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# SOLVABILITY OF A CLASS OF THERMAL DYNAMICAL CONTACT PROBLEMS WITH SUBDIFFERENTIAL CONDITIONS

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**ABSTRACT.** We study a class of dynamic thermal sub-differential contact problems with friction, for long memory visco-elastic materials, which can be put into a general model of system defined by a second order evolution inequality, coupled with a first order evolution equation. We present and establish an existence and uniqueness result, by using general results on first order evolution inequality, with monotone operators and fixed point methods.

**1. Introduction.** Despite of numerous recent progress, contact mechanics still remains a rich domain of various new problems, and the literature devoted to the subject is more and more extensive. An early attempt at the study of contact problems for elastic viscoelastic materials within the mathematical analysis framework was introduced in the pioneering reference works [5, 7, 12, 13, 15]. More recently, the dynamic Signorini's problem for a cracked viscoelastic body is studied in [6]. Mathematical analysis of unilateral contact problems involving static, quasi-static and dynamic process may be found in the recent self-contained book [8]. Further extensions to non convex contact conditions with non-monotone and possible multi-valued constitutive laws led to the active domain of non-smooth mechanic within the framework of the so-called hemivariational inequalities, for a mathematical as well as mechanical treatment we refer to [9, 16].

Quasi-static contact problems, by taking into account the parameter of the temperature field, were analyzed in [1], where the friction is described by a general normal damped response condition, there the existence and uniqueness of weak solution has been established. The existence of solutions for thermal viscoelastic dynamic contact problems with Coulomb friction law were treated in [8].

This work is a companion paper of the results obtained in [4]. In [4] we studied a class of dynamical long memory viscoelastic problems, where the contact is bilateral and frictional obeying the Tresca's law. The variational formulation of the problem leads to a second order evolution inequality on the displacement field. Then an existence and uniqueness result of "weak" solution on displacement has been proved.

Here we consider a class of dynamical long memory viscoelastic problems, the new feature in this paper is that the contact is defined by a general sub-differential condition on the velocity, and with heat exchange, putting then the problem into a coupled system, defined by a second order evolution inequality on the displacement field, and a differential equation which governs the evolution of the temperature field. On the other hand we investigate an existence and uniqueness result of "strong"

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solution on displacement and temperature fields, i.e. solution with more regularity. For this proposal we specify stronger assumptions on the data and operators, and use a new method based on a version of first order evolution inequality with monotonicity, convexity and fixed point theory.

The paper is organized as follows. In Section 2 we describe the mechanical problem, specify the assumptions on the data to derive the variational formulation, and then we state our main existence and uniqueness result. In Section 3, we give the proof of the claimed result.

**2. Statement of the problem.** In this section we study a class of thermal contact problems with sub-differential conditions, for long memory visco-elastic materials. We describe the mechanical problems, list the assumptions on the data and derive the corresponding variational formulations. Then we state an existence and uniqueness result on displacement and temperature fields, which we will prove in the next section.

The physical setting is as follows. A visco-elastic body occupies the domain  $\Omega$  with surface  $\Gamma$  that is partitioned into three disjoint measurable parts  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$ , such that  $\text{meas}(\Gamma_1) > 0$ . Let  $[0, T]$  be the time interval of interest, where  $T > 0$ . The body is clamped on  $\Gamma_1 \times (0, T)$  and therefore the displacement field vanishes there. We also assume that a volume force of density  $\mathbf{f}_0$  acts in  $\Omega \times (0, T)$  and that surface tractions of density  $\mathbf{f}_2$  act on  $\Gamma_2 \times (0, T)$ . The body may come in contact with an obstacle, the foundation, over the potential contact surface  $\Gamma_3$ . The model of the contact is specified by a general sub-differential boundary condition, where thermal effects may occur in the frictional contact with the basis. We are interested in the dynamic evolution of the body.

Let us recall now some classical notations, see e.g. [7, 14] for further details. We denote by  $S_d$  the space of second order symmetric tensors on  $\mathbb{R}^d$  ( $d = 2, 3$ ), while “ $\cdot$ ” and  $|\cdot|$  will represent the inner product and the Euclidean norm on  $S_d$  and  $\mathbb{R}^d$ . Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with a Lipschitz boundary  $\Gamma$  and let  $\boldsymbol{\nu}$  denote the unit outer normal on  $\Gamma$ . Everywhere in the sequel the indexes  $i$  and  $j$  run from 1 to  $d$ , summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable. We also use the following notation:

$$H = \left( L^2(\Omega) \right)^d, \quad \mathcal{H} = \{ \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \},$$

$$H_1 = \{ \mathbf{u} \in H \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H} \}, \quad \mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H} \mid \text{Div } \boldsymbol{\sigma} \in H \}.$$

Here  $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$  and  $\text{Div} : \mathcal{H}_1 \longrightarrow H$  are the deformation and the divergence operators, respectively, defined by :

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j})_H.$$

The spaces  $H, \mathcal{H}, H_1$  and  $\mathcal{H}_1$  are real Hilbert spaces endowed with the canonical inner products given by :

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_H &= \int_{\Omega} u_i v_i \, dx, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx, \\ (\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\text{Div } \boldsymbol{\sigma}, \text{Div } \boldsymbol{\tau})_H. \end{aligned}$$

We recall that  $C$  denotes the class of continuous functions; and  $C^m$ ,  $m \in \mathbb{N}^*$  the set of  $m$  times differentiable functions.

Finally  $\mathcal{D}(\Omega)$  denotes the set of infinitely differentiable real functions with compact support in  $\Omega$ ; and  $W^{m,p}$ ,  $m \in \mathbb{N}$ ,  $1 \leq p \leq +\infty$  for the classical Sobolev spaces; and

$$H_0^m(\Omega) := \{ w \in W^{m,2}(\Omega), w = 0 \text{ on } \Gamma \}, \quad m \geq 1.$$

To continue, the mechanical problem is then formulated as follows.

