 \leq s \leq t})$ of the whole path $(\nu_s^i - \nu_s^j)_{0 \leq s \leq t}$, G being bounded and locally Lipschitz continuous w.r.t. the L^∞ -norm. We then write

$$d\nu_t^i = \frac{1}{N} \sum_{j \neq i} G((\nu_s^i - \nu_s^j)_{0 \leq s \leq t}) dt + \sum_{k=1}^{+\infty} \sigma_k(\chi_t^i) dW_t^k, \quad i \in \{1, \dots, d\}, \quad (84)$$

for $t \in [0, \rho]$, with $\rho = \inf_{i \neq j} \rho^{i,j}$. (Obviously, the equation is also satisfied by ζ' .) Eq. (84) reads as a functional equation driven by bounded and locally Lipschitz-continuous coefficients, the Lipschitz constants of the coefficients on any balls being finite random variables depending upon ω through the initial condition Z_0 only. Lemma 48 easily follows by proving that $\mathbb{P}\{\sup_{0 \leq s \leq \rho} |\nu_s - \nu'_s| = 0 | Z_0\} = 1$.

Step 4. We have proven weak existence and strong uniqueness for a.e. initial condition $z \in \Gamma_N$. Following the proof by Yamada and Watanabe in the finite-dimensional case, we deduce that both strong existence and strong uniqueness hold for a.e. initial condition $z \in \Gamma_N$. ■

5.3 No coalescence for any initial condition in Γ_N

We now establish the main result of the paper. To do so, we first prove

Lemma 49 *For any $z \in \Gamma_N$, there exists a unique solution $(\varphi_t(z))_{0 \leq t \leq \tau_z}$ to (46), with z as initial condition, on the interval $[0, \tau_z]$, where $\tau_z = \inf\{t \geq 0 : \varphi_t(z) \in \Gamma_N^c\}$. Moreover, the mapping $\Gamma_N \ni z \mapsto (\varphi_{t \wedge \tau_z}(z))_{t \geq 0} \in \mathcal{C}([0, +\infty), \mathbb{R}^{2Nd})$ is measurable.*

Proof. The whole difficulty is here to handle the possible coalescence of the particles in the space of positions. By induction, we build a non-decreasing sequence of stopping times $(\tau_z^n)_{n \in \mathbb{N}}$ such that $\tau_z^n \rightarrow \tau_z$ almost-surely as n tends to $+\infty$ and (46) has a unique solution $(\varphi_t(z))_{0 \leq t \leq \tau_z^n}$ on each $[0, \tau_z^n]$ with z as initial solution for any $n \in \mathbb{N}$. The stopping time τ_0 is set equal to 0. Given $(\varphi_t(z) = (\chi_t(z), \nu_t(z)))_{0 \leq t \leq \tau_z^n}$, for some $n \in \mathbb{N}$, we can follow the proof of Lemma 48 and build a (unique) solution $(\varphi_t(z))_{\tau_z^n \leq t \leq \tau_z^{n+1}}$ to (46) on $[\tau_z^n, \tau_z^{n+1}]$, where $\tau_z^{n+1} = \tau_z \wedge \tau_z'^{n+1}$, where $\tau_z'^{n+1} = \inf_{i \neq j} \rho^{i,j,n+1}$, with $\rho^{i,j,n+1} = \max(\rho^{1,i,j,n+1}, \rho^{2,i,j,n+1})$,

$$\rho^{1,i,j,n+1} = \inf \{t \geq \tau_z^n : |\chi_t^i(z) - \chi_t^j(z)| \leq |\chi_{\tau_z^n}^i(z) - \chi_{\tau_z^n}^j(z)| / 2\},$$

and,

$$\rho^{2,i,j,n+1} = \inf \{t > \tau_z^n : |\chi_t^i(z) - \chi_t^j(z)| \leq (t - \tau_z^n) |\nu_{\tau_z^n}^i(z) - \nu_{\tau_z^n}^j(z)| / 2\}.$$

Clearly, the sequence $(\tau_z^n)_{n \in \mathbb{N}}$ is non-decreasing.

On each step, existence and uniqueness hold since Eq. (46) can be written as a functional SDE on the interval $[\tau_z^n, \tau_z^{n+1}]$ with bounded and locally Lipschitz-continuous coefficients. (As already emphasized in the proof of Lemma 48, the Lipschitz constants of the coefficients on bounded sets depend on the initial position $\varphi_{\tau_z^n}(z)$. Anyhow, this has no consequences on the existence and uniqueness of a solution on the interval $[\tau_z^n, \tau_z^{n+1}]$.)

