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Stéphane Gaussent, Yves Guiraud, Philippe Malbos

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# COHERENT PRESENTATIONS OF ARTIN GROUPS

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STÉPHANE GAUSSENT      YVES GUIRAUD      PHILIPPE MALBOS

**Abstract** – A coherent presentation of a monoid is an extension of a presentation of this monoid by a homotopy basis, making a natural cellular complex associated to the presentation contractible. In the case of Artin monoids, we show that the usual presentation defined by Artin, using braid relations, can be completed in a coherent presentation that we give in an explicit way. To be able to handle presentations that are not confluent, we develop a homotopical completion-reduction procedure that combines and extends methods of rewriting systems introduced by Squier and by Knuth and Bendix. Since any Artin monoid embeds in its Artin group, the coherent presentation of the monoid gives a coherent presentation of the group. In addition, the category of actions of a monoid on categories is equivalent to the category of 2-functors from a coherent presentation of the monoid to  $\text{Cat}$ . In this vein, our procedure gives also a new proof of a theorem of Deligne concerning the action of an Artin monoid on a category in terms of a presentation based on the Garside structure.

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**Coherent presentations and actions on categories.** In [13, 1.3.2], Deligne uses a formulation in terms of actions of monoids on categories, instead of coherent presentations: an action  $T$  of a monoid  $\mathbf{M}$  on a category  $\mathbf{C}$  is a family of endofunctors  $T(u)$  of  $\mathbf{C}$ , one for every element  $u$  of  $\mathbf{M}$ , such that  $T$  is compatible with the product of  $\mathbf{M}$ . This compatibility is expressed by natural isomorphisms

$$T(u)T(v) \xrightarrow{T_{u,v}} T(uv)$$

that, in turn, satisfy coherence conditions with respect to the associativity of the product of  $\mathbf{M}$ :

$$\begin{array}{ccc}
 & T_{u,v}T(w) \xrightarrow{\quad} & T(uv)T(w) \xrightarrow{\quad} T_{uv,w} \\
 & \nearrow & \searrow \\
 T(u)T(v)T(w) & = & T(uvw) \\
 & \searrow & \nearrow \\
 & T(u)T_{v,w} \xrightarrow{\quad} & T(u)T(vw) \xrightarrow{\quad} T_{u,vw}
 \end{array} \tag{1}$$

This definition of action has an interpretation in terms of coherent presentations. Indeed, every monoid  $\mathbf{M}$  admits a canonical presentation  $\text{Can}(\mathbf{M})$  whose generators are the elements of  $\mathbf{M}$  and whose relations are given by the multiplication table of the monoid  $\mathbf{M}$ . This presentation  $\text{Can}(\mathbf{M})$  can be extended into a coherent presentation, denoted by  $\text{Can}_+(\mathbf{M})$ , by adjunction of the 3-cells corresponding to the associativity of the product: the images of these 3-cells under  $T$  are exactly the coherence conditions (1).

In fact, one gets the same definition of action if one replaces the coherent presentation  $\text{Can}_+(\mathbf{M})$  by any other coherent presentation  $\Sigma$  of  $\mathbf{M}$ . More precisely, up to equivalence, an action of  $\mathbf{M}$  on  $\mathbf{C}$  is specified by an endofunctor  $T(x)$  of  $\mathbf{C}$  for every generator  $x$  of  $\Sigma_1$  and a natural isomorphism between  $T(u)$  and  $T(v)$  for every relation  $u = v$  of  $\Sigma_2$ , such that the elements of  $\Sigma_3$  are mapped to commutative diagrams by  $T$ .

**Theorem 3.4.3.** *If  $\Sigma$  is a coherent presentation of a monoid  $\mathbf{M}$ , then the category  $\text{Act}(\mathbf{M})$  of actions of  $\mathbf{M}$  on categories is equivalent to the category of 2-functors from  $\Sigma^\top$  to  $\text{Cat}$  that send the elements of the homotopy basis to commutative diagrams.*