**Problem Q :** Find a displacement field  $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$  and a stress field  $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow S_d$  and a temperature field  $\theta : \Omega \times [0, T] \rightarrow \mathbb{R}_+$  such that for a.e.  $t \in (0, T)$ :

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}(s))ds - \theta(t)C_e \quad \text{in } \Omega \quad (2.1)$$

$$\ddot{\mathbf{u}}(t) = \text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } \Omega \quad (2.2)$$

$$\mathbf{u}(t) = 0 \quad \text{on } \Gamma_1 \quad (2.3)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2 \quad (2.4)$$

$$\mathbf{u}(t) \in U, \quad \varphi(\mathbf{w}) - \varphi(\dot{\mathbf{u}}(t)) \geq -\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\mathbf{w} - \dot{\mathbf{u}}(t)) \quad \forall \mathbf{w} \in U \quad \text{on } \Gamma_3 \quad (2.5)$$

$$\dot{\theta}(t) - \text{div}(K_c \nabla \theta(t)) = -c_{ij} \frac{\partial \dot{u}_i}{\partial x_j}(t) + q(t) \quad \text{on } \Omega \quad (2.6)$$

$$-k_{ij} \frac{\partial \theta}{\partial x_j}(t) n_i = k_e (\theta(t) - \theta_R) \quad \text{on } \Gamma_3 \quad (2.7)$$

$$\theta(t) = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \quad (2.8)$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega \quad (2.9)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0 \quad \text{in } \Omega \quad (2.10)$$

Here, (2.1) is the Kelving Voigt's long memory thermo-visco-elastic constitutive law of the body,  $\boldsymbol{\sigma}$  the stress tensor,  $\mathcal{A}$  is the viscosity operator,  $\mathcal{G}$  for the elastic operator,  $C_e := (c_{ij})$  represents the thermal expansion tensor, and  $\mathcal{B}$  is the so called tensor of relaxation which defines the long memory of the material, as an important particular case, when  $\mathcal{B} \equiv 0$ , we find again the usual visco-elasticity of short memory. In (2.2) is the dynamic equation of motion where the mass density  $\rho \equiv 1$ . On the contact surface, the general relation (2.5) is a sub-differential boundary condition, where

$$\mathcal{D}(\Omega)^d \subset U \subset H_1$$

represents the set of contact admissible test functions,  $\boldsymbol{\sigma}\boldsymbol{\nu}$  denotes the Cauchy stress vector on the contact boundary and  $\varphi : \Gamma_3 \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a given function. Various situations may be modelled by such a condition, and some concrete examples will be recalled below. The differential equation (2.6) describes the evolution of the temperature field, where  $K_c := (k_{ij})$  represents the thermal conductivity tensor,  $q(t)$  the density of volume heat sources. The associated temperature boundary condition is given by (2.7), where  $\theta_R$  is the temperature of the foundation, and  $k_e$  is the heat exchange coefficient between the body and the obstacle, and  $p_\tau : \Gamma_3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a given tangential function. Finally,  $\mathbf{u}_0$ ,  $\mathbf{v}_0$ ,  $\theta_0$  represents the initial displacement, velocity and temperature, respectively.

To derive the variational formulation of the mechanical problems (2.1)–(2.10) we need additional notations. Thus, let  $V$  denote the closed subspace of  $H_1$  defined by

$$\mathcal{D}(\Omega)^d \subset V = \{ \mathbf{v} \in H_1 \mid \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_1 \} \cap U;$$

$$E = \{ \eta \in H^1(\Omega), \eta = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \}.$$

Since  $\text{meas } \Gamma_1 > 0$ , Korn's inequality holds : there exists  $C_K > 0$  which depends only on  $\Omega$  and  $\Gamma_1$  such that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \geq C_K \|\mathbf{v}\|_{H_1} \quad \forall \mathbf{v} \in V$$

A proof of Korn's inequality may be found in [14], p.79.

On  $V$  we consider the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

and let  $\|\cdot\|_V$  be the associated norm, i.e.

$$\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \quad \forall \mathbf{v} \in V.$$

It follows that  $\|\cdot\|_{H_1}$  and  $\|\cdot\|_V$  are equivalent norms on  $V$  and therefore  $(V, \|\cdot\|_V)$  is a real Hilbert space. Moreover, by the Sobolev's trace theorem, we have a constant  $C_0 > 0$  depending only on  $\Omega$ ,  $\Gamma_1$ , and  $\Gamma_3$  such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)} \leq C_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V.$$

In the study of the mechanical problem (2.1)-(2.10), we assume that the viscosity operator  $\mathcal{A} : \Omega \times S_d \rightarrow S_d$ ,  $(\mathbf{x}, \boldsymbol{\tau}) \mapsto (a_{ijkh}(\mathbf{x}) \tau_{kh})$  is linear on the second variable and satisfies the usual properties of ellipticity and symmetry, i.e.

$$\left\{ \begin{array}{l} \text{(i) } a_{ijkh} \in W^{1,\infty}(\Omega); \\ \text{(ii) } \mathcal{A}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{A}\boldsymbol{\tau} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \text{ a.e. in } \Omega; \\ \text{(iii) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad \mathcal{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{A}} |\boldsymbol{\tau}|^2 \quad \forall \boldsymbol{\tau} \in S_d, \text{ a.e. in } \Omega \end{array} \right. \quad (2.11)$$

The elasticity operator  $\mathcal{G} : \Omega \times S_d \rightarrow S_d$  satisfies :

$$\left\{ \begin{array}{l} \text{(i) there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{G}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega; \\ \text{(ii) } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \forall \boldsymbol{\varepsilon} \in S_d; \\ \text{(iii) the mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}) \in \mathcal{H} \end{array} \right. \quad (2.12)$$

The relaxation tensor  $\mathcal{B} : [0, T] \times \Omega \times S_d \rightarrow S_d$ ,  $(t, \mathbf{x}, \boldsymbol{\tau}) \mapsto (B_{ijkh}(t, \mathbf{x}) \tau_{kh})$  satisfies

$$\left\{ \begin{array}{l} \text{(i) } B_{ijkh} \in W^{1,\infty}(0, T; L^\infty(\Omega)); \\ \text{(ii) } \mathcal{B}(t)\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{B}(t)\boldsymbol{\tau} \\ \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \text{ a.e. } t \in (0, T), \text{ a.e. in } \Omega \end{array} \right. \quad (2.13)$$

