Well-posedness of The Prandtl Equation in Sobolev Spaces

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Abstract. We develop a new approach to study the well-posedness theory of the Prandtl equation in Sobolev spaces by using a *direct energy method* under a monotonicity condition on the tangential velocity field instead of using the Crocco transformation. Precisely, we firstly investigate the linearized Prandtl equation in some weighted Sobolev spaces when the tangential velocity of the background state is monotonic in the normal variable. Then to cope with the loss of regularity of the perturbation with respect to the background state due to the degeneracy of the equation, we apply the Nash-Moser-Hörmander iteration to obtain a well-posedness theory of classical solutions to the nonlinear Prandtl equation when the initial data is a small perturbation of a monotonic shear flow.

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1. Introduction

In this work, we study the well-posedness of the Prandtl equation which is the foundation of the boundary layer theory introduced by Prandtl in 1904, [20]. It describes the behavior of the flow near a boundary in the small viscosity limit of

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an incompressible viscous flow with the non-slip boundary condition. We consider the following initial-boundary value problem,

$$\begin{cases} u_t + uu_x + vu_y + p_x = u_{yy}, \quad t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = U(t,x), \\ u|_{t=0} = u_0(x,y)\,, \end{cases} \tag{1.1}$$

where $u(t,x,y)$ and $v(t,x,y)$ represent the tangential and normal velocities of the boundary layer, with y being the scaled normal variable to the boundary, while $U(t,x)$ and $p(t,x)$ are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U\partial_x U + \partial_x p = 0.$$

The well-posedness theories and the justification of the Prandtl equation remain as the challenging problems in the mathematical theory of fluid mechanics. Up to now, there are only a few rigorous mathematical results. Under a monotonic assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial-boundary value problems in [18], and this result together with some of her works with collaborators were well presented in the monograph [19]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so called favorable condition on the pressure, Xin and Zhang obtained the existence of global weak solutions to the Prandtl equation in [22]. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by degeneracy and mixed type of the equation.

Without the monotonicity assumption, E and Engquist in [5] constructed some finite time blowup solutions to the Prandtl equation. And in [21], Sammartino and Caflisch obtained the local existence of analytic solutions to the Prandtl equation, and a rigorous theory on the stability of boundary layers with analytic data in the framework of the abstract Cauchy-Kowaleskaya theory. This result was extended to the function space which is only analytic in the tangential variable in [14].

In recent years, there have been some interesting works concerning the linear and nonlinear instability of the Prandtl equation in the Sobolev spaces. In [8], Grenier showed that the unstable Euler shear flow $(u_s(y), 0)$ with $u_s(y)$ having an inflection point (the well-known Rayleigh's criterion) yields instability for the Prandtl equation. In the spirit of Grenier's approach, in [6], Gérard-Varet and Dormy showed that if the shear flow profile $(u^s(t,y), 0)$ of the Prandtl equation has a non-degenerate critical point, then it leads to a strong linear ill-posedness of the Prandtl equation in the Sobolev framework. In a similar approach, in [7] Gérard-Varet and Nguyen strengthened the ill-posedness result of [6] for the linearized Prandtl equation for an unstable shear flow. Moreover, they also showed that if a solution, as a small perturbation of the unstable shear flow, to the nonlinear Prandtl equation exists in the Sobolev setting, then it cannot be Lipschitz continuous. Along this direction, Guo and Nguyen in [9] proved that the nonlinear Prandtl equation is ill-posed near non-stationary and non-monotonic shear flows, and showed that the asymptotic boundary-layer expansion is not valid for non-monotonic shear layer flows in Sobolev spaces. Hong and Hunter [11] studied the formation of singularities and instability in the unsteady inviscid Prandtl equation. All these works show what happens in an unsteady boundary layer separation. For the related mathematical results and discussions, also see the review papers [2, 4].

As mentioned in [2, 6, 7], the local in time well-posedness of the Prandtl equation for initial data in Sobolev space remains an open problem. Therefore, it would be very interesting to recover Oleinik's well-posedness results simply through direct energy estimates. And this is the goal of this paper.

In this paper we consider the case of an uniform outflow $U = 1$, which implies p being a constant. Consider the following problem for the Prandtl equation,

$$\begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, \quad t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x, y)\,. \end{cases} \tag{1.2}$$

We are going to study the well-posedness of the Prandtl equation around a monotonic shear flow. More precisely, assume that

$$u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y),$$

where u_0^s is monotonic in y

$$\partial_y u_0^s(y) > 0, \quad \forall y \geq 0. \tag{1.3}$$

To state the main result, we first introduce the following notations. Set $\Omega_T = [0, T] \times \mathbb{R}^2_+$. For any non-negative integer k and real number ℓ, define

$$\|u\|_{\mathcal{A}^k_\ell(\Omega_T)} = \Big(\sum_{k_1+[\frac{k_2+1}{2}]\leq k} \|\langle y\rangle^\ell \partial^{k_1}_{(t,x)} \partial^{k_2}_y u\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \Big)^{1/2},$$

and

$$\|u\|_{\mathcal{D}^k_\ell(\Omega_T)} = \sum_{k_1+[\frac{k_2+1}{2}]\leq k} \|\langle y\rangle^\ell \partial^{k_1}_{(t,x)} \partial^{k_2}_y u\|_{L^\infty_y(L^2_{t,x})},$$

with $\langle y\rangle = (1 + y^2)^{\frac{1}{2}}$. When the function is independent of t (or x) variable, we use the same notations for the non-isotropic norms as above under the convention that we do not take integration with respect to this variable.

The main result of this paper can be stated as follows.

Theorem 1.1. *Concerning the problem* (1.2)*, we have the following existence and stability results.*

(1) Given any integer $k \geq 5$ and real number $\ell > \frac{1}{2}$, let the initial data $u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y)$ satisfy the compatibility conditions of the initial boundary value problem (1.2) *up to order $k + 4$. Assume the following two conditions:*

(i) The monotonicity condition (1.3) *holds for u_0^s, and*

$$\begin{cases} \partial_y^{2j} u_0^s(0) = 0, \qquad \forall\ 0 \leq j \leq k + 4, \\ \|\langle y\rangle^\ell (u_0^s - 1)\|_{H^{2k+9}(\mathbb{R}^2_+)} + \|\frac{\partial_y^2 u_0^s}{\partial_y u_0^s}\|_{H^{2k+7}(\mathbb{R}^2_+)} \leq C\,, \end{cases} \tag{1.4}$$

for a fixed constant $C > 0$.

(ii) There exists a small constant $\epsilon > 0$ depending only on u_0^s, such that

$$\|\tilde{u}_0\|_{\mathcal{A}^{2k+9}_\ell(\mathbb{R}^2_+)} + \|\frac{\partial_y \tilde{u}_0}{\partial_y u_0^s}\|_{\mathcal{A}^{2k+9}_\ell(\mathbb{R}^2_+)} \leq \epsilon. \tag{1.5}$$

Then there is $T > 0$, such that the problem (1.2) *admits a classical solution u satisfying*

$$u - u^s \in \mathcal{A}^k_\ell(\Omega_T),\ \frac{\partial_y(u-u^s)}{\partial_y u^s_0} \in \mathcal{A}^k_\ell(\Omega_T),\ v \in \mathcal{D}^{k-1}_0(\Omega_T). \tag{1.6}$$

Here, note that $(u^s(t,y),0)$ is the shear flow of the Prandtl equation defined by the initial data $(u^s_0(y),0)$.

(2) The solution is unique in the function space described in (1.6)*. Moreover, we have the stability with respect to the initial data in the following sense. For any given two initial data*

$$u^1_0 = u^s_0 + \tilde{u}^1_0, \quad u^2_0 = u^s_0 + \tilde{u}^2_0,$$

if u^s_0 satisfy (1.3), (1.4)*, and $\tilde{u}^1_0, \tilde{u}^2_0$ satisfy* (2.5)*, then the corresponding solutions (u^1, v^1) and (u^2, v^2) of* (1.2) *satisfy*

$$\|u^1 - u^2\|_{\mathcal{A}^p_\ell(\Omega_T)} + \|v^1 - v^2\|_{\mathcal{D}^{p-1}_0(\Omega_T)} \le C \|\frac{\partial}{\partial y}\left(\frac{u^1_0 - u^2_0}{\partial_y u^s_0}\right)\|_{\mathcal{A}^p_\ell(\mathbb{R}^2_+)},$$

for all $p \le k-1$, where the constant $C > 0$ depends only on T and the upper bounds of the norms of u^1_0, u^2_0.

Remark 1.2. *Note that the solutions obtained in the above theorem are less regular than the initial data mainly due to the degeneracy with respect to the tangential variable x of the Prandtl equation.*

Remark 1.3. *(1) It is not difficulty to see from the proof of Theorem 1.1 that the above main result holds also for the problem* (1.2) *defined in the torus $\mathbb{T}^1$ for the $x-$variable.*

(2) It can be seen by using our approach that the above well-posedness result can be generalized to the nonlinear problem (1.1) *with a non-trivial Euler outflow. For example, for a smooth positive tangential velocity $U(t,x)$ of the outflow, if the monotonic initial data $u_0(x,y)$ converges to $U(0,x)$ exponentially fast as $y \to +\infty$, and there is $\alpha > 0$ such that $u_0(x,y) - U(0,x)(1-e^{-\alpha y})$ is small in some weighted Sobolev spaces, then a similar local well-posedness result holds for the nonlinear Prandtl equation* (1.1)*, with the role of u^s_0 being replaced by $U(0,x)(1-e^{-\alpha y})$.*

The rest of the paper is organized as follows. In Section 2, we will introduce some weighted non-isotropic Sobolev spaces which will be used later, and give the properties of the monotonic shear flow produced by the initial data $(u^s_0(y),0)$. Then, in Section 3, we will study the well-posedness of the linearized problem of the Prandtl equation (1.2) in the Sobolev spaces by a direct energy method, when the background tangential velocity is monotonic in the normal variable. Again, note that without using the Crocco transformation, the approach introduced here is completely new and will have further applications. The proof of Theorem 1.1 will be presented in Section 4, Section 5 and Section 6. As we mentioned earlier, from the energy estimates obtained in Section 3 for the linearized Prandtl equation, there is a loss of regularity of solutions with respect to the background state and the initial data. For this, we apply the Nash-Moser-Hörmander iteration scheme to study the nonlinear Prandtl equation. In Section 4, we will first construct the iteration scheme for the problem (1.2), then in Section 5 we will prove the convergence of this iteration scheme by a series of estimates and then conclude the existence of solutions to the Prandtl equation. The uniqueness and stability of the solution will

be proved in Section 6. Finally, Section 7 is devoted to the proof of several technical estimates used in Section 5.

2. Preliminary

As a preparation, in this section we introduce some functions spaces which will be used later. And the properties of a monotonic shear flow will also be given.

2.1. **Weighted non-isotropic Sobolev spaces.** Since the Prandtl equation given in (1.2) is a degenerate parabolic equation coupled with the divergence-free condition, it is natural to work in some weighted anisotropic Sobolev spaces.

In addition to the spaces $\mathcal{A}^k_\ell$ and $\mathcal{D}^k_\ell$ introduced already in Section 1, we also need the following function spaces.

Denote by $\partial^k_{\mathcal{T}}$ the summation of tangential derivatives $\partial^\beta_{\mathcal{T}} = \partial_t^{\beta_0}\partial_x^{\beta_1}$ for all $\beta = (\beta_0, \beta_1) \in \mathbb{N}^2$ with $|\beta| \le k$. Recall $\langle y\rangle = (1+|y|^2)^{1/2}$. For any given $k, k_1, k_2 \in \mathbb{N}, \lambda \ge 0, \ell \in \mathbb{R}$ and $0 < T < +\infty$, we introduce

$$\|f\|_{\mathcal{B}^{k_1,k_2}_{\lambda,\ell}} = \Big(\sum_{0\le m\le k_1, 0\le q\le k_2} \|e^{-\lambda t}\langle y\rangle^\ell \partial^m_{\mathcal{T}}\partial^q_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\Big)^{1/2},$$

$$\|f\|_{\tilde{\mathcal{B}}^{k_1,k_2}_{\lambda,\ell}} = \Big(\sum_{0\le m\le k_1, 0\le q\le k_2} \|e^{-\lambda t}\langle y\rangle^\ell \partial^m_{\mathcal{T}}\partial^q_y f\|^2_{L^\infty([0,T];L^2(\mathbb{R}^2_+))}\Big)^{1/2},$$

and

$$\|u\|_{\mathcal{C}^k_\ell} = \sum_{k_1+[\frac{k_2+1}{2}]\le k} \|\langle y\rangle^\ell \partial^{k_1}_{\mathcal{T}}\partial^{k_2}_y u\|_{L^2_y(L^\infty_{t,x})}.$$

For the space $\mathcal{A}^k_\ell$ introduced in Section 1, obviously, we have

$$\mathcal{A}^k_\ell = \bigcap_{j=0}^k \mathcal{B}^{k-j,2j}_{0,\ell}.$$

As mentioned in Section 1, when the function is independent of t (or x) variable, we use the same notations for the non-isotropic norms defined above under the convention that we do not take integration or supremum with respect to this variable. Note that the parameter λ is associated to the variable t and the parameter ℓ to the variable y.

The homogeneous norms $\|\cdot\|_{\dot{\mathcal{A}}^k_\ell}, \|\cdot\|_{\dot{\mathcal{C}}^k_\ell}, \|\cdot\|_{\dot{\mathcal{D}}^k_\ell}$ correspond to the summation $1 \le k_1 + [\frac{k_2+1}{2}] \le k$ in the definitions.

For $1 \le p \le +\infty$, we will also use $\|f\|_{L^p_\ell(\mathbb{R}^2_+)} = \|\langle y\rangle^\ell f\|_{L^p(\mathbb{R}^2_+)}$.

By classical theory, it is easy to get the following Sobolev type embeddings

$$\|u\|_{\mathcal{C}^k_\ell} \le C_s\|u\|_{\mathcal{A}^{k+2}_\ell}, \qquad \|u\|_{\mathcal{D}^k_\ell} \le C_s\|u\|_{\mathcal{A}^{k+1}_\ell}. \tag{2.1}$$

Moreover, for any $\ell \ge 0$ and $k \ge 2$, the space $\mathcal{A}^k_\ell$ is continuously embedded into C^{k-2}_b, the space of $(k-2)-$th order continuously differentiable functions with all derivatives being bounded.

In the following, we will use some Morse-type inequalities for the above four function spaces, which are consequences of interpolation inequality and the fact that the space $L^2 \cap L^\infty$ is an algebra.

Lemma 2.1. *For any proper functions f and g, we have*

$$\|f\,g\|_{\mathcal{A}^k_\ell} \le M_k\{\|f\|_{\mathcal{A}^k_\ell}\|g\|_{L^\infty} + \|f\|_{L^\infty}\,\|g\|_{\dot{\mathcal{A}}^k_\ell}\}\,,$$

and

$$\|f\,g\|_{\mathcal{A}^k_\ell} \le M_k\{\|f\|_{\mathcal{C}^k_\ell}\|g\|_{\mathcal{D}^0_0} + \|f\|_{\mathcal{C}^0_\ell}\|g\|_{\dot{\mathcal{D}}^k_0}\}\,.$$

Similar inequalities hold for the norms $\|\cdot\|_{\mathcal{B}^{k_1,k_2}_\ell}, \|\cdot\|_{\mathcal{C}^k_\ell}$ and $\|\cdot\|_{\mathcal{D}^k_\ell}$. Here, $M_k > 0$ is a constant depending only on k.

This result can be obtained in a way similar to that given in [15].

2.2. **Properties of a monotonic shear flow.** Let $u^s(t,y)$ be the solution of the following initial boundary value problem

$$\begin{cases} \partial_t u^s = \partial_y^2 u^s, \quad t>0, \quad y>0, \\ u^s|_{y=0} = 0, \quad \lim\limits_{y\to+\infty} u^s(t,y) = 1, \quad t>0, \\ u^s|_{t=0} = u_0^s(y), \quad y>0. \end{cases} \tag{2.2}$$

Note that $(u^s(t,y),0)$ is a shear flow for the Prandtl equation (1.2).

Proposition 2.2. *For any fixed $k \ge 2$, assume that the initial data $u_0^s(y)$ satisfies the monotonicity condition*

$$\partial_y u_0^s(y) > 0, \quad \forall y \ge 0\,,$$

and the compatibility conditions for (2.2) up to order k, i.e.

$$\lim_{y\to+\infty} u_0^s(y) = 1, \qquad \partial_y^{2j} u_0^s(0) = 0, \quad \forall\ 0 \le j \le k.$$

Moreover, assume that

$$\|u_0^s - 1\|_{L^2(\mathbb{R}_+)} + \|u_0^s\|_{L^\infty(\mathbb{R}_+)} + \|u_0^s\|_{\dot{\mathcal{C}}^k_\ell} + \|\frac{\partial_y^2 u_0^s}{\partial_y u_0^s}\|_{\mathcal{C}^{k-1}_0} \le C\,,$$

for a constant $C>0$, and a fixed $\ell > 0$. Then, we have

$$\partial_y u^s(t,y) > 0, \quad \forall\ t,y \ge 0, \tag{2.3}$$

for any fixed $T>0$, there is a constant $C(T)$ such that

$$\|u^s\|_{L^\infty([0,T]\times\mathbb{R}_+)} + \|u^s\|_{\dot{\mathcal{C}}^k_\ell} + \|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^{k-1}_0} \le C(T). \tag{2.4}$$

Moreover, for a fixed $0 < R_0 < T$, there is a constant $C(T,R_0) > 0$ such that

$$\max_{0\le\bar{y}\le R_0} \|\frac{\partial_y u^s(t,y+\bar{y})}{\partial_y u^s(t,y)}\|_{\mathcal{C}^{k-1}_0([0,T]\times\mathbb{R}^+_y)} \le C(T,R_0), \tag{2.5}$$

and

$$\max_{0\le\bar{t}\le R_0} \|\frac{\partial_y u^s(t+\bar{t},y)}{\partial_y u^s(t,y)}\|_{\mathcal{C}^{k-2}_0([0,T-R_0]\times\mathbb{R}^+_y)} \le C(T,R_0). \tag{2.6}$$

Proof. Obviously, the solution of (2.2) can be written as

$$\begin{aligned} u^s(t,y) &= \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde{y})^2}{4t}} - e^{-\frac{(y+\tilde{y})^2}{4t}}\Big)u_0^s(\tilde{y})d\tilde{y} \\ &= \frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2}u_0^s(2\sqrt{t}\xi+y)d\xi - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2}u_0^s(2\sqrt{t}\xi-y)d\xi\Big)\,, \end{aligned}$$

which gives

$$\partial_t u^s(t,y) = \frac{1}{\sqrt{\pi t}}\Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty} \xi\, e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi + y) d\xi - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} \xi\, e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi - y) d\xi \Big).$$

By using $\partial_y^{2j} u_0^s(0) = 0$ for $0 \le j \le k$, it follows

$$\partial_y^p u^s(t,y) = \frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y^p u_0^s)(2\sqrt{t}\xi + y) d\xi + (-1)^{p+1} \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y^p u_0^s)(2\sqrt{t}\xi - y) d\xi \Big),$$

for all $1 \le p \le 2k$, from which one immediately deduces the monotonicity property (2.3).

