Plates made of piezoelectric materials: When are they really piezoelectric?

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Abstract. This paper aims at presenting in a synthetic way mathematical results that have been rigorously and recently derived by the authors. These results deal with the simplified but accurate modeling of linearly piezoelectric thin plates. It is shown how mathematical tools lead to two different situations linked to either sensors or actuators devices. Moreover we enlighten the fact that for some piezoelectric crystal classes, the coupling between the electrical and the mechanical effects disappear in the plate. Finally, we furnish a detailled example of such models in the case of a plate constitued by a 222 symmetry class material.

1 Introduction

Piezoelectric materials are widely used in the design of smart structures (see [1] for a widespread presentation of such structures). For example (see [2] and [3]) thin piezoelectric plates devices can be either bonded or embedded in these structures to determine strains and displacements (sensing effect) or to provide localized strains through which the deformation of the structure can be controlled (actuation effect). It is thus of major technological interest to provide efficient modelings of such structures. In [4], [5], [6] and [7], taking advantage of the particular shapes of such devices, the authors have derived new models by a rigorous study of the asymptotic behavior of a three dimensional body whose thickness, considered as a parameter, tends to zero. The important result is that two different models appear at the limit, enlightening the different behavior of piezoelectric sensors and actuators devices. In particular, it is shown how the kind of electrical loading is connected to the obtained limit model.

Starting from a general three-dimensional piezoelectric problem denoted by $\mathcal{P}_{3D}$, we outline the method that lead to simplified but accurate plate models denoted by $\mathcal{P}_{2D}$ because they are two-dimensional in essence. The ground of the method is to view the thickness of the plate as a small normalizing parameter whose aim is to tend to zero. Then, in particular, it can be shown that the two limit generalized kinematics do not have the same number of variables. Moreover, we investigate the influence of crystalline symmetries on the properties of our models and show that some crystal classes lead to a striking structural switch-off of the piezoelectric effect: *even if the material is piezoelectric, it is not anymore the case for the thin plate*. More precisely, this switch-off does not depend only on the crystal class of the piezoelectric material that constitutes the plate but also on the electrical boundary conditions, *i.e.* crystal symmetries do not have the same influence on sensors and actuators. It is shown that if the plate is used as a sensor, the decoupling occurs for the classes

$$2, 222, 2mm, 4, \bar{4}, 422, 4mm, \bar{4}2m, 6, 622, 6mm, 23 \text{ and } \bar{4}3m,$$

while, if the plate is used as an actuator, the decoupling takes place with the classes

$$m, 32, 422, \bar{6}, 622 \text{ and } \bar{6}m2.$$

We therefore observe that the decoupling occurs for sensors *and* actuators for only two classes: 422 and 622. These results are detailled in Section 4.

Finally, we give an example of our results in the case of a 222 symmetry class material. The distinction between the two purely mechanical behaviors is also made explicit by showing that the difference lies in a non negligible inplane shear coefficient.

2 Theoretical considerations

It is useful to recall the basic equations governing the electroelastic behavior of piezoelectric continua. It is the starting point of many problems that can either be of mathematical or numerical nature.

2.1 Basic piezoelectric equations

Latin lower indexes run from 1 to 3 and the lower index $'\,,i\,'$ stands for the derivation with respect to the i^{th} coordinate. Moreover, the convention of summation over repeated indexes is used.

The equilibrium of a piezoelectric body whose reference configuration is a $3D$ domain Ω with boundary S leads to:

$$\sigma_{ij,j} + f_i = 0, \quad D_{i,i} - q = 0, \quad rot\, E = 0. \tag{1}$$

Obviously, the electric field E being irrotational, it derives from an electric scalar potential φ. In the equations above, f_i, q are the mechanical body force components and the electric body charge, while σ_{ij} and D_i stand for the Cauchy stress tensor and the electric displacement vector components. These latter components are related to those of small strain tensor $e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ (u denotes the displacement vector field) and of electric vector field $E_i = -\varphi_{,i}$ by the constitutive equations (see [8] for example):

$$\begin{aligned} \sigma_{ij} &= \mathsf{a}_{ijkh}\, e_{kh}(u) - \mathsf{b}_{ijk}\, \varphi_{,k}, \\ D_p &= \mathsf{b}_{pqr}\, e_{qr}(u) \;\; + \mathsf{c}_{pq}\, \varphi_{,q}. \end{aligned} \tag{2}$$

In this expression, a_{ijkh}, b_{ijk} and c_{pq} denote the elastic, piezoelectric and dielectric material constants, respectively.

Remark 1. It is possible to define an operator $M : \mathbb{R}^{12} \longrightarrow \mathbb{R}^{12}$ governing the constitutive law (2) by writing that[1]

$$\begin{pmatrix} \sigma \\ D \end{pmatrix} = M \begin{pmatrix} e(u) \\ \nabla\varphi \end{pmatrix}. \tag{3}$$

The couple $\begin{pmatrix} \sigma \\ D \end{pmatrix}$ is called the generalized stress, while $\begin{pmatrix} e(u) \\ \nabla\varphi \end{pmatrix}$ is the generalized strain.