In [13, Theorem 1.5], Deligne already observes that this equivalence holds for a specific presentation of spherical Artin monoids. This presentation  $\text{Del}(\mathbf{W})$  of  $\mathbf{B}^+(\mathbf{W})$ , based on the Garside structure of  $\mathbf{B}^+(\mathbf{W})$ , has the elements of  $\mathbf{W} \setminus \{1\}$  as generators and the

$$u|v = uv \quad \text{if } l(uv) = l(u) + l(v)$$

as relations. The notation  $u|v$  stands for the product in the free monoid over  $\mathbf{W} \setminus \{1\}$  and  $l(u)$  is the length of  $u$  in  $\mathbf{W}$ . In Deligne's theorem on actions of  $\mathbf{B}^+(\mathbf{W})$ , the coherence conditions correspond to the following 3-cells of  $\text{Del}(\mathbf{W})^\top$ :

$$\begin{array}{ccc}
 & uv|w & \\
 & \nearrow & \searrow \\
 u|v|w & A_{u,v,w} & uvw \\
 & \searrow & \nearrow \\
 & u|vw &
 \end{array} \quad \text{if } l(uvw) = l(u) + l(v) + l(w).$$

## 1. Introduction

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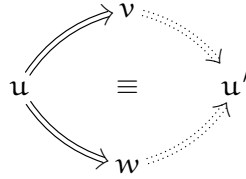
We prove that the 3-cells  $A_{u,v,w}$  extend  $\text{Del}(\mathbf{W})$  into a coherent presentation, denoted by  $\text{Del}_+(\mathbf{W})$ , so that Deligne's result also holds for non-spherical Artin monoids.

**Theorem 5.1.4.** *For every Coxeter group  $\mathbf{W}$ , the Artin monoid  $\mathbf{B}^+(\mathbf{W})$  admits  $\text{Del}_+(\mathbf{W})$  as a coherent presentation.*

**Rewriting methods for coherence.** As initiated by Squier in [35], rewriting theory provides methods to compute homotopy bases for specific presentations. Rewriting is a combinatorial theory of equalities in algebraic structures that consists in replacing equalities  $u = v$  by oriented relations  $u \Rightarrow v$  to define decision procedures. In particular, a presentation  $\Sigma$  of a monoid  $\mathbf{M}$  is *convergent* if it satisfies the following two conditions:

- *termination, i.e.*, there is no infinite reduction  $u_1 \Rightarrow u_2 \Rightarrow u_3 \Rightarrow (\dots)$ ,
- *confluence, i.e.*, two different reductions  $u \Rightarrow v$  and  $u \Rightarrow w$  can always be joined by some  $v \Rightarrow u'$  and  $w \Rightarrow u'$ .

*Squier's completion* is a procedure that extends a convergent presentation  $\Sigma$  of a monoid  $\mathbf{M}$  into a coherent presentation  $\mathcal{S}(\Sigma)$  of  $\mathbf{M}$ . It is based on the study of the *critical branchings* of  $\Sigma$ : those are the minimal overlaps of the left members of the relations. The coherent presentation  $\mathcal{S}(\Sigma)$  is obtained from  $\Sigma$  by adjunction of all the 3-cells



where the plain part is a critical branching and the dotted part is obtained by the confluence hypothesis.

For example, the 3-cells  $A_{u,v,w}$  of Deligne's coherent presentation  $\text{Del}_+(\mathbf{W})$  are obtained by this construction if the relations of  $\text{Del}(\mathbf{W})$  are directed as in  $u|v \Rightarrow uv$ . However, in general, Squier's completion does not apply directly to the presentations  $\text{Art}(\mathbf{W})$  or  $\text{Del}(\mathbf{W})$  because they lack confluence. To apply the rewriting methods to the case of Artin monoids, we extend Squier's completion to non-confluent presentations, yielding a *homotopical completion-reduction* procedure. The homotopical completion-reduction  $\mathcal{R}(\Sigma)$  of a terminating presentation  $\Sigma$  is obtained as follows:

1. *Homotopical completion* combines Squier's completion and *Knuth-Bendix completion procedure*, a method from rewriting theory that adds sufficiently many relations to a terminating presentation to achieve confluence.
2. *Homotopical reduction* eliminates some unnecessary generators, relations and cells of the homotopy basis. In particular, the *critical triple branchings* are used as redundancy relations between the 3-cells.