To estimate the last term given in (2.4), denote

$$\alpha(t,y) = \frac{\partial_y^2 u^s}{\partial_y u^s}(t,y).$$

Then, from (2.2), we know that $\alpha(t,y)$ satisfies the following initial boundary value problem for the Burgers equation,

$$\begin{cases} \partial_t \alpha = \partial_y^2 \alpha + 2\alpha \partial_y \alpha, \\ \alpha|_{y=0} = 0, \quad t > 0, \\ \alpha|_{t=0} = \alpha^0(y) := \frac{\partial_y^2 u_0^s}{\partial_y u_0^s}(y), \quad y > 0. \end{cases} \tag{2.7}$$

It is easy to verify that the compatibility conditions of (2.7) hold up to order $k-1$, the estimates on $\|\alpha\|_{\mathcal{C}_0^{k-1}}$ can be obtained by standard energy method after an odd extension of the initial data to the whole $\mathbb{R}$.

To prove (2.5), note that

$$\frac{\partial_y u^s(t, y+\bar{y})}{\partial_y u^s(t,y)} = \frac{\partial_y u^s(t,y+\bar{y}) - \partial_y u^s(t,y)}{\partial_y u^s(t,y)} + 1, \tag{2.8}$$

and

$$\partial_y u^s(t,y+\bar{y}) - \partial_y u^s(t,y) = \int_0^{\bar{y}} \partial_y^2 u^s(t,y+z) dz.$$

Hence, from (2.8), we have by using (2.4) that

$$\| \frac{\partial_y u^s(t,y+\bar{y})}{\partial_y u^s(t,y)} \|_{\mathcal{C}_0^{k-1}([0,T]\times\mathbb{R}_y^+)} \le 1 + C \int_0^{\bar{y}} \| \frac{\partial_y u^s(t,y+z)}{\partial_y u^s(t,y)} \|_{\mathcal{C}_0^{k-1}([0,T]\times\mathbb{R}_y^+)} dz. \tag{2.9}$$

By applying the Gronwall inequality to (2.9), it follows

$$\| \frac{\partial_y u^s(t,y+\bar{y})}{\partial_y u^s(t,y)} \|_{\mathcal{C}_0^{k-1}([0,T]\times\mathbb{R}_y^+)} \le e^{C\bar{y}}.$$

The estimate (2.6) can be proved similarly by noting that $\partial_t u^s = \partial_y^2 u^s$.

□

3. Well-posedness of the linearized Prandtl equation

In this section, we study the well-posedness of a linearized problem of the Prandtl equation (1.2) in the Sobolev spaces by the energy method, when the background tangential velocity is monotonic in the normal variable. Again, the main novelty here is that unlike most of the previous works, the Crocco transformation will not be used.

In the estimates on the solutions to the linearized problem, we will see that there is a loss of regularity with respect to both the source term and the background state. And this inspires us to use the Nash-Moser-Hörmander iteration scheme to study the nonlinear Prandtl equation in next section.

Let $(\tilde{u},\tilde{v})$ be a smooth background state satisfying

$$\partial_y\tilde{u}(t,x,y)>0, \qquad \partial_x\tilde{u}+\partial_y\tilde{v}=0,$$

and other conditions that will be specified later. Consider the following linearized problem of (1.2) around $(\tilde{u},\tilde{v})$,

$$\begin{cases} \partial_t u+\tilde{u}\partial_x u+\tilde{v}\partial_y u+u\partial_x\tilde{u}+v\partial_y\tilde{u}-\partial_y^2 u=f,\\ \partial_x u+\partial_y v=0,\\ u|_{y=0}=v|_{y=0}=0, \quad \lim\limits_{y\to+\infty}u(t,x,y)=0,\\ u|_{t\le 0}=0\,. \end{cases} \tag{3.1}$$

Unlike using the Crocco transformation, our main idea is to rewrite the problem of (3.1) into a degenerate parabolic equation with an integral term without changing the independent variables, for which we can perform the energy estimates directly. For this purpose, we introduce the following change of unknown function:

$$w(t,x,y)=\left(\frac{u}{\partial_y\tilde{u}}\right)_y(t,x,y), \quad \text{that is,} \quad u(t,x,y)=(\partial_y\tilde{u})\int_0^y w(t,x,\tilde{y})d\tilde{y}.$$

By a direct calculation, we get that for classical solutions, the problem (3.1) is equivalent to

$$\begin{cases} \partial_t w+\partial_x(\tilde{u}w)+\partial_y(\tilde{v}w)-2\partial_y(\eta w)\\ \qquad\qquad +\partial_y(\zeta\int_0^y w(t,x,\tilde{y})d\tilde{y})-\partial_y^2 w=\partial_y\tilde{f},\\ \big(\partial_y w+2\eta w\big)|_{y=0}=-\tilde{f}|_{y=0},\\ w|_{t\le 0}=0, \end{cases} \tag{3.2}$$

where

$$\eta=\frac{\partial_y^2\tilde{u}}{\partial_y\tilde{u}}, \quad \zeta=\frac{\big(\partial_t+\tilde{u}\partial_x+\tilde{v}\partial_y-\partial_y^2\big)\partial_y\tilde{u}}{\partial_y\tilde{u}}, \quad \tilde{f}=\frac{f}{\partial_y\tilde{u}}.$$

To simplify the notations in the estimates on solutions to the problem (3.2), denote

$$\begin{aligned} \lambda_{k_1,k_2} \;=\;& \|\tilde{u}-u^s\|_{\mathcal{B}^{k_1,k_2}_{0,0}}+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}u^s\|_{L^2_{t,y}}+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\tilde{v}\|_{L^\infty_y(L^2_{t,x})}\\ &+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\overline{\eta}\|_{L^2_y(L^\infty_{t,x})}+\|\eta-\overline{\eta}\|_{\mathcal{B}^{k_1,k_2}_{0,0}}+\|\zeta\|_{\mathcal{B}^{k_1,k_2}_{0,\ell}}, \end{aligned}$$

and

$$\lambda_k=\sum_{k_1+[\frac{k_2+1}{2}]\le k}\lambda_{k_1,k_2},$$

where

$$\overline{\eta}=\frac{\partial_y^2 u^s}{\partial_y\tilde{u}}.$$

The following estimate for the problem (3.2) can be obtained by a direct energy method.

Theorem 3.1. *For a given positive integer k, suppose that the compatibility conditions of the problem* (3.2) *hold up to order k. Then for any fixed $\ell > 1/2$, we have the following estimate*

$$\|w\|_{\mathcal{A}^k_\ell} \le C_1(\lambda_3)\|\tilde{f}\|_{\mathcal{A}^k_\ell} + C_2(\lambda_3)\lambda_k\|\tilde{f}\|_{\mathcal{A}^3_\ell}, \tag{3.3}$$

where $C_1(\lambda_3), C_2(\lambda_3)$ are polynomials of λ_3 of order less or equal to k.

Remark 3.2. *(1) It is easy to see that the compatibility conditions for the problem* (3.2) *up to order k follow immediately from the corresponding conditions of the problem* (3.1).

(2) From the estimate (3.3)*, one can easily deduce the estimates on the solution (u, v) to the problem* (3.1) *in some weighted Sobolev spaces. Hence, from these estimates we can obtain the well-posedness of the linearized Prandtl equation in the Sobolev spaces.*

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.3. (L^2-estimate) *Under the assumptions of Theorem 3.1, for any fixed $T > 0$, there is a constant $C(T) > 0$ such that*

$$\|w\|^2_{\tilde{\mathcal{B}}^{0,0}_{\lambda,\ell}} + \lambda\|w\|^2_{\mathcal{B}^{0,0}_{\lambda,\ell}} + \|\partial_y w\|^2_{\mathcal{B}^{0,0}_{\lambda,\ell}} \le C(T)\|\tilde{f}\|^2_{\mathcal{B}^{0,0}_{\lambda,\ell}}.$$

Proof. Multiplying (3.2) by $e^{-2\lambda t}\langle y\rangle^{2\ell} w$ and integrating over $\mathbb{R}^2_+$, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t\|e^{-\lambda t}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + \lambda\|e^{-\lambda t}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + \|e^{-\lambda t}\partial_y w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)}\\
&\le \Big(2\ell + 2\|\eta\|_{L^\infty} + \|\zeta\|_{L^\infty(\mathbb{R}_x; L^2_\ell(\mathbb{R}_{+,y}))}\Big)\|e^{-\lambda t}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\\
&+ \ell\|\tilde{v}\|_{L^\infty_{-1}}\|e^{-\lambda t}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + 2\ell\|e^{-\lambda t}\tilde{f}(t)\|_{L^2_{\ell-1}(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\\
&+ \|e^{-\lambda t}\tilde{f}(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)},
\end{aligned}$$

by using the boundary condition given in (3.2).

Using the classical Sobolev embedding theorem, it follows

$$4\Big(2\ell + 2\|\eta\|_{L^\infty} + \|\zeta\|_{L^\infty_{t,x}(L^2_{\ell,y})}\Big)^2 + 2\ell\|\tilde{v}\|_{L^\infty} \le (4\ell(1+\lambda_{3,0}))^2.$$

Thus, by taking $\lambda \ge (4\ell(1+\lambda_{3,0}))^2$, we get

$$\begin{aligned}
\partial_t\|e^{-\lambda t}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + \lambda\|e^{-\lambda t}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + \|e^{-\lambda t}\partial_y w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)}\\
\le 4\|e^{-\lambda t}\tilde{f}(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)},
\end{aligned}$$

which implies

$$\|w\|^2_{\tilde{\mathcal{B}}^{0,0}_{\lambda,\ell}} + \lambda\|w\|^2_{\mathcal{B}^{0,0}_{\lambda,\ell}} + \|\partial_y w\|^2_{\mathcal{B}^{0,0}_{\lambda,\ell}} \le C(T)\|\tilde{f}\|^2_{\mathcal{B}^{0,0}_{\lambda,\ell}},$$

for all fixed $0 < T < +\infty$. □

Lemma 3.4. (Energy estimate for tangential derivatives) *Under the assumptions of Theorem 3.1, for any fixed $T > 0$, there is a constant $C(T) > 0$ such*

that

$$
\begin{aligned}
\|w\|^2_{\tilde{\mathcal{B}}^{k,0}_{\lambda,\ell}} + \lambda\|w\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}} + \|\partial_y w\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}} \le C(T)\bigg(&\|\tilde{f}\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}} \\
&+\Big(\|\zeta\|^2_{\mathcal{B}^{k,0}_{0,\ell}} + \|\partial^k_{\mathcal{T}}\tilde{v}\|^2_{L^2_y(L^2_{t,x})} + \|\partial^k_{\mathcal{T}}\overline{\eta}\|^2_{L^2_y(L^\infty_{t,x})}\Big)\|w\|^2_{\mathcal{B}^{2,1}_{\lambda,\ell}} \\
&+\Big(\|\tilde{u}-u^s\|^2_{\mathcal{B}^{k,0}_{0,0}} + \|\partial^k_{\mathcal{T}}u^s\|_{L^2_{t,y}} + \|\eta-\overline{\eta}\|^2_{\mathcal{B}^{k,0}_{0,0}}\Big)\|w\|^2_{\mathcal{B}^{3,1}_{\lambda,\ell}}\bigg).
\end{aligned}
\tag{3.4}
$$

Proof. Taking the differentiation $\partial^\beta_{\mathcal{T}}$ ($|\beta| \le k$) on the equation in (3.2), multiplying it by $e^{-2\lambda t}\langle y\rangle^{2\ell}\partial^\beta_{\mathcal{T}}w$ and integrating over $\mathbb{R}^2_+$, as in the proof of Lemma 3.3, for $\lambda > (4\ell(1+\lambda_{3,0}))^2$, we have

$$
\begin{aligned}
&\partial_t\|e^{-\lambda t}\partial^\beta_{\mathcal{T}}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + \lambda\|e^{-\lambda t}\partial^\beta_{\mathcal{T}}w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} + \|e^{-\lambda t}\partial^\beta_{\mathcal{T}}\partial_y w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)} \\
&\qquad\le 4\|e^{-\lambda t}\partial^\beta_{\mathcal{T}}\tilde{f}(t)\|^2_{L^2_\ell} + A_1 + A_2 + A_3,
\end{aligned}
\tag{3.5}
$$

where we have used the compatibility conditions of the problem (3.2) and

$$
\partial^\beta_{\mathcal{T}}\big(\tilde{v}w\big)|_{y=0} = 0,
$$

by noting $\tilde{v}(t,x,y) = -\int_0^y \partial_x\tilde{u}(t,x,\tilde{y})d\tilde{y}$, and A_1, A_2 and A_3 come from the commutators between $\partial^\beta_{\mathcal{T}}$ and the nonlinear terms in (3.2). For brevity, the precise definitions of A_i, $i = 1,2,3$ are given as follows respectively.

Firstly,

$$
\begin{aligned}
A_1 = &\sum_{\beta_1+\beta_2\le\beta;|\beta_2|<|\beta|} C^{\beta_1}_\beta \left|\int_{\mathbb{R}^2_+} e^{-2\lambda t}\langle y\rangle^{2\ell}(\partial^{\beta_1}_{\mathcal{T}}\tilde{u})(\partial^{\beta_2}_{\mathcal{T}}\partial_x w)(\partial^\beta_{\mathcal{T}}w)dxdy\right| \\
&+ \sum_{\beta_1+\beta_2\le\beta;|\beta_2|<|\beta|} C^{\beta_1}_\beta \left|\int_{\mathbb{R}^2_+} e^{-2\lambda t}\langle y\rangle^{2\ell}(\partial^{\beta_1}_{\mathcal{T}}\tilde{v})(\partial^{\beta_2}_{\mathcal{T}}w)(\partial^\beta_{\mathcal{T}}\partial_y w)dxdy\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_1 \lesssim &\|e^{-\lambda t}\partial^\beta_{\mathcal{T}}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\big\{\|\tilde{u}(t)\|_{L^\infty(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial^{|\beta|}_{\mathcal{T}}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)} \\
&+ \|\partial^{|\beta|}_{\mathcal{T}}(\tilde{u}(t)-u^s(t))\|_{L^2(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_x w(t)\|_{L^\infty_\ell(\mathbb{R}^2_+)} \\
&+ \|\partial^{|\beta|}_{\mathcal{T}}u^s(t)\|_{L^2_y}\|e^{-\lambda t}\partial_x w(t)\|_{L^\infty_{y,\ell}(L^2_x)}\big\} \\
&+ \|e^{-\lambda t}\partial^\beta_{\mathcal{T}}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\big\{\|\tilde{v}(t)\|_{L^\infty(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial^{|\beta|-1}_{\mathcal{T}}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)} \\
&+ \|\partial^{|\beta|}_{\mathcal{T}}\tilde{v}(t)\|_{L^\infty_y(L^2_x)}\|e^{-\lambda t}w(t)\|_{L^2_{y,\ell}(L^\infty_x)}\big\},
\end{aligned}
$$

where we have used Lemma 2.1. Here and in the sequel, for simplicity, we shall use the notation $A \lesssim B$ when there exists a generic positive constant C such that $A \le CB$. Similar definition holds for $A \gtrsim B$.