Most of the time, the mapping (2) is seen in a 9×9 matrix-form representation. In this direction, we introduce

$$M = \begin{pmatrix} a_{IJ} & -b_{Kl} \\ b_{lK} & c_{mn} \end{pmatrix}, \tag{4}$$

with $c_{mn} = \mathsf{c}_{mn}$ and where indexes l, m and n take their values in $\{1, 2, 3\}$ while I, J and K satisfy the Voigt contraction convention, taking their values in $\{1, 2, 3, 4, 5, 6\}$. We recall that the Voigt contraction convention is a mapping which associates to a couple of indexes (i, j) a sole index I such that

(i, j)	(1, 1)	(2, 2)	(3, 3)	(2, 3)	(3, 1)	(1, 2)
I	1	2	3	4	5	6

Thus, the elastic tensor a can be seen as a 6×6 real matrix which is written in another font by a. In the same way, the piezoelectric tensor b takes the form of a 6×3 real matrix denoted by b. However, due to the scalar product, it is necessary to adjust the physical constants:

$$\begin{array}{ll} a_{IJ} = \mathsf{a}_{ijkh} & \text{if } 1 \le I, J \le 3, \\ a_{IJ} = \sqrt{2}\,\mathsf{a}_{ijkh} & \text{if } 1 \le I, J \le 3 \text{ and } 4 \le J, I \le 6, \\ a_{IJ} = 2\,\mathsf{a}_{ijkh} & \text{if } 4 \le I, J \le 6, \\ b_{Ik} = \mathsf{b}_{ijk} & \text{if } 1 \le I \le 3, \\ b_{Ik} = \sqrt{2}\,\mathsf{b}_{ijk} & \text{if } 4 \le I \le 6. \end{array}$$

The relation (3), which governs the linearly piezoelectric constitutive law, then takes the form

[1] We recall that $\nabla\varphi = \begin{pmatrix} \varphi_{,1} \\ \varphi_{,2} \\ \varphi_{,3} \end{pmatrix}$.

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{31} \\ \sqrt{2}\,\sigma_{12} \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \left(\begin{array}{cccccc|ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & -b_{11} & -b_{12} & -b_{13} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & -b_{21} & -b_{22} & -b_{23} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} & -b_{31} & -b_{32} & -b_{33} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} & -b_{41} & -b_{42} & -b_{41} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} & -b_{51} & -b_{52} & -b_{53} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} & -b_{61} & -b_{62} & -b_{63} \\ \hline b_{11} & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} & c_{11} & c_{12} & c_{13} \\ b_{12} & b_{22} & b_{32} & b_{42} & b_{52} & b_{62} & c_{12} & c_{22} & c_{23} \\ b_{13} & b_{23} & b_{33} & b_{43} & b_{53} & b_{63} & c_{13} & c_{23} & c_{33} \end{array}\right) \cdot \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ e_{33}(u) \\ \sqrt{2}\,e_{23}(u) \\ \sqrt{2}\,e_{31}(u) \\ \sqrt{2}\,e_{12}(u) \\ \varphi_{,1} \\ \varphi_{,2} \\ \varphi_{,3} \end{pmatrix}. \quad (5)$$

We therefore note that the generalized three dimensional kinematics of a piezoelectric solid is described by *nine variables* (six mechanical and three electrical).

The piezoelectric body can be submitted to either essential or natural mechanical and electrical boundary conditions, or a combination of them, on S:

$$\begin{array}{lll} u_i = U_i & , & \sigma_{ij}\,n_j = F_i, \\ \varphi = V & , & D_i\,n_i = Q. \end{array} \quad (6)$$

We do not precise the associated partitions of S, the reader is refered to [4] for the details. Here U_i, F_i, V, Q and n_i denote the specified mechanical displacement and surface force components, the electrical potential, the flux through S of the electric induction and the outward unit normal vector components, respectively.

The local three-dimensional electroelastic problem $\mathcal{P}_{3D}(\Omega)$ consists of finding the electromechanical state $s = (u, \varphi)$ satisfying equations (1), (2) and (6).

2.2 Variational piezoelectric equations

It is necessary to rewrite $\mathcal{P}_{3D}(\Omega)$ in another form in order to gather useful informations on the electromechanical state $s = (u, \varphi)$. Multiplying by sufficiently smooth[2] kinematically admissible virtual displacements v_i and electrical potential ψ, equation (1) becomes equivalent to

$$\int_\Omega (\sigma_{ij,j} + f_i)\,v_i\;d\Omega + \int_\Omega (D_{i,i} - q)\,\psi\;d\Omega = 0. \quad (7)$$

Integrating by part this expression leads to

$$-\int_\Omega \sigma_{ij,j}\,v_{i,j}\;d\Omega + \int_S \sigma_{ij}\,n_j v_i\;dS + \int_\Omega f_i\,v_i\;d\Omega$$
$$-\int_\Omega D_i\,\psi_{,i}\;d\Omega + \int_S D_i\,n_i\,\psi\;d\Omega - \int_\Omega q\,\psi\;d\Omega = 0. \quad (8)$$

[2] In the sequel, we do not precise the exact mathematical background of such a formulation.