**Theorem 4.5.1.** *If  $\Sigma$  is a terminating presentation of a monoid  $\mathbf{M}$ , then the homotopical completion-reduction  $\mathcal{R}(\Sigma)$  of  $\Sigma$  is a coherent presentation of  $\mathbf{M}$ .*

For every Coxeter group  $\mathbf{W}$ , the homotopical completion-reduction of Deligne's presentation  $\text{Del}(\mathbf{W})$  of  $\mathbf{B}^+(\mathbf{W})$  is Artin's coherent presentation  $\text{Art}_+(\mathbf{W})$ . This proves Theorem 2.4.5 for Artin monoids. Moreover, the homotopical completion-reduction process also yields Deligne's coherent presentation  $\text{Del}_+(\mathbf{W})$ , thus proving Theorem 5.1.4. Finally, the homotopical completion-reduction procedure is also used to adapt  $\text{Art}_+(\mathbf{W})$  into a coherent presentation of the Artin group  $\mathbf{B}(\mathbf{W})$ , yielding Theorem 2.4.5 for Artin groups.

As a final remark, in [21, Theorem 4.5.3], Squier's completion is extended in higher dimensions to produce *polygraphic resolutions* of monoids, of which coherent presentations form the first three dimensions. From that point of view, the present work is a first step towards the construction of polygraphic resolutions  $\text{Del}_*(\mathbf{W})$  and  $\text{Art}_*(\mathbf{W})$  of Artin monoids and Artin groups, extending the coherent presentations  $\text{Del}_+(\mathbf{W})$  and  $\text{Art}_+(\mathbf{W})$ . Moreover, the abelian resolutions obtained from  $\text{Del}_*(\mathbf{W})$  and  $\text{Art}_*(\mathbf{W})$  by [21, Theorem 5.4.3] should be related to the abelian resolutions introduced in [10].

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## 2. COHERENT PRESENTATIONS OF CATEGORIES AND GROUPOIDS

A *coherent presentation* of a monoid or, more generally, of a category is a data made of generators, generating relations and generating coherence conditions. This is formalised in terms of *polygraphs*, which are presentations of higher-dimensional categories introduced by Burroni in [7] and by Street in [36], under the name of *computad*. We refer the reader to [30] for more details on higher categories. In this section, we recall rewriting methods, initiated by Squier [35], to compute coherent presentations and we formulate the main theorem about the coherent presentations of Artin monoids and Artin groups.

### 2.1. Higher-dimensional categories

If  $\mathcal{C}$  is an  $n$ -category (we always consider strict, globular  $n$ -categories), we denote by  $\mathcal{C}_k$  the set (and the  $k$ -category) of  $k$ -cells of  $\mathcal{C}$ . If  $f$  is a  $k$ -cell of  $\mathcal{C}$ , then  $s_i(f)$  and  $t_i(f)$  respectively denote the  $i$ -source and  $i$ -target of  $f$ ; we drop the suffix  $i$  if  $i = k - 1$ . The source and target maps satisfy the *globular relations*:

$$s_i \circ s_{i+1} = s_i \circ t_{i+1} \quad \text{and} \quad t_i \circ s_{i+1} = t_i \circ t_{i+1}.$$

We respectively denote by  $f : u \rightarrow v$ ,  $f : u \rightrightarrows v$ ,  $f : u \Rrightarrow v$  and  $f : u \Rrightarrow v$  a 1-cell, a 2-cell, a 3-cell and a 4-cell  $f$  with source  $u$  and target  $v$ . If  $f$  and  $g$  are  $i$ -composable  $k$ -cells, that is if  $t_i(f) = s_i(g)$ , we denote by  $f \star_i g$  their  $i$ -composite; we simply write  $fg$  if  $i = 0$ . The compositions satisfy the *exchange relations* given, for every  $i \neq j$  and every possible cells  $f, g, h$  and  $k$ , by:

$$(f \star_i g) \star_j (h \star_i k) = (f \star_j h) \star_i (g \star_j k).$$

If  $f$  is a  $k$ -cell, we denote by  $1_f$  its identity  $(k + 1)$ -cell. If  $1_f$  is composed with cells of dimension  $k + 1$  or higher, we simply denote it by  $f$ .