Secondly, for

$$
A_2 = \left|\int_{\mathbb{R}^2_+} e^{-2\lambda}\langle y\rangle^{2\ell}\partial^\beta_{\mathcal{T}}(\eta w)(\partial^\beta_{\mathcal{T}}\partial_y w)dxdy\right|,
$$

we have

$$\begin{aligned}A_2 \lesssim & \|\eta(t)\|_{L^\infty(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{|\beta|}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{\beta}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\\ &+\|\partial_{\mathcal{T}}^{|\beta|}(\eta-\overline{\eta})(t)\|_{L^2(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|_{L^\infty_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{\beta}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\\ &+\|\partial_{\mathcal{T}}^{|\beta|}\overline{\eta}(t)\|_{L^\infty(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{\beta}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}.\end{aligned}$$

Finally, for

$$A_3=\left|\int_{\mathbb{R}^2_+} e^{-2\lambda}\langle y\rangle^{2\ell}\partial_{\mathcal{T}}^{\beta}\Big(\zeta\int_0^y w d\tilde{y}\Big)(\partial_{\mathcal{T}}^{\beta}\partial_y w)dxdy\right|,$$

we get for $\ell>1/2$,

$$\begin{aligned}A_3 \lesssim & \|\zeta(t)\|_{L^2_{y,\ell}(L^\infty_x)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{|\beta|}w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{\beta}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\\ &+\|\partial_{\mathcal{T}}^{|\beta|}\zeta(t)\|_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|_{L^2_{y,\ell}(L^\infty_x)}\|e^{-\lambda t}\partial_{\mathcal{T}}^{\beta}\partial_y w(t)\|_{L^2_\ell(\mathbb{R}^2_+)}.\end{aligned}$$

Substituting these estimates of A_1, A_2, A_3 into (3.5), and taking summation over all $|\beta|\le k$, it follows

$$\begin{aligned}&\partial_t\|e^{-\lambda t}\partial_{\mathcal{T}}^k w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)}+\lambda\|e^{-\lambda t}\partial_{\mathcal{T}}^k w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)}+\|e^{-\lambda t}\partial_{\mathcal{T}}^k\partial_y w(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)}\\ &\lesssim \|e^{-\lambda t}\partial_{\mathcal{T}}^k\tilde{f}(t)\|^2_{L^2_\ell}+\|\partial_{\mathcal{T}}^k(\tilde{u}-u^s)\|^2_{L^2(\mathbb{R}^2_+)}\|e^{-\lambda t}\partial_x w(t)\|^2_{L^\infty_\ell(\mathbb{R}^2_+)}\\ &\quad+\|\partial_{\mathcal{T}}^k u^s\|^2_{L^2_y}\|e^{-\lambda t}\partial_x w(t)\|^2_{L^\infty_{y,\ell}(L^2_x)}+\|\partial_{\mathcal{T}}^k\tilde{v}\|^2_{L^\infty_y(L^2_x)}\|e^{-\lambda t}w(t)\|^2_{L^2_{y,\ell}(L^\infty_x)}\\ &\quad+\|\partial_{\mathcal{T}}^k\overline{\eta}(t)\|^2_{L^2_y(L^\infty_x)}\|e^{-\lambda t}w(t)\|^2_{L^\infty_{y,\ell}(L^2_x)}+\|\partial_{\mathcal{T}}^k(\eta-\overline{\eta})(t)\|^2_{L^2(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|^2_{L^\infty_\ell(\mathbb{R}^2_+)}\\ &\quad+\|\partial_{\mathcal{T}}^k\zeta(t)\|^2_{L^2_\ell(\mathbb{R}^2_+)}\|e^{-\lambda t}w(t)\|^2_{L^2_{y,\ell}(L^\infty_x)},\end{aligned}$$

for any $\lambda>(4\ell(1+\lambda_{3,0}))^2$. Integrating the above inequality on $[0,T]$, and using the compatibility condition

$$(\partial^k_{t,x,y}w)|_{t=0}=0,$$

we get for any fixed $T>0$,

$$\begin{aligned}&\|w\|^2_{\tilde{\mathcal{B}}^{k,0}_{\lambda,\ell}}+\lambda\|w\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}}+\|\partial_y w\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}}\le C(T)\Big\{\|\tilde{f}\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}}\\ &+\Big(\|\zeta\|^2_{\mathcal{B}^{k,0}_{0,\ell}}+\|\partial^k_{\mathcal{T}}\tilde{v}\|^2_{L^2([0,T]\times\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\Big)\|e^{-\lambda t}w\|^2_{L^\infty([0,T]\times\mathbb{R}_x;L^2_{y,\ell}(\mathbb{R}_+))}\\ &+\|\tilde{u}-u^s\|^2_{\mathcal{B}^{k,0}_{0,0}}\|e^{-\lambda t}\partial_x w\|^2_{L^\infty_\ell(\mathbb{R}^2_+)}+\|\partial^k_{\mathcal{T}}u^s\|^2_{L^2_{t,y}}\|e^{-\lambda t}\partial_x w\|^2_{L^\infty_{y,\ell}(L^\infty([0,T],L^2_x(\mathbb{R})))}\\ &+\|\partial^k_{\mathcal{T}}\overline{\eta}\|^2_{L^2_y(L^\infty_{t,x})}\|e^{-\lambda t}w\|^2_{L^\infty_{y,\ell}(L^2_{t,x})}\\ &+\|\eta-\overline{\eta}\|^2_{\mathcal{B}^{k,0}_{0,0}}\|e^{-\lambda t}w\|^2_{L^\infty_\ell([0,T]\times\mathbb{R}^2_+)}\Big\}.\end{aligned}$$

Using the Sobolev embedding theorem in the above inequality, the estimate (3.4) follows immediately. □

Remark 3.5. *The estimate* (3.4) *implies*

$$\|w\|^2_{\mathcal{B}^{k,1}_{\lambda,\ell}}\lesssim\|\tilde{f}\|^2_{\mathcal{B}^{k,0}_{\lambda,\ell}}+\lambda^2_{k,0}\|w\|^2_{\mathcal{B}^{3,1}_{\lambda,\ell}}. \tag{3.6}$$

Moreover, using the same argument as in the above proof together with Lemma 3.3, when $\lambda>(4\ell(1+\lambda_{3,0}))^2$, *we can obtain*

$$\|w\|_{\mathcal{B}^{k,1}_{\lambda,\ell}}\lesssim\|\tilde{f}\|_{\mathcal{B}^{k,0}_{\lambda,\ell}},\qquad 0\le k\le 3. \tag{3.7}$$

Proof of Theorem 3.1: Recall from (3.2) that

$$\partial_y^2 w = \partial_t w + \tilde{u}\partial_x w + \tilde{v}\partial_y w - 2\partial_y(\eta w) + \partial_y\Big(\zeta\int_0^y w(t,x,\tilde{y})d\tilde{y}\Big) - \partial_y\tilde{f}. \tag{3.8}$$

By applying $\partial_{\mathcal{T}}^{k_1}$ to this equation and using Lemma 2.1, we get

$$\begin{aligned}
\|w\|_{\mathcal{B}^{k_1,2}_{\lambda,\ell}} &\le \|w\|_{\mathcal{B}^{k_1+1,0}_{\lambda,\ell}} + \|\tilde{u}\partial_x w\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \|\tilde{v}\partial_y w\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + 2\|\eta w\|_{\mathcal{B}^{k_1,1}_{\lambda,\ell}}\\
&\quad + \|(\partial_y\zeta)\int_0^y w(t,x,\tilde{y})d\tilde{y}\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \|\zeta w\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \|\partial_y\tilde{f}\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}}\\
&\lesssim (1+\|\tilde{u}\|_{L^\infty})\|w\|_{\mathcal{B}^{k_1+1,0}_{\lambda,\ell}} + (\|\tilde{v}\|_{L^\infty}+\|\eta\|_{L^\infty})\|w\|_{\mathcal{B}^{k_1,1}_{\lambda,\ell}}\\
&\quad + \|\zeta\|_{L^\infty}\|w\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \|\tilde{u}-u^s\|_{\mathcal{B}^{k_1,0}_{0,0}}\|\partial_x w\|_{L^\infty_{\lambda,\ell}}\\
&\quad + \|\partial_{\mathcal{T}}^{k_1}u^s\|_{L^2_y(L^\infty_t)}\|\partial_x w\|_{L^\infty_{y,\ell}(L^2_{t,x,\lambda})} + \|\partial_{\mathcal{T}}^{k_1}\tilde{v}\|_{L^\infty_y(L^2_{t,x})}\|\partial_y w\|_{L^2_{y,\ell}(L^\infty_{t,x,\lambda})}\\
&\quad + \|\partial_{\mathcal{T}}^{k_1}\partial_y\overline{\eta}\|_{L^2_y(L^\infty_{t,x})}\|w\|_{L^\infty_{y,\ell}(L^2_{t,x,\lambda})} + \|\eta-\overline{\eta}\|_{\mathcal{B}^{k_1,1}_{0,0}}\|w\|_{L^\infty_{\lambda,\ell}}\\
&\quad + \|\partial_y\tilde{f}\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \|\partial_y\zeta\|_{L^\infty_{t,x}(L^2_\ell)}\|w\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \|\zeta\|_{\mathcal{B}^{k_1,1}_{0,\ell}}\|w\|_{L^\infty_{t,x}(L^2_{\lambda,\ell})}.
\end{aligned}$$

Then by using the Sobolev embedding theorem, it follows that

$$\|w\|_{\mathcal{B}^{k_1,2}_{\lambda,\ell}} \lesssim \lambda_{2,1}(\|w\|_{\mathcal{B}^{k_1+1,0}_{\lambda,\ell}} + \|w\|_{\mathcal{B}^{k_1,1}_{\lambda,\ell}}) + \|\partial_y\tilde{f}\|_{\mathcal{B}^{k_1,0}_{\lambda,\ell}} + \lambda_{k_1,1}\|w\|_{\mathcal{B}^{3,1}_{\lambda,\ell}}.$$

And from (3.6) and (3.7), we get

$$\|w\|_{\mathcal{B}^{k_1,2}_{\lambda,\ell}} \lesssim \lambda_{2,1}\Big(\|\tilde{f}\|_{\mathcal{B}^{k_1+1,0}_{\lambda,\ell}} + \|\tilde{f}\|_{\mathcal{B}^{k_1,1}_{\lambda,\ell}}\Big) + (\lambda_{2,1}\lambda_{k_1+1,0} + \lambda_{k_1,1})\|\tilde{f}\|_{\mathcal{B}^{3,0}_{\lambda,\ell}}.$$

For $k_2 > 2$, differentiating the equation (3.8) by $\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2-2}$, we have

$$\begin{aligned}
\|w\|_{\mathcal{B}^{k_1,k_2}_{\lambda,\ell}} &\le \|w\|_{\mathcal{B}^{k_1+1,k_2-2}_{\lambda,\ell}} + \|\tilde{u}\partial_x w\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}} + \|\tilde{v}\partial_y w\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}}\\
&\quad + 2\|\eta w\|_{\mathcal{B}^{k_1,k_2-1}_{\lambda,\ell}} + \|(\partial_y\zeta)\int_0^y w(t,x,\tilde{y})d\tilde{y}\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}}\\
&\quad + \|\zeta w\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}} + \|\partial_y\tilde{f}\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}}\\
&\lesssim (1+\|\tilde{u}\|_{L^\infty})\|w\|_{\mathcal{B}^{k_1+1,k_2-2}_{\lambda,\ell}} + (\|\tilde{v}\|_{L^\infty}+\|\eta\|_{L^\infty})\|w\|_{\mathcal{B}^{k_1,k_2-1}_{\lambda,\ell}}\\
&\quad + \|\zeta\|_{L^\infty}\|w\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}} + \|\partial_y\zeta\|_{L^\infty_{t,x}(L^2_\ell)}\|w\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}} + \|\partial_y\tilde{f}\|_{\mathcal{B}^{k_1,k_2-2}_{\lambda,\ell}}\\
&\quad + \|\tilde{u}-u^s\|_{\mathcal{B}^{k_1,k_2-2}_{0,0}}\|\partial_x w\|_{L^\infty_{\lambda,\ell}} + \|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2-2}u^s\|_{L^2_y(L^\infty_t)}\|\partial_x w\|_{L^\infty_{y,\ell}(L^2_{t,x,\lambda})}\\
&\quad + \|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2-2}\tilde{v}\|_{L^\infty_y(L^2_{t,x})}\|\partial_y w\|_{L^2_{y,\ell}(L^\infty_{t,x,\lambda})}\\
&\quad + \|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2-1}\overline{\eta}\|_{L^2_y(L^\infty_{t,x})}\|w\|_{L^\infty_{y,\ell}(L^2_{t,x,\lambda})} + \|\eta-\overline{\eta}\|_{\mathcal{B}^{k_1,k_2-1}_{0,0}}\|w\|_{L^\infty_{\lambda,\ell}}\\
&\quad + \|\zeta\|_{\mathcal{B}^{k_1,k_2-1}_{0,\ell}}\|w\|_{L^\infty_{y,\ell}(L^2_{t,x,\lambda})}.
\end{aligned}$$

Therefore, we get

$$\|w\|_{\mathcal{B}^{k_1,k_2}_{\lambda,\ell}} \lesssim \lambda_{2,1}(\|w\|_{\mathcal{B}^{k_1+1,k_2-2}_{\lambda,\ell}} + \|w\|_{\mathcal{B}^{k_1,k_2-1}_{\lambda,\ell}}) + \lambda_{k_1,k_2-1}\|w\|_{\mathcal{B}^{3,1}_{\lambda,\ell}} + \|\tilde{f}\|_{\mathcal{B}^{k_1,k_2-1}_{\lambda,\ell}},$$

which immediately implies

$$\|w\|_{\mathcal{B}^{k_1,k_2}_{0,\ell}} \lesssim \lambda_{2,1}(\|w\|_{\mathcal{B}^{k_1+1,k_2-2}_{0,\ell}} + \|w\|_{\mathcal{B}^{k_1,k_2-1}_{0,\ell}}) + \lambda_{k_1,k_2-1}\|w\|_{\mathcal{B}^{3,1}_{0,\ell}} + \|\tilde{f}\|_{\mathcal{B}^{k_1,k_2-1}_{0,\ell}},$$

by fixing $\lambda > (4\ell(1+\lambda_{3,0}))^2$.

The proof of Theorem 3.1 can then be completed by induction on k_2.

4. Iteration scheme for the nonlinear Prandtl equation

From the estimate (3.3) given in Theorem 3.1, we see that there is a loss of regularity in the solutions to the linearized Prandtl equation with respect to the source term and the background state. In order to take care of this loss, we are going to apply the Nash-Moser-Hörmander iteration scheme, cf. [1, 10, 13, 16, 17], to study the nonlinear problem (1.2).

4.1. **The smoothing operators.** For a function f defined on $\Omega = [0, +\infty[\times\mathbb{R}_x \times \mathbb{R}^+_y$, let $\tilde{f}$ be its extension to $\mathbb{R}^3$ by 0. Then for a large constant θ, introduce a family of smoothing operators S_θ:

$$(S_\theta f)(t,x,y) = \int \rho_\theta(\tau)\rho_\theta(\xi)\rho_\theta(\eta)\tilde{f}(t-\tau+\theta^{-1}, x-\xi, y-\eta+\theta^{-1})d\tau d\xi d\eta,$$

where $\rho_\theta(\tau) = \theta\,\rho(\theta\tau)$, $\rho \in C_0^\infty(\mathbb{R})$ with Supp $\rho \subseteq [-1,1]$ and $\|\rho\|_{L^1} = 1$. One has

$$\{S_\theta\}_{\theta>0}:\ \mathcal{A}^0_\ell(\Omega) \longrightarrow \cap_{s\geq 0}\mathcal{A}^s_\ell(\Omega),$$

together with

$$\begin{cases} \|S_\theta u\|_{\mathcal{A}^s_\ell} \leq C_\rho \theta^{(s-\alpha)_+}\|u\|_{\mathcal{A}^\alpha_\ell}, \quad \text{for all } s,\alpha \geq 0, \\ \|(1-S_\theta)u\|_{\mathcal{A}^s_\ell} \leq C_\rho \theta^{s-\alpha}\|u\|_{\mathcal{A}^\alpha_\ell}, \quad \text{for all } 0 \leq s \leq \alpha, \end{cases} \tag{4.1}$$

where the constant C_ρ depends only on the function ρ and the orders of differentiations s and α. For the smoothing parameter, we set $\theta_n = \sqrt{\theta_0^2 + n}$ for any $n \geq 1$ and a large fixed constant θ_0. We have also

$$\|(S_{\theta_n} - S_{\theta_{n-1}})u\|_{\mathcal{A}^s_\ell} \leq C_\rho \theta_n^{s-\alpha}\Delta\theta_n\|u\|_{\mathcal{A}^\alpha_\ell}, \quad \text{for all } s,\alpha \geq 0, \tag{4.2}$$

where $\Delta\theta_n = \theta_{n+1} - \theta_n$.

The operator S_θ acting on the other three spaces introduced in Section 2.1 shares the same properties.

The following commutator estimates will be used frequently later,

Lemma 4.1. *For any proper function f, we have*

$$\|[\frac{1}{\partial_y u^s}, S_\theta](\partial_y f)\|_{\mathcal{A}^k_\ell} \leq C_k\|\frac{f}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}, \tag{4.3}$$

and

$$\|\partial_y[\frac{1}{\partial_y u^s}, \partial_y S_\theta]f\|_{\mathcal{A}^k_\ell} \leq C_k\theta\|\frac{f}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}, \tag{4.4}$$

with the constant C_k depends on the constant in (2.4)*. Similar inequalities hold for the norms $\|\cdot\|_{\mathcal{B}^{k_1,k_2}_\ell}, \|\cdot\|_{\mathcal{C}^k_\ell}$ and $\|\cdot\|_{\mathcal{D}^k_\ell}$.*

Proof. This lemma can be proved in a classical way, cf. [3, 12]. To be self-contained, we give a brief proof of the estimate (4.3) here. Note that (4.4) can be proved similarly.

From the definition, we have

$$
\begin{aligned}
&[\tfrac{1}{\partial_y u^s}, S_\theta]b = \tfrac{S_\theta(b)}{\partial_y u^s}(t,x,y) - S_\theta\left(\tfrac{b}{\partial_y u^s}\right)(t,x,y)\\
&= \int \rho_\theta(\tau)\rho_\theta(\xi)\rho_\theta(\eta)\left(\tfrac{\partial_y u^s(t-\tau+\theta^{-1},y-\eta+\theta^{-1})-\partial_y u^s(t,y)}{\partial_y u^s(y)}\right)\\
&\qquad\qquad \times\left(\tfrac{\tilde b}{\partial_y u^s}\right)(t-\tau+\theta^{-1},x-\xi,y-\eta+\theta^{-1})d\tau d\xi d\eta \qquad (4.5)\\
&= \theta^{-1}\int \rho_\theta(\xi)\Big(\tilde\rho_\theta(\tau)\rho_\theta(\eta)a_1(t,\tau,y,\eta,\theta)+\rho_\theta(\tau)\tilde\rho_\theta(\eta)a_2(t,\tau,y,\eta,\theta)\Big)\\
&\qquad\qquad \times\left(\tfrac{\tilde b}{\partial_y u^s}\right)(t-\tau+\theta^{-1},x-\xi,y-\eta+\theta^{-1})d\tau d\xi d\eta,
\end{aligned}
$$

with $\tilde\rho_\theta(\tau)=\theta\,\rho(\theta\,\tau)(1-\theta\,\tau)$, $\tilde\rho_\theta(\eta)=\theta\,\rho(\theta\,\eta)(1-\theta\,\eta)$, and

$$
a_1(t,\tau,y,\eta,\theta)=\int_0^1 \frac{\partial_t\partial_y u^s(t+\lambda(\theta^{-1}-\tau),y+\lambda(\theta^{-1}-\eta))}{\partial_y u^s(t,y)}d\lambda,
$$

and

$$
a_2(t,\tau,y,\eta,\theta)=\int_0^1 \frac{\partial_y^2 u^s(t+\lambda(\theta^{-1}-\tau),y+\lambda(\theta^{-1}-\eta))}{\partial_y u^s(t,y)}d\lambda.
$$

Using (2.5) and (2.6), for $j=1,2$, we have

$$
\sup_{0\le|\tau|,|\eta|\le\theta^{-1}\le R_0}\|a_j(\,\cdot\,,\tau,\,\cdot\,,\eta,\,\theta)\|_{\mathcal{C}_0^{k+1}}\le\tilde C(T). \qquad (4.6)
$$

Thus, from (4.5) we get the estimate (4.3) immediately. □

4.2. **The iteration scheme.** Denote by

$$
\mathcal{P}(u,v)=\partial_t u+u\partial_x u+v\partial_y u-\partial_y^2 u,
$$

the nonlinear operator associated with problem (1.2), and its linearized operator around $(\tilde u,\tilde v)$ by

$$
\mathcal{P}'_{(\tilde u,\tilde v)}(u,v)=\partial_t u+\tilde u\partial_x u+\tilde v\partial_y u+u\partial_x\tilde u+v\partial_y\tilde u-\partial_y^2 u.
$$

In this subsection, we introduce an iteration scheme in order to construct an approximate solution sequence $\{(u^n,v^n)\}$ to the problem (1.2).