The symmetry of the stress tensor ($\sigma_{ij} = \sigma_{ji}$), the definition of the small strain tensor and the boundary conditions (6) thus give

$$-\int_{\Omega} \sigma_{ij,j}\, e_{ij}(v)\; d\Omega + \int_S F_i\, v_i\; dS + \int_{\Omega} f_i\, v_i\; d\Omega \\ - \int_{\Omega} D_i\, \psi_{,i}\; d\Omega + \int_S Q\, \psi\; dS - \int_{\Omega} q\, \psi\; d\Omega = 0. \quad (9)$$

For any kinematically admissible virtual electromechanical state $r = (v, \psi)$, we introduce the linear form

$$L(r) = \int_S F_i\, v_i\; dS + \int_{\Omega} f_i\, v_i\; d\Omega + \int_S Q\, \psi\; dS - \int_{\Omega} q\, \psi\; d\Omega. \quad (10)$$

We notice that

$$-\int_{\Omega} \sigma_{ij,j}\, e_{ij}(v)\; d\Omega - \int_{\Omega} D_i\, \psi_{,i}\; d\Omega = -\int_{\Omega} M \begin{pmatrix} e(u) \\ \nabla\varphi \end{pmatrix} \cdot \begin{pmatrix} e(v) \\ \nabla\psi \end{pmatrix}\; d\Omega, \quad (11)$$

and, for brevity, define the bilinear form m associated with the electroelastic potential $\frac{1}{2}(\sigma \cdot e + D \cdot E)$

$$m(s, r) = m((u, \varphi), (v, \psi)) = \int_{\Omega} M \begin{pmatrix} e(u) \\ \nabla\varphi \end{pmatrix} \cdot \begin{pmatrix} e(v) \\ \nabla\psi \end{pmatrix}\; d\Omega. \quad (12)$$

It is then possible to reformulate the problem of determining the electromechanical state at equilibrium:

$$\mathcal{P}_{3D}(\Omega) \begin{cases} \text{Find } s = (u, \varphi) \text{ sufficiently smooth such that} \\ m(s, r) = L(r), \text{ for all virtual electromechanical state } r = (v, \psi). \end{cases}$$

This expression of the electromechanical problem is at the starting point of either finite element formulations or mathematical questions in (linear) piezoelectricity (see [9] and [10] for example).

3 The problem of piezoelectric plates

In pure elasticity, the problem of finding simplified but accurate plate models is quite old. Recently however, [11] has presented a rigorous mathematical justification of the Kirchhoff-Love model in the homogeneous and isotropic case. This method has been successfully generalized to (possibly heterogeneous) linearly piezoelectric plates (see [4], [5], [6] and [7]), both in the static and transient siuations. At this point, it has been made possible to reconciliate some existing divergent models (see [12], [13], [14], [15]) or to precise the mathematical framework of some existing results (see [16]).

In the plate models derivation, a crucial role is played by the thickness direction (also called the outplane direction). For commodity, this direction corresponds to the third coordinate axis. The role played by the thickness is crucial because it is very small compared to the other dimensions of the plate : $\Omega = \omega \times (-\varepsilon/2, +\varepsilon/2)$, where ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary and where ε denotes the thickness of the plate. That leads to the idea of considering ε as a *small parameter* and of connecting this parameter to the data of our problem, *i.e.* the electromechanical coefficients, loading and state. In a sense, by this way, plates models can be interpreted as a peculiar electromechanical state resulting of a given class of electromechanical loading imposed to a thin flat piezoelectric plate.

From the mathematical point of view, the method consists in studying what does happen to the unique solution of $\mathcal{P}_{3D}(\Omega)$ when Ω is the reference configuration of a flat piezoelectric body whose thickness goes to zero (this is the reason why this method belongs to the field of *asymptotic analysis*). The striking fact is that two models, *i.e.* two different kinds of behavior, appear at the limit. These two models are intimately connected to the type of electric loading subjected to the plate, thus giving a rigorous theoretical background to the study of sensors and actuators plate-like devices. In order to emphasize the fact that the models we get are arising through a dimension reduction process, they will be denoted by $\mathcal{P}_{2D}(\Omega)$. More precisely, by different averagings through the thickness, it is possible to show that our limit models can be fully described by taking into account only the inplane coordinates.