## 2. Coherent presentations of categories and groupoids

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**2.1.1.  $(n, p)$ -categories.** In an  $n$ -category  $\mathcal{C}$ , a  $k$ -cell  $f$ , with source  $x$  and target  $y$ , is *invertible* if there exists a (necessarily unique)  $k$ -cell  $f^-$  in  $\mathcal{C}$ , with source  $y$  and target  $x$  in  $\mathcal{C}$ , called the *inverse of  $f$* , such that

$$f \star_{k-1} f^- = 1_x \quad \text{and} \quad f^- \star_{k-1} f = 1_y.$$

An  $(n, p)$ -category is an  $n$ -category whose  $k$ -cells are invertible for every  $k > p$ . In particular, an  $(n, n)$ -category is an ordinary  $n$ -category and an  $(n, 0)$ -category is an  $n$ -groupoid.

**2.1.2. Spheres.** Let  $\mathcal{C}$  be an  $n$ -category. A  $0$ -sphere of  $\mathcal{C}$  is a pair  $\gamma = (f, g)$  of  $0$ -cells of  $\mathcal{C}$  and, for  $1 \leq k \leq n$ , a  $k$ -sphere of  $\mathcal{C}$  is a pair  $\gamma = (f, g)$  of parallel  $k$ -cells of  $\mathcal{C}$ , i.e., with  $s(f) = s(g)$  and  $t(f) = t(g)$ ; we call  $f$  the *source* of  $\gamma$  and  $g$  its *target* and we write  $s(\gamma) = f$  and  $t(\gamma) = g$ . If  $f$  is a  $k$ -cell of  $\mathcal{C}$ , for  $1 \leq k \leq n$ , the *boundary of  $f$*  is the  $(k-1)$ -sphere  $(s(f), t(f))$ .

**2.1.3. Cellular extensions.** Let  $\mathcal{C}$  be an  $n$ -category. A *cellular extension of  $\mathcal{C}$*  is a set  $\Gamma$  equipped with a map from  $\Gamma$  to the set of  $n$ -spheres of  $\mathcal{C}$ , whose value on  $\gamma$  is denoted by  $(s(\gamma), t(\gamma))$ . By considering all the formal compositions of elements of  $\Gamma$ , seen as  $(n+1)$ -cells with source and target in  $\mathcal{C}$ , one builds the *free  $(n+1)$ -category generated by  $\Gamma$  over  $\mathcal{C}$* , denoted by  $\mathcal{C}[\Gamma]$ . The *quotient of  $\mathcal{C}$  by  $\Gamma$* , denoted by  $\mathcal{C}/\Gamma$ , is the  $n$ -category one gets from  $\mathcal{C}$  by identification of the  $n$ -cells  $s(\gamma)$  and  $t(\gamma)$ , for every  $n$ -sphere  $\gamma$  of  $\Gamma$ . If  $\mathcal{C}$  is an  $(n, 1)$ -category and  $\Gamma$  is a cellular extension of  $\mathcal{C}$ , then the *free  $(n+1, 1)$ -category generated by  $\Gamma$  over  $\mathcal{C}$*  is denoted by  $\mathcal{C}(\Gamma)$  and defined as follows:

$$\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \check{\Gamma}] / \text{Inv}(\Gamma)$$

where  $\check{\Gamma}$  contains the same  $(n+1)$ -cells as  $\Gamma$ , with source and target reversed, and  $\text{Inv}(\Gamma)$  is the cellular extension made of two  $(n+2)$ -cells

$$\check{\gamma} \star_n \gamma \xrightarrow{\lambda_\gamma} 1_g \quad \text{and} \quad \gamma \star_n \check{\gamma} \xrightarrow{\rho_\gamma} 1_f$$

for each  $(n+1)$ -cell  $\gamma$  from  $f$  to  $g$  in  $\Gamma$ .

**2.1.4. Homotopy bases.** Let  $\mathcal{C}$  be an  $n$ -category. A *homotopy basis of  $\mathcal{C}$*  is a cellular extension  $\Gamma$  of  $\mathcal{C}$  such that, for every  $n$ -sphere  $\gamma$  of  $\mathcal{C}$ , there exists an  $(n+1)$ -cell with boundary  $\gamma$  in  $\mathcal{C}(\Gamma)$  or, equivalently, if the quotient  $n$ -category  $\mathcal{C}/\Gamma$  has  $n$ -spheres of shape  $(f, f)$  only. For example, the  $n$ -spheres of  $\mathcal{C}$  form a homotopy basis of  $\mathcal{C}$ .