For a fixed integer $k\ge 0$, suppose that the initial data in the Prandtl equation (1.2) satisfies the compatibility conditions up to order k, and that (u,v) is a classical solution. If we set $\tilde u=u-u^s$ with $u^s(t,y)$ being the heat profile defined in Section 2.2, then it is easy to see that

$$
\begin{cases}
\partial_t\tilde u+(\tilde u+u^s)\tilde u_x+v\partial_y(\tilde u+u^s)-\tilde u_{yy}=0, \quad (x,y)\in\mathbb{R}^2_+,\ t>0,\\
\partial_x\tilde u+\partial_y v=0,\\
\tilde u|_{y=0}=v|_{y=0}=0, \quad \lim\limits_{y\to+\infty}\tilde u=0,\\
\tilde u|_{t=0}=\tilde u_0(x,y).
\end{cases} \qquad (4.7)
$$

The compatibility conditions for (4.7) follow from those for (1.2) immediately.

The zero-th order approximate solution: Denote

$$
\tilde u_0^j(x,y)=\partial_t^j\tilde u|_{t=0}, \qquad v_0^j(x,y)=\partial_t^j v|_{t=0}.
$$

Then from the compatibility conditions for (4.7), $\{\tilde u_0^j,v_0^j\}_{j\le k}$ are defined directly by $\tilde u_0(x,y)$. We are going to construct the zero-th order approximate solution $(\tilde u^0,v^0)$ of (4.7), such that

$$
\partial_t^j\tilde u^0|_{t=0}=\tilde u_0^j(x,y), \qquad \partial_t^j v^0|_{t=0}=v_0^j(x,y), \quad 0\le j\le k,
$$

and $(u^0, v^0) = (u^s + \tilde{u}^0, v^0)$ satisfying

$$\begin{cases} \partial_x u^0 + \partial_y v^0 = 0, \quad (x,y) \in \mathbb{R}^2_+, \ t \geq 0, \\ u^0|_{y=0} = v^0|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u^0 = 1, \\ u^0|_{t=0} = u_0(x,y)\,. \end{cases} \tag{4.8}$$

Other properties of (u^0, v^0) will be studied in more details in Section 5.1.

The Nash-Moser iteration scheme: Assume that for all $k = 0, \ldots, n$, we have constructed the approximate solutions (u^k, v^k) of (1.2) satisfying the same conditions given in (4.8) for (u^0, v^0). We now construct the $(n+1)$-th approximation solution (u^{n+1}, v^{n+1}) as follows. Set

$$u^{n+1} = u^n + \delta u^n = u^s + \tilde{u}^n + \delta u^n, \qquad v^{n+1} = v^n + \delta v^n\,, \tag{4.9}$$

where the increment $(\delta u^n, \delta v^n)$ is the solution of the following initial-boundary value problem,

$$\begin{cases} \mathcal{P}'_{(u^n_{\theta_n}, v^n_{\theta_n})}(\delta u^n, \delta v^n) = f^n, \\ \partial_x(\delta u^n) + \partial_y(\delta v^n) = 0, \\ \delta u^n|_{y=0} = \delta v^n|_{y=0} = 0, \qquad \lim\limits_{y\to+\infty} \delta u^n = 0, \\ \delta u^n|_{t\leq 0} = 0, \end{cases} \tag{4.10}$$

where $u^n_{\theta_n} = u^s + S_{\theta_n}\tilde{u}^n$ and $v^n_{\theta_n} = S_{\theta_n} v^n$.

Now, we define the source term f^n for the problem (4.10) in order to have the convergence of the approximate solution sequence (u^n, v^n) to the solution of the Prandtl equation (1.2) as n goes to infinity. Obviously, we have the following identity,

$$\mathcal{P}(u^{n+1}, v^{n+1}) - \mathcal{P}(u^n, v^n) = \mathcal{P}'_{(u^n_{\theta_n}, v^n_{\theta_n})}(\delta u^n, \delta v^n) + e_n\,, \tag{4.11}$$

where

$$e_n = e_n^{(1)} + e_n^{(2)}.$$

Here

$$\begin{aligned} e_n^{(1)} &= \mathcal{P}(u^n + \delta u^n, v^n + \delta v^n) - \mathcal{P}(u^n, v^n) - \mathcal{P}'_{(u^n, v^n)}(\delta u^n, \delta v^n) \\ &= \delta u^n \partial_x(\delta u^n) + \delta v^n \partial_y(\delta u^n), \end{aligned}$$

is the error from the Newton iteration scheme, and

$$\begin{aligned} e_n^{(2)} &= \mathcal{P}'_{(u^n, v^n)}(\delta u^n, \delta v^n) - \mathcal{P}'_{(u^n_{\theta_n}, v^n_{\theta_n})}(\delta u^n, \delta v^n) \\ &= \big((1 - S_{\theta_n})(u^n - u^s)\big)\partial_x(\delta u^n) + \delta u^n \partial_x\big((1 - S_{\theta_n})(u^n - u^s)\big) \\ &\quad + \delta v^n \partial_y\big((1 - S_{\theta_n})(u^n - u^s)\big) + \big((1 - S_{\theta_n})v^n\big)\partial_y(\delta u^n), \end{aligned}$$

is the error coming from mollifying the coefficients.

From (4.11), we have

$$\mathcal{P}(u^{n+1}, v^{n+1}) = \sum_{j=0}^{n} (\mathcal{P}'_{(S_{\theta_j} u^j, S_{\theta_j} v^j)}(\delta u^j, \delta v^j) + e_j) + f^a, \tag{4.12}$$

with

$$f^a = \mathcal{P}(u^0, v^0) := \partial_t u^0 + u^0 \partial_x u^0 + v^0 \partial_y u^0 - \partial_y^2 u^0.$$

Note that if the approximate solution (u^n, v^n) converges to a solution of the problem (1.2), then the right hand side in the equation (4.12) should go to zero

when $n \to +\infty$. Thus, it is natural to require that $(\delta u^n, \delta v^n)$ satisfies the following equation for all $n \geq 0$,

$$\mathcal{P}'_{(u^n_{\theta_n}, v^n_{\theta_n})}(\delta u^n, \delta v^n) = f^n,$$

where f^n is defined by

$$\sum_{j=0}^{n} f^j = -S_{\theta_n}(\sum_{j=0}^{n-1} e_j) - S_{\theta_n} f^a,$$

by induction on n. Obviously, we have

$$\begin{cases} f^0 = -S_{\theta_0} f^a, \qquad f^1 = (S_{\theta_0} - S_{\theta_1}) f^a + S_{\theta_1} e_0, \\ f^n = (S_{\theta_{n-1}} - S_{\theta_n})\big(\sum\limits_{j=0}^{n-2} e_j\big) - S_{\theta_n} e_{n-1} + (S_{\theta_{n-1}} - S_{\theta_n}) f^a, \quad \forall n \geq 2. \end{cases} \tag{4.13}$$

5. Existence of the classical solutions

In this section, we study the iteration scheme (4.9)-(4.10) with f^n being given in (4.13), by using the estimate (3.3) given in Theorem 3.1. To do this, let us first state the main assumption (MA) on the initial data $\tilde{u}_0(x, y)$ of (4.7) as follows:

> **(MA)** For any fixed integers $\tilde{k} \geq 7$, $k_0 \geq \tilde{k} + 2$, and a real number $\ell > \frac{1}{2}$, suppose that $\tilde{u}_0 \in \mathcal{A}^{2k_0+1}(\mathbb{R}^2_+)$ satisfies the compatibility conditions for the problem (4.7) up to order k_0, and
>
> $$\|\tilde{u}_0\|_{\mathcal{A}^{2k_0+1}_\ell(\mathbb{R}^2_+)} + \|\frac{\partial_y \tilde{u}_0}{\partial_y u_0^s}\|_{\mathcal{A}^{2k_0+1}_\ell(\mathbb{R}^2_+)} \leq \epsilon,$$
>
> for a small quantity $\epsilon > 0$ depending on the norms of $u_0^s(y)$.

5.1. **The zero-th order approximation.** Let us construct the zero-th order approximate solution (u^0, v^0) satisfying (4.8) to the problem (1.2). As mentioned in Section 4.2, from the equation (4.7) one can easily obtain $\tilde{u}_0^j(x, y) = \partial_t^j \tilde{u}(0, x, y)$ and $v_0^j(x, y) = \partial_t^j v(0, x, y)$ in terms of $\tilde{u}_0(x, y)$ for all $0 \leq j \leq k_0$, and then have the following relations

$$\begin{cases} u_0^j(x, y) = \partial_y^2 u_0^{j-1} - \sum\limits_{k=0}^{j-1} C_{j-1}^k \Big(u_0^k \partial_x u_0^{j-1-k} + v_0^k \partial_y u_0^{j-1-k} \Big), \\ v_0^j(x, y) = -\int_0^y \partial_x u_0^j(x, \xi) d\xi, \end{cases} \tag{5.1}$$

by induction on j, with $u_0^j(x, y) = \tilde{u}_0^j(x, y) + (\partial_t^j u^s)(0, y)$.

To construct $(\tilde{u}^0, v^0)$ satisfying

$$\partial_t^j \tilde{u}^0|_{t=0} = \tilde{u}_0^j(x, y), \qquad \partial_t^j v^0|_{t=0} = v_0^j(x, y), \quad 0 \leq j \leq k_0,$$

we can simply define

$$\tilde{u}^0(t, x, y) = \sum_{j=0}^{k_0} \frac{t^j}{j!} \tilde{u}_0^j(x, y), \quad v^0(t, x, y) = \sum_{j=0}^{k_0} \frac{t^j}{j!} v_0^j(x, y)\,. \tag{5.2}$$

For this approximate solution, we have

Lemma 5.1. *Under the assumption (MA), for any fixed $T > 0$, there is a constant $C = C(k_0, T)$ depends only on k_0 and T such that,*

$$\|\tilde{u}^0\|_{\tilde{\mathcal{A}}^{k_0+1}_\ell([0,T]\times\mathbb{R}^2_+)} \leq C\epsilon, \quad \|v^0\|_{\mathcal{D}^{k_0}([0,T]\times\mathbb{R}^2_+)} \leq C\epsilon, \tag{5.3}$$

and

$$\|f^a\|_{\tilde{\mathcal{A}}^{k_0}_\ell([0,T]\times\mathbb{R}^2_+)} \le C\epsilon. \tag{5.4}$$

Here, we have used the notations

$$\|\tilde{u}^0\|_{\tilde{\mathcal{A}}^k_\ell([0,T]\times\mathbb{R}^2_+)} := \sum_{j=0}^{k} \|\tilde{u}^0\|_{W^{j,\infty}(0,T;\mathcal{A}^{k-j}_\ell(\mathbb{R}^2_+))},$$

and

$$f^a = \partial_t\tilde{u}^0 + (\tilde{u}^0+u^s)\partial_x\tilde{u}^0 + v^0\partial_y(\tilde{u}^0+u^s) - \partial_y^2\tilde{u}^0.$$

Proof. Since $\tilde{u}_0 \in \mathcal{A}^{2k_0+1}_\ell(\mathbb{R}^2_+)$, it follows immediately that

$$v_0(x,y) = -\int_0^y \partial_x\tilde{u}_0(x,\eta)d\eta \in \mathcal{D}^{2k_0}(\mathbb{R}^2_+),$$

which implies

$$\tilde{u}_0^1 = -(\tilde{u}_0+u_0^s)\partial_x\tilde{u}_0 - v_0\partial_y(\tilde{u}_0+u_0^s) + \partial_y^2\tilde{u}_0 \in \mathcal{A}^{2k_0}_\ell(\mathbb{R}^2_+),$$

and

$$v_0^1(x,y) = -\int_0^y \partial_x\tilde{u}_0^1(x,\eta)d\eta \in \mathcal{D}^{2k_0-1}(\mathbb{R}^2_+).$$

In this way, using (5.1) and by induction on j we can deduce

$$\tilde{u}_0^j \in \mathcal{A}^{2k_0+1-j}_\ell(\mathbb{R}^2_+), \qquad v_0^j \in \mathcal{D}^{2k_0-j}(\mathbb{R}^2_+),$$

for all $j \le k_0$, and

$$\|\tilde{u}_0^j\|_{\mathcal{A}^{2k_0-j+1}_\ell(\mathbb{R}^2_+)} \le C(j)\epsilon,$$

for a constant $C(j)$ depending only on j. Then, (5.3) and (5.4) follow immediately from the construction (5.2). □

Remark 5.2. *Denoting $u^0 = \tilde{u}^0 + u^s$, it is easy to see that (u^0, v^0) is an approximate solution to the original problem* (1.2) *satisfying*

$$\begin{cases} u_t^0 + u^0u_x^0 + v^0u_y^0 - u_{yy}^0 = f^a, \quad t>0, \quad (x,y)\in\mathbb{R}^2_+, \\ \partial_x u^0 + \partial_y v^0 = 0, \quad (x,y)\in\mathbb{R}^2_+,\ t\ge 0, \\ (u^0,v^0)|_{y=0} = 0, \quad \lim\limits_{y\to+\infty} u^0 = 1, \\ u^0|_{t=0} = u_0(x,y), \end{cases} \tag{5.5}$$

with $\partial_t^j f^a|_{t=0} = 0$ for all $0 \le j \le k_0 - 1$.

Lemma 5.3. *Under the assumptions (MA), for any fixed $T > 0$, there is a constant $C(k,T)$ such that,*

$$\|\frac{\partial_y\tilde{u}^0}{\partial_y u^s}\|_{\mathcal{A}^{k_0+1}_\ell([0,T]\times\mathbb{R}^2_+)} + \|\frac{f^a}{\partial_y u^s}\|_{\mathcal{A}^{k_0}_\ell([0,T]\times\mathbb{R}^2_+)} \le C\epsilon. \tag{5.6}$$

This result can be proved in a way similar to the proof of Lemma 5.1, so we omit it for brevity.

Remark 5.4. *From the smallness of the first term given in* (5.6)*, it is easy to see that for a fixed $T > 0$, there is a small ϵ such that when the conditions of Lemmas 5.1 and 5.5 hold, we have the monotonicity*

$$\partial_y u^0(t,x,y) > 0, \qquad (t,x,y)\in[0,T]\times\mathbb{R}^2_+.$$

5.2. **Estimates of the approximate solutions.** Obviously, the problem (4.10) can be written as

$$\left\{\begin{array}{l} \partial_t(\delta u^n)+u^n_{\theta_n}\partial_x(\delta u^n)+v^n_{\theta_n}\partial_y(\delta u^n)+\delta u^n\partial_x(u^n_{\theta_n})\\ \quad +\delta v^n\partial_y(u^n_{\theta_n})-\partial_y^2(\delta u^n)=f^n,\\ \partial_x(\delta u^n)+\partial_y(\delta v^n)=0,\\ \delta u^n|_{y=0}=\delta v^n|_{y=0}=0,\quad \lim\limits_{y\to+\infty}\delta u^n=0,\\ \delta u^n|_{t=0}=0\,, \end{array}\right. \tag{5.7}$$

where

$$u^n_{\theta_n}=u^s+S_{\theta_n}(\tilde{u}^0+\sum_{0\le j\le n-1}\delta u^j),\quad v^n_{\theta_n}=S_{\theta_n}(v^0+\sum_{0\le j\le n-1}\delta v^j).$$

Set

$$w^n=\partial_y\left(\frac{\delta u^n}{\partial_y u^n_{\theta_n}}\right).$$

As in Section 3, from (5.7), we know that w^n satisfies

$$\left\{\begin{array}{l} \partial_t w^n+\partial_x(u^n_{\theta_n}w^n)+\partial_y(v^n_{\theta_n}w^n)-2\partial_y(\eta^n w^n)\\ \quad +\partial_y\left(\zeta^n\int_0^y w^n(t,x,\tilde{y})d\tilde{y}\right)-\partial_y^2 w^n=\partial_y\tilde{f}^n,\\ \left(\partial_y w^n+2\eta^n w^n\right)\big|_{y=0}=-\tilde{f}^n\big|_{y=0},\\ w^n|_{t=0}=0\,, \end{array}\right. \tag{5.8}$$

where

$$\eta^n=\frac{\partial_y^2 u^n_{\theta_n}}{\partial_y u^n_{\theta_n}},\qquad \zeta^n=\frac{(\partial_t+u^n_{\theta_n}\partial_x+v^n_{\theta_n}\partial_y-\partial_y^2)\partial_y u^n_{\theta_n}}{\partial_y u^n_{\theta_n}},$$

and

$$\tilde{f}^n=\frac{f^n}{\partial_y u^n_{\theta_n}}=\frac{(S_{\theta_{n-1}}-S_{\theta_n})\big(\sum_{j=0}^{n-2}e_j\big)-S_{\theta_n}e_{n-1}+(S_{\theta_{n-1}}-S_{\theta_n})f^a}{\partial_y u^n_{\theta_n}}. \tag{5.9}$$

From the above main assumption (MA) and the construction of the approximate solution $(\tilde{u}^0,v^0)$ to the problem (4.7), it is easy to show by induction on n that the compatibility conditions for the problem (5.8) up to order k_0 hold for all $n\ge 0$.

Similar to Section 3, set

$$\begin{array}{rcl}\lambda^n_{k_1,k_2} &=& \|u^n_{\theta_n}-u^s\|_{\mathcal{B}^{k_1,k_2}_{0,0}}+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}u^s\|_{L^2_y(L^\infty_t)}+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}v^n_{\theta_n}\|_{L^\infty_y(L^2_{t,x})}\\ &&+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\bar{\eta}^n\|_{L^2_y(L^\infty_{t,x})}+\|\eta^n-\bar{\eta}^n\|_{\mathcal{B}^{k_1,k_2}_{0,0}}+\|\zeta^n\|_{\mathcal{B}^{k_1,k_2}_{0,l}},\end{array}$$

with $\bar{\eta}^n=\frac{\partial_y^2 u^s}{\partial_y u^n_{\theta_n}}$, and

$$\lambda^n_k=\sum_{k_1+[\frac{k_2+1}{2}]\le k}\lambda^n_{k_1,k_2}.$$

That is,

$$\begin{array}{rcl}\lambda^n_k &=& \|u^n_{\theta_n}-u^s\|_{\mathcal{A}^k_0}+\|u^s\|_{\mathcal{C}^k_0}+\|v^n_{\theta_n}\|_{\mathcal{D}^k_0}\\ &&+\|\bar{\eta}^n\|_{\mathcal{C}^k_0}+\|\eta^n-\bar{\eta}^n\|_{\mathcal{A}^k_0}+\|\zeta^n\|_{\mathcal{A}^k_l}.\end{array} \tag{5.10}$$

By applying Theorem 3.1 to the problem (5.8), we have

Proposition 5.5. *Under the main assumption (MA), the solution w^n to the problem* (5.8) *satisfies*

$$\|w^n\|_{\mathcal{A}^k_\ell} \le C_1(\lambda_3^n)\|\tilde{f}^n\|_{\mathcal{A}^k_\ell} + C_2(\lambda_3^n)\lambda_k^n\|\tilde{f}^n\|_{\mathcal{A}^3_\ell}, \tag{5.11}$$

where $C_1(\lambda_3^n), C_2(\lambda_3^n)$ are polynomials of λ_3^n of order less or equal to k.