We suppose the body forces and surface tractions satisfy

$$\mathbf{f}_0 \in W^{1,2}(0, T; H), \quad \mathbf{f}_2 \in W^{1,2}(0, T; L^2(\Gamma_2)^d) \quad (2.14)$$

For the thermal tensors and the heat sources density, we suppose that

$$C_e = (c_{ij}), \quad c_{ij} = c_{ji} \in L^\infty(\Omega), \quad q \in W^{1,2}(0, T; L^2(\Omega)) \quad (2.15)$$

The boundary thermic data satisfy

$$k_e \in L^\infty(\Omega; \mathbb{R}^+), \quad \theta_R \in W^{1,2}(0, T; L^2(\Gamma_3)) \quad (2.16)$$

The thermal conductivity tensor verifies the usual symmetry and ellipticity : for some  $c_k > 0$  and for all  $(\xi_i) \in \mathbb{R}^d$ ,

$$K_c = (k_{ij}), \quad k_{ij} = k_{ji} \in L^\infty(\Omega), \quad k_{ij} \xi_i \xi_j \geq c_k \xi_i \xi_i. \quad (2.17)$$

Finally we have to put technical assumptions on the initial data and the sub-differential condition on the contact surface, in order to use classical results on first order set valued evolution equations. Many various possibilities could be considered (see e.g. ([2, 3, 11])). Here we use a general theorem taken in ([11]) p. 46, in a simplified case, which is enough for our proposal and applications.

We assume that the initial data satisfy the conditions

$$\mathbf{u}_0 \in V, \quad \mathbf{v}_0 \in V \cap H_0^2(\Omega)^d, \quad \theta_0 \in E \cap H_0^2(\Omega) \quad (2.18)$$

On the contact surface, the following frictional contact function

$$\psi(\mathbf{w}) := \int_{\Gamma_3} \varphi(\mathbf{w}) da$$

verifies

$$\left\{ \begin{array}{l}
\text{(i) } \psi : V \longrightarrow \mathbb{R} \text{ is well defined, continuous and convex;} \\
\text{(ii) there exists a sequence of differentiable convex functions} \\
\quad (\psi_n) : V \longrightarrow \mathbb{R} \text{ such that } \forall \mathbf{w} \in L^2(0, T; V), \\
\quad \int_0^T \psi_n(\mathbf{w}(t)) dt \longrightarrow \int_0^T \psi(\mathbf{w}(t)) dt, n \longrightarrow +\infty; \\
\text{(iii) for all sequence } (\mathbf{w}_n) \text{ and } \mathbf{w} \text{ in } W^{1,2}(0, T; V) \text{ such that} \\
\quad \mathbf{w}_n \rightharpoonup \mathbf{w}, \mathbf{w}'_n \rightharpoonup \mathbf{w}' \text{ weakly in } L^2(0, T; V), \\
\quad \text{then } \liminf_{n \rightarrow +\infty} \int_0^T \psi_n(\mathbf{w}_n(t)) dt \geq \int_0^T \psi(\mathbf{w}(t)) dt \\
\text{(iv) } \forall \mathbf{w} \in V, \quad (\mathbf{w} = 0 \text{ on } \Gamma_3 \implies \forall n \in \mathbb{N}, \psi'_n(\mathbf{w}) = 0_{V'})
\end{array} \right. \quad (2.19)$$

Here  $\psi'_n(\mathbf{v})$  denotes the Fréchet derivative of  $\psi_n$  at  $\mathbf{v}$ .

To continue, using Green's formula, we obtain the variational formulation of the mechanical problem  $Q$  in abstract form as follows.

**Problem  $QV$**  : Find  $\mathbf{u} : [0, T] \rightarrow V$ ,  $\theta : [0, T] \rightarrow E$  satisfying a.e.  $t \in (0, T)$ :

$$\left\{ \begin{array}{l}
\langle \ddot{\mathbf{u}}(t) + A \dot{\mathbf{u}}(t) + B \mathbf{u}(t) + C \theta(t), \mathbf{w} - \dot{\mathbf{u}}(t) \rangle_{V' \times V} \\
+ \left( \int_0^t B(t-s) \boldsymbol{\varepsilon}(\mathbf{u}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) \right)_{\mathcal{H}} + \psi(\mathbf{w}) - \psi(\dot{\mathbf{u}}(t)) \\
\geq \langle \mathbf{f}(t), \mathbf{w} - \dot{\mathbf{u}}(t) \rangle_{V' \times V} \quad \forall \mathbf{w} \in V; \\
\dot{\theta}(t) + K \theta(t) = R \dot{\mathbf{u}}(t) + Q(t) \quad \text{in } E'; \\
\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0.
\end{array} \right.$$

Here, the operators and functions  $A, B : V \rightarrow V'$ ,  $C : E \rightarrow V'$ ,  $\psi : V \rightarrow \mathbb{R}$ ,  $K : E \rightarrow E'$ ,  $R : V \rightarrow E'$ ,  $\mathbf{f} : [0, T] \rightarrow V'$ , and  $Q : [0, T] \rightarrow E'$  are defined by  $\forall \mathbf{v} \in V, \forall \mathbf{w} \in V, \forall \tau \in E, \forall \eta \in E$ :

$$\begin{aligned}
\langle A \mathbf{v}, \mathbf{w} \rangle_{V' \times V} &= (\mathcal{A}(\boldsymbol{\varepsilon} \mathbf{v}), \boldsymbol{\varepsilon} \mathbf{w})_{\mathcal{H}}; \\
\langle B \mathbf{v}, \mathbf{w} \rangle_{V' \times V} &= (\mathcal{G}(\boldsymbol{\varepsilon} \mathbf{v}), \boldsymbol{\varepsilon} \mathbf{w})_{\mathcal{H}}; \\
\langle C \tau, \mathbf{w} \rangle_{V' \times V} &= -(\tau C_e, \boldsymbol{\varepsilon} \mathbf{w})_{\mathcal{H}}; \\
\langle \mathbf{f}(t), \mathbf{w} \rangle_{V' \times V} &= (\mathbf{f}_0(t), \mathbf{w})_H + (\mathbf{f}_2(t), \mathbf{w})_{(L^2(\Gamma_2))^d}; \\
\langle Q(t), \eta \rangle_{E' \times E} &= \int_{\Gamma_3} k_e \theta_R(t) \eta dx + \int_{\Omega} q(t) \eta dx; \\
\langle K \tau, \eta \rangle_{E' \times E} &= \sum_{i,j=1}^d \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \cdot \eta da; \\
\langle R \mathbf{v}, \eta \rangle_{E' \times E} &= - \int_{\Omega} c_{ij} \frac{\partial v_i}{\partial x_j} \eta dx.
\end{aligned}$$