Almost-surely, the sequence $(\tau_z^n)_{n \in \mathbb{N}}$ cannot have an accumulation point before $\varphi(z)$ hits Γ_N^c , as otherwise there would be a blow-up for the modulus of continuity of $\varphi(z)$. Again, the precise argument goes back to the proof of Lemma 48: the length $\tau_z^{n+1} - \tau_z^n$ depends on the modulus of continuity of the path $(\varphi_t(z))_{\tau_z^n \leq t \leq \tau_z^{n+1}}$ —the length of the interval is controlled from below when the modulus is controlled from above—, on the distance $\text{dist}(\varphi_{\tau_z^n}(z), \Gamma_N^c)$ —the length of the interval is controlled from below when the distance is away

from zero– and on the norm $|\varphi_{\tau_z^n}(z)|$ –the length of the interval is controlled from below when the norm is away from $+\infty$ –. The modulus of continuity is controlled in terms of the bounds of the coefficients by Kolmogorov’s criterion, the norm of $\varphi(z)$ is controlled in terms of the bounds of the coefficients as well, and the distance from $\varphi(z)$ to $\Gamma_N^{\mathbb{C}}$ is bounded from below on any $[0, \tilde{\tau}_z^\epsilon]$, with $\tilde{\tau}_z^\epsilon = \inf\{t \geq 0 : \text{dist}(\varphi_t(z), \Gamma_N^{\mathbb{C}}) \leq \epsilon\}$, for $\epsilon > 0$. This proves that, a.s., $\sup_{n \geq 1} \tau_z^n \geq \tilde{\tau}_z^\epsilon$ for any $\epsilon > 0$, that is $\sup_{n \geq 1} \tau_z^n \geq \tilde{\tau}_z^\epsilon = \tau_z$. ■

Proof. (Theorem 41.) We now complete the proof. We add a point Δ to \mathbb{R}^{2Nd} and set $\varphi_t(z) = \Delta$ for $t \geq \tau_z$, $z \in \Gamma_N$, when $\tau_z < \infty$. The resulting family of processes $(\varphi_t(z))_{t \geq 0}$, $z \in \Gamma_N$, has $\Gamma_N \cup \Delta$ as state space. It is a homogeneous Markov process. By Theorem 47, $\mathbb{P}\{\tau_z = +\infty\} = 1$ for a.e. $z \in \mathbb{R}^{2Nd}$. In particular, we can write, for any $0 < \varepsilon < T$,

$$\mathbb{P}\{\varphi_{[\varepsilon, T]}(z) \in \Gamma_N\} = \int_{\Gamma_N \cup \{\Delta\}} \mathbb{P}\{\varphi_{[0, T-\varepsilon]}(z') \in \Gamma_N\} \mu_{\varphi_\varepsilon(z)}(dz')$$

where $\{\varphi_{[\varepsilon, T]}(z) \in \Gamma_N\} = \{\omega \in \Omega : \varphi_t(z)(\omega) \in \Gamma_N, \text{ for any } t \in [\varepsilon, T]\}$ and $\mu_{\varphi_\varepsilon(z)}$ is the law of $\varphi_\varepsilon(z)$. By Theorem 47, we can find a Borel subset $\mathcal{N} \subset \mathbb{R}^{2Nd}$ of zero Lebesgue measure such that

$$\mathbb{P}\{\varphi_{[0, T-\varepsilon]}(z) \in \Gamma_N\} = 1$$

for all $z \in \mathcal{N}^c$. Then

$$\begin{aligned} \mathbb{P}\{\varphi_{[\varepsilon, T]}(z) \in \Gamma_N\} &\geq \int_{\mathcal{N}^c} \mathbb{P}\{\varphi_{[0, T-\varepsilon]}(z') \in \Gamma_N\} \mu_{\varphi_\varepsilon(z)}(dz') \\ &= 1 - \mu_{\varphi_\varepsilon(z)}(\mathcal{N}). \end{aligned} \quad (85)$$