The key step in proving the convergence of the Nash-Moser-Hörmander iteration scheme (4.9)-(4.10) is given by the following result.

Theorem 5.6. *Under the main assumption (MA), there exists a positive constant C_0, such that*

$$\|w^n\|_{\mathcal{A}^k_\ell} \le C_0\,\epsilon\,\theta_n^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_n, \tag{5.12}$$

holds for all $n \ge 0$, $0 \le k \le k_0$ where $\theta_n = \sqrt{\theta_0^2 + n}$ and $\Delta\theta_n = \theta_{n+1} - \theta_n$.

Theorem 5.6 will be proved by induction on n. First of all, to apply Proposition 5.5, we need to estimate λ_k^n and $\tilde{f}^n$ by induction on n also. For this purpose, we first give the following estimates, some of the proofs being postponed to Section 7. The proof of Theorem 5.6 will be completed at the end of this subsection.

Lemma 5.7. *Suppose that the main assumption (MA), and* (5.12) *for w^j, $0 \le j \le n-1$, hold. Then there is a constant $C_1 > 0$, such that*

$$\|\delta u^j\|_{\mathcal{A}^k_\ell} \le C_1\epsilon\,\theta_j^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_j, \qquad 0 \le k \le k_0, \tag{5.13}$$

$$\|\frac{\delta u^j}{\partial_y u^s}\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} \le C_1\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j, \tag{5.14}$$

and

$$\|\frac{\delta u^j}{\partial_y u^s}\|_{\mathcal{D}_0^{k-1}} \le C_1\epsilon\,\theta_j^{\max\{3-\tilde{k},\,k-1-\tilde{k}\}}\Delta\theta_j, \qquad 1 \le k \le k_0, \tag{5.15}$$

hold for all $0 \le j \le n-1$.

Since the proof of this lemma is technical, it will be given in Section 7.

As $\delta v^j(t,x,y) = -\int_0^y (\partial_x \delta u^j)(t,x,\tilde{y})d\tilde{y}$, from (5.13) we immediately have

Lemma 5.8. *Under the same assumptions as for Lemma 5.7, there is a constant $C_2 > 0$, such that*

$$\|\delta v^j\|_{\mathcal{D}_0^k} \le C_2\epsilon\theta_j^{\max\{3-\tilde{k},k+1-\tilde{k}\}}\Delta\theta_j, \qquad 0 \le k \le k_0 - 1, \tag{5.16}$$

holds for all $0 \le j \le n-1$.

Based on Lemma 5.7 and Lemma 5.8, we have

Lemma 5.9. *Under the same assumptions as for Lemma 5.7, there is a constant $C_3 > 0$, such that*

$$\|u^n - u^s\|_{\mathcal{A}^k_l} \le C_3\epsilon\theta_n^{\max\{0,\,k+1-\tilde{k}\}}, \qquad 0 \le k \le k_0, \tag{5.17}$$

$$\|v^n_{\theta_n}\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} + \|v^n_{\theta_n}\|_{L^\infty([0,\infty)_y, L^2([0,T]\times\mathbb{R}_x)} \le C_3, \tag{5.18}$$

and

$$\|v^n_{\theta_n}\|_{\mathcal{D}_0^k} \le \begin{cases} C_\rho\|v^n\|_{\mathcal{D}_0^k} \le C_3\epsilon\,\theta_n^{\max\{0,\,k+2-\tilde{k}\}}, & 0 \le k \le k_0 - 1, \\ C_\rho\theta_n\|v^n\|_{\mathcal{D}_0^{k_0-1}} \le C_3\epsilon\,\theta_n^{\max\{1,\,k_0+2-\tilde{k}\}}, & k = k_0, \end{cases} \tag{5.19}$$

hold, where $C_\rho > 0$ is given in (4.1).

Proof. From the identity

$$u^n - u^s = \tilde{u}^0 + \sum_{j=0}^{n-1} \delta u^j$$

we have immediately by using (5.3) and Lemma 5.7 that

$$\begin{aligned} \|u^n - u^s\|_{\mathcal{A}_l^k} &\le \|\tilde{u}^0\|_{\mathcal{A}_l^k} + \sum_{j=0}^{n-1} \|\delta u^j\|_{\mathcal{A}_l^k} \\ &\le C^a\epsilon + C_1\epsilon \sum_{j=0}^{n-1} \theta_j^{\max\{3-\tilde{k},\, k-\tilde{k}\}} \Delta\theta_j \\ &\le C^a\epsilon + C_1\tilde{C}\epsilon\theta_n^{\max\{0,\, k+1-\tilde{k}\}}. \end{aligned} \tag{5.20}$$

Here, we have used the fact that

$$\sum_{p=0}^{j-1} \theta_p^{k-\tilde{k}} \Delta\theta_p \le \begin{cases} \tilde{C}\theta_j^{k+1-\tilde{k}}, & \text{as } k-\tilde{k} \ge 0, \\ \tilde{C}, & \text{as } k-\tilde{k} \le -2, \end{cases} \tag{5.21}$$

for an absolute constant $\tilde{C}$.

From (5.20), we obtain the estimate (5.17) immediately. Similarly, from the identity

$$v^n = v^0 + \sum_{j=0}^{n-1} \delta v^j,$$

we can easily deduce the estimates (5.18) and (5.19) by using Lemma 5.8. □

As a direct consequence of the estimate (5.17), there is a constant $\tilde{C}_3 > 0$ such that

$$\|u_{\theta_n}^n - u^s\|_{\mathcal{A}_\ell^k} \le \tilde{C}_3\epsilon\theta_n^{\max\{0,\, k+1-\tilde{k}\}}, \qquad 0 \le k \le k_0. \tag{5.22}$$

To get the estimate of λ_k^n, we need to estimate the norms of $\eta^n = \frac{\partial_y^2 u_{\theta_n}^n}{\partial_y u_{\theta_n}^n}$ and

$$\zeta^n = \frac{(\partial_t + u_{\theta_n}^n \partial_x + v_{\theta_n}^n \partial_y - \partial_y^2)\partial_y u_{\theta_n}^n}{\partial_y u_{\theta_n}^n}, \tag{5.23}$$

which are given as follows. Again, the proofs of the next two Lemmas will be given in Section 7.

Lemma 5.10. *Under the same assumptions as for Lemma 5.7, there is a constant $C_4 > 0$, such that*

$$\|\eta^n - \bar{\eta}^n\|_{\mathcal{A}_\ell^k} \le \begin{cases} C_4\epsilon\, \theta_n^{\max\{1,k+2-\tilde{k}\}}, & 4 \le k \le k_0, \\ C_4\epsilon, & k = 3, \end{cases} \tag{5.24}$$

and

$$\|\bar{\eta}^n\|_{\mathcal{C}_\ell^k} \le C_4(1 + \epsilon\theta_n^{\max\{0,\, k+3-\tilde{k}\}}), \qquad 0 \le k \le k_0, \tag{5.25}$$

where $\bar{\eta}^n = \frac{\partial_y^2 u^s}{\partial_y u_{\theta_n}^n}$.

Lemma 5.11. *Under the same assumptions as for Lemma 5.7, for ζ^n defined in (5.23), there is a positive constant C_5 such that*

$$\|\zeta^n\|_{\mathcal{A}^k_\ell} \le \begin{cases} C_5\theta_n^{\max\{1,\,k+3-\tilde{k}\}}, & 4\le k\le k_0,\\ C_5, & k=3. \end{cases} \tag{5.26}$$

By plugging the estimates (5.19), (5.24), (5.25), (5.26) and (5.22) into the definition (5.10) of λ_k^n, we conclude

Proposition 5.12. *Suppose that the main assumption (MA), and (5.12) for w^j, $0\le j\le n-1$, hold. There exists a positive constant $C_6>0$ depending on C_p $(1\le p\le 5)$ given in Lemmas 5.7-5.11, such that*

$$\lambda_k^n \le \begin{cases} C_6\theta_n^{\max\{1,\,k+3-\tilde{k}\}}, & 4\le k\le k_0,\\ C_6, & k=3. \end{cases}$$

To estimate $\tilde{f}^n$ defined in (5.9), we will need the following two estimates whose proofs will be given in Section 7.

Lemma 5.13. *Under the same assumptions as for Lemma 5.7, there is a constant $C_7>0$, such that*

$$\|\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)^{-1}\|_{L^\infty}\le 2, \qquad \|\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)^{-1}\|_{\dot{\mathcal{A}}^k_0}\le C_7\epsilon\,\theta_n^{\max\{0,k+1-\tilde{k}\}}, \tag{5.27}$$

hold, with $1\le k\le k_0$.

Lemma 5.14. *Under the same assumptions as for Lemma 5.7, there is a constant $C_8>0$, such that for the error terms $e_j^{(1)}=\delta u^j\partial_x\delta u^j+\delta v^j\partial_y\delta u^j$ and*

$$\begin{aligned} e_j^{(2)} = \ & \big((1-S_{\theta_j})(u^j-u^s)\big)\partial_x(\delta u^j)+\delta u^j\partial_x\big((1-S_{\theta_j})(u^j-u^s)\big)\\ & +\delta v^j\partial_y\big((1-S_{\theta_j})(u^j-u^s)\big)+\big((1-S_{\theta_n})v^j\big)\partial_y(\delta u^j), \end{aligned}$$

the following estimates

$$\|\frac{e_j^{(1)}}{\partial_y u^s}\|_{\mathcal{A}^{k_1}_\ell}\le C_8\epsilon^2\theta_j^{\max\{6-2\tilde{k},\,k_1+3-2\tilde{k}\}}\Delta\theta_j, \tag{5.28}$$

and

$$\|\frac{e_j^{(2)}}{\partial_y u^s}\|_{\mathcal{A}^{k_1}_l}\le C_8\epsilon^2\theta_j^{\max(3-\tilde{k},k_1+5-2\tilde{k})}\Delta\theta_j, \tag{5.29}$$

hold for all $k_1\le k_0-1$ and $0\le j\le n-1$.

In summary, for

$$\tilde{f}^n=\frac{f^n}{\partial_y u^n_{\theta_n}}=\frac{(S_{\theta_{n-1}}-S_{\theta_n})\big(\sum_{j=0}^{n-2}e_j\big)-S_{\theta_n}e_{n-1}+(S_{\theta_{n-1}}-S_{\theta_n})f^a}{\partial_y u^n_{\theta_n}},$$

with $e_n=e_n^{(1)}+e_n^{(2)}$, we have

Proposition 5.15. *Suppose that the main assumption (MA), and (5.12) for w^j, $0\le j\le n-1$, hold, there exists a constant $C_9>0$ such that*

$$\|\tilde{f}^n\|_{\mathcal{A}^k_\ell}\le C_9\epsilon\theta_n^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_n\,, \qquad 0\le k\le k_0. \tag{5.30}$$

Proof. From $\tilde{f}^n = \frac{f^n}{\partial_y u^n_{\theta_n}} = \frac{f^n}{\partial_y u^s}\left(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\right)^{-1}$, by using Lemma 2.1 and Lemma 5.13 we have

$$\|\tilde{f}^n\|_{\mathcal{A}^k_\ell} \le M_k\Big\{2\|\frac{f^n}{\partial_y u^s}\|_{\mathcal{A}^k_\ell} + \|\frac{f^n}{\partial_y u^s}\|_{L^\infty_\ell} C_7\epsilon\,\theta_n^{\max\{0,k+1-\tilde{k}\}}\Big\}.$$

On the other hand, using (4.1) and (4.2), for any $k, k_j \ge 0$ $(j = 1, 2, 3)$, we have

$$\|\frac{f^n}{\partial_y u^s}\|_{\mathcal{A}^k_\ell} \le C_\rho\Big\{\sum_{j=0}^{n-2}\|\frac{e_j}{\partial_y u^s}\|_{\mathcal{A}^{k_1}_\ell}\theta_n^{k-k_1}\Delta\theta_n + \|\frac{e_{n-1}}{\partial_y u^s}\|_{\mathcal{A}^{k_2}_\ell}\theta_n^{(k-k_2)_+} + \|\frac{f^a}{\partial_y u^s}\|_{\mathcal{A}^{k_3}_\ell}\theta_n^{k-k_3}\Delta\theta_n\Big\}. \tag{5.31}$$

Thus, by using (5.28), (5.29) in (5.31), we get

$$\begin{aligned}\|\frac{f^n}{\partial_y u^s}\|_{\mathcal{A}^k_\ell} \le C_\rho\Big\{&2C_8\sum_{j=0}^{n-2}\epsilon^2\theta_j^{\max\{3-\tilde{k},\,k_1+5-2\tilde{k}\}}\Delta\theta_j\theta_n^{k-k_1}\Delta\theta_n\\ &+ 2C_8\epsilon^2\theta_{n-1}^{\max\{3-\tilde{k},\,k_2+5-2\tilde{k}\}}\Delta\theta_{n-1}\theta_n^{(k-k_2)_+} + C^a\epsilon\,\theta_n^{k-k_3}\Delta\theta_n\Big\},\end{aligned} \tag{5.32}$$

for $k_1 \le k_0 - 1$ and $k_2 \le k_0 - 1$, provided $\|\frac{f^a}{\partial_y u^s}\|_{\mathcal{A}^{k_3}_l} \le C^a\epsilon$.

When $k = 3$, by setting $k_1 = k_3 = \tilde{k}$ and $k_2 = 3$ in (5.32) we get

$$\|\frac{f^n}{\partial_y u^s}\|_{\mathcal{A}^3_\ell} \le C_\rho(2C_8(1+\tilde{C})\epsilon^2 + C^a\epsilon)\theta_n^{3-\tilde{k}}\Delta\theta_n. \tag{5.33}$$

When $4 \le k \le k_0$, by choosing $k_1 \ge 1 + \tilde{k}$, $k_2 = \tilde{k} - 2$ and $k_3 = \tilde{k}$ in (5.32) we obtain

$$\|\frac{f^n}{\partial_y u^s}\|_{\mathcal{A}^k_\ell} \le C_\rho(2C_8(1+\tilde{C})\epsilon^2 + C^a\epsilon)\theta_n^{k-\tilde{k}}\Delta\theta_n. \tag{5.34}$$

Here, we have used the fact that $(k-k_2)_+ + k_2 + 5 - \tilde{k} \le k$ for all $4 \le k \le k_0$.

Combining (5.33) with (5.34), we conclude the estimate (5.30). □

Proof of Theorem 5.6:

We are now ready to conclude the proof of Theorem 5.6 by induction on n.

For $n = 0$, from the main assumption (MA), Lemma 5.1 and Lemma 5.5, we get immediately that for any fixed $T > 0$, there is a constant $C^a = C^a(k_0, T)$ such that

$$\|\tilde{u}^0\|_{\mathcal{A}^{k_0+1}_\ell} + \|\frac{\partial_y\tilde{u}^0}{\partial_y u^s}\|_{\mathcal{A}^{k_0+1}_\ell} + \|\frac{f^a}{\partial_y u^s}\|_{\mathcal{A}^{k_0}_\ell} \le C^a\epsilon.$$

This implies that $\tilde{f}^0 = \frac{f^a}{\partial_y(u^s + S_{\theta_0}\tilde{u}^0)}$ satisfies

$$\|\tilde{f}^0\|_{\mathcal{A}^{k_0}_\ell} \le \tilde{C}^a\epsilon,$$

for a constant $\tilde{C}^a$.

A direct calculation yields

$$\lambda^0_k \le \|u^s\|_{\mathcal{C}^k_0} + \|\frac{\partial^2_y u^s}{\partial_y u^s}\|_{\mathcal{C}^k_0} + \bar{C}^a\epsilon \le C_{k_0}, \qquad \forall k \le k_0,$$

for a constant $\bar{C}^a$ depending on C^a and $\tilde{C}^a$ given at above.

By applying Proposition 5.5 for w^0, and using the above estimates, it follows

$$\|w^0\|_{\mathcal{A}^k_\ell} \le \bar{C}_{k_0}\epsilon, \qquad \forall k \le k_0, \tag{5.35}$$

for a constant $\bar{C}_{k_0}$ depending on $\tilde{C}^a$ and C_{k_0} given above. Hence, the estimate (5.12) for the case $n = 0$ follows immediately from (5.35) with a constant C_0 depends on $\bar{C}_{k_0}, k_0, \tilde{k}$ and θ_0.

Now, assuming that (5.12) holds for all w^j with $0 \le j \le n-1$, we are going to prove it for w^n. In fact, from the estimates (5.11) and (5.30), we get

$$\|w^n\|_{\mathcal{A}^k_\ell} \le C_1(\lambda_3^n)C_9\epsilon\theta_n^{\max\{3-\tilde{k},k-\tilde{k}\}} + C_2(\lambda_3^n)\lambda_k^n C_9\epsilon\theta_n^{3-\tilde{k}},$$

which implies by Proposition 5.12 that

$$\|w^n\|_{\mathcal{A}^3_\ell} \le C_0\epsilon\theta_n^{3-\tilde{k}},$$

and

$$\|w^n\|_{\mathcal{A}^k_\ell} \le C_0\epsilon\theta_n^{k-\tilde{k}},$$

for all $4 \le k \le k_0$, with the constant $C_0 \ge (C_1(C_6) + C_2(C_6)C_6)C_9$.

5.3. **Convergence of the iteration scheme.** In this subsection, we will prove the convergence of the iteration scheme and this immediately yields the existence of classical solutions to the Prandtl equation (1.2).