In the sequel, we consider the following four electromechanical boundary conditions on the set $\Gamma^{\pm}$ constituted by the lower and the upper faces of the flat thin plate occupying Ω:

$$(BC)_{sensor} \quad : D \cdot n = Q \quad \text{on} \quad \Gamma^{\pm},$$

$$(BC)_{actuator} : \quad V = V_0^{\pm} \quad \text{on} \quad \Gamma^{\pm}.$$

Here, we focus on the presentation of the obtained models. For the mathematical arguments underlying the whole analysis of this problem, we refer the reader to [4], [6] and [7].

3.1 The sensor model

Three kinds of information are needed to fully describe an electromechanical model. These are the generalized kinematics (or generalized strain), the inner loading (or generalized stress) and the constitutive equations (which link them).

The generalized kinematics The generalized kinematics involves the tensor of small strains and the electrical potential gradient. In [6] and [7], we have shown that these two mathematical objects appear in reduced forms in the limit models.

The displacements field We obtain a Kirchhoff-Love displacements field, which in particular means that the model cannot render shear effects (see [17], [18] [2] and [3] in which shear or thickness effects are rendered with different methods). More precisely, a Kirchhoff-Love displacement v satisfies:

$$e(v) = \begin{pmatrix} e_{11}(v) & e_{12}(v) & 0 \\ e_{12}(v) & e_{22}(v) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

It is possible to show that Kirchhoff-Love displacements can be decomposed into a membrane and a flexural parts (see [11], [6] and [7]). This result is a very classical one in pure elasticity. However, it is important to note that because of the symmetry of the small strain tensor, the number of purely mechanical variables comes down from six to three.

The electrical field The asymptotic analysis of the three-dimensional problem shows a crucial difference between the actuator and the sensor cases: in the actuator case, the electric potential intervenes only (at the first order) through the outplane direction while, in the sensor case, the electric potential does not depend (at the first order) on the outplane direction. Focusing here on the sensor case, we are in the situation for which the outplane direction plays no role, *i.e.* the electrical potential does not depend on x_3 so that the limit model only takes into account two variables (the inplane electrical ones).

We can therefore conclude that the limit generalized kinematics is described by only five variables instead of nine in the full $3D$ situation. It is represented by the $\widetilde{e}_1(u, \varphi)$ vector:

$$\widetilde{e}_1(u, \varphi) = \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ \sqrt{2}\, e_{12}(u) \\ \varphi_{,1} \\ \varphi_{,2} \end{pmatrix}. \tag{14}$$

The generalized stress The generalized stress involves the stress tensor and the electrical displacement vector. As a result of the asymptotic analysis one finds that this mathematical object reduces to its inplane components, so that it takes the reduced form:

$$\widetilde{\sigma}_1 = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\, \sigma_{12} \\ D_1 \\ D_2 \end{pmatrix}. \tag{15}$$

The sensor constitutive law We now have to identify the mathematical object $\widetilde{M}_{sensor}$ which link the generalized stress to the generalized strain that live on the plate. The algebraic arguments that lead to the exact formula of the limit constitutive law are presented and justified in [6] and [7]. In fact, the limit constitutive equations emerge from a recombination of the electromechanical components of M (see (5)). This recombination is imposed by the structure of $\widetilde{e}_1(u, \varphi)$ and $\widetilde{\sigma}_1$ described above. In the sensor case, the recombination lead to rewrite (5) as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\,\sigma_{12} \\ D_1 \\ D_2 \\ \hline \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{31} \\ D_3 \end{pmatrix} = M_1' \cdot \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ \sqrt{2}\,e_{12}(u) \\ \varphi_{,1} \\ \varphi_{,2} \\ \hline e_{33}(u) \\ \sqrt{2}\,e_{23}(u) \\ \sqrt{2}\,e_{31}(u) \\ \varphi_{,3} \end{pmatrix}. \tag{16}$$

with

$$M_1' = \left(\begin{array}{ccccc|cccc} a_{11} & a_{12} & a_{16} & -b_{11} & -b_{12} & a_{13} & a_{14} & a_{15} & -b_{13} \\ a_{12} & a_{22} & a_{26} & -b_{21} & -b_{22} & a_{23} & a_{24} & a_{25} & -b_{23} \\ a_{16} & a_{26} & a_{66} & -b_{61} & -b_{62} & a_{36} & a_{46} & a_{56} & -b_{63} \\ b_{11} & b_{21} & b_{61} & c_{11} & c_{12} & b_{31} & b_{41} & b_{51} & c_{13} \\ b_{12} & b_{22} & b_{62} & c_{12} & c_{22} & b_{32} & b_{42} & b_{52} & c_{23} \\ \hline a_{13} & a_{23} & a_{36} & -b_{31} & -b_{32} & a_{33} & a_{34} & a_{35} & -b_{33} \\ a_{14} & a_{24} & a_{46} & -b_{41} & -b_{42} & a_{34} & a_{44} & a_{45} & -b_{43} \\ a_{15} & a_{25} & a_{56} & -b_{51} & -b_{52} & a_{35} & a_{45} & a_{55} & -b_{53} \\ b_{13} & b_{23} & b_{63} & c_{13} & c_{23} & b_{33} & b_{43} & b_{53} & c_{33} \end{array}\right).$$