Now, assume that $z = (x, v)$ is such that $x \in \Gamma_{x, N}$ (so that $z \in \Gamma_N$). Then, there exists $\varepsilon^* > 0$ such that $\inf_{i \neq j} |x^i - x^j| > \varepsilon^*$. Defining $\tau_z^* = \inf\{t \geq 0 : \inf_{i \neq j} |\pi_{i, x}(\varphi_t(z)) - \pi_{j, x}(\varphi_t(z))| \leq \varepsilon^*\}$, we have

$$\begin{aligned} \mu_{\varphi_\varepsilon(z)}(\mathcal{N}) &= \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}\} \\ &= \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \tau_z^* > \varepsilon\} + \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \tau_z^* \leq \varepsilon\} \\ &\leq \mathbb{P}\{\varphi_\varepsilon^*(z) \in \mathcal{N}, \tau_z^* > \varepsilon\} + \mathbb{P}\{\tau_z^* \leq \varepsilon\} \\ &\leq \mathbb{P}\{\varphi_\varepsilon^*(z) \in \mathcal{N}\} + \mathbb{P}\{\tau_z^* \leq \varepsilon\}, \end{aligned} \quad (86)$$

where $(\varphi_t^*(z))_{t \geq 0}$ stands for the solution to (46) when driven by a Lipschitz drift that coincides with the original one on $\{(x^1, \dots, x^N) \in \mathbb{R}^{2Nd} : \inf_{i \neq j} |x^i -$

$x^j | > \varepsilon^*$. Since \mathcal{N} is Lebesgue-negligible and the law of $\varphi_\varepsilon^*(z)$ on \mathbb{R}^{2Nd} is absolutely continuous with respect to Lebesgue measure when $x \in \Gamma_{x,N}$, we deduce that $\mathbb{P}\{\varphi_\varepsilon^*(z) \in \mathcal{N}\} = 0$. (See Proposition 40.) Letting ε tend to zero, we deduce that $\mathbb{P}\{\varphi_{[0,T]} \in \Gamma_N\} = 1$. Indeed, $\bigcap_{\varepsilon>0}\{\varphi_{[\varepsilon,T]}(z) \in \Gamma_N\} = \{\varphi_{(0,T]}(z) \in \Gamma_N\} = \{\varphi_{[0,T]}(z) \in \Gamma_N\}$ since $\varphi_0(z) \in \Gamma_N^c$ implies $\varphi_t(z) = \Delta$ for any $t > 0$.

Assume now that $z \in \Gamma_N$ but $x \notin \Gamma_{x,N}$. Following the proof of Lemma 48, we know that, \mathbb{P} almost-surely, there exists a non-empty interval $(0, \rho(\omega))$ such that $\varphi_t(z) \in \Gamma_{x,N}$, ρ standing for a stopping-time. (When $x^i = x^j$, $|\pi_{i,x}(\varphi_t(z)) - \pi_{j,x}(\varphi_t(z))| \geq (t/2)|v^i - v^j|$ for $t > 0$ small and is thus non-zero for $t > 0$ small.) In particular, $\tau_z(\omega) \geq \rho(\omega)$. For any $\delta \in (0, \varepsilon)$, we have

$$\begin{aligned} \mu_{\varphi_\varepsilon(z)}(\mathcal{N}) &= \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}\} \\ &= \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \delta < \rho\} + \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \rho \leq \delta\} \\ &\leq \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \delta < \rho\} + \mathbb{P}\{\rho \leq \delta\} \\ &\leq \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \Pi_x(\varphi_\delta(z)) \in \Gamma_{x,N}\} + \mathbb{P}\{\rho \leq \delta\}. \end{aligned} \tag{87}$$

By Markov property, we then claim

$$\begin{aligned} &\mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \Pi_x(\varphi_\delta(z)) \in \Gamma_{x,N}\} \\ &= \int_{\mathbb{R}^{2Nd}} \mathbb{P}\{\varphi_{\varepsilon-\delta}(z') \in \mathcal{N}\} \mathbf{1}_{\{\Pi_x(z') \in \Gamma_{x,N}\}} \mu_{\varphi_\delta(z)}(dz'). \end{aligned} \tag{88}$$

Going back to (85), we write ε as $\varepsilon = \varepsilon_1 + \varepsilon_2$, with $\varepsilon_1, \varepsilon_2 > 0$. Choosing $\delta = \varepsilon_1$ in (87), we have $\varepsilon - \delta = \varepsilon_2$ in (88). By (86) and by Lebesgue's dominated convergence Theorem, we know that the right-hand side in (88) tends to 0 as $\varepsilon_2 = \varepsilon - \delta$ tends to 0. Passing to the limit in (85) and using (87), this says that $\mathbb{P}\{\varphi_{[\varepsilon_1,T]}(z) \in \Gamma_N\} \geq 1 - \mathbb{P}\{\rho \leq \varepsilon_1\}$, for any $\varepsilon_1 > 0$. Letting ε_1 tend to zero, we obtain $\mathbb{P}\{\varphi_{[0,T]}(z) \in \Gamma_N\} = 1$. ■

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