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# FINITE-TIME LYAPUNOV STABILITY ANALYSIS OF EVOLUTION VARIATIONAL INEQUALITIES

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**ABSTRACT.** Using Lyapunov's stability and LaSalle's invariance principle for nonsmooth dynamical systems, we establish some conditions for finite-time stability of evolution variational inequalities. The theoretical results are illustrated by some examples drawn from electrical circuits involving nonsmooth elements like diodes.

**1. Introduction.** Stability analysis of dynamical systems in the sense of Lyapunov is one of the most important topic in control theory. There are many concrete systems in engineering which have nonsmooth dynamics. This the case e.g. of mechanical systems subject to unilateral constraints and/or Coulomb friction and/or impacts or electrical circuits with switches, diodes, DC/DC converters or hybrid dynamical systems in control and engineering. There are also some applications in Genetic Regulatory Networks, Neural firing Networks, transportation science and economy. It seems that the formalism of evolution variational inequalities represents a large class of unilateral dynamical systems, [1], [2], [3], , [10], [11]. From a mathematical point of view, it is important to rigorously analyze, for such nonsmooth systems, the existence of stationary points, their stability, their asymptotic stability and their finite-time stability as well. In contrast to the commonly used notion of asymptotic stability, the finite-time stability requires at the same time the stability and the convergence of the trajectory to the stationary point in finite-time (called the settling time). Usually asymptotic stability is enough for practical applications in control theory and engineering, but there are some situations where the infinite settling time is not acceptable. Some early results on finite-time stabilization for ordinary differential equations can be found in [6], [13], [23] and references cited therein. Concerning discontinuous dynamical systems, we can cite e.g. the works of [16],[17] and [18].

Our aim in this paper is to present some conditions ensuring the finite-time stability of a general class of evolution variational inequalities. These results, based

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*Key words and phrases.* Finite-Time stability, Lyapunov stability, LaSalle invariance principle, nonsmooth analysis, variational inequality, complementarity problem, differential inclusions, convex analysis.

on a Lyapunov stability theorem developed in [3] for this kind of problems, provide a basic tool for the analysis and the synthesis of nonlinear and nonsmooth dynamical systems. More precisely, we are concerned with the study of the finite-time stability of first-order nonsmooth dynamical systems given by

$$\dot{u}(t) + F(u(t)) \in -\partial\varphi(u(t)) \quad \text{a.e. } t \geq 0, \quad (1)$$

where  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is an extended real-valued proper convex and lower semicontinuous function. Here  $\partial$  stands for the subdifferential of convex analysis and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $k$ -Lipschitzian vector field, for some constant  $k > 0$ .

A major difference with the classical Filippov theory is that the set appearing in the right hand side of (1) is not necessarily compact and, so, may be unbounded due to the fact that  $\varphi$  is an extended real-valued function. In particular, when the function  $\varphi$  is of class  $C^1$  then (1) becomes an ordinary differential equation (ODE, for short). Also, if  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex, then it is continuous and hence  $\partial\varphi(u)$  is a nonempty, compact and convex set for every  $u \in \mathbb{R}^n$ . In this case (1) reduces to a ordinary differential inclusion (ODI, for short) known as Filippov's differential inclusion. An other important case is obtained when the function  $\varphi = \psi_C$  (the indicator function of a closed convex cone in  $\mathbb{R}^n$ , i.e.  $\psi_C(x) = 0$  if  $x \in C$  and  $\psi_C(x) = +\infty$  if  $x \notin C$ ). In this case, problem (1) reduces to the well-known complementarity problem defined by

$$\begin{cases} \text{For } u_0 \in C, \text{ find } u \in C^0([0, +\infty); \mathbb{R}^n) \text{ such that } \dot{u} \in L_{loc}^\infty([0, +\infty); \mathbb{R}^n) \text{ and} \\ C \ni u(t) \perp (\dot{u}(t) + F(u(t))) \in C^+, \text{ a.e. } t \geq 0, \\ u(0) = u_0. \end{cases}$$

where  $C^+ = \{p \in \mathbb{R}^n : \langle p, x \rangle \geq 0, \forall x \in C\}$  stands for the positive dual cone of  $C$ .

The sufficient and necessary conditions for finite-time stability are explicitly given in term of Lyapunov functions, and may easily be verified in concrete situations such as in non-regular electronic circuits and unilateral mechanics. The finite-time stability was considered in [6] and [13] for ODE. See also [16], where a sufficient and a necessary conditions are used to obtain finite-time convergence for ODI with right hand side verifying the Filippov conditions. Our approach is mainly based on the works of Adly & Goeleven [3] and Moulay & Perruquetti [16]. We study sufficient and necessary conditions for the stability and the finite-time convergence of the stationary solutions of (1). As a particular case, we consider the complementarity problem given when the function  $\varphi$  coincides with an indicator function of a closed convex cone of  $\mathbb{R}^n$ . These results could be seen as a unification and an extension of the results obtained by Moulay & Perruquetti [16] in an other framework. Finally, we show that our theoretical results are applicable to some examples in nonregular electrical circuits involving nonsmooth devices like ideal diodes, practical diodes or Zener diodes.

**2. Notations and preliminaries.** Throughout this paper  $\Gamma_0(\mathbb{R}^n)$  will denote the set of all proper, convex and lower semicontinuous (l.s.c. for short) extended real valued functions  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ . For a function  $\varphi \in \Gamma_0(\mathbb{R}^n)$ , the sets  $D(\varphi)$  and  $D(\partial\varphi)$  stand for the domains of  $\varphi$  and the subdifferential  $\partial\varphi$  of  $\varphi$  respectively, i.e.

$$D(\varphi) = \{u \in \mathbb{R}^n \mid \varphi(u) < +\infty\} \text{ and } D(\partial\varphi) = \{u \in \mathbb{R}^n \mid \partial\varphi(u) \neq \emptyset\}.$$

In addition, for a function  $u(t)$ , the notation  $\dot{u}(t)$  means  $\frac{du}{dt}(t)$  and for a function  $V(u)$ , the notation  $V'(u)$  means  $\frac{dV}{du}$ . Here  $\|\cdot\|$  stands for the Euclidean norm of  $\mathbb{R}^n$  associated to the product scalar  $\langle \cdot, \cdot \rangle$ . We denote by  $\mathbb{B}_\rho$  the closed ball of center the origin and radius  $\rho > 0$ .