Now, let

$$\begin{gathered} M_1^{00} = \begin{pmatrix} a_{11} & a_{12} & a_{16} & -b_{11} & -b_{12} \\ a_{12} & a_{22} & a_{26} & -b_{21} & -b_{22} \\ a_{16} & a_{26} & a_{66} & -b_{61} & -b_{62} \\ b_{11} & b_{21} & b_{61} & c_{11} & c_{12} \\ b_{12} & b_{22} & b_{62} & c_{12} & c_{22} \end{pmatrix}, M_1^{0-} = \begin{pmatrix} a_{13} & a_{14} & a_{15} & -b_{13} \\ a_{23} & a_{24} & a_{25} & -b_{23} \\ a_{36} & a_{46} & a_{56} & -b_{63} \\ b_{31} & b_{41} & b_{51} & c_{13} \\ b_{32} & b_{42} & b_{52} & c_{23} \end{pmatrix}, \\ M_1^{-0} = \begin{pmatrix} a_{13} & a_{23} & a_{36} & -b_{31} & -b_{32} \\ a_{14} & a_{24} & a_{46} & -b_{41} & -b_{42} \\ a_{15} & a_{25} & a_{56} & -b_{51} & -b_{52} \\ b_{13} & b_{23} & b_{63} & c_{13} & c_{23} \end{pmatrix}, M_1^{--} = \begin{pmatrix} a_{33} & a_{34} & a_{35} & -b_{33} \\ a_{34} & a_{44} & a_{45} & -b_{43} \\ a_{35} & a_{45} & a_{55} & -b_{53} \\ b_{33} & b_{43} & b_{53} & c_{33} \end{pmatrix}. \end{gathered} \tag{17}$$

Because the asymptotic analysis of $\mathcal{P}_{3D}(\Omega)$ associated with the boundary conditions $(BC)_{sensor}$ shows that the vector $\begin{pmatrix} \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{31} \\ D_3 \end{pmatrix}$ can be neglected, the sensor constitutive equations appears as the *Schur complement* (or the condensation) of the block M_1^{--} of M_1':

$$\boxed{\widetilde{M}_{sensor} = \widetilde{M}_1 = M_1^{00} - M_1^{0-}(M_1^{--})^{-1}M_1^{-0}.} \tag{18}$$

Introducing the mechanical (m) and electrical (e) components of the generalized stress and strain, we associate to $\widetilde{M}_{sensor}$ the sub-operators $\widetilde{M}_{1_{\mathsf{mm}}}$, $\widetilde{M}_{1_{\mathsf{me}}}$, $\widetilde{M}_{1_{\mathsf{em}}}$ and $\widetilde{M}_{1_{\mathsf{ee}}}$:

$$\widetilde{M}_{sensor} = \begin{pmatrix} \widetilde{M}_{1_{\mathsf{mm}}} & \widetilde{M}_{1_{\mathsf{me}}} \\ \widetilde{M}_{1_{\mathsf{em}}} & \widetilde{M}_{1_{\mathsf{ee}}} \end{pmatrix}. \tag{19}$$

It is shown in [6] that M and $\widetilde{M}_{sensor}$ share the same inner structure, that is:

$$\widetilde{M}_{1_{\mathsf{mm}}} = \widetilde{M}^T_{1_{\mathsf{mm}}}, \quad \widetilde{M}_{1_{\mathsf{me}}} = -\widetilde{M}^T_{1_{\mathsf{em}}}, \quad \widetilde{M}_{1_{\mathsf{ee}}} = \widetilde{M}^T_{1_{\mathsf{ee}}}. \tag{20}$$

Variational formulation of the sensor model Similarly to (12), we define

$$\widetilde{m}_1(s,r) = \widetilde{m}_1((u,\varphi),(v,\psi)) = \int_\Omega \widetilde{M}_{sensor}\, \widetilde{e}_1(u,\varphi)\cdot \widetilde{e}_1(v,\psi)\ d\Omega. \tag{21}$$

Our proposed model which allows us to determine the electromechanical state of a plate-like sensor at equilibrium then reads as:

$$\mathcal{P}^{sensor}_{2D}(\Omega) \begin{cases} \text{Find } s=(u,\varphi) \text{ sufficiently smooth such that} \\ \widetilde{m}_1(s,r) = L(r), \text{ for all virtual electromechanical state } r=(v,\psi). \end{cases}$$

As an *asymptotic result*, the thinner the plate (compared to its other dimensions), the more accurate the model is. In this direction, the degree of accuracy of this result is precised in [4]. Mathematically speaking, it is of importance to precise that the function space on which live the (limit) admissible electromechanical state is not the same that in the three-dimensional case. This is the reason why it is often spoken of "singular perturbations" problems.