**Theorem 1.** Assume that (2.11)–(2.19) hold, then there exists a unique solution  $\{\mathbf{u}, \theta\}$  to problem  $QV$  with the regularity :

$$\left\{ \begin{array}{l}
\mathbf{u} \in W^{2,2}(0, T; V) \cap W^{2,\infty}(0, T; H) \\
\theta \in W^{1,2}(0, T; E) \cap W^{1,\infty}(0, T; F).
\end{array} \right. \quad (2.20)$$

Before giving the proof, we show in the following two typical examples of sub-differential conditions.

**Example 1. Tresca's friction law.**

The contact condition on  $\Gamma_3$  is bilateral, and satisfies (see e.g. [7, 15]):

$$\begin{cases} u_\nu = 0, & |\boldsymbol{\sigma}_\tau| \leq g, \\ |\boldsymbol{\sigma}_\tau| < g \implies \dot{\mathbf{u}}_\tau = \mathbf{0}, \\ |\boldsymbol{\sigma}_\tau| = g \implies \dot{\mathbf{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, & \text{for some } \lambda \geq 0, \end{cases} \quad \text{on } \Gamma_3 \times (0, T).$$

Here  $g$  represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins, with  $g \in L^\infty(\Gamma_3)$ ,  $g \geq 0$  a.e. on  $\Gamma_3$ . We deduce the admissible displacement space:

$$V := \{\mathbf{w} \in H_1; \text{ with } \mathbf{w} = \mathbf{0} \text{ on } \Gamma_1; w_\nu = 0 \text{ on } \Gamma_3\},$$

and the sub-differential contact function:

$$\varphi(\mathbf{x}, \mathbf{y}) = g(\mathbf{x})|\mathbf{y}_{\tau(\mathbf{x})}| \quad \forall \mathbf{x} \in \Gamma_3, \mathbf{y} \in \mathbb{R}^d,$$

where  $\mathbf{y}_{\tau(\mathbf{x})} := \mathbf{y} - y_{\nu(\mathbf{x})}\boldsymbol{\nu}(\mathbf{x})$ ,  $y_{\nu(\mathbf{x})} := \mathbf{y} \cdot \boldsymbol{\nu}(\mathbf{x})$ , with  $\boldsymbol{\nu}(\mathbf{x})$  the unit normal at  $\mathbf{x} \in \Gamma_3$ . We have then

$$\psi(\mathbf{v}) := \int_{\Gamma_3} g|\mathbf{v}_\tau| da, \quad \forall \mathbf{v} \in V$$

is well defined on  $V$  with the property: for some  $c > 0$ ,

$$|\psi(\mathbf{w}) - \psi(\mathbf{v})| \leq c \|\mathbf{v} - \mathbf{w}\|_{L^2(\Gamma_3)^d}, \quad \forall \mathbf{v}, \mathbf{w} \in V.$$

Let us show that assumptions in (2.19) are verified.

Firstly it is clear that  $\psi : V \rightarrow \mathbb{R}$  is convex. By using the continuous embedding from  $V$  into  $L^2(\Gamma_3)^d$  and the last inequality, we find that

$$\psi \text{ is Lipschitz continuous on } V.$$

This gives (2.19)(i). To approximate the function  $\psi$ , we use the sequence

$$\psi_n(\mathbf{v}) := \int_{\Gamma_3} g \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}} da, \quad \forall \mathbf{v} \in V, \forall n \in \mathbb{N}^*.$$

Now we claim that  $\psi_n$  ( $n \in \mathbb{N}^*$ ) is Frechet differentiable and the Frechet derivative of  $\psi_n$  is given by

$$\psi'_n(\mathbf{v}) \cdot \mathbf{h} = \int_{\Gamma_3} g \frac{(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da, \quad \forall \mathbf{h} \in V. \quad (2.21)$$

Indeed. Let fix  $n \in \mathbb{N}^*$  and  $\mathbf{v} \in V$ . For any  $\mathbf{h} \in V$ :

$$\psi_n(\mathbf{v} + \mathbf{h}) - \psi_n(\mathbf{v}) = \int_{\Gamma_3} g \left( \sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} - \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}} \right) da,$$

thus

$$\psi_n(\mathbf{v} + \mathbf{h}) - \psi_n(\mathbf{v}) = \int_{\Gamma_3} g \frac{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 - |\mathbf{v}_\tau|^2}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da,$$

and

$$\psi_n(\mathbf{v} + \mathbf{h}) - \psi_n(\mathbf{v}) = \int_{\Gamma_3} g \frac{2(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d} + |\mathbf{h}_\tau|^2}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da.$$

As

$$\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}} \geq \frac{2}{\sqrt{n}},$$

we have

$$\int_{\Gamma_3} g \frac{|\mathbf{h}_\tau|^2}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da \in O(\|\mathbf{h}\|_{L^2(\Gamma_3)^d}^2) \quad \text{as } \mathbf{h} \rightarrow \mathbf{0}_V.$$

Hence

$$\int_{\Gamma_3} g \frac{|\mathbf{h}_\tau|^2}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da \in O(\|\mathbf{h}\|_V^2) \quad \text{as } \mathbf{h} \rightarrow \mathbf{0}_V.$$

We obtain as  $\mathbf{h} \rightarrow \mathbf{0}_V$ :

$$\frac{2(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} - \frac{(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} = (\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d} \left( \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}} - \sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} \right) O(1),$$

where  $O(1)$  denotes some scalar function of  $\mathbf{h}$  bounded in a neighborhood of  $\mathbf{0}_V$ .