From the iteration scheme (4.9)-(4.10) with f^n defined in (4.13), we know that the approximate solution

$$u^{n+1} = u^s + \tilde{u}^0 + \sum_{j=0}^{n}\delta u^j, \quad v^{n+1} = v^0 + \sum_{j=0}^{n}\delta v^j,$$

satisfies

$$\begin{cases} \mathcal{P}(u^{n+1}, v^{n+1}) = (1 - S_{\theta_n})\sum_{j=0}^{n} e_j + S_{\theta_n}e_n + (1 - S_{\theta_n})f^a, \\ \partial_x u^{n+1} + \partial_y v^{n+1} = 0, \\ u^{n+1}|_{y=0} = v^{n+1}|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u^{n+1} = 1, \\ u^{n+1}|_{t=0} = u_0(x, y)\,. \end{cases} \tag{5.36}$$

From the estimates (5.13) and (5.16), we know that there exist $u \in u^s + \mathcal{A}^{\tilde{k}-2}_\ell$ and $v \in \mathcal{D}^{\tilde{k}-3}_0$, such that

$$\lim_{n\to+\infty}\|u^n - u\|_{\mathcal{A}^{\tilde{k}-2}_\ell} = 0, \quad \lim_{n\to+\infty}\|v^n - v\|_{\mathcal{D}^{\tilde{k}-3}_0} = 0.$$

To verify that the limit (u, v) is a classical solution to the problem (1.2), it is enough to show that the right hand side of the equation in (5.36) converges to zero as $n \to +\infty$.

Obviously, we have

$$\|(1 - S_{\theta_n})(f^a + \sum_{j=0}^{n} e_j)\|_{\mathcal{A}^k_\ell} \le \theta_n^{-1}(\|f^a\|_{\mathcal{A}^{k+1}_\ell} + \sum_{j=0}^{n}\|e_j\|_{\mathcal{A}^{k+1}_\ell}).$$

Thus, it is enough to prove the convergence of series $\sum_{j=0}^{+\infty}\|e_j\|_{\mathcal{A}^{k+1}_\ell}$. Recall

$$e_j = e_j^{(1)} + e_j^{(2)},$$

with

$$e_j^{(1)} = \delta u^j\partial_x(\delta u^j) + \delta v^j\partial_y(\delta u^j),$$

and

$$e_j^{(2)} = \partial_y\Big(\delta v^j\big((1-S_{\theta_j})(u^j-u^s)\big) + \big((1-S_{\theta_j})v^j\big)(\delta u^j)\Big).$$

By using Lemma 2.1, it follows that

$$\begin{aligned}\|e_j^{(1)}\|_{\mathcal{A}_\ell^{k+1}} &\le M_k\Big(\|\delta u^j\|_{L^\infty}\|\delta u^j\|_{\mathcal{A}_\ell^{k+2}} + \|\delta v^j\|_{L^\infty}\|\delta u^j\|_{\mathcal{A}_\ell^{k+2}}\\ &\qquad + \|\delta v^j\|_{\mathcal{D}_0^{k+2}}\|\delta u^j\|_{L^2_{y,\ell}(L^\infty_{t,x})}\Big)\\ &\le C_{10}\epsilon^2\,\theta_j^{k+5-2\tilde{k}}\Delta\theta_j,\end{aligned}$$

and

$$\begin{aligned}\|e_j^{(2)}\|_{\mathcal{A}_\ell^{k+1}} &\le M_k\Big(\|\delta v^j\|_{L^\infty}\|u^j-u^s\|_{\mathcal{A}_\ell^{k+2}} + \|\delta v^j\|_{\mathcal{D}_0^{k+2}}\|u^j-u^s\|_{L^2_{y,\ell}(L^\infty_{t,x})}\\ &\quad + \|v^j\|_{L^\infty}\|\delta u^j\|_{\mathcal{A}_\ell^{k+2}} + \|v^j\|_{\mathcal{D}_0^{k+2}}\|\delta u^j\|_{L^2_{y,\ell}(L^\infty_{t,x})}\Big)\\ &\le C_{10}\epsilon^2\,\theta_j^{k+3-\tilde{k}}\Delta\theta_j,\end{aligned}$$

for a positive constant $C_{10} > 0$, where we have used (5.13), (5.16) and (5.17). Therefore, we obtain

$$\sum_{j=0}^{+\infty}\|e_j\|_{\mathcal{A}_\ell^{k+1}} \le C\sum_{j=0}^{+\infty}\theta_j^{k+3-\tilde{k}}\Delta\theta_j \le C\tilde{C},$$

for all $k \le \tilde{k}-5$. And this concludes the convergence of the iteration scheme and the existence of classical solutions to the Prandtl equation (1.2).

6. Uniqueness and stability

In this section, we study the stability of classical solutions to the Prandtl equation (1.2), and thus the uniqueness of the classical solution obtained in Section 5 will follow immediately.

Let (u^1, v^1) and (u^2, v^2) be two classical solutions to the problem (1.2) in the solution spaces given in Theorem 1.1 with the initial data $u_0^1(x,y)$ and $u_0^2(x,y)$ as two small perturbations of $u_0^s(y)$ as stated in Theorem 1.1. Denoting by

$$u = u^1 - u^2, \quad v = v^1 - v^2, \quad \tilde{u} = \frac{u^1+u^2}{2}, \quad \tilde{v} = \frac{v^1+v^2}{2},$$

then from (1.2), we deduce that (u, v) satisfies

$$\begin{cases}\partial_t u + \tilde{u}\partial_x u + \tilde{v}\partial_y u + u\partial_x\tilde{u} + v\partial_y\tilde{u} - \partial_y^2 u = 0,\\ \partial_x u + \partial_y v = 0,\\ u|_{y=0} = v|_{y=0} = 0, \quad \lim\limits_{y\to+\infty} u = 0,\\ u|_{t=0} = u_0(x,y) := u_0^1 - u_0^2.\end{cases} \tag{6.1}$$

As in Section 3, set

$$w(t,x,y) = \left(\frac{u}{\partial_y\tilde{u}}\right)_y(t,x,y) \quad \text{that is,} \quad u(t,x,y) = (\partial_y\tilde{u})\int_0^y w(t,x,y')dy'. \tag{6.2}$$

Then, from (6.1) we know that $w(t,x,y)$ satisfies

$$\begin{cases} \partial_t w + \partial_x(\tilde{u}w) + \partial_y(\tilde{v}w) - 2\partial_y(\eta w) + \partial_y(\zeta \int_0^y w(t,x,\tilde{y})d\tilde{y}) - \partial_y^2 w = 0, \\ \big(\partial_y w + 2\eta w\big)|_{y=0} = 0, \\ w|_{t=0} = w_0(x,y) = \Big(\frac{u_0}{\partial_y \tilde{u}}\Big)_y(x,y), \end{cases} \tag{6.3}$$

where

$$\eta = \frac{\partial_y^2 \tilde{u}}{\partial_y \tilde{u}}, \quad \zeta = \frac{\big(\partial_t + \tilde{u}\partial_x + \tilde{v}\partial_y - \partial_y^2\big)\partial_y \tilde{u}}{\partial_y \tilde{u}}.$$

Similar to the proof for (3.3) in the problem (6.3), it follows

$$\|w\|_{\mathcal{A}^k_\ell([0,T]\times\mathbb{R}^2_+)} \le C(T)\|w_0\|_{\mathcal{A}^k_\ell(\mathbb{R}^2_+)}, \qquad k \le \tilde{k} - 3, \tag{6.4}$$

for a constant $C(T)$ depending on $T > 0$ and the norms of the initial data u_0^1, u_0^2 in the spaces given in the existence part of Theorem 1.1.

From (6.4), and the transformation (6.2), we deduce

$$\|u^1 - u^2\|_{\mathcal{A}^k_\ell([0,T]\times\mathbb{R}^2_+)} + \|v^1 - v^2\|_{\mathcal{D}_0^{k-1}([0,T]\times\mathbb{R}^2_+)} \le C\|\frac{\partial}{\partial y}\left(\frac{u_0^1 - u_0^2}{\partial_y u_0^s}\right)\|_{\mathcal{A}^k_\ell(\mathbb{R}^2_+)},$$

for all $k \le \tilde{k} - 3$. And this concludes the uniqueness and stability results stated in Theorem 1.1.

7. Proof of some technical estimates

Finally, in this section, we give the proofs for Lemmas 5.7, 5.10, 5.11, 5.13 and 5.14 stated in Section 5 about the iteration scheme (4.9)-(4.10).

We start with the proof for Lemma 5.7.

Proof of Lemma 5.7: Let us first prove the estimate (5.13). First of all, it holds true for δu^0. Indeed, from

$$\delta u^0 = \partial_y(u^s + S_{\theta_0}\tilde{u}^0)\int_0^y w^0 d\tilde{y},$$

by using Lemma 2.1 and the Sobolev embedding theorem, we have

$$\|\delta u^0\|_{\mathcal{A}^k_l} \le C_k^0\|w^0\|_{\mathcal{A}^k_l} \le C_k^0 C_0 \epsilon \theta_0^{\max\{3-\tilde{k},\, k-\tilde{k}\}}\Delta\theta_0, \qquad k \ge 2,$$

with

$$C_k^0 = \|u^s\|_{\mathcal{C}^{k+1}_\ell} + \|\tilde{u}^0\|_{\mathcal{A}^{k+1}_\ell}.$$

The estimate (5.13) holds obviously for δu^0 when $k = 0, 1$.

Now, suppose that (5.13) holds for δu^p, $0 \le p \le j-1$. We estimate δu^j as follows.

Recalling $w^j = (\frac{\delta u^j}{\partial_y(u^j_{\theta_j})})_y$, we have

$$\delta u^j = \partial_y(u^j_{\theta_j})\int_0^y w^j d\tilde{y}.$$

Using again Lemma 2.1, the Sobolev embedding theorem and (5.21), it follows that, for $k \geq 4$ and $\tilde{k} \geq 6$,

$$\begin{aligned}
\|\delta u^j\|_{\mathcal{A}^k_l} &\leq M_k((C^0_k + \|\partial_y(u^j_{\theta_j} - u^s)\|_{\mathcal{A}^k_l})\|w^j\|_{\mathcal{A}^2_l} + \|\partial_y u^j_{\theta_j}\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|w^j\|_{\mathcal{A}^k_l}) \\
&\leq M_k((C^0_k + C_\rho\theta_j \sum_{p=0}^{j-1}\|\delta u^p\|_{\mathcal{A}^k_l})\|w^j\|_{\mathcal{A}^2_l} + \|\partial_y u^j\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|w^j\|_{\mathcal{A}^k_l}) \\
&\leq C_0M_kC_\rho(C^0_k + C_1\epsilon\,\theta_j\textstyle\sum_{p=0}^{j-1}\theta_p^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_p)\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j \\
&\quad + C_0C_\rho M_k(C^0_3 + C_1\epsilon\textstyle\sum_{p=0}^{j-1}\theta_p^{3-\tilde{k}}\Delta\theta_p)\epsilon\theta_j^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_j \\
&\leq \epsilon C_0C_\rho M_k(C^0_k + C_1\tilde{C}\epsilon\,\theta_j^{1+\max\{0,\,k+1-\tilde{k}\}})\theta_j^{3-\tilde{k}}\Delta\theta_j \\
&\quad + \epsilon C_0C_\rho M_k(C^0_3 + C_1\tilde{C}\epsilon)\theta_j^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_j \\
&\leq \epsilon C_0C_\rho M_k(C^0_k\theta_0^{-1} + C^0_3 + 2C_1\tilde{C}\epsilon)\theta_j^{k-\tilde{k}}\Delta\theta_j,
\end{aligned}$$

where the constant C_ρ comes from (4.1).

By setting $C_1 = 4C_0C_\rho M_kC^0_3$, we can choose $0 < \epsilon \leq \epsilon_0$ and $\theta_0 > 0$ large enough such that

$$C_0C_\rho M_k(C^0_k\theta_0^{-1} + C^0_3 + 2C_1\tilde{C}\epsilon) \leq C_1.$$

Therefore, we get

$$\|\delta u^j\|_{\mathcal{A}^k_l} \leq C_1\epsilon\theta_j^{k-\tilde{k}}\Delta\theta_j,$$

for $k \geq 4$. On the other hand, we have

$$\begin{aligned}
\|\delta u^j\|_{\mathcal{A}^3_\ell} &\leq M_k(C^0_3 + C_\rho\sum_{p=0}^{j-1}\|\delta u^p\|_{\mathcal{A}^4_\ell})\|w^j\|_{\mathcal{A}^2_\ell} + M_k\|\partial_y u^j\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|w^j\|_{\mathcal{A}^3_\ell} \\
&\leq C_0M_kC_\rho(C^0_3 + C_1\epsilon\textstyle\sum_{p=0}^{j-1}\theta_p^{4-\tilde{k}}\Delta\theta_p)\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j \\
&\quad + C_0C_\rho M_k(C^0_3 + C_1\epsilon\textstyle\sum_{p=0}^{j-1}\theta_p^{3-\tilde{k}}\Delta\theta_p)\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j \\
&\leq C_1\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j,
\end{aligned}$$

for $\tilde{k} \geq 6$, by choosing a proper constant $C_1 > 0$. And this completes the proof of the estimate (5.13).

We now turn to the estimates (5.14) and (5.15). When $j = 0$, from

$$\frac{\delta u^0}{\partial_y u^s} = (1 + \frac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s})\int_0^y w^0(t,x,\tilde{y})d\tilde{y}, \tag{7.1}$$

we have, by using the Sobolev embedding theorem and for some $\ell > 1/2$, that

$$\begin{aligned}
\|\tfrac{\delta u^0}{\partial_y u^s}\|_{L^\infty} &\leq (1 + \|\tfrac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s}\|_{L^\infty})\bar{C}_\ell\|w^0\|_{L^2_{y,\ell}(L^\infty_{t,x})} \\
&\leq (1 + \|\tfrac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s}\|_{L^\infty})\bar{C}_\ell\|w^0\|_{\mathcal{A}^2_\ell},
\end{aligned}$$

where $\bar{C}_\ell = (\int_0^{+\infty}(1+y^2)^{-\ell}dy)^{\frac{1}{2}}$. The estimate (5.14) with $j = 0$ follows immediately by choosing

$$C_1 \geq C_0\bar{C}_\ell(1 + \|\frac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s}\|_{L^\infty}).$$

Applying Lemma 2.1 to (7.1) gives

$$\sum_{k_1+[\frac{k_2+1}{2}]\le k-1} \|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left(\frac{\delta u^0}{\partial_y u^s}\right)\|_{L_y^\infty(L^2_{t,x})}$$
$$\le M_k\bar{C}_\ell\left\{\sum_{1\le k_1+[\frac{k_2+1}{2}]\le k-1}\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left(\frac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s}\right)\|_{L^\infty}\|w^0\|_{L^2_\ell}\right.$$
$$\left.+(1+\|\frac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s}\|_{L^\infty})\|w^0\|_{\mathcal{A}_\ell^{k-1}}\right\}.$$

Then this yields the estimate (5.15) with $j=0$ by choosing

$$C_1 \ge M_k C_0\bar{C}_\ell\Big\{1+\sum_{0\le k_1+[\frac{k_2+1}{2}]\le k-1}\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left((1+\frac{\partial_y S_{\theta_0}(\tilde{u}^0)}{\partial_y u^s})\right)\|_{L^\infty}\Big\}.$$

When $1\le j\le n-1$, by definition, we have

$$\frac{\delta u^j}{\partial_y u^s} = \frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\int_0^y w^j(t,x,\tilde{y})d\tilde{y}. \tag{7.2}$$

By using Lemma 2.1 and the assumptions on w^j, it is sufficient to obtain the bounds of

$$\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s} = 1+\frac{\partial_y S_{\theta_j}(\tilde{u}^0)}{\partial_y u^s}+\sum_{p=0}^{j-1}\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s}$$

in the spaces L^∞ and $\mathcal{D}_0^{k-1}$ respectively.

To obtain the estimate (5.14) with index j, suppose that (5.14) holds for $\frac{\delta u^p}{\partial_y u^s}$ with $0\le p\le j-1$ and prove it by induction. Note that $j=0$ holds by the above argument.