Practically speaking, this case corresponds to a device which is able to measure (directly or indirectly) the flux of the electric induction, so that the linear form L is perfectly determined. A numerical treatment of $\mathcal{P}^{sensor}_{2D}(\Omega)$ then gives the piezoelectric state in the plate. That is why we can call this model sensor.

3.2 The actuator model

As it has been specified earlier, the difference between sensor and actuator models lies in the informations that the electrical potential can take into account. Here, in the actuator case, these informations are collected only upon the outplane direction, while in the sensor case these informations were collected upon the two inplane directions. Of course, the purely mechanical informations do not change, but the fact that only $\varphi_{,3}$ appears in the actuator model radically changes the generalized kinematics and stress together with the constitutive law.

The generalized kinematics and stress As it has just been pointed out, displacements field is always of Kirchhoff-Love type (see (13)). As to the electrical potential, it can be shown that only E_3 appears. The generalized kinematics is therefore described by four variables at the limit. It is represented by the vector:

$$\widetilde{e}_2(u,\varphi) = \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ \sqrt{2}\, e_{12}(u) \\ \varphi_{,3} \end{pmatrix}. \tag{22}$$

Similarly, the generalized stress takes the form:

$$\widetilde{\sigma}_2 = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\, \sigma_{12} \\ D_3 \end{pmatrix}. \tag{23}$$

The actuator constitutive law The method to find the constitutive relations is similar to the one presented in the sensor case. However, the difficulty lies in an adequate electromechanical coefficients recombination. We precise this point here.

First of all, we rewrite (5) as

$$\left(\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\,\sigma_{12} \\ D_3 \\ \hline \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{31} \\ \hline D_1 \\ D_2 \end{array}\right) = M_2' \cdot \left(\begin{array}{c} e_{11}(u) \\ e_{22}(u) \\ \sqrt{2}\, e_{12}(u) \\ \partial_3\varphi \\ \hline e_{33}(u) \\ \sqrt{2}\, e_{23}(u) \\ \sqrt{2}\, e_{31}(u) \\ \hline \partial_1\varphi \\ \partial_2\varphi \end{array}\right), \tag{24}$$

so that:

$$M_2' = \left(\begin{array}{cccc|ccc|cc} a_{11} & a_{12} & a_{16} & -b_{13} & a_{13} & a_{14} & a_{15} & -b_{11} & -b_{12} \\ a_{12} & a_{22} & a_{26} & -b_{23} & a_{23} & a_{24} & a_{25} & -b_{21} & -b_{22} \\ a_{16} & a_{26} & a_{66} & -b_{63} & a_{36} & a_{46} & a_{56} & -b_{61} & -b_{62} \\ b_{13} & b_{23} & b_{63} & c_{33} & b_{33} & b_{43} & b_{53} & c_{13} & c_{23} \\ \hline a_{13} & a_{23} & a_{36} & -b_{33} & a_{33} & a_{34} & a_{35} & -b_{31} & -b_{32} \\ a_{14} & a_{24} & a_{46} & -b_{43} & a_{34} & a_{44} & a_{45} & -b_{41} & -b_{42} \\ a_{15} & a_{25} & a_{56} & -b_{53} & a_{35} & a_{45} & a_{55} & -b_{51} & -b_{52} \\ \hline b_{11} & b_{21} & b_{61} & c_{13} & b_{31} & b_{41} & b_{51} & c_{11} & c_{12} \\ b_{12} & b_{22} & b_{62} & c_{23} & b_{32} & b_{42} & b_{52} & c_{12} & c_{22} \end{array}\right).$$

and define

$$M_2^{00} = \begin{pmatrix} a_{11} & a_{12} & a_{16} & -b_{13} \\ a_{12} & a_{22} & a_{26} & -b_{23} \\ a_{16} & a_{26} & a_{66} & -b_{63} \\ b_{13} & b_{23} & b_{63} & c_{33} \end{pmatrix}, M_2^{0-} = \begin{pmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \\ a_{36} & a_{46} & a_{56} \\ b_{33} & b_{43} & b_{53} \end{pmatrix},$$
$$M_2^{-0} = \begin{pmatrix} a_{13} & a_{23} & a_{36} & -b_{33} \\ a_{14} & a_{24} & a_{46} & -b_{43} \\ a_{15} & a_{25} & a_{56} & -b_{53} \end{pmatrix}, M_2^{--} = \begin{pmatrix} a_{33} & a_{34} & a_{35} \\ a_{34} & a_{44} & a_{45} \\ a_{35} & a_{45} & a_{55} \end{pmatrix}. \tag{25}$$

Here, the asymptotic analysis of $\mathcal{P}_{3D}(\Omega)$ associated with the boundary conditions $(BC)_{actuator}$ shows that the two vectors $\begin{pmatrix} \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{31} \end{pmatrix}$ and $\begin{pmatrix} \partial_1\phi \\ \partial_2\phi \end{pmatrix}$ can be neglected, so that the actuator constitutive equations reads as

$$\boxed{\widetilde{M}_{actuator} = \widetilde{M}_2 = M_2^{00} - M_2^{0-}(M_2^{--})^{-1}M_2^{-0}.} \tag{26}$$

This operator shares the same structure and symmetry properties as those exhibited in (19)-(20).