Previous computations show that

$$\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}} - \sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} = (\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d} O(1) \quad \text{as } \mathbf{h} \rightarrow \mathbf{0}_V.$$

Thus

$$\frac{2(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} - \frac{(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} = O(\|\mathbf{h}_\tau\|_{\mathbb{R}^d}^2) \quad \text{as } \mathbf{h} \rightarrow \mathbf{0}_V.$$

Consequently

$$\int_{\Gamma_3} g \frac{2(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau + \mathbf{h}_\tau|^2 + \frac{1}{n}} + \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da = \int_{\Gamma_3} g \frac{(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da + O(\|\mathbf{h}\|_V^2) \quad \text{as } \mathbf{h} \rightarrow \mathbf{0}_V.$$

This ends the proof of the claim as

$$\mathbf{h} \in V \mapsto \int_{\Gamma_3} g \frac{(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da$$

is a linear continuous functional. □

To continue, we deduce from the Frechet derivative's formula that  $\psi_n$  is of class  $C^1$ . Direct algebraic computations show that for all  $\alpha \geq 0$ ,  $\beta \geq 0$  such that  $\alpha + \beta = 1$ , and for all reals  $x$  and  $y$ ,  $n \geq 1$ :

$$\sqrt{(\alpha x + \beta y)^2 + \frac{1}{n}} \leq \alpha \sqrt{x^2 + \frac{1}{n}} + \beta \sqrt{y^2 + \frac{1}{n}}. \quad (2.22)$$

Then  $\psi_n$  is convex for all  $n \geq 1$ . The convergence property in (2.19)(ii) follows from Lebesgue's dominated convergence theorem.

To justify (2.19)(iii), consider a sequence  $(\mathbf{w}_n)$  and  $\mathbf{w}$  in  $W^{1,2}(0, T; V)$  satisfying

$$\mathbf{w}_n \rightharpoonup \mathbf{w}, \quad \mathbf{w}'_n \rightharpoonup \mathbf{w}' \quad \text{weakly in } L^2(0, T; V).$$

As

$$\psi_n(\mathbf{w}_n) \geq \psi(\mathbf{w}_n), \quad \forall n \in \mathbb{N}^*,$$

then

$$\liminf_{n \rightarrow +\infty} \int_0^T \psi_n(\mathbf{w}_n(t)) dt \geq \liminf_{n \rightarrow +\infty} \int_0^T \psi(\mathbf{w}_n(t)) dt.$$

Now using the compact embedding theorem, we obtain

$$\mathbf{w}_n \rightharpoonup \mathbf{w}, \quad \mathbf{w}'_n \rightharpoonup \mathbf{w}' \quad \text{weakly in } L^2(0, T; V) \implies \mathbf{w}_n \rightarrow \mathbf{w} \quad \text{strongly in } L^2(0, T; L^2(\Gamma_3)^d).$$

Because of the continuous embedding from  $V$  into  $L^2(\Gamma_3)^d$ , we deduce that

$$\mathbf{w}_n \rightarrow \mathbf{w} \quad \text{strongly in } L^2(0, T; V).$$



As  $\psi$  is Lipschitz continuous on  $V$ , we have that the functional

$$\mathbf{w} \in L^2(0, T; V) \longmapsto \int_0^T \psi(\mathbf{w}(t)) dt$$

is well defined and is continuous. Consequently

$$\liminf_{n \rightarrow +\infty} \int_0^T \psi(\mathbf{w}_n(t)) dt = \lim_{n \rightarrow +\infty} \int_0^T \psi(\mathbf{w}_n(t)) dt = \int_0^T \psi(\mathbf{w}(t)) dt.,$$

which gives the property (2.19)(iii).

Finally (2.19)(iv) is immediate from the formula stated for the Frechet derivative of  $\psi_n$  (see (2.21)).

□

**Example 2. Thermal contact problem with normal damped response and Tresca's friction law**

The normal damped response contact condition with Tresca's friction law (see e.g. [10]) is defined by:

$$\begin{cases} -\sigma_\nu = k_0 |\dot{u}_\nu|^{r-1} \dot{u}_\nu, & |\boldsymbol{\sigma}_\tau| \leq g, \\ |\boldsymbol{\sigma}_\tau| < g \implies \dot{\mathbf{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = g \implies \dot{\mathbf{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, & \text{for some } \lambda \geq 0, \end{cases} \quad \text{on } \Gamma_3 \times (0, T).$$

Here  $0 < r < 1$  and  $g, k_0 \in L^\infty(\Gamma_3)$ ,  $g \geq 0$ ,  $k_0 \geq 0$ . The coefficient  $k_0$  represents the hardness of the foundation, and  $g$  the friction threshold. The admissible displacement space is given by:

$$V := \{\mathbf{w} \in H_1; \text{ with } \mathbf{w} = \mathbf{0} \text{ on } \Gamma_1\}$$

and the sub-differential contact function

$$\varphi(\mathbf{x}, \mathbf{y}) = \frac{1}{r+1} k_0(\mathbf{x}) |y_\nu(\mathbf{x})|^{r+1} + g(\mathbf{x}) |\mathbf{y}_\tau(\mathbf{x})| \quad \forall \mathbf{x} \in \Gamma_3, \mathbf{y} \in \mathbb{R}^d.$$

Then denoting by  $p := r + 1$ , we have the contact function well defined on  $V$  by

$$\psi(\mathbf{v}) := \int_{\Gamma_3} \frac{k_0}{p} |v_\nu|^p da, + \int_{\Gamma_3} g |\mathbf{v}_\tau| da, \quad \forall \mathbf{v} \in V.$$

Here  $1 < p < 2$ . Then the mapping  $x \geq 0 \longmapsto x^p$  is convex, which implies that  $\psi$  is convex on  $V$ . Using the continuous embeddings from  $V$  into  $L^2(\Gamma_3)^d$ , and from  $L^2(\Gamma_3)$  into  $L^p(\Gamma_3)$ , we verify also that

$\psi$  is Lipschitz continuous on  $V$ .