The sets of class  $\mathcal{K}$  and  $\mathcal{K}_\infty$  are respectively defined by

$$\mathcal{K} \triangleq \left\{ g : [0, \sigma] \rightarrow \mathbb{R}^+ \text{ continuous} \mid g(0) = 0 \text{ and } g \text{ is strictly increasing on } [0, \sigma] \right\}.$$

$$\mathcal{K}_\infty \triangleq \left\{ g : [0, +\infty[ \rightarrow \mathbb{R}^+ \text{ continuous} \mid g(0) = 0, \quad g \text{ is strictly increasing on } [0, +\infty[ \text{ and } \lim_{x \rightarrow +\infty} g(x) = +\infty \right\}.$$

### 3. Finite-time stability of evolution variational inequalities.

**3.1. Stability and preliminary results.** In this section, we are interested in the following evolution variational inequality problem: For  $u_0 \in D(\partial\varphi)$ ,

$$(\mathcal{P}) \begin{cases} \text{Find } u \in C^0([0, +\infty); \mathbb{R}^n) \text{ such that } \dot{u} \in L_{loc}^\infty([0, +\infty); \mathbb{R}^n) \text{ and} \\ \langle \dot{u}(t) + F(u(t)), v - u(t) \rangle + \varphi(v) - \varphi(u(t)) \geq 0, \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq 0, \\ u(t) \in D(\partial\varphi), t \geq 0, \\ u(0) = u_0. \end{cases}$$

Here, for  $k > 0$ ,  $F$  is a  $k$ -Lipschitzian vector field and the function  $\varphi \in \Gamma_0(\mathbb{R}^n)$ .

Under these assumptions, the existence and uniqueness of the solution of the problem  $(\mathcal{P})$  is a direct consequence of Kato's theorem given in [14]. For a given initial condition  $u_0 \in D(\partial\varphi)$ , we denote by  $u(t) \equiv u(t; u_0)$  the unique solution of problem  $(\mathcal{P})$ .

Remark that problem  $(\mathcal{P})$  is equivalent to the following differential inclusion problem:

$$(\mathcal{P}) \equiv \begin{cases} \dot{u}(t) + F(u(t)) \in -\partial\varphi(u(t)) \text{ a.e. } t \geq 0 \\ u(0) = u_0. \end{cases}$$

By shifting the origin 0 of our system, we may suppose that the equilibrium point of interest is located at the origin 0. Therefore, if the following condition holds

$$0 \in D(\partial\varphi) \text{ and } F(0) \in -\partial\varphi(0), \quad (2)$$

then  $u(t; 0) \equiv 0$  is the (unique) trivial solution of  $(\mathcal{P})$ .

A particular and interesting case of problem  $(\mathcal{P})$  is given when the function  $\varphi = \psi_C$  is the indicator function of a closed convex subset  $C$  of  $\mathbb{R}^n$ . We recall the indicator function  $\psi_C$  is defined by

$$\psi_C(u) = \begin{cases} 0 & \text{if } u \in C \\ +\infty & \text{else.} \end{cases}$$

The subdifferential  $\partial\psi_C$  of  $\psi_C$  is given by

$$\partial\psi_C(u) = N_C(u)$$

where  $N_C(u)$  is the normal cone of  $C$  at  $u \in C$  defined by

$$N_C(u) = \{p \in \mathbb{R}^n : \langle p, v - u \rangle \leq 0 \quad \forall v \in C\}.$$

The tangent cone  $T_C(u)$  of  $C$  at  $u \in C$  is defined by

$$T_C(u) = \{p \in \mathbb{R}^n : \langle p, v \rangle \leq 0 \quad \forall v \in N_C(u)\}.$$

Hence, problem  $(\mathcal{P})$  can be written equivalently as

$$(\mathcal{P}_C) \begin{cases} \text{For } u_0 \in C, \text{ find } u \in C^0([0, +\infty); \mathbb{R}^n) \text{ such that } \dot{u} \in L_{loc}^\infty([0, +\infty); \mathbb{R}^n) \\ \text{and } \dot{u}(t) + F(u(t)) \in -N_C(u(t)), \text{ a.e. } t \geq 0, \\ u(0) = u_0. \end{cases}$$

If  $C$  is a closed convex cone of  $\mathbb{R}^n$ , then problem  $(\mathcal{P}_C)$  is equivalent to the following complementarity problem:

$$(\mathcal{P}_C) \begin{cases} \text{For } u_0 \in C, \text{ find } u \in C^0([0, +\infty); \mathbb{R}^n) \text{ such that } \dot{u} \in L_{loc}^\infty([0, +\infty); \mathbb{R}^n) \\ \text{and } C \ni u(t) \perp (\dot{u}(t) + F(u(t))) \in C^+, \text{ a.e. } t \geq 0, \\ u(0) = u_0. \end{cases}$$

with  $\perp$  indicating orthogonality and  $C^+$  standing for the positive dual cone of  $C$  defined by

$$C^+ = \{p \in \mathbb{R}^n : \langle p, x \rangle \geq 0, \quad \forall x \in C\}.$$

Note that in this case condition (2) simply reads

$$-F(0) \in N_C(0). \quad (3)$$

For a given function  $V \in C^1(\mathbb{R}^n; \mathbb{R})$ , we set

$$\mathcal{E}_V \triangleq \{u \in D(\partial\varphi) \mid \langle F(u), V'(u) \rangle + \varphi(u) - \varphi(u - V'(u)) = 0\}.$$

We recall the definition of the stability (in the sense of Lyapunov) of the trivial solution of  $(\mathcal{P})$ .

**Definition 3.1.** The equilibrium point  $u = 0$  of  $(\mathcal{P})$  is said:

1. Stable if, for every  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that for any  $u_0 \in D(\partial\varphi) \cap \mathbb{B}_\delta$ , the solution  $u(t; u_0)$  of problem  $(\mathcal{P})$  satisfies  $\|u(t; u_0)\| \leq \varepsilon$ .
2. Attractive if there exists  $\delta > 0$  such that for any  $u_0 \in D(\partial\varphi) \cap \mathbb{B}_\delta$ , we have  $\lim_{t \rightarrow +\infty} \|u(t; u_0)\| = 0$ .
3. Asymptotically stable if it is stable and attractive.