Similarly to the sensor case, in order to get the variational formulation of the plate-like actuator problem, we define

$$\widetilde{m}_2(s,r) = \widetilde{m}_2((u,\varphi),(v,\psi)) = \int_\Omega \widetilde{M}_2\, \widetilde{e}_2(u,\varphi)\cdot\widetilde{e}_2(v,\psi)\; d\Omega, \tag{27}$$

and the problem of determining the electromechanical state of a plate-like actuator at equilibrium then takes the form:

$$\mathcal{P}_{2D}^{actuator}(\Omega) \begin{cases} \text{Find } s=(u,\varphi) \text{ sufficiently smooth such that} \\ \widetilde{m}_2(s,r) = L(r), \text{ for all virtual electromechanical state } r=(v,\psi), \end{cases}$$

which is also a singularly perturbed problem.

Remark 2. To be more precise, in the expression of the model $\mathcal{P}_{2D}^{actuator}(\Omega)$, the terms 'sufficiently smooth' mean that s has to satisfy $(BC)_2$ while r has to satisfy $(BC)_2$ with $V_0^{\pm} = 0$, see [4] for the technical details.

This case corresponds to a device subjected to given electric potential at its boundary. A numerical treatment of $\mathcal{P}_{2D}^{actuator}(\Omega)$ supplies the piezoelectric state in the plate. Therefore, the mechanical state can be controlled through electric loading. That is why we call this model actuator.

4 Influence of crystalline symmetries

It is interesting to give some properties of the operator $\widetilde{M}_p$ $(p=1,2)$, which supplies the constitutive equations of the piezoelectric thin plates. As we saw, the fundamental coupling property of M remains true for $\widetilde{M}_p$:

$$\widetilde{M}_{p_{\mathsf{me}}} = -(\widetilde{M}_{p_{\mathsf{em}}})^T, \tag{28}$$

where m and e respectively denote the mechanical and electrical components of the generalized kinematics and stress.

Considering the influence of crystalline symmetries on the three-dimensional constitutive law (see [8] for example), we can deduce, in the case of a polarization normal to the plate, that[3]:

- $\widetilde{M}_{2_{\mathsf{mm}}}$ involves mechanical terms only,
- $\widetilde{M}_{1_{\mathsf{mm}}} = \widetilde{M}_{2_{\mathsf{mm}}}$ for the crystalline classes m, 32, 422, $\bar{6}$, 622 and $\bar{6}m2$,
- $\widetilde{M}_{1_{\mathsf{mm}}}$ involves electrical terms except for these previous classes,
- when $p = 1$, there is an electromechanical decoupling ($\widetilde{M}_{p_{\mathsf{me}}} = 0$) for the classes 2, 222, $2mm$, 4, $\bar{4}$, 422, $4mm$, $\bar{4}2m$, 6, 622, $6mm$, 23 and $\bar{4}3m$, when $p = 2$, this decoupling occurs with the classes m, 32, 422, $\bar{6}$, 622 and $\bar{6}m2$, nevertheless the operators $\widetilde{M}_{p_{\mathsf{mm}}}$ and $\widetilde{M}_{p_{\mathsf{ee}}}$ involve a mixture of elastic, piezoelectric and dielectric coefficients. In these cases, the plate can be considered as no more piezoelectric. We are then in a situation of a *structural switch-off* of the piezoelectric effect.

We then enlighten situations for which piezoelectric materials lead to non-piezoelectric structures. For recent results concerning the reverse situation, that is the possibility of conceiving piezoelectric composites without using piezoelectric materials, the reader is refered to [19].

5 Application and example: 222 crystalline class

Let's consider a thin piezoelectric plate constituted by a material whose crystalline symmetry class is 222. Then (5) takes the form:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{31} \\ \sqrt{2}\,\sigma_{12} \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \left(\begin{array}{cccccc|ccc} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 & -b_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & -b_{52} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 & -b_{63} \\ \hline 0 & 0 & 0 & b_{41} & 0 & 0 & c_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{52} & 0 & 0 & c_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{63} & 0 & 0 & c_{33} \end{array}\right) \cdot \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ e_{33}(u) \\ \sqrt{2}\,e_{23}(u) \\ \sqrt{2}\,e_{31}(u) \\ \sqrt{2}\,e_{12}(u) \\ \varphi_{,1} \\ \varphi_{,2} \\ \varphi_{,3} \end{pmatrix}. \tag{29}$$

Therefore, (18) and (26) respectively lead to

[3] In the following, the letter m in Sans Serif font stands for 'mechanical' while the same letter m in italic stands for 'mirror', as it is usually understood in crystallography.