To approximate the function  $\psi$  we use the sequence

$$\psi_n(\mathbf{v}) := \int_{\Gamma_3} \frac{k_0}{p} \left( |v_\nu|^2 + \frac{1}{n} \right)^{\frac{p}{2}} da + \int_{\Gamma_3} g \sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}} da, \quad \forall \mathbf{v} \in V, \forall n \in \mathbb{N}^*.$$

We verify that the Frechet derivative of  $\psi_n$  is given by

$$\psi'_n(\mathbf{v}) \cdot \mathbf{h} := \int_{\Gamma_3} k_0 \frac{v_\nu h_\nu}{(|v_\nu|^2 + \frac{1}{n})^{1-\frac{p}{2}}} da + \int_{\Gamma_3} g \frac{(\mathbf{v}_\tau, \mathbf{h}_\tau)_{\mathbb{R}^d}}{\sqrt{|\mathbf{v}_\tau|^2 + \frac{1}{n}}} da, \quad \forall \mathbf{h} \in V.$$

From the fact that the mapping  $x \geq 0 \longmapsto x^p$  is convex increasing, and using (2.22), we verify that  $\psi_n$  is convex for all  $n \geq 1$ . Similarly we have the conditions (ii)-(iv) in (2.19) by the same arguments in the previous example.

Then we conclude that the assumptions in (2.19) are also satisfied in this example.

□

**3. Proof of Theorem 1.** The idea is to bring the second order inequality to a first order inequality, using monotone operator, convexity and fixed point arguments, and will be carried out in several steps.

Let us introduce the velocity variable

$$\mathbf{v} = \dot{\mathbf{u}}.$$

The system in Problem  $QV$  is then written for a.e.  $t \in (0, T)$ :

$$\left\{ \begin{array}{l} \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds; \\ \langle \dot{\mathbf{v}}(t) + A \mathbf{v}(t) + B \mathbf{u}(t) + C \theta(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} \\ + \left( \int_0^t B(t-s) \varepsilon(\mathbf{u}(s)) ds, \varepsilon(\mathbf{w}) - \varepsilon(\mathbf{v}(t)) \right)_{\mathcal{H}} + \psi(\mathbf{w}) - \psi(\mathbf{v}(t)) \\ \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} \quad \forall \mathbf{w} \in V; \\ \dot{\theta}(t) + K \theta(t) = R \mathbf{v}(t) + Q(t) \quad \text{in } E'; \\ \mathbf{v}(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0, \end{array} \right.$$

with the regularity

$$\left\{ \begin{array}{l} \mathbf{v} \in W^{1,2}(0, T; V) \cap W^{1,\infty}(0, T; H) \\ \theta \in W^{1,2}(0, T; E) \cap W^{1,\infty}(0, T; F). \end{array} \right.$$

Various abstract formulations concerning the existence and uniqueness result on parabolic variational inequalities of the second kind, could be found in the literature, depending on the assumptions on the operators and data (see e.g. [3, 7, 11]). We will use a version taken in ([7]) p. 46, which is sufficient for our proposal, and which we recall as follows.

**Theorem 2.** Let  $A : V \rightarrow V'$  be linear continuous coercive,  $\{\psi; (\psi_n)\}$  verify the hypotheses (2.19),  $F \in W^{1,2}(0, T; V')$ , and  $\mathbf{v}_0 \in V$  satisfy: there exists a sequence  $(\mathbf{v}_0^n)$  in  $V$ , there exists a bounded sequence  $(\mathbf{h}_n)$  in  $H$  such that  $\mathbf{v}_0^n \rightarrow \mathbf{v}_0$  in  $V$  and for all  $n \in \mathbb{N}$ ,

$$\langle A \mathbf{v}_0^n, \mathbf{w} \rangle_{V' \times V} + \langle \psi'_n(\mathbf{v}_0^n), \mathbf{w} \rangle_{V' \times V} = \langle \mathbf{h}_n, \mathbf{w} \rangle_H, \quad \forall \mathbf{w} \in V.$$

Then there exists an unique

$$\mathbf{v} \in W^{1,2}(0, T; V) \cap W^{1,\infty}(0, T; H)$$

satisfying:

$$\left\{ \begin{array}{l} \langle \dot{\mathbf{v}}(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} + \langle A \mathbf{v}(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} + \psi(\mathbf{w}) - \psi(\mathbf{v}(t)) \\ \geq \langle F(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V}, \quad \forall \mathbf{w} \in V, \quad \text{a.e. } t \in (0, T); \\ \text{and } \mathbf{v}(0) = \mathbf{v}_0. \end{array} \right.$$

□

We can apply the Theorem 2 by using (2.11); the conditions on  $\mathbf{v}_0$  in (2.18) imply that there exists a sequence  $(\mathbf{v}_0^n)$  in  $\mathcal{D}(\Omega)^d$  such that  $\mathbf{v}_0^n \rightarrow \mathbf{v}_0$  for  $\|\cdot\|_{H^2(\Omega)^d}$ . Then

$$\psi'_n(\mathbf{v}_0^n) = 0_{V'}, \quad \forall n \in \mathbb{N}$$

and

$$\mathbf{h}_n := A \mathbf{v}_0^n = -\text{Div}(\mathcal{A}(\varepsilon \mathbf{v}_0^n)) \rightarrow -\text{Div}(\mathcal{A}(\varepsilon \mathbf{v}_0)) = A \mathbf{v}_0 \quad \text{in } H.$$

Thus  $(\mathbf{h}_n)$  defines a bounded sequence in  $H$ .