As shows Definition 3.1, to prove the stability (resp. the asymptotic stability) of an equilibrium point, it is necessary to find the solution  $u(t; u_0)$  of problem  $(\mathcal{P})$  for a given initial condition  $u_0 \in D(\partial\varphi) \cap \mathbb{B}_\delta$ . This is not an easy task for most problems (even for ODE's). The Lyapunov's direct method (known also as the second method of Lyapunov) allows us to determine the stability or the instability of an equilibrium point without integrating the corresponding differential inclusion  $(\mathcal{P})$ . It consists to find a positive definite function  $V$  of class  $C^1$  such that its orbital derivative along the trajectory  $u(t)$  defined by

$$\dot{V}(u) = \langle V'(u(t)), \dot{u}(t) \rangle,$$

is a negative semidefinite (resp. negative definite) function.

In [9], the authors extend this method to the case of the problem  $(\mathcal{P})$ . The following result is a particular case of the one proved in [9].

**Theorem 3.2.** *If there exist  $\rho > 0$  and a  $C^1$  positive definite function  $V$ , such that*

$$\langle F(u), V'(u) \rangle + \varphi(u) - \varphi(u - V'(u)) \geq 0, \quad \forall u \in D(\partial\varphi) \cap \mathbb{B}_\rho, \quad (4)$$

*then the trivial equilibrium  $u = 0$  of  $(\mathcal{P})$  is stable.*

Various applications of Theorem 3.2 for engineering problems can be found in [2] and [3].

Applying Theorem 3.2 to problem  $(\mathcal{P}_C)$ , we have the following corollary.

**Corollary 1.** *Let  $C$  be a closed convex subset of  $\mathbb{R}^n$  containing the origin  $0$ . Suppose that there exist  $\rho > 0$  and a  $C^1$  positive definite function  $V$  such that:*

1.  $\langle F(u), V'(u) \rangle \geq 0, \forall u \in C \cap \mathbb{B}_\rho;$
2.  $-V'(u) \in T_C(u), u \in \partial C \cap \mathbb{B}_\rho,$  (where  $\partial C$  is the boundary of  $C$ ).

*Then the trivial equilibrium  $u = 0$  of  $(\mathcal{P}_C)$  is stable.*

**Remark 1.** Theorem 3.2 gives only a sufficient condition ensuring the stability of the trivial stationary point of problem  $(\mathcal{P})$  and does not say about how to determine the Lyapunov function  $V$  satisfying the sufficient condition. In general, finding a Lyapunov function  $V$  satisfying (4) is not an easy task. However, there are some cases e.g. like mechanical systems, where the energy of the system could be used as a Lyapunov function. We note also that the asymptotic stability of the stationary point is not guaranteed by Theorem 3.2. To achieve this aim, we will use LaSalle's invariance principle adapted to evolution variational inequalities. In [3], the authors extend LaSalle's invariance principle to problem  $(\mathcal{P})$ . The following result is in this sense.

**Theorem 3.3.** ([3]) *Let us assume that the assumptions of Theorem 3.2 together with condition (2) are satisfied. In addition, suppose that*

1.  $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$  is l.s.c. on  $D(\partial\varphi)$ ;
2.  $D(\partial\varphi)$  is closed;
3.  $\mathcal{E}_V = \{0\}$ .

*Then the equilibrium  $u = 0$  of  $(\mathcal{P})$  is asymptotically stable.*

**3.2. Finite-time stability.** This subsection is divided into two parts. First, we will give two sufficient conditions for the finite-time stability, where the final time, at which the equilibrium is attained, is estimated. Second, we will establish a necessary condition for the finite-time convergence. The complementarity problem will be also examined.

Following the lines of [6] (Definition 2.2) for autonomous continuous ordinary differential equations, let us first recall the definition of the finite-time stability of the trivial equilibrium point of problem  $(\mathcal{P})$ .

**Definition 3.4.** The equilibrium point  $u = 0$  of problem  $(\mathcal{P})$  is said finite-time stable if

1. it is asymptotically stable for  $(\mathcal{P})$ , and
2. it is finite-time convergent for  $(\mathcal{P})$ , i.e. there exists  $\delta > 0$  and a function  $T : \mathbb{B}_\delta \cap D(\partial\varphi) \setminus \{0\} \rightarrow ]0, +\infty[$  with  $T(0) = 0$  (called the settling-time function) such that for every  $u_0 \in \mathbb{B}_\delta \cap D(\partial\varphi)$ , we have

$$u(t; u_0) = 0 \text{ for all } t \geq T(u_0).$$

**Remark 2.** The equilibrium point  $u = 0$  of problem  $(\mathcal{P})$  is said globally finite-time stable if it is globally asymptotically stable and globally finite-time convergent for  $(\mathcal{P})$  i.e. for any initial condition  $u_0 \in D(\partial\varphi)$ , there is a settling time  $T(u_0) \in ]0, +\infty[$  such that the unique solution  $u(t; u_0)$  of problem  $(\mathcal{P})$  satisfies

$$u(t; u_0) = 0 \text{ for all } t \geq T(u_0).$$

### 3.2.1. Sufficient Conditions for Finite-Time Stability.

**Theorem 3.5.** *Suppose that the assumptions of Theorem 3.3 are satisfied. If there exists a function  $g \in \mathcal{K}_\infty$  satisfying:*

$$\int_0^\varepsilon \frac{dz}{g(z)} < +\infty \quad \forall \varepsilon > 0, \quad (5)$$

and, if we have

$$\langle F(u), V'(u) \rangle + \varphi(u) - \varphi(u - V'(u)) \geq g(V(u)), \forall u \in D(\partial\varphi), \quad (6)$$

then, the equilibrium  $u = 0$  is finite-time stable.

*Proof.* Thanks to Theorem 3.3, the origin is asymptotically stable i.e. 0 is Lyapunov stable and attractive. Therefore, there exists  $\delta > 0$  such that for every  $u_0 \in \mathbb{B}_\delta \cap D(\partial\varphi)$ , we have  $\lim_{t \rightarrow +\infty} \|u(t; u_0)\| = 0$ .

For  $u_0 \in \mathbb{B}_\delta \cap D(\partial\varphi)$ , let  $u(t) := u(t; u_0)$  be the unique solution of  $(\mathcal{P})$  which tends to the origin with a settling time  $0 \leq T(u_0) \leq +\infty$ . It remains to prove that  $T(u_0) < +\infty$ .