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\,\sigma_{12} \\ D_1 \\ D_2 \end{pmatrix} = \left(\begin{array}{ccc|cc} a_{11} - \frac{a_{13}^2}{a_{33}} & a_{12} - \frac{a_{13}a_{23}}{a_{33}} & 0 & 0 & 0 \\ a_{12} - \frac{a_{13}a_{23}}{a_{33}} & a_{22} - \frac{a_{23}^2}{a_{33}} & 0 & 0 & 0 \\ 0 & 0 & a_{66} + \frac{b_{63}^2}{c_{33}} & 0 & 0 \\ \hline 0 & 0 & 0 & c_{11} + \frac{b_{41}^2}{a_{44}} & 0 \\ 0 & 0 & 0 & 0 & c_{22} + \frac{b_{52}^2}{a_{55}} \end{array}\right) \cdot \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ \sqrt{2}\,e_{12}(u) \\ \varphi_{,1} \\ \varphi_{,2} \end{pmatrix} \tag{30}$$

in the sensor case and to

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\,\sigma_{12} \\ D_3 \end{pmatrix} = \left(\begin{array}{ccc|c} a_{11} - \frac{a_{13}^2}{a_{33}} & a_{12} - \frac{a_{13}a_{23}}{a_{33}} & 0 & 0 \\ a_{12} - \frac{a_{13}a_{23}}{a_{33}} & a_{22} - \frac{a_{23}^2}{a_{33}} & 0 & 0 \\ 0 & 0 & a_{66} & -b_{63} \\ \hline 0 & 0 & b_{63} & c_{33} \end{array}\right) \cdot \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ \sqrt{2}\,e_{12}(u) \\ \varphi_{,3} \end{pmatrix} \tag{31}$$

in the actuator case.

As outlined in the preceeding section, the relation (30) shows that σ and D respectively depend solely on $e(u)$ and $\nabla\varphi$ when the plate acts as a sensor, so that it can be considered as no more piezoelectric. However, when the same plate acts as an actuator, the piezoelectric coupling does not vanish as it can be seen in (31). Moreover, we observe that the difference between both $M_{p_{\mathrm{mm}}}$ lies in the inplane shear coefficient: if $p = 1$, this coefficient is equal to $a_{66} + \frac{b_{63}^2}{c_{33}}$ while it is equal to a_{66} if $p = 2$. Since the order of magnitude of the permittivity c_{33} is most of the time very low compared to the piezoelectric constants (see [8] p. 146), the term $\frac{b_{63}^2}{c_{33}}$ cannot be neglected. Therefore, from the purely mechanical point of view, it appears a significative difference between the two models.

6 Conclusion

Through classical tools of variational analysis, two of the authors derived in [4] the mathematical modeling of linearly piezoelectric plates. In this paper, the properties of the two models that arise from this derivation are detailed. These two models are related to the kind of electrical boundary conditions imposed to the plate (*i.e.* whether the piezoelectric plate is electroded or not). When the upper and lower faces are not covered with electrodes, the plate acts as a sensor whose model uses a richer generalized kinematics[4] than the one of the actuator obtained when the upper and lower faces are electroded. The procedure that leads to the constitutive laws of our models is presented and detailled.

[4] In the sense that the generalized kinematics has more components in this situation. Note, however, that the two electric components $\varphi_{,1}$ and $\varphi_{,2}$ do not depend on x_3 while, for the actuator, the electric potential may depend on x_1, x_2 and x_3 even if the sole (vertical) electric component appears in the generalized kinematics.

As our results come from the field of theoretical mathematical modeling, our paper does not directly deals with a numerical treatment of the studied problem. However, the theorems proven in [4] and presented here can be straightforwardly used in the field of numerical studies through finite element method. It has to be emphasized that standard 3D mechanical problems are often prohibitive from the numerical point of view. It is one of the many reasons for which the of asymptotic analysis of dimension reduction problems is so important: in the case presented here, our limit models are bidimensional. Therefore, the numerical treatment is made much more easier. Of course, the accuracy of our models is clearly related to the order of magnitude of the thickness of the plate : the thinner the plate, the more accurate the modeling. We also emphasize on the point that our modeling is carried out for every crystal classes that are piezoelectrically compatible. The phenomenon of structural switch-off is theoretically enlightened for the first time.

To improve the modeling of smart structures, it is however necessary to look more precisely at a rigorous modeling of multiparameterized multiphysical structures, in order to take into account the thickness of the electrodes, but also the fact that piezoelectric devices are often glued as patches on host structures. This kind of multiphysical structures has to be very carefully studied. The difficulty lies on the fact that many different orders of magnitude of geometrical and mechanical natures appear in such problem : the thicknesses of the plate, of the electrodes and of the bonded joint, but also the orders of magnitude of the stiffness of the glue, of the electrodes and, eventually, of the dielectric constants of the plate.

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