## 2.2. Coherent presentations of categories and groupoids

**2.2.1. Polygraphs.** A  $1$ -polygraph is a pair  $\Sigma = (\Sigma_0, \Sigma_1)$  made of a set  $\Sigma_0$  and a cellular extension  $\Sigma_1$  of  $\Sigma_0$ . The free category  $\Sigma^*$  over  $\Sigma$  is  $\Sigma^* = \Sigma_0[\Sigma_1]$ . A  $2$ -polygraph is a pair  $\Sigma = (\Sigma_1, \Sigma_2)$  where  $\Sigma_1$  is a  $1$ -polygraph and  $\Sigma_2$  is a cellular extension of the free category  $\Sigma_1^*$ . The free  $2$ -category  $\Sigma^*$  over  $\Sigma$ , the free  $(2, 1)$ -category  $\Sigma^\top$  over  $\Sigma$  and the category  $\bar{\Sigma}$  presented by  $\Sigma$  are respectively defined by

$$\Sigma^* = \Sigma_1^*[\Sigma_2], \quad \Sigma^\top = \Sigma_1^*(\Sigma_2) \quad \text{and} \quad \bar{\Sigma} = \Sigma_1^*/\Sigma_2.$$

A  $(3, 1)$ -polygraph is a pair  $\Sigma = (\Sigma_2, \Sigma_3)$  made of a  $2$ -polygraph  $\Sigma_2$  and a cellular extension  $\Sigma_3$  of the free  $(2, 1)$ -category  $\Sigma_2^\top$ . The free  $(3, 1)$ -category  $\Sigma^\top$  over  $\Sigma$  and the  $(2, 1)$ -category presented by  $\Sigma$  are

## 2.2. Coherent presentations of categories and groupoids

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defined by

$$\Sigma^\top = \Sigma_2^\top(\Sigma_3) \quad \text{and} \quad \bar{\Sigma} = \Sigma_2^\top/\Sigma_3.$$

The category presented by a  $(3, 1)$ -polygraph  $\Sigma$  is the one presented by its underlying 2-polygraph, namely  $\bar{\Sigma}_2 = \Sigma_1^*/\Sigma_2$ . If  $\Sigma$  is a polygraph, we identify its underlying  $k$ -polygraph  $\Sigma_k$  and the set of  $k$ -cells of the corresponding cellular extension. We say that  $\Sigma$  is *finite* if it has finitely many cells in every dimension. A  $(3, 1)$ -polygraph  $\Sigma$  can be summarised by a diagram representing the cells and the source and target maps of the free  $(3, 1)$ -category  $\Sigma^\top$  it generates:

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1^* \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{t_1} \end{array} \Sigma_2^\top \begin{array}{c} \xleftarrow{s_2} \\ \xrightarrow{t_2} \end{array} \Sigma_3^\top.$$

**2.2.2. Coherent presentations of categories.** Let  $\mathbf{C}$  be a category. A *presentation of  $\mathbf{C}$*  is a 2-polygraph  $\Sigma$  whose presented category  $\bar{\Sigma}$  is isomorphic to  $\mathbf{C}$ . We usually commit the abuse to identify  $\mathbf{C}$  and  $\bar{\Sigma}$  and we denote by  $\bar{u}$  the image of a 1-cell  $u$  of  $\Sigma^*$  through the canonical projection onto  $\mathbf{C}$ . An *extended presentation of  $\mathbf{C}$*  is a  $(3, 1)$ -polygraph  $\Sigma$  whose presented category is isomorphic to  $\mathbf{C}$ . A *coherent presentation of  $\mathbf{C}$*  is an extended presentation  $\Sigma$  of  $\mathbf{C}$  such that the cellular extension  $\Sigma_3$  of  $\Sigma_2^\top$  is a homotopy basis. Example 2.3.8 proves that every category admits a coherent presentation.