First let us prove that the function  $(V \circ u) : [0, +\infty) \rightarrow \mathbb{R}$  defined by  $(V \circ u)(t) = V(u(t))$  is strictly decreasing for  $t \geq 0$ . Indeed, for  $T > 0$ , the function  $(V \circ u)$  is a.e. strongly differentiable on  $[0, T]$  and we have for a.e.  $t \in [0, T]$

$$\frac{d(V \circ u)}{dt}(t) = \langle V'(u(t)), \dot{u}(t) \rangle.$$

Since  $u(t)$  is the solution of  $(\mathcal{P})$  there exists  $\omega(t) \in \partial\varphi(u(t))$  a.e.  $t \geq 0$ , such that

$$\dot{u}(t) + F(u(t)) + \omega(t) = 0. \quad (7)$$

Using the fact that  $\omega(t) \in \partial\varphi(u(t))$ , we have

$$\langle \omega(t), v - u(t) \rangle \leq \varphi(v) - \varphi(u(t)), \forall v \in \mathbb{R}^n.$$

In particular for  $v = u(t) - V'(u(t))$  one obtains,

$$\langle \omega(t), -V'(u(t)) \rangle \leq \varphi(u(t) - V'(u(t))) - \varphi(u(t)). \quad (8)$$

By using (6) and (8) we get for a.e.  $t \in [0, T]$ .

$$\begin{aligned} \langle \dot{u}(t), V'(u(t)) \rangle &= -\langle F(u(t)), V'(u(t)) \rangle - \langle \omega(t), V'(u(t)) \rangle \\ &\leq -\langle F(u(t)), V'(u(t)) \rangle + \varphi(u(t) - V'(u(t))) - \varphi(u(t)) \\ &\leq -g(V(u(t))) \end{aligned} \quad (9)$$

$$< 0. \quad (10)$$

From (10) we deduce

$$\langle \dot{u}(t), V'(u(t)) \rangle < 0 \text{ for a.e. } t \in [0, T]$$

thus

$$\frac{d(V \circ u)}{dt}(t) < 0 \text{ for a.e. } t \in [0, T].$$

We know that  $u \in C^0([0, T]; \mathbb{R}^n)$ ,  $\dot{u} \in L^\infty([0, T]; \mathbb{R}^n)$  and  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  thus  $(V \circ u) \in W^{1,1}([0, T]; \mathbb{R})$ . According to the Gronwall lemma given in [14], we obtain that  $(V \circ u)$  is strictly decreasing on  $[0, T]$ . Since  $T$  is arbitrary, one obtains that  $(V \circ u)$  is strictly decreasing on  $[0, +\infty)$ .

By using the change of variables:  $[0, T(u_0)] \rightarrow [0, V(u_0)]$  defined by  $z = V(u(t))$ , we have

$$\int_{V(u_0)}^0 \frac{dz}{-g(z)} = \int_0^{T(u_0)} \frac{\langle V'(u(t)), \dot{u}(t) \rangle}{-g(V(u(t)))} dt. \quad (11)$$

From (9), we have

$$\frac{\langle \dot{u}(t), V'(u(t)) \rangle}{-g(V(u(t)))} \geq 1. \quad (12)$$

Using (11), (12) and (5), we have

$$T(u_0) = \int_0^{T(u_0)} dt \leq \int_0^{T(u_0)} \frac{\langle \dot{u}(t), V'(u(t)) \rangle}{-g(V(u(t)))} dt = \int_0^{V(u_0)} \frac{dz}{g(z)} < +\infty.$$

Thus, the origin of  $(\mathcal{P})$  is finite-time stable, which completes the proof of Theorem 3.5.  $\square$

The following simple example illustrates the previous theorem.

**Example 1.** Let us suppose that  $F \equiv 0$  and the function  $\varphi(x) = |x|$ ,  $x \in \mathbb{R}$ . The subdifferential of  $\varphi$  is given by

$$\partial\varphi(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

In this case problem  $(\mathcal{P})$  reduces to

$$\dot{u}(t) \in -\partial\varphi(u(t)). \quad (13)$$

First let us remark that all assumptions of Theorem 3.3 are satisfied using the Lyapunov function  $V(x) = \frac{1}{2}x^2$ ,  $x \in \mathbb{R}$ . Hence, 0 is asymptotically stable for problem (13).

In order to study the finite-time stability of the origin, we apply Theorem 3.5. Let us set  $g(x) = x^{\frac{1}{2}}$  for  $x \geq 0$ . It is clear that the function  $g \in \mathcal{K}_\infty$ . On the other hand, we have

$$\langle F(u), V'(u) \rangle + \varphi(u) - \varphi(u - V'(u)) = |u| \geq \frac{\sqrt{2}}{2} |u| = g(V(u)).$$

Thus, the equilibrium  $u = 0$  is finite-time stable for problem (13).

Applying Theorem 3.5 to problem  $(\mathcal{P}_C)$ , we obtain the following corollary:

**Corollary 2.** *Assume that the assumptions of Corollary 1 are satisfied. In addition, if we have:*

$$\mathcal{E}_C \triangleq \{u \in C \mid \langle F(u), V'(u) \rangle = 0\} = \{0\},$$

together with the existence of  $g \in \mathcal{K}_\infty$  satisfying:

$$(\alpha_1) \langle F(u), V'(u) \rangle \geq g(V(u)), \quad \forall u \in C,$$

$$(\alpha_2) \text{ for all } \varepsilon > 0 \text{ we have } \int_0^\varepsilon \frac{dz}{g(z)} < +\infty.$$

Then, the equilibrium  $u = 0$  of  $(\mathcal{P}_C)$  is finite-time stable.