□

To continue, we assume in the sequel that the conditions (2.11)–(2.19) of the Theorem 1 are satisfied. Let define

$$\mathcal{W} := \{\eta \in W^{1,2}(0, T; \mathcal{H}), \eta(0) = \mathcal{G}(\varepsilon \mathbf{u}_0) - \theta_0 C_e\}.$$

We begin by

**Lemma 1.** For all  $\eta \in \mathcal{W}$ , there exists an unique

$$\mathbf{v}_\eta \in W^{1,2}(0, T; V) \cap W^{1,\infty}(0, T; H)$$

satisfying

$$\begin{cases} \langle \dot{\mathbf{v}}_\eta(t) + A \mathbf{v}_\eta(t), \mathbf{w} - \mathbf{v}_\eta(t) \rangle_{V' \times V} + (\eta(t), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\mathbf{v}_\eta(t)))_{\mathcal{H}} \\ \quad + \psi(\mathbf{w}) - \psi(\mathbf{v}_\eta(t)) \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{v}_\eta(t) \rangle_{V' \times V}, \\ \quad \forall \mathbf{w} \in V, \quad \text{a.e. } t \in (0, T); \\ \mathbf{v}_\eta(0) = \mathbf{v}_0. \end{cases} \quad (3.1)$$

Moreover,  $\exists c > 0$  such that  $\forall \eta_1, \eta_2 \in \mathcal{W}$ :

$$\|\mathbf{v}_{\eta_2}(t) - \mathbf{v}_{\eta_1}(t)\|_H^2 + \int_0^t \|\mathbf{v}_{\eta_1} - \mathbf{v}_{\eta_2}\|_V^2 \leq c \int_0^t \|\eta_1 - \eta_2\|_{\mathcal{H}}^2, \quad \forall t \in [0, T]. \quad (3.2)$$

**Proof.** Let  $\eta \in \mathcal{W}$ . The existence and uniqueness of  $\mathbf{v}_\eta$  follows straightly from Theorem 2, where we apply  $F$  defined by for all  $t \in [0, T]$ ,

$$\langle F(t), \mathbf{w} \rangle_{V' \times V} := \langle f(t), \mathbf{w} \rangle_{V' \times V} - (\eta(t), \boldsymbol{\varepsilon}(\mathbf{w}))_{\mathcal{H}}, \quad \forall \mathbf{w} \in V.$$

The assumptions in (2.14) imply that  $F \in W^{1,2}(0, T; V')$ .

Now let  $\eta_1, \eta_2 \in L^2(0, T; V')$ . In (3.1) we take  $(\eta = \eta_1, \mathbf{w} = \mathbf{v}_{\eta_2}(t))$ , then  $(\eta = \eta_2, \mathbf{w} = \mathbf{v}_{\eta_1}(t))$ . Adding the two inequalities, we deduce that for a.e.  $t \in (0, T)$ :

$$\begin{aligned} & \langle \dot{\mathbf{v}}_{\eta_2}(t) - \dot{\mathbf{v}}_{\eta_1}(t), \mathbf{v}_{\eta_2}(t) - \mathbf{v}_{\eta_1}(t) \rangle_{V' \times V} + \langle A \mathbf{v}_{\eta_2}(t) - A \mathbf{v}_{\eta_1}(t), \mathbf{v}_{\eta_2}(t) - \mathbf{v}_{\eta_1}(t) \rangle_{V' \times V} \\ & \leq -(\eta_2(t) - \eta_1(t), \boldsymbol{\varepsilon}(\mathbf{v}_{\eta_2}(t)) - \boldsymbol{\varepsilon}(\mathbf{v}_{\eta_1}(t)))_{\mathcal{H}}. \end{aligned}$$

Then integrating over  $(0, t)$ , from (2.11)(iii) and from the initial condition on the velocity, we obtain:

$$\begin{aligned} \forall t \in [0, T], \quad & \|\mathbf{v}_{\eta_2}(t) - \mathbf{v}_{\eta_1}(t)\|_H^2 + m_A \int_0^t \|\mathbf{v}_{\eta_2}(s) - \mathbf{v}_{\eta_1}(s)\|_V^2 ds \\ & \leq - \int_0^t (\eta_2(s) - \eta_1(s), \boldsymbol{\varepsilon}(\mathbf{v}_{\eta_2}(s)) - \boldsymbol{\varepsilon}(\mathbf{v}_{\eta_1}(s)))_{\mathcal{H}} ds. \end{aligned}$$

Then (3.2) follows. □

Here and below, we denote by  $c > 0$  a generic constant, whose value may change from lines to lines.

**Lemma 2.** For all  $\eta \in \mathcal{W}$ , there exists an unique

$$\theta_\eta \in W^{1,2}(0, T; E) \cap W^{1,\infty}(0, T; F)$$

satisfying

$$\begin{cases} \dot{\theta}_\eta(t) + K \theta_\eta(t) = R \mathbf{v}_\eta(t) + Q(t), \quad \text{in } E', \quad \text{a.e. } t \in (0, T), \\ \theta_\eta(0) = \theta_0. \end{cases} \quad (3.3)$$

Moreover,  $\exists c > 0$  such that  $\forall \eta_1, \eta_2 \in \mathcal{W}$ :

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_F^2 \leq c \int_0^t \|\mathbf{v}_{\eta_1} - \mathbf{v}_{\eta_2}\|_V^2, \quad \forall t \in [0, T]. \quad (3.4)$$

and

$$\|\dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t)\|_F^2 \leq c \int_0^t \|\mathbf{v}_{\eta_1} - \mathbf{v}_{\eta_2}\|_V^2, \quad \text{a.e. } t \in (0, T). \quad (3.5)$$

**Proof.** The existence and uniqueness result verifying (3.3) follows from classical result on first order evolution equation, which can be seen as a particular case of Theorem 2 applied to the Gelfand evolution triple (see e.g. [17] p . 416).

$$E \subset F \equiv F' \subset E'.$$

We verify that the operator  $K : E \rightarrow E'$  is linear continuous and strongly monotone, and from the expression of the operator  $R$ ,

$$\mathbf{v}_\eta \in W^{1,2}(0, T; V) \implies R \mathbf{v}_\eta \in W^{1,2}(0, T; F),$$

as  $Q \in W^{1,2}(0, T; E')$  then  $R \mathbf{v}_\eta + Q \in W^{1,2}(0, T; E')$ .

Now for  $\eta_1, \eta_2 \in \mathcal{W}$ , we have for a.e.  $t \in (0, T)$ :

$$\begin{aligned} & \langle \dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} + \langle K \theta_{\eta_1}(t) - K \theta_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} \\ & = \langle R \mathbf{v}_{\eta_1}(t) - R \mathbf{v}_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E}. \end{aligned}$$

Then integrating the last property over  $(0, t)$ , using the strong monotonicity of  $K$  and the Lipschitz continuity of  $R : V \rightarrow E'$ , we deduce (3.4).