For this, first we show

Lemma 7.1. *Suppose that the estimate* (5.12) *holds for* w^j, $0\le j\le n-1$, *with* $\tilde{k}\ge 7$, *then there exists a constant* M_s, *such that for all* $0\le j\le n$,

$$\|\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\|_{L^\infty}\le M_s \tag{7.3}$$

For continuity of the presentation, we postpone the proof of Lemma 7.1 later. Then by Lemma 7.1, with (7.2), it follows

$$\|\frac{\delta u^j}{\partial_y u^s}\|_{L^\infty}\le M_s\bar{C}_\ell C_0\epsilon\,\theta_j^{3-\tilde{k}}\Delta\theta_j \le C_1\epsilon\,\theta_j^{3-\tilde{k}}\Delta\theta_j,$$

with

$$C_1\ge M_s\bar{C}_\ell C_0.$$

To prove (5.15) with index j, we also suppose that it holds for $\frac{\delta u^p}{\partial_y u^s}$, $0\le p\le j-1$. Then, obviously, we have the identity

$$\begin{aligned}\sum_{p=0}^{j-1}\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s} &= \sum_{p=0}^{j-1}\Big\{S_{\theta_j}(\tfrac{\partial_y(\delta u^p)}{\partial_y u^s})+[\tfrac{1}{\partial_y u^s},S_{\theta_j}](\partial_y(\delta u^p))\Big\}\\ &= \sum_{p=0}^{j-1}\Big\{S_{\theta_j}(\partial_y\tfrac{\delta u^p}{\partial_y u^s})+S_{\theta_j}(\tfrac{\delta u^p}{\partial_y u^s}\tfrac{\partial_y^2 u^s}{\partial_y u^s})+[\tfrac{1}{\partial_y u^s},S_{\theta_j}](\partial_y(\delta u^p))\Big\}.\end{aligned} \tag{7.4}$$

Thus, we have

$$\begin{aligned}
&\|\sum_{p=0}^{j-1}\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left(\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s}\right)\|_{L^\infty_y(L^2_{t,x})}\\
&\le \tilde{B}\sum_{p=0}^{j-1}\Big\{\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}S_{\theta_j}\partial_y\Big(\frac{\delta u^p}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})}+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\delta u^p}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})}\Big\}\\
&\le \tilde{B}\sum_{p=0}^{j-1}\Big\{\theta_j\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\delta u^p}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})}+\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\delta u^p}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})}\Big\},
\end{aligned}$$

where the constant $\tilde{B}$ depends on the commutators in (7.4) which is independent of j and p. Using the induction hypothesis for (5.15), we have

$$\begin{aligned}
&\sum_{p=0}^{j-1}\sum_{k_1+[\frac{k_2+1}{2}]\le k-1}\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\delta u^p}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})}\\
&\qquad\le C_1\epsilon\sum_{p=0}^{j-1}\theta_p^{\max\{3-\tilde{k},k-1-\tilde{k}\}}\Delta\theta_p\le C_1\tilde{C}\epsilon\,\theta_j^{\max\{0,k-\tilde{k}\}}.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
&\sum_{k_1+[\frac{k_2+1}{2}]\le k-1}\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})}\\
&\qquad\le\sum_{k_1+[\frac{k_2+1}{2}]\le k-1}\Big\{\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left(\frac{\partial_y S_{\theta_j}\tilde{u}^0}{\partial_y u^s}\right)\|_{L^\infty_y(L^2_{t,x})}\\
&\qquad\qquad+\|\sum_{p=0}^{j-1}\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left(\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s}\right)\|_{L^\infty_y(L^2_{t,x})}\Big\}\qquad(7.5)\\
&\qquad\le C^a\epsilon+2\tilde{B}\tilde{C}C_1\epsilon\theta_j^{\max\{1,k+1-\tilde{k}\}}.
\end{aligned}$$

Now using Lemma 2.1, it follows

$$\begin{aligned}
&\sum_{k_1+[\frac{k_2+1}{2}]\le k-1}\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\left(\frac{\delta u^j}{\partial_y u^s}\right)\|_{L^\infty_y(L^2_{t,x})}\qquad(7.6)\\
&\le M_k\Big\{\sum_{k_1+[\frac{k_2+1}{2}]\le k-1}\bar{C}_\ell\Big\|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\Big)\Big\|_{L^\infty_y(L^2_{t,x})}\|w^j\|_{L^2_{y,\ell}(L^\infty_{t,x})}\\
&+\sum_{k_1+[\frac{k_2+1}{2}]\le k-1}\Big\|\Big(\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\Big)\Big\|_{L^\infty}\Big\|\Big(\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\int_0^y w^j(t,x,\tilde{y})d\tilde{y}\Big)\Big\|_{L^\infty_y(L^2_{t,x})}\Big\}\\
&\le M_k\bar{C}_\ell\Big((C^a+2\tilde{B}\tilde{C}C_1\theta_j^{\max\{1,\,k+1-\tilde{k}\}})\epsilon\,\|w^j\|_{\mathcal{A}^2_\ell}+M_s\|w^j\|_{\mathcal{A}^{k-1}_\ell}\Big)\\
&\le C_1\epsilon\,\theta_j^{\max\{4-\tilde{k},\,k-1-\tilde{k}\}}\Delta\theta_j,
\end{aligned}$$

by choosing C_1 satisfying

$$C_1\ge M_kC_0\bar{C}_\ell((C^a+2\tilde{B}\tilde{C}C_1)\epsilon+M_s).$$

Then, we have proved (5.15) for $4 \le k-1 \le k_0-1$.

On the other hand, similar to the argument to derive (7.5) from (7.4), one can deduce

$$\sum_{k_1+[\frac{k_2+1}{2}]\le 3} \|\partial_{\mathcal{T}}^{k_1}\partial_y^{k_2}\Big(\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\Big)\|_{L^\infty_y(L^2_{t,x})} \le C\epsilon,$$

for a positive constant $C>0$. Thus, as in (7.6), we obtain

$$\|\frac{\delta u^j}{\partial_y u^s}\|_{\mathcal{D}^3_0} \le C_1\epsilon\,\theta_j^{3-\tilde{k}}\Delta\theta_j.$$

This completes the proof of Lemma 5.7.

We now turn to the proof of Lemma 7.1.

Proof of Lemma 7.1: The case when $0 \le j \le 1$ is obvious. From (7.4), we have

$$\begin{aligned}
\|\sum_{p=0}^{j-1}\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s}\|_{L^\infty} &\le \sum_{p=0}^{j-1}\Big\{\|S_{\theta_j}(\partial_y\frac{\delta u^p}{\partial_y u^s})\|_{L^\infty}\\
&\quad + \|S_{\theta_j}(\delta u^p\frac{\partial_y^2 u^s}{(\partial_y u^s)^2})\|_{L^\infty} + \|[\frac{1}{\partial_y u^s}, S_{\theta_j}](\partial_y(\delta u^p))\|_{L^\infty}\Big\}\\
&\le \sum_{p=0}^{j-1}\Big\{\|S_{\theta_j}(\partial_y\frac{\delta u^p}{\partial_y u^s})\|_{L^\infty} + B\|\frac{\delta u^p}{\partial_y u^s}\|_{L^\infty}\Big\},
\end{aligned}$$

where we have used the estimate of commutators associated with the mollifier S_{θ_j} given in Lemma 4.1.

Suppose now (7.3) holds for $0 \le p \le j-1$. Let us check the case when $p=j$. Obviously, we have

$$\begin{aligned}
S_{\theta_j}(\partial_y\frac{\delta u^p}{\partial_y u^s}) &= S_{\theta_j}\partial_y\left(\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\int_0^y w^p(t,x,\tilde{y})d\tilde{y}\right)\\
&= S_{\theta_j}\left(\Big(\partial_y\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\Big)\int_0^y w^p(t,x,\tilde{y})d\tilde{y}\right) + S_{\theta_j}\left(\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}w^p\right).
\end{aligned} \tag{7.7}$$

For the second term given on the right hand side of (7.7), using the induction hypothesis, we get

$$\begin{aligned}
\sum_{p=0}^{j-1}\|S_{\theta_j}(\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}w^p)\|_{L^\infty} &\le \sum_{p=0}^{j-1}\|\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\|_{L^\infty}\|w^p\|_{L^\infty}\\
&\le M_s\sum_{p=0}^{j-1}\|w^p\|_{\mathcal{A}^2_0} \le M_s C_0\epsilon\sum_{p=0}^{j-1}\theta_p^{3-\tilde{k}}\Delta\theta_p \le M_s C_0\tilde{C}\epsilon\,.
\end{aligned}$$

For the first term on the right hand side of (7.7), using Lemma 4.1 gives when $\tilde{k} \geq 7$,

$$
\begin{aligned}
&\sum_{p=0}^{j-1} \|S_{\theta_j}\left(\Big(\partial_y \frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\Big)\int_0^y w^p(t,x,\tilde{y})d\tilde{y}\right)\|_{L^\infty} \\
&= \sum_{p=0}^{j-1} \|S_{\theta_j}\left\{\partial_y\left(\frac{\partial_y S_{\theta_p}\tilde{u}^0}{\partial_y u^s} + \sum_{q=0}^{p-1}\frac{\partial_y S_{\theta_p}\delta u^q}{\partial_y u^s}\right)\int_0^y w^p(t,x,\tilde{y})d\tilde{y}\right\}\|_{L^\infty} \\
&\leq \sum_{p=0}^{j-1}\Big\{\bar{C}_\ell \|w^p\|_{L^2_{y,\ell}(L^\infty_{t,x})}\Big(\|\partial_y\frac{\partial_y S_{\theta_p}\tilde{u}^0}{\partial_y u^s}\|_{L^\infty} + \sum_{q=0}^{p-1}\big(\theta_p^2\|\frac{\delta u^q}{\partial_y u^s}\|_{L^\infty} \\
&\qquad + \|\partial_y[\frac{1}{\partial_y u^s}, \partial_y S_{\theta_p}]\delta u^q\|_{L^\infty}\big)\Big)\Big\} \\
&\leq \tilde{C}\bar{C}_\ell C_0\epsilon \sum_{p=0}^{j-1}\theta_p^{3-\tilde{k}}\Delta\theta_p \leq \tilde{C}^2\bar{C}_\ell C_0\epsilon\,.
\end{aligned}
$$

In summary, we conclude

$$
\begin{aligned}
\|\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\|_{L^\infty} &\leq 1 + \|\frac{\partial_y \tilde{u}^0_{\theta_j}}{\partial_y u^s}\|_{L^\infty} + \|\sum_{p=0}^{j-1}\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s}\|_{L^\infty} \\
&\leq 1 + C^a\epsilon + \tilde{C}^2\bar{C}_\ell C_0\epsilon + M_s C_0\tilde{C}\epsilon + BC_1\tilde{C}\epsilon,
\end{aligned} \tag{7.8}
$$

which implies the estimate (7.3) by choosing

$$
M_s \geq 2\Big(1 + C^a\epsilon + \tilde{C}^2\bar{C}_\ell C_0\epsilon + BC_1\tilde{C}\epsilon\Big),
$$

and

$$
0 < \epsilon \leq \frac{1}{2C_0\tilde{C}}.
$$

To prove Lemma 5.10 for η^n, we need the following

Lemma 7.2. *Under the assumptions of Lemma 7.1, there exists a constant C_{11}, such that for all $0 \leq j \leq n$,*

$$
\|\frac{\partial_y(u^j_{\theta_j} - u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell} \leq C_{11}\epsilon\theta_j^{\max\{0,k+1-\tilde{k}\}}. \tag{7.9}
$$

Proof: This lemma can also be proved by induction on j. Suppose that it holds for $0 \leq p \leq j-1$, let us study the case when $p = j$.

From the identity

$$
\frac{\partial_y(u^j_{\theta_j} - u^s)}{\partial_y u^s} = \sum_{p=0}^{j-1}\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s} + \frac{\partial_y S_{\theta_j}\tilde{u}^0}{\partial_y u^s},
$$

we have

$$
\begin{aligned}
\|\frac{\partial_y(u^j_{\theta_j}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell} &\le \|\sum_{p=0}^{j-1}\frac{\partial_y S_{\theta_j}\delta u^p}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}+\|\frac{\partial_y S_{\theta_j}\tilde{u}^0}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}\\
&\le \sum_{p=0}^{j-1}\Big\{\|S_{\theta_j}(\partial_y\frac{\delta u^p}{\partial_y u^s})\|_{\mathcal{A}^k_\ell}+\|S_{\theta_j}(\delta u^p\frac{\partial_y^2 u^s}{(\partial_y u^s)^2})\|_{\mathcal{A}^k_\ell}\\
&\quad+\|[\frac{1}{\partial_y u^s},S_{\theta_j}](\partial_y(\delta u^p))\|_{\mathcal{A}^k_\ell}\Big\}+\|\frac{\partial_y S_{\theta_j}\tilde{u}^0}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}\\
&\le \sum_{p=0}^{j-1}\Big\{\|S_{\theta_j}(\partial_y\frac{\delta u^p}{\partial_y u^s})\|_{\mathcal{A}^k_\ell}+\tilde{B}\|\frac{\delta u^p}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}\Big\}+\|\frac{\partial_y S_{\theta_j}\tilde{u}^0}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}.
\end{aligned}
\tag{7.10}
$$

To estimate $\|S_{\theta_j}(\partial_y\frac{\delta u^p}{\partial_y u^s})\|_{\mathcal{A}^k_\ell}$, we use the relation (7.7). For the second term on the right hand side of (7.7), by using (7.3) we have

$$
\begin{aligned}
&\sum_{p=0}^{j-1}\|S_{\theta_j}(\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}w^p)\|_{\mathcal{A}^k_\ell}\le M_k\sum_{p=0}^{j-1}\Big\{\|\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\|_{L^\infty}\|w^p\|_{\mathcal{A}^k_\ell}+\|\frac{\partial_y(u^p_{\theta_p}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}\|w^p\|_{L^\infty}\Big\}\\
&\le M_k\sum_{p=0}^{j-1}\Big\{M_sC_0\epsilon\,\theta_p^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_p+C_{11}C_0\epsilon^2\theta_p^{\max\{3-\tilde{k},k+4-2\tilde{k}\}}\Delta\theta_p\Big\}\\
&\le M_k\tilde{C}\,C_0\epsilon(M_s+C_{11}\epsilon)\theta_j^{\max\{0,k+1-\tilde{k}\}}.
\end{aligned}
$$

For the first term on the right hand side of (7.7), we have for $k\ge 4$,

$$
\begin{aligned}
&\sum_{p=0}^{j-1}\|S_{\theta_j}\left(\Big(\partial_y\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\Big)\int_0^y w^p(t,x,\tilde{y})d\tilde{y}\right)\|_{\mathcal{A}^k_\ell}\\
&\le M_k\theta_j\bar{C}_\ell\sum_{p=0}^{j-1}\Big\{\|w^p\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|\partial_y\frac{\partial_y(u^p_{\theta_p}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^{k-1}_\ell}+\|w^p\|_{\mathcal{A}^{k-1}_\ell}\|\partial_y\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\|_{L^2_{y,\ell}(L^\infty_{t,x})}\Big\}\\
&\le M_k\theta_j\bar{C}_\ell\sum_{p=0}^{j-1}\Big\{\|w^p\|_{\mathcal{A}^2_\ell}\|\frac{\partial_y(u^p_{\theta_p}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}+\|w^p\|_{\mathcal{A}^k_\ell}\|\frac{\partial_y(u^p_{\theta_p}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^3_\ell}\Big\}\\
&\le M_k\theta_j\bar{C}_\ell\sum_{p=0}^{j-1}\Big\{C_0\theta_p^{3-\tilde{k}}\Delta\theta_pC_{11}\epsilon^2\theta_p^{\max\{0,k+1-\tilde{k}\}}+C_0\theta_p^{\max\{3-\tilde{k},\,k-\tilde{k}\}}\Delta\theta_p\,C_{11}\epsilon^2\Big\}\\
&\le 2M_k\bar{C}_\ell C_0\tilde{C}C_{11}\epsilon^2\theta_j^{\max\{1,\,k+2-\tilde{k}\}},
\end{aligned}
$$

and

$$
\sum_{p=0}^{j-1}\|S_{\theta_j}\left(\Big(\partial_y\frac{\partial_y u^p_{\theta_p}}{\partial_y u^s}\Big)\int_0^y w^p(t,x,\tilde{y})d\tilde{y}\right)\|_{\mathcal{A}^3_\ell}\le B_1\epsilon^2,
$$

for a positive constant B_1.

By plugging the above three estimates into the first term on the right hand side of (7.10), (7.9) follows by choosing a proper constant $C_{11}>0$.

Remark 7.3. *From the above argument, it is easy to see that the estimates* (7.3) *and* (7.9) *hold without mollifying* $(\,\cdot\,)_{\theta_j}$*, but for* $k\le k_0-1$.

We are now ready to prove Lemma 5.13.

Proof of Lemma 5.13: From the estimate (7.8), we get immediately that there is $\epsilon_0 > 0$ such that when $0 < \epsilon \leq \epsilon_0$, it holds that

$$\|\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\|_{L^\infty} \leq 2, \quad \inf|\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}| \geq \frac{1}{2}, \tag{7.11}$$

for $0 \leq j \leq n$. Hence, the first estimate given in (5.27) follows.

By using the following Fà Di Bruno formula,

$$\partial^m(g(f)) = m! \sum_{1\leq r\leq m} \frac{1}{r!} g^{(r)}(f) \prod_{m_1+\cdots m_r=m, m_j\geq 1} \frac{1}{m_j!}\partial^{m_j} f,$$

we have,

$$\begin{aligned}
\|(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s})^{-1}\|_{\dot{\mathcal{A}}^k_0} \leq & B_k \sum_{1\leq k_1+[\frac{k_2+1}{2}]\leq k} \sum_{1\leq r\leq k_1+k_2} \|\prod_{1\leq j\leq r} \partial_{\mathcal{T}}^{m_j^1}\partial_y^{m_j^2}\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)\|_{L^2} \\
\leq & B_k \sum_{1\leq k_1+[\frac{k_2+1}{2}]\leq k} \sum_{1\leq r\leq k_1+k_2} \|\partial_{\mathcal{T}}^{m_r^1}\partial_y^{m_r^2}\Big(\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\Big)\|_{L^2} \\
& \times \prod_{1\leq j\leq r-1} \|\partial_{\mathcal{T}}^{m_j^1}\partial_y^{m_j^2}\Big(\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\Big)\|_{L^\infty} \\
\leq & B_k \sum_{1\leq k_1+[\frac{k_2+1}{2}]\leq k} \sum_{1\leq r\leq k_1+k_2} \|\Big(\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\Big)\|_{\mathcal{A}_0^{m_r^1+[\frac{m_r^2+1}{2}]}} \\
& \times \prod_{1\leq j\leq r-1} \|\Big(\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\Big)\|_{\mathcal{A}_0^{m_j^1+2+[\frac{m_j^2+2}{2}]}},
\end{aligned}$$

where $m_1^1+\cdots m_r^1 = k_1, m_1^2+\cdots m_r^2 = k_2, m_j^1+m_j^2 \geq 1$, with m_r^1 and m_r^2 being supposed to be the largest integers in the corresponding group of indices, respectively. Then, by using (7.9) in the above inequality, it follows

$$\|(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s})^{-1}\|_{\dot{\mathcal{A}}^k_0} \leq C_7\epsilon\,\theta_n^{\max\{0,k+1-\tilde{k}\}}, \tag{7.12}$$

for a positive constant $C_7 > 0$. This completes the proof of the lemma.

Now, we turn to the estimates on $\eta^n = \frac{\partial_y^2 u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}$ stated in Lemma 5.10.

Proof of Lemma 5.10: Obviously, we have

$$\begin{aligned}
\|\eta^n - \bar{\eta}^n\|_{\mathcal{A}^k_\ell} &= \|\frac{\partial_y^2(u^n_{\theta_n}-u^s)}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^k_\ell} \\
&\leq M_k\Big(\|\frac{\partial_y^2(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}\|(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s})^{-1}\|_{L^\infty} \\
&\quad + \|\frac{\partial_y^2(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{L^\infty_\ell}\|\frac{\partial_y u^s}{\partial_y u^n_{\theta_n}}\|_{\dot{\mathcal{A}}^k_0}\Big).
\end{aligned} \tag{7.13}$$

We now estimate term by term on the right hand side of (7.13). Since

$$\frac{\partial_y^2(u^n_{\theta_n}-u^s)}{\partial_y u^s}=\partial_y\Big(\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\Big)+\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\frac{\partial_y^2 u^s}{\partial_y u^s},$$

by using (7.9), we have for $k\geq 4$

$$\begin{aligned}
&\|\frac{\partial_y^2(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}\\
&\leq M_k\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^k_0}\|\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}+\|\partial_y\Big(\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\Big)\|_{\mathcal{A}^k_\ell}\\
&\leq M_k\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^k_0}\|\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^k_\ell}+C_\rho\theta_n\|\Big(\frac{\partial_y(u^n-u^s)}{\partial_y u^s}\Big)\|_{\mathcal{A}^k_\ell}\\
&\leq \tilde{C}_{11\epsilon}\,\theta_n^{1+\max\{0,k+1-\tilde{k}\}}.
\end{aligned}\tag{7.14}$$

On the other hand, by using (7.9), it holds that

$$\begin{aligned}
\|\frac{\partial_y^2(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^3_\ell}&\leq M_3\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^3_0}\|\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^3_\ell}+\|\frac{\partial_y(u^n_{\theta_n}-u^s)}{\partial_y u^s}\|_{\mathcal{A}^4_\ell}\\
&\leq (M_3\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^3_0}+1)C_{11\epsilon}.
\end{aligned}\tag{7.15}$$

Plugging the estimates (7.14), (7.15) and (7.12) into (7.13), we obtain the estimate (5.24) given in Lemma 5.10.