### 3.2.2. Necessary Condition for Finite-Time Stability.

**Theorem 3.6.** *Suppose that assumptions of Theorem 3.3 are satisfied. If the origin of  $(\mathcal{P})$  is globally finite-time stable and if there is  $g \in \mathcal{K}_\infty$  such that*

$$\langle F(u), V'(u) \rangle + \varphi(u + V'(u)) - \varphi(u) \leq g(V(u)), \quad \forall u \in D(\partial\varphi), \quad (14)$$

then

$$\int_0^{V(u_0)} \frac{dz}{g(z)} < +\infty, \quad \text{for every } u_0 \in D(\partial\varphi).$$

*Proof.* Since the origin is globally finite-time stable for problem  $(\mathcal{P})$ , then it is globally asymptotically stable and finite-time convergent for problem  $(\mathcal{P})$ . For  $u_0 \in D(\partial\varphi)$ , let  $u(t)$  be the solution of  $(\mathcal{P})$  with a settling time  $0 \leq T(u_0) < +\infty$ . In addition, the function  $(V \circ u) : [0, +\infty) \rightarrow \mathbb{R}$  defined by  $(V \circ u)(t) = V(u(t))$  is strictly decreasing for  $t \geq 0$  (see proof of Theorem 3.5).

By using the change of variables :  $[0, T(u_0)] \rightarrow [0, V(u_0)]$  defined by  $z = V(u(t))$ , we have

$$\int_{V(u_0)}^0 \frac{dz}{-g(z)} = \int_0^{T(u_0)} \frac{\langle V'(u(t)), \dot{u}(t) \rangle}{-g(V(u(t)))} dt. \quad (15)$$

Since that  $u(t)$  is the solution of  $(\mathcal{P})$  there exist  $\omega(t) \in \partial\varphi(u(t))$ , such that

$$\dot{u}(t) + F(u(t)) + \omega(t) = 0. \quad (16)$$

We have

$$\omega(t) \in \partial\varphi(u(t)) \iff \langle \omega(t), v - u(t) \rangle \leq \varphi(v) - \varphi(u(t)), \forall v \in \mathbb{R}^n.$$

In particular for  $v = u(t) + V'(u(t))$  we obtain,

$$\langle \omega(t), V'(u(t)) \rangle \leq \varphi(u(t) + V'(u(t))) - \varphi(u(t)), \forall v \in \mathbb{R}^n. \quad (17)$$

By using (14), (16) and (17), we have

$$\begin{aligned} \langle \dot{u}(t), V'(u(t)) \rangle &= -\langle F(u(t)), V'(u(t)) \rangle - \langle \omega(t), V'(u(t)) \rangle \\ &\geq -\langle F(u(t)), V'(u(t)) \rangle + \varphi(u(t)) - \varphi(u(t) + V'(u(t))) \\ &\geq -g(V(u(t))). \end{aligned}$$

Therefore

$$\frac{\langle \dot{u}(t), V'(u(t)) \rangle}{-g(V(u(t)))} \leq 1. \quad (18)$$

By using (15) and (18), we have

$$\int_0^{V(u_0)} \frac{dz}{g(z)} = \int_0^{T(u_0)} \frac{\langle \dot{u}(t), V'(u(t)) \rangle}{-g(V(u(t)))} dt \leq \int_0^{T(u_0)} dt = T(u_0) < +\infty.$$

□

As a consequence from Theorem 3.6, the following corollary gives a necessary condition for the trivial solution of the problem  $(\mathcal{P}_C)$ .

**Corollary 3.** *With the assumptions of Corollary 1, suppose that the following conditions hold:*

- ( $\beta_1$ ) *there is some  $g \in \mathcal{K}_\infty$  such that  $\langle F(u), V'(u) \rangle \leq g(V(u))$ ,  $\forall u \in C$ ;*
- ( $\beta_2$ )  *$V'(u) \in T_C(u)$ ;*
- ( $\beta_3$ )  *$\mathcal{E}_C = \{0\}$ .*

If the origin is globally finite-time stable for  $(\mathcal{P}_C)$ , then

$$\int_0^{V(u_0)} \frac{dz}{g(z)} < +\infty, \text{ for every } u_0 \in C.$$

**3.3. Examples.** In this section, we give some applications of Theorem 3.5 and Corollary 2 in electrical circuits containing nonsmooth devices like diodes. A diode is a device that constitutes a rectifier which permits the easy flow of charges in one direction and restrains the flow in the opposite direction. Figure 1 illustrates the ampere-volt characteristic of an ideal diode.

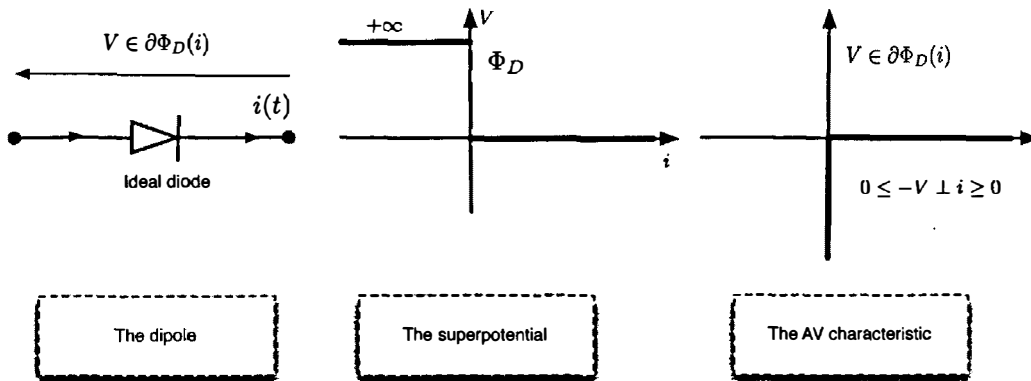


FIGURE 1. Ideal diode model

The ideal model diode is a simple switch. When the diode is forward-biased, it acts like a closed switch and when it is reverse-biased, it acts like an open switch i.e. if  $V < 0$  then  $i = 0$  and the diode is blocking. If  $i > 0$  then  $V = 0$  and the diode is conducting. We note that the ideal diode is described by the complementarity relation

$$V \leq 0, \quad i \geq 0, \quad Vi = 0$$

Figure 2 illustrates the superpotential and the ampere-volt characteristic of a practical diode model. There is a voltage point, called the knee voltage  $V_1$ , at which the diode begins to conduct and a maximum reverse voltage, called the peak reverse voltage  $V_2$ , that will not force the diode to conduct. When this voltage is exceeded, the depletion may breakdown and allow the diode to conduct in the reverse direction. Note that usually  $|V_2| \gg |V_1|$ .