To continue, from

$$\dot{\theta}_\eta \in L^2(0, T; E)$$

and from (3.3) which implies

$$\ddot{\theta}_\eta(t) + K \dot{\theta}_\eta(t) = R \mathbf{u}_\eta(t) + \dot{Q}(t) \text{ a.e. } t \in (0, T) \implies \ddot{\theta}_\eta \in L^2(0, T; E'),$$

we deduce that

$$\dot{\theta}_\eta \in C([0, T]; F).$$

Then that for  $\eta_1, \eta_2 \in \mathcal{W}$ , we have for a.e.  $t \in (0, T)$ :

$$\begin{aligned} & \langle \ddot{\theta}_{\eta_1}(t) - \ddot{\theta}_{\eta_2}(t), \dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t) \rangle_{E' \times E} + \langle K \dot{\theta}_{\eta_1}(t) - K \dot{\theta}_{\eta_2}(t), \dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t) \rangle_{E' \times E} \\ & = \langle R \mathbf{u}_{\eta_1}(t) - R \mathbf{u}_{\eta_2}(t), \dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t) \rangle_{E' \times E}. \end{aligned}$$

Integrating the last property over  $(0, t)$ , and with similar arguments we deduce

$$\|\dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t)\|_F^2 \leq c \int_0^t \|\mathbf{u}_{\eta_1} - \mathbf{u}_{\eta_2}\|_V^2, \quad \text{a.e. } t \in (0, T).$$

And (3.5) follows. □

**Proof of Theorem 1.** We have now all the ingredients to prove the Theorem 1. Consider the operator  $\Lambda : \mathcal{W} \rightarrow \mathcal{W}$  defined by for all  $\eta \in \mathcal{W}$ :

$$\Lambda \eta(t) = \mathcal{G}(\varepsilon(\mathbf{u}_\eta(t))) + \int_0^t B(t-s) \varepsilon(\mathbf{u}_\eta(s)) ds - \theta_\eta(t) C_e, \quad \forall t \in [0, T],$$

where

$$\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_\eta(s) ds, \quad \forall t \in [0, T]; \quad \mathbf{u}_\eta \in W^{2,2}(0, T; V).$$

Then from (2.12), (2.13), and Lemma 2, we deduce that for all  $\eta_1, \eta_2 \in \mathcal{W}$ , for all  $t \in [0, T]$ :

$$\begin{aligned} \|\Lambda \eta_1(t) - \Lambda \eta_2(t)\|_{\mathcal{H}}^2 & \leq c \|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_F^2 + c \int_0^t \|\mathbf{v}_{\eta_1}(s) - \mathbf{v}_{\eta_2}(s)\|_V^2 ds \\ & \leq c \int_0^t \|\mathbf{v}_{\eta_1}(s) - \mathbf{v}_{\eta_2}(s)\|_V^2 ds. \end{aligned} \tag{3.6}$$

Again from (2.13) and (2.12), we have

$$\begin{aligned} & \left\| \frac{d}{dt} \left( \int_0^t B(t-s) \boldsymbol{\varepsilon}(\mathbf{u}_{\eta_1}(s)) ds - \int_0^t B(t-s) \boldsymbol{\varepsilon}(\mathbf{u}_{\eta_2}(s)) ds \right) \right\|_{\mathcal{H}}^2 \\ & \leq c \|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V^2 + c \int_0^t \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_V^2 ds \\ & \leq c \int_0^t \|\mathbf{v}_{\eta_1}(s) - \mathbf{v}_{\eta_2}(s)\|_V^2 ds. \end{aligned}$$

and

$$\left\| \frac{d}{dt} \left( \mathcal{G}(\boldsymbol{\varepsilon}(\mathbf{u}_{\eta_1}(t))) - \mathcal{G}(\boldsymbol{\varepsilon}(\mathbf{u}_{\eta_2}(t))) \right) \right\|_{\mathcal{H}}^2 \leq c \|\mathbf{v}_{\eta_1}(t) - \mathbf{v}_{\eta_2}(t)\|_V^2.$$

Then

$$\left\| \frac{d}{dt} \left( \Lambda \eta_1(t) - \Lambda \eta_2(t) \right) \right\|_{\mathcal{H}}^2 \leq c \|\mathbf{v}_{\eta_1}(t) - \mathbf{v}_{\eta_2}(t)\|_V^2 + c \int_0^t \|\mathbf{v}_{\eta_1}(s) - \mathbf{v}_{\eta_2}(s)\|_V^2 ds. \quad (3.7)$$

Now using (3.6) and (3.7), after some algebraic manipulations, we have for any  $\beta > 0$ :

$$\begin{aligned} & \int_0^t e^{-\beta\tau} \left( \|\Lambda \eta_1(\tau) - \Lambda \eta_2(\tau)\|_{\mathcal{H}}^2 + \|\dot{\Lambda} \eta_1(\tau) - \dot{\Lambda} \eta_2(\tau)\|_{\mathcal{H}}^2 \right) d\tau \\ & \leq \frac{c}{\beta} \int_0^t e^{-\beta\tau} \left( \|\eta_1(\tau) - \eta_2(\tau)\|_{\mathcal{H}}^2 + \|\dot{\eta}_1(\tau) - \dot{\eta}_2(\tau)\|_{\mathcal{H}}^2 \right) d\tau. \end{aligned}$$

We conclude from the last inequality by contracting principle that the operator  $\Lambda$  has a unique fixed point  $\eta^* \in \mathcal{W}$ . We verify then that the functions

$$\mathbf{u}(t) := \mathbf{u}_0 + \int_0^t \mathbf{v}_{\eta^*}, \quad \forall t \in [0, T], \quad \theta := \theta_{\eta^*}$$

are solutions to problem  $QV$  with the regularity (2.20), the uniqueness follows from the uniqueness in Lemma 1 and Lemma 2.  $\square$

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