To derive the estimate (5.25), from the definition of $\bar{\eta}^n$, we have

$$\begin{aligned}
\|\bar{\eta}^n\|_{\mathcal{C}^k_\ell} &\leq M_k\left(\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^k_\ell}\|\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)^{-1}\|_{L^\infty}+\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{L^\infty_\ell}\|\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)^{-1}\|_{\dot{\mathcal{C}}^k_0}\right)\\
&\leq M_k\left(\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^k_\ell}\|\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)^{-1}\|_{L^\infty}+\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{L^\infty_\ell}\|\Big(\frac{\partial_y u^n_{\theta_n}}{\partial_y u^s}\Big)^{-1}\|_{\dot{\mathcal{A}}^{k+2}_0}\right)\\
&\leq M_k(2\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}^k_\ell}+\|\frac{\partial_y^2 u^s}{\partial_y u^s}\|_{L^\infty_\ell}C_7\epsilon\theta_n^{\max\{0,\,k+3-\tilde{k}\}}),
\end{aligned}$$

where we have used (7.12) and (7.11). Thus, we get the estimate (5.25) immediately. And this completes the proof of the lemma.

To estimate ζ^n, similar to the proof for Lemma 7.1, from

$$\begin{aligned}
\tfrac{\partial^2_{xy}S_{\theta_j}u^j}{\partial_y u^s} &= \tfrac{\partial^2_{xy}(S_{\theta_j}\tilde{u}^0)}{\partial_y u^s}+\sum_{p=0}^{j-1}\Big(S_{\theta_j}(\tfrac{\partial^2_{xy}\delta u^p}{\partial_y u^s})+[\tfrac{1}{\partial_y u^s},S_{\theta_j}]\partial^2_{xy}\delta u^p\Big)\\
&= \tfrac{\partial^2_{xy}(S_{\theta_j}\tilde{u}^0)}{\partial_y u^s}+\sum_{p=0}^{j-1}\Big(S_{\theta_j}\left(\partial_x w^p+\partial_x(\tfrac{\delta u^p}{\partial_y u^s})\tfrac{\partial^2_y u^s}{\partial_y u^s}\right)+[\tfrac{1}{\partial_y u^s},S_{\theta_j}]\partial^2_{xy}\delta u^p\Big),
\end{aligned}$$

we have

Lemma 7.4. *Under the assumptions of Lemma 7.1, there is a constant $C_{12}>0$, such that*

$$\|\frac{\partial^2_{xy}u^j_{\theta_j}}{\partial_y u^j_{\theta_j}}\|_{\mathcal{A}^k_\ell}\leq C_{12\epsilon}\theta_j^{1+\max\{0,k+1-\tilde{k}\}},$$

and

$$\|\frac{\partial^2_{xy}u^j_{\theta_j}}{\partial_y u^j_{\theta_j}}\|_{\mathcal{A}^3_\ell}\leq C_{12\epsilon},$$

hold for all $0 \le j \le n$.

With this, we are now ready to prove Lemma 5.11.

Proof of Lemma 5.11: First of all, note that

$$\|\frac{v^n_{\theta_n}\partial_y^2 u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^k_\ell} \le M_k(\|v^n_{\theta_n}\|_{L^\infty}\|\eta^n-\bar\eta^n\|_{\mathcal{A}^k_\ell} + \|v^n_{\theta_n}\|_{L^\infty_y(L^2_{t,x})}\|\bar\eta^n\|_{\mathcal{C}^k_\ell} + \|\eta^n\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|v^n_{\theta_n}\|_{\mathcal{D}^k_0}).$$

By using Lemma 5.9 and Lemma 5.10, we get

$$\begin{aligned}\|\frac{v^n_{\theta_n}\partial_y^2 u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^k_\ell} &\le M_k\Big(C_3C_4\epsilon\theta_n^{\max\{1,k+2-\tilde k\}} + C_3C_4(1+\epsilon\theta_n^{\max\{0,k+3-\tilde k\}}) \\ &\qquad + 2C_4C_3\epsilon\theta_n^{\max\{1,k+2-\tilde k\}}\Big) \\ &\le C_5\theta_n^{\max\{1,k+3-\tilde k\}},\end{aligned}$$

when $k \ge 4$ for a positive constant C_5. Moreover,

$$\|\frac{v^n_{\theta_n}\partial_y^2 u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^3_\ell} \le M_3(C_3C_4\epsilon + C_3C_4(1+\epsilon) + 2C_3C_4\epsilon) \le C_8.$$

Similarly, by using (2.1) and Lemma 7.4, we can show that

$$\begin{aligned}\|\tfrac{u^n_{\theta_n}\partial^2_{xy}u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^k_\ell} &\le M_k(\|u^n_{\theta_n}\|_{L^\infty}\|\tfrac{\partial^2_{xy}u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^k_\ell} + \|\tfrac{\partial^2_{xy}u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{L^\infty}\|u^n_{\theta_n}-u^s\|_{\mathcal{A}^k_\ell} \\ &\qquad + \|\tfrac{\partial^2_{xy}u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{L^\infty_y(L^2_{t,x})}\|u^s\|_{\mathcal{C}^k_\ell}) \\ &\le M_k\Big(C_s(C^a+C_3\epsilon)C_{12}\epsilon\theta_n^{1+\max\{0,k+1-\tilde k\}} \\ &\qquad + C_s(C^a+C_3\epsilon\theta_n^{\max\{0,k+1-\tilde k\}})C_{12}\epsilon\Big) \\ &\le C_5\theta_n^{\max\{1,k+2-\tilde k\}}.\end{aligned}$$

By noticing that

$$\frac{(\partial_t-\partial_y^2)\partial_y u^n_{\theta_n}}{\partial_y u^n_{\theta_n}} = \frac{(\partial_t-\partial_y^2)\partial_y u^n_{\theta_n}}{\partial_y u^s}\cdot\frac{\partial_y u^s}{\partial_y u^n_{\theta_n}} = \frac{(\partial_t-\partial_y^2)\partial_y (u^n_{\theta_n}-u^s)}{\partial_y u^s}\frac{\partial_y u^s}{\partial_y u^n_{\theta_n}},$$

using (7.9) and (7.12), we have

$$\|\frac{(\partial_t-\partial_y^2)\partial_y u^n_{\theta_n}}{\partial_y u^n_{\theta_n}}\|_{\mathcal{A}^k_\ell} \le C_5\theta_n^{\max\{0,\,k+2-\tilde k\}}.$$

Plugging the above estimates into the definition of ζ^n in (5.23), it leads to the (5.26) given in Lemma 5.11 and then completes its proof.

Finally, let us give the proof of Lemma 5.14.

Proof of Lemma 5.14: Recall the definition

$$e_n^{(1)} = \delta u^n\partial_x\delta u^n + \delta v^n\partial_y\delta u^n,$$

and

$$\begin{aligned}e_n^{(2)} = \ &\big((1-S_{\theta_n})(u^n-u^s)\big)\partial_x(\delta u^n) + \delta u^n\partial_x\big((1-S_{\theta_n})(u^n-u^s)\big) \\ &+\delta v^n\partial_y\big((1-S_{\theta_n})(u^n-u^s)\big) + \big((1-S_{\theta_n})v^n\big)\partial_y(\delta u^n).\end{aligned}$$

We get

$$\begin{aligned}\|\frac{e_j^{(1)}}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}} &\le M_{k_1}\Big\{\|\partial_x\delta u^j\|_{\mathcal{A}_\ell^{k_1}}\|\frac{\delta u^j}{\partial_y u^s}\|_{L^\infty}+\|\partial_x\delta u^j\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|\frac{\delta u^j}{\partial_y u^s}\|_{\mathcal{D}_\ell^{k_1}}\\ &\quad+\|\delta v^j\|_{L^\infty}\|\frac{\partial_y\delta u^j}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}+\|\frac{\partial_y\delta u^j}{\partial_y u^s}\|_{L^2_{y,l}(L^\infty_{t,x})}\|\delta v^j\|_{\mathcal{D}_0^{k_1}}\Big\}.\end{aligned}\tag{7.16}$$

Obviously, from

$$\frac{\partial_y\delta u^j}{\partial_y u^s}=\frac{\partial_y^2 u^j_{\theta_j}}{\partial_y u^s}\int_0^y w^j(t,x,\tilde{y})d\tilde{y}+\frac{\partial_y u^j_{\theta_j}}{\partial_y u^s}w^j,$$

we have

$$\begin{aligned}&\|\tfrac{\partial_y\delta u^j}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}\le\|\tfrac{\partial_y^2(u^j_{\theta_j}-u^s)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}\|w^j\|_{\mathcal{C}_\ell^0}+\|\tfrac{\partial_y^2 u^s}{\partial_y u^s}\|_{\mathcal{C}_\ell^{k_1}}\|w^j\|_{L^2_\ell}\\ &+\|\tfrac{\partial_y^2 u^j_{\theta_j}}{\partial_y u^s}\|_{\mathcal{C}_\ell^0}\|w^j\|_{\mathcal{A}_\ell^{k_1}}+\|\tfrac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\|_{L^\infty}\|w^j\|_{\mathcal{A}_\ell^{k_1}}+\|\tfrac{\partial_y u^j_{\theta_j}}{\partial_y u^s}\|_{\dot{\mathcal{A}}^{k_1}}\|w^j\|_{L^\infty_\ell}\\ \le\;& C_0C_{11}\epsilon^2\theta_j^{\max\{3-\tilde{k},k_1+5-2\tilde{k}\}}\Delta\theta_j+(C_{11}\epsilon+C^a)C_0\epsilon\theta_j^{\max\{3-\tilde{k},\,k_1-\tilde{k}\}}\Delta\theta_j\\ &+C^aC_0\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j+2C_0\epsilon\theta_j^{\max\{3-\tilde{k},\,k_1-\tilde{k}\}}\Delta\theta_j+(C_{11}\epsilon\theta_j^{\max\{0,\,k_1+1-\tilde{k}\}}+C^a)C_0\epsilon\theta_j^{3-\tilde{k}}\Delta\theta_j,\end{aligned}\tag{7.17}$$

where we have used (7.9), (7.11) and (5.12).

If we choose a constant

$$\tilde{C}_{11}\ge C_0(2+3C^a+3C_{11}\epsilon_0),$$

then from (7.17), we get

$$\|\frac{\partial_y\delta u^j}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}\le\tilde{C}_{11}\epsilon\theta_j^{\max\{3-\tilde{k},k_1-\tilde{k}\}}\Delta\theta_j,\tag{7.18}$$

by using $\tilde{k}\ge 7$.

By using (7.18), Lemmas 5.7 and 5.8, from (7.16), there exists a constant $C_8>0$ such that

$$\|\frac{e_j^{(1)}}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}\le C_8\epsilon^2\theta_j^{\max\{6-2\tilde{k},\,k_1+3-2\tilde{k}\}}\Delta\theta_j,$$

for all $k_1\le k_0-1$.

Similarly,

$$\begin{aligned}\|\frac{e_j^{(2)}}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}&\le\|\Big((1-S_{\theta_j})(u^j-u^s)\Big)\frac{\delta u^j}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1+1}}\\ &\quad+\|\delta v^j\,\frac{\partial_y((1-S_{\theta_j})(u^j-u^s))}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}+\|\Big((1-S_{\theta_j})v^j\Big)\frac{\partial_y(\delta u^j)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}},\end{aligned}$$

by using Lemma 2.1, the second formula of (4.1) and (5.21), we get

$$
\begin{aligned}
&\|\frac{e_j^{(2)}}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}} \le M_{k_1}\Big\{\|u^j-u^s\|_{\mathcal{A}_\ell^{k_1+1}}\|\frac{\delta u^j}{\partial_y u^s}\|_{L^\infty}\\
&+\|(1-S_{\theta_j})(u^j-u^s)\|_{L^2_{y,\ell}(L^\infty_{t,x})}\|\frac{\delta u^j}{\partial_y u^s}\|_{\mathcal{D}_0^{k_1+1}}\\
&+\|\delta v^j\|_{L^\infty}\|\frac{\partial_y(u^j-u^s)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}+\|\delta v^j\|_{\mathcal{D}_0^{k_1}}\|(1-S_{\theta_j})(\frac{\partial_y(u^j-u^s)}{\partial_y u^s})\|_{L^2_{y,\ell}(L^\infty_{t,x})}\\
&+\|(1-S_{\theta_j})v^j\|_{L^\infty}\|\frac{\partial_y(\delta u^j)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}+\|v^j\|_{\mathcal{D}_0^{k_1}}\|\frac{\partial_y(\delta u^j)}{\partial_y u^s}\|_{L^2_{y,\ell}(L^\infty_{t,x})}\Big\}\\
&\le M_{k_1}\Big\{\|u^j-u^s\|_{\mathcal{A}_\ell^{k_1+1}}\|\frac{\delta u^j}{\partial_y u^s}\|_{L^\infty}+\theta_j^{2-k'}\|u^j-u^s\|_{\mathcal{A}_\ell^{k'}}\|\frac{\delta u^j}{\partial_y u^s}\|_{\mathcal{D}_0^{k_1+1}}\\
&+\|\delta v^j\|_{L^\infty}\|\frac{\partial_y(u^j-u^s)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}+\theta_j^{2-k'}\|\delta v^j\|_{\mathcal{D}_0^{k_1}}\|\frac{\partial_y(u^j-u^s)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k'}}\\
&+\theta_j^{2-k'}\|v^j\|_{\mathcal{D}_0^{k'}}\|\frac{\partial_y(\delta u^j)}{\partial_y u^s}\|_{\mathcal{A}_\ell^{k_1}}+\|v^j\|_{\mathcal{D}_0^{k_1}}\|\frac{\partial_y(\delta u^j)}{\partial_y u^s}\|_{L^2_{y,\ell}(L^\infty_{t,x})}\Big\},
\end{aligned}
$$

for a fixed integer $2 \le k' \le k_0 - 2$.

By applying Lemma 5.7, Lemma 5.8 and estimates (7.9), (7.18) to the above inequality, and by setting $k' \ge \tilde{k} - 2$, it follows that for $k_1 \le k_0 - 1$,

$$
\|\frac{e_j^{(2)}}{\partial_y u^s}\|_{\mathcal{A}_l^{k_1}} \le C_8\epsilon^2\theta_j^{\max(3-\tilde{k},k_1+5-2\tilde{k})}\Delta\theta_j,
$$

for a positive constant C_8. And this completes the proof of the lemma.

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References

[1] Alinhac, S. & Gérard, P.: *Pseudo-Differential Operators and The Nash-Moser Theorem*. Translated from the 1991 French original by Stephen S. Wilson. Graduate Studies in Mathematics, 82. American Mathematical Society, Providence, RI, 2007. viii+168 pp.

[2] Caflisch, R.E. & Sammartino, M.: Existence and singularities for the Prandtl boundary layer equations, *Z. Angew. Math. Mech.*, 80(2000), 733-744.

[3] Chazarain, J. & Piriou, A.: *Introduction to The Theory of Linear Partial Differential Equations*, North-Holland Publishing Company, Amsterdam, 1982.

[4] E, W.: Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)* 16(2000), 207-218.

[5] E, W. & Engquist, B.: Blow up of solutions of the unsteady Prandtl's equation, *Comm. Pure Appl. Math.*, 50(1997), 1287-1293.

[6] Gérard-Varet, D. & Dormy, E.: On the ill-posedness of the Prandtl equation, *J. Amer. Math. Soc.*, 23(2010), 591-609.
[7] Gérard-Varet, D. & Nguyen, T.: Remarks on the ill-posedness of the Prandtl equation, preprint 2010. arXiv:1008.0532v1.
[8] Grenier, E.: On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* 53(2000), 1067-1091.
[9] Guo, Y. & Nguyen, T.: A note on the Prandtl boundary layers, *Comm. Pure Appl. Math.* 64 (2011) 1416-1438, doi: 10.1002/cpa.20377
[10] Hamilton, R.: The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.*, 7(1982), 65-222.
[11] Hong, L. & Hunter, J. K.: Singularity formation and instability in the unsteady inviscid and viscous Prandtl equations, *Commun. Math. Sci.* 1 (2003), 293-316.
[12] Hörmander, L.: *The Analysis of Linear Partial Differential Operators* (Vol. I-IV), Springer Verlag 1985.
[13] Hörmander, L.: The boundary problems of physical geodesy. *Arch. Rational Mech. Anal.*, 62(1982), 1-52.
[14] Lombardo, M. C., Cannone, M. & Sammartino, M.: Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.*, 35(2003), 987-1004 (electronic).
[15] Métivier, G.: *Small Viscosity and Boundary Layer Methods. Theory, Stability Analysis, and Applications*. Modeling and Simulation in Science, Engineering and Technology. Birkhauser Boston, Inc., Boston, MA, 2004. xxii+194 pp.
[16] Moser, J.: A new technique for the construction of solutions of nonlinear differential equations, *Proc. Nat. Acad. Sci.*, 47(1961), 1824-1831.
[17] Nash, J.: The imbedding problem for Riemannian manifolds, *Ann. of Math.*, 63(1956), 20-63.
[18] Oleinik, O. A.: The Prandtl system of equations in boundary layer theory. *Soviet Math Dokl*, 4(1963), 583-586.
[19] Oleinik, O. A. & Samokhin, V. N.: *Mathematical Models in Boundary Layers Theory*. Chapman & Hall/CRC, 1999.
[20] Prandtl, L.: Über Flüssigkeitsbewegungen bei sehr kleiner Reibung. In "*Verh. Int. Math. Kongr., Heidelberg 1904*", Teubner 1905, 484-494.
[21] Sammartino, M. & Caflisch, R. E.: Zero viscosity limit for analytic solutions of the Navier-Stokes equations on a half-space, I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(1998), 433-461; II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(1998), 463-491.
[22] Xin, Z. & Zhang, L.: On the global existence of solutions to the Prandtl's system, *Adv. Math.*, 181(2004), 88-133.

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