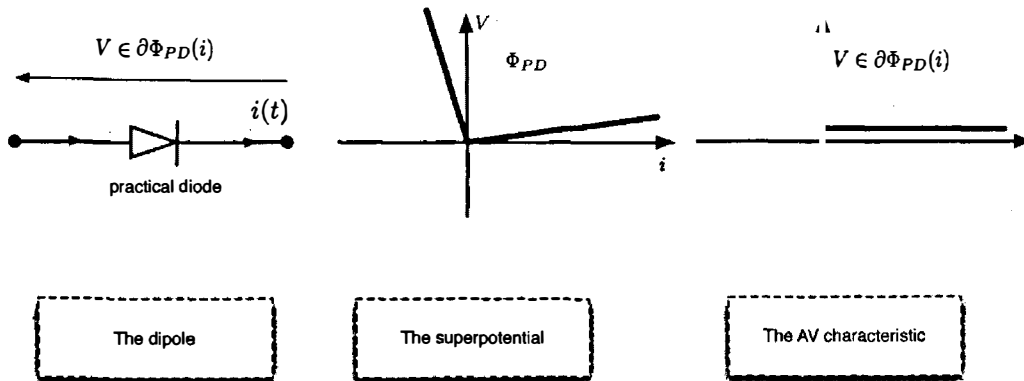


FIGURE 2. Practical diode model

**Example 2. [RLD Circuit]**

Let us consider the circuit of Figure 3 involving a load resistance  $R > 0$ , an input-

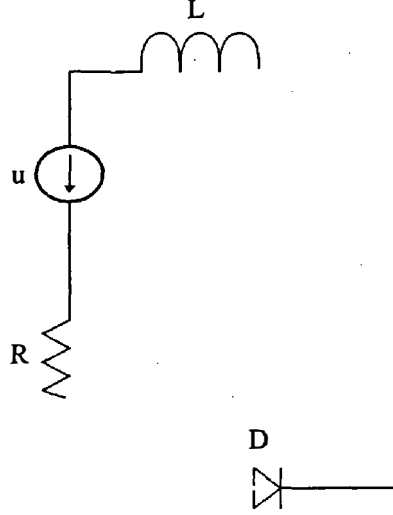


FIGURE 3. RLD Circuit

signal source  $u$  and corresponding instantaneous current  $i$ , a diode  $D$  and an inductor  $L$ .

Suppose that the electrical superpotential of the diode is given by

$$\Phi_D(i) \triangleq \frac{\sqrt{2}}{2} |i|.$$

Using the Kirchhoff's voltage law, we have

$$u - U_R - U_L = U_D,$$

where  $U_R = Ri$  denotes the difference of potential across the resistor,  $U_L = L \frac{di}{dt}$  and  $U_D \in \partial \Phi_D(i)$  is the difference of potential across the diode. Thus

$$L \frac{di}{dt} + Ri - u \in -\partial \Phi_D(i). \quad (19)$$

Without loss of generality, let us suppose that the input-signal  $u = 0$  and the value of  $L = 1$ . Then we obtain,

$$\frac{di}{dt} + Ri \in -\partial \Phi_D(i). \quad (20)$$

To prove the finite-time stability of the equilibrium  $i = 0$  of (20), we apply Theorem 3.5.

Indeed, let us consider the function  $g(x) = x^{\frac{1}{2}}$ ,  $x \geq 0$  and the function  $V(i) = \frac{1}{2} i^2$ . Then  $V'(i) = i$ .

The function  $V$  is positive definite and  $V$  verifies all the assumptions of Theorem 3.3. Then, the equilibrium  $i = 0$  is asymptotically stable. It remains to show that conditions (5) and (6) of Theorem 3.5 are satisfied.

First, it is clear to see that the function  $g \in \mathcal{K}_\infty$  and verifies condition (5). Second, for  $F(i) = Ri$ , one obtains

$$\begin{aligned} \langle F(i), V'(i) \rangle + \Phi_D(i) - \Phi_D(i - V'(i)) &= \langle Ri, i \rangle + \frac{\sqrt{2}}{2} |i| \\ &= Ri^2 + \frac{\sqrt{2}}{2} |i| \\ &\geq g(V(i)) = \frac{\sqrt{2}}{2} |i|. \end{aligned}$$

Then condition (6) is satisfied.

Thus, the equilibrium  $i = 0$  of (20) is finite-time stable.

**Example 3.** [RLDC Circuit with an ideal and a Practical Diode]

Let us consider the following dynamics that corresponds to the circuit depicted in Figure 4:

$$\begin{aligned} \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} &= \overbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}}^A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \overbrace{\begin{pmatrix} 0 \\ -\frac{1}{L} \end{pmatrix}}^B y_L + \overbrace{\begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}}^D u, \\ y &= \overbrace{\begin{pmatrix} 0 & -1 \end{pmatrix}}^C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

and

$$y_L \in \partial\Phi(y).$$

where  $R > 0$  is a resistor,  $L > 0$  an inductor,  $C > 0$  a capacitor,  $u$  is the voltage supply,  $x_1(t)$  is the time integral of the current across the capacitance,  $x_2(t) = i(t)$  is the current across the circuit,  $y_L$  is the voltage of the silicon controlled rectifier,  $\Phi$  is the electrical superpotential of the diode. .

For simplicity, let us set  $L = C = R = 1$  and  $u \equiv 0$ . Note that in this case:  $B = C^T$ . Hence, the dynamic of the system is then equivalent to

$$\dot{x}(t) \in Ax(t) - C^T \partial\Phi(Cx(t)).$$

By setting

$$\varphi = \Phi \circ C,$$

we have  $\partial\varphi(x) = C^T \partial\Phi(Cx)$ . Therefore, the dynamic of the system is

$$\dot{x}(t) \in Ax(t) - \partial\varphi(x(t)).$$

If we suppose that the diode in Figure 4 is ideal (see Figure 1) then its superpotential is given by

$$\Phi(x) = \Phi_D(x) = \psi_{\mathbb{R}_+}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ +\infty & \text{elsewhere} \end{cases}$$

and its subdifferential is given by

$$\partial\Phi_D(x) = \begin{cases} \emptyset & \text{if } x < 0 \\ ]-\infty, 0] & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

If we suppose that the diode in Figure 4 is a practical diode (see Figure 2) then its

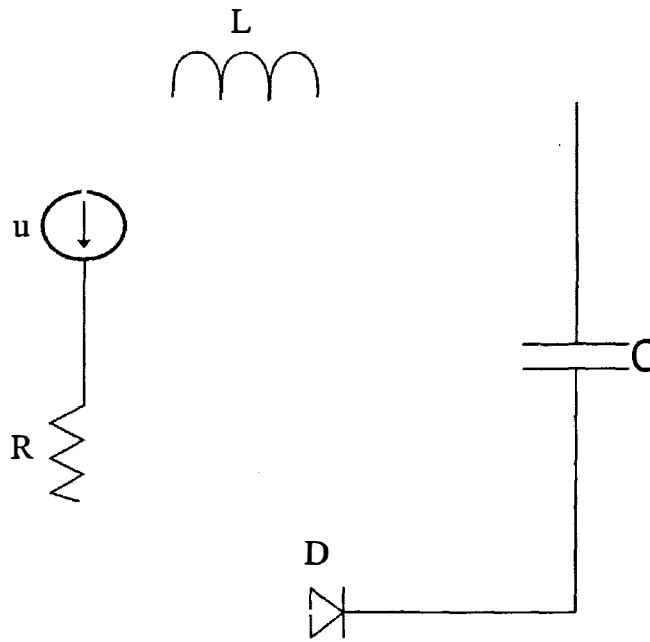


FIGURE 4. Circuit of the example 3

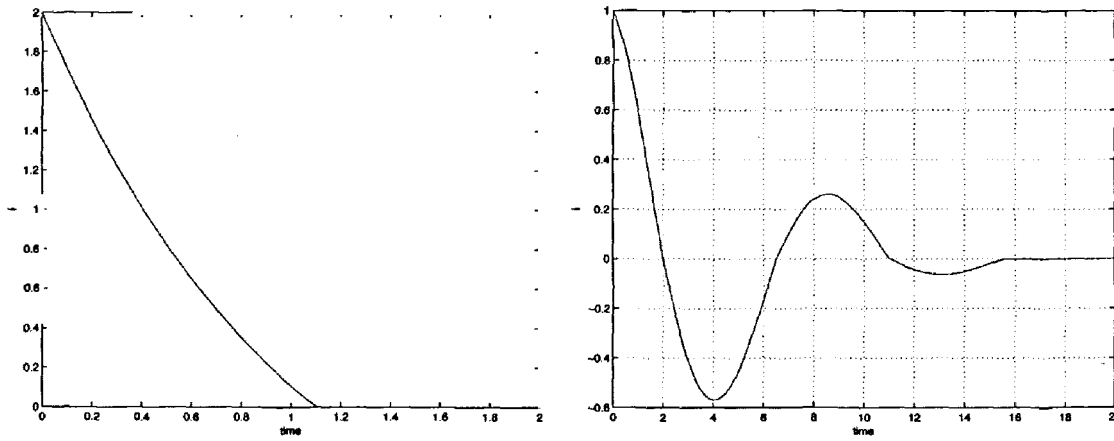


FIGURE 5. Simulation of the circuit in Fig. 4 with an ideal diode and a practical diode respectively

*superpotential is given by*

$$\varphi_{PD}(x) = \begin{cases} V_1 x & \text{if } x \geq 0 \\ V_2 x & \text{if } x < 0 \end{cases}, \quad (x \in \mathbb{R}).$$

and its subdifferential is given by

$$\partial\varphi_{PD}(x) = \begin{cases} V_2 & \text{if } x < 0 \\ [V_2, V_1] & \text{if } x = 0 \\ V_1 & \text{if } x > 0 \end{cases}, \quad (x \in \mathbb{R}).$$

The simulation results in Fig. 5 show the output signal of the circuit 4. The settling time can be easily seen there.

#### Example 4. [RLDC Circuit with an Zener and a Practical Diode]

The Zener diodes are made to permit current to flow in the reverse direction if the voltage is larger than the rated breakdown or "Zener voltage". In this example we consider the Ampere-Volt characteristic of Fig. 6.

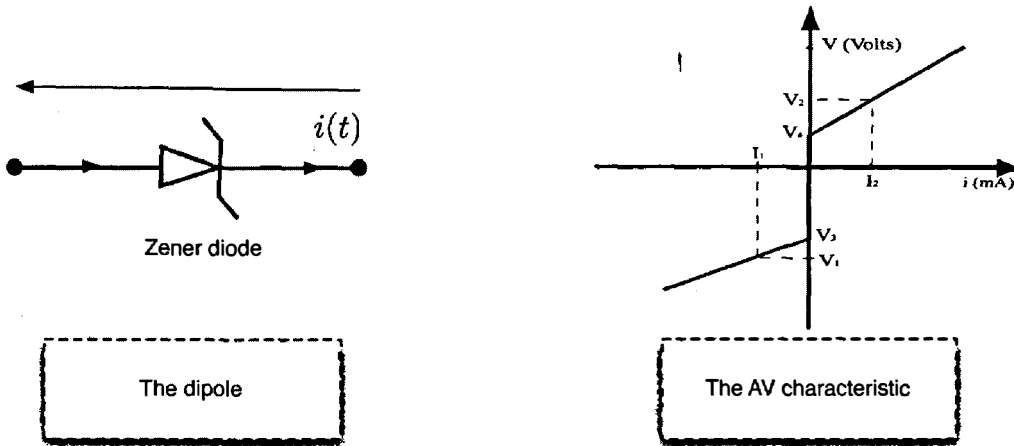


FIGURE 6. Zener diode model

$$V \in \partial\Phi_Z(i)$$

Let us consider the following dynamics that corresponds to the circuit depicted in Figure 7:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{L_3 C_4} & -\frac{(R_1 + R_3)}{L_3} & \frac{R_1}{L_3} \\ 0 & \frac{R_1}{L_2} & \frac{(R_1 + R_2)}{L_2} \end{pmatrix}}^A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

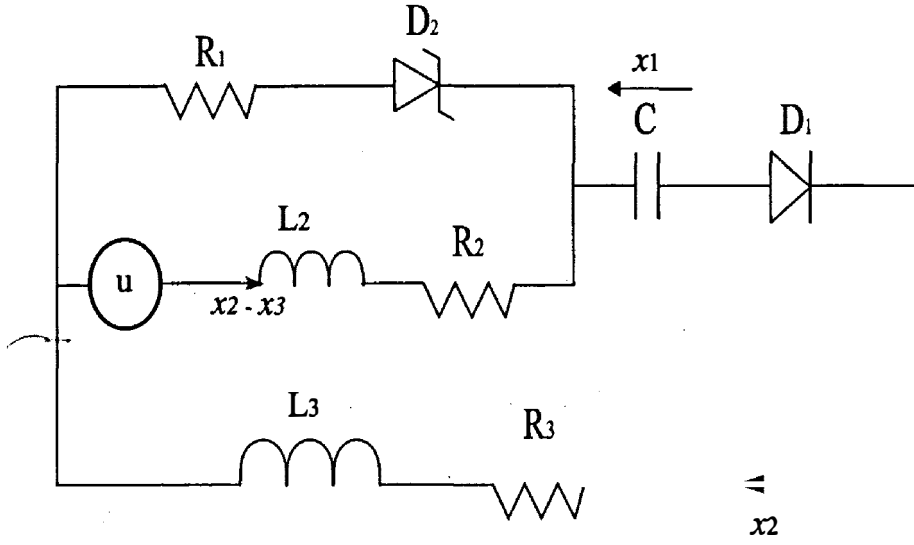


FIGURE 7. Circuit of the example 3.4

$$- \overbrace{\begin{pmatrix} 0 & 0 \\ \frac{1}{L_3} & \frac{1}{L_2} \\ -\frac{1}{L_2} & 0 \end{pmatrix}}^B \begin{pmatrix} y_{L,1} \\ y_{L,2} \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{1}{L_2} \end{pmatrix}}^D u,$$

and

$$\begin{cases} y_{L,1} \in \partial\Phi_D(-x_3 + x_2) \\ y_{L,2} \in \partial\Phi_Z(x_2) \end{cases} \quad (21)$$

where  $R_1 > 0, R_2 > 0, R_3 > 0$  are resistors,  $L_2 > 0, L_3 > 0$  are inductors,  $C_4 > 0$  is a capacitor,  $x_1$  is the time integral of the current across the capacitor,  $x_2$  is the current across the capacitor,  $x_3$  is the current across the inductor  $L_2$  and resistor  $R_2$ ,  $y_{L,1}$  is the voltage of the Zener diode,  $y_{L,2}$  is the voltage of the diode,  $\Phi_Z$  is the electrical superpotential of the Zener diode and  $\Phi_D$  is the electrical superpotential of the diode. Setting

$$y = \overbrace{\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}^C \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and defining the function  $\Phi_{ZD} : \mathbb{R}^2 \rightarrow \mathbb{R}; X \mapsto \Phi_{ZD}(X)$  by the formula:

$$\Phi_{ZD}(X) = \Phi_Z(X_1) + \Phi_D(X_2),$$

we may write the relations in (21) equivalently as:

$$y_L \in \partial\Phi_{ZD}(Cx).$$

For simplicity, let us set  $L_3 = L_2 = 1$  and  $u \equiv 0$ . Note that in this case:  $B = C^T$ . Hence, the dynamic of the system is then equivalent to

$$\dot{x}(t) \in Ax(t) - C^T \partial\Phi_{ZD}(Cx(t)).$$

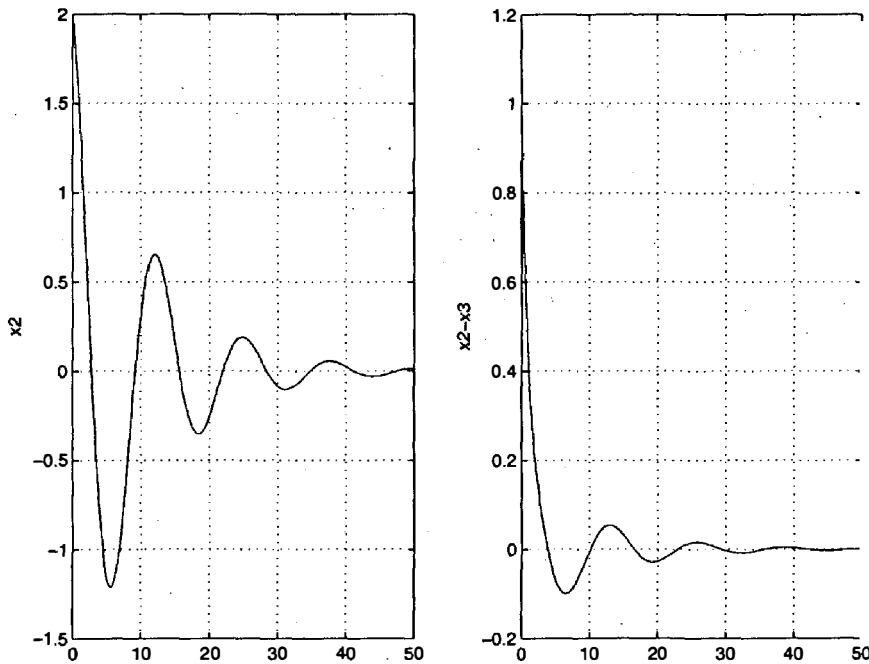


FIGURE 8. Simulation of the circuit in Fig. 7

By setting

$$\varphi = \Phi_{ZD} \circ C,$$

we have  $\partial\varphi(x) = C^T \partial\Phi_{ZD}(Cx)$ . Therefore, the dynamic of the system is

$$\dot{x}(t) \in Ax(t) - \partial\varphi(x(t)).$$

Fig. 8 shows the current behavior in the two main meshes of the circuit as defined in Fig. 7.

**Conclusion.** In this paper we have provided some conditions ensuring the finite-time stability for a general first order evolution variational inequality. These results extend and unify some existing result [16] for Filippov's differential inclusion. Our approach is quite different from the one presented in [16] and is based on an extension of the LaSalle's invariance principle for evolution variational inequalities [3]. The theoretical results are supported by some numerical simulation for electrical circuits containing nonsmooth electrical devices like an ideal diode, an practical diode or Zener diode. All the results announced in this paper assume the differentiability of the Lyapunov function. It is well-known that for some problems in Control Theory [20] the construction of a smooth Lyapunov function is not possible, so it would be interesting to obtain some similar results for nonsmooth Lyapunov functions. This is out of the scope of this paper and will be probably the subject of another work.

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