**2.2.3. Coherent presentations of groupoids.** If  $\mathbf{G}$  is a groupoid, a *presentation of  $\mathbf{G}$*  is a 2-polygraph

$$\Sigma_0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} (\Sigma_1 \amalg \check{\Sigma}_1)^* \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} (\Sigma_2 \amalg \text{Inv}(\Sigma_1))^\top$$

that is a presentation of  $\mathbf{G}$ , seen as a category. We denote such a presentation of  $\mathbf{G}$  by  $(\Sigma_0, \Sigma_1, \Sigma_2)$ . A *coherent presentation of  $\mathbf{G}$*  is a  $(3, 1)$ -polygraph

$$\Sigma_0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} (\Sigma_1 \amalg \check{\Sigma}_1)^* \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} (\Sigma_2 \amalg \text{Inv}(\Sigma_1))^\top \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} (\Sigma_3 \amalg \text{Inv}_+(\Sigma_1))^\top$$

that is a coherent presentation of  $\mathbf{G}$ , seen as a category, where  $\text{Inv}_+(\Sigma_1)$  contains, for every 1-cell  $x$  of  $\Sigma_1$ , the following two 3-cells:

$$\begin{array}{ccc} \begin{array}{c} \lambda_x \check{x} \\ \check{x} x \check{x} \\ \check{x} \rho_x \end{array} & \begin{array}{c} \Downarrow I_x \\ \Downarrow \\ \Downarrow \end{array} & \check{x} \\ \begin{array}{c} \rho_x x \\ x \check{x} x \\ x \lambda_x \end{array} & \begin{array}{c} \Downarrow J_x \\ \Downarrow \\ \Downarrow \end{array} & x \end{array}$$

We denote by  $(\Sigma_2, \Sigma_3)$  such a coherent presentation of  $\mathbf{G}$ . The justification of this definition is that, if one denotes by

$$(\Sigma_1 \amalg \check{\Sigma}_1) \xrightarrow{\pi} (\Sigma_1 \amalg \check{\Sigma}_1)/\text{Inv}(\Sigma_1)$$

the canonical projection, then  $\pi(\Sigma_3)$  forms a homotopy basis of the 2-groupoid

$$\left( (\Sigma_1 \amalg \check{\Sigma}_1)/\text{Inv}(\Sigma_1) \right) (\pi(\Sigma_2)).$$



## 2. Coherent presentations of categories and groupoids

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### 2.3. Squier's completion for convergent presentations

We recall notions of rewriting theory for 2-polygraphs from [20, 4.1] and [21, 4.1], together with Squier's completion to compute coherent presentations from convergent presentations. We detail this already-known material because it is central in the methods we develop in Section 4. Let us fix a 2-polygraph  $\Sigma$ .

**2.3.1. Rewriting and normal forms.** A *rewriting step* of  $\Sigma$  is a 2-cell of the free 2-category  $\Sigma^*$  with shape

$$\begin{array}{ccccc} & & u & & \\ & & \curvearrowright & & \\ y & \xrightarrow{w} & x & & x' & \xrightarrow{w'} & y' \\ & & \Downarrow \varphi & & \\ & & v & & \end{array}$$

where  $\varphi : u \Rightarrow v$  is a 2-cell of  $\Sigma$  and  $w$  and  $w'$  are 1-cells of  $\Sigma^*$ . A *rewriting sequence* of  $\Sigma$  is a finite or infinite sequence

$$u_1 \xrightarrow{f_1} u_2 \xrightarrow{f_2} (\dots) \xrightarrow{f_{n-1}} u_n \xrightarrow{f_n} (\dots)$$

of rewriting steps. If  $\Sigma$  has a non-empty rewriting sequence from  $u$  to  $v$ , we say that  $u$  *rewrites into*  $v$ . Let us note that every 2-cell  $f$  of  $\Sigma^*$  decomposes into a finite rewriting sequence of  $\Sigma$ , this decomposition being unique up to exchange relations. A 1-cell  $u$  of  $\Sigma^*$  is a *normal form* if  $\Sigma$  has no rewriting step with source  $u$ . A *normal form of*  $u$  is a 1-cell  $v$  that is a normal form and such that  $u$  rewrites into  $v$ .

**2.3.2. Termination.** We say that  $\Sigma$  *terminates* if it has no infinite rewriting sequence. In that case, every 1-cell has at least one normal form and *Noetherian induction* allows definitions and proofs of properties of 1-cells by induction on the maximum size of the 2-cells leading to normal forms. If  $\Sigma$  is a 2-polygraph, a *termination order* on  $\Sigma$  is an order relation  $\leq$  on parallel 1-cells of  $\Sigma^*$  such that the following properties are satisfied:

- the composition of 1-cells of  $\Sigma^*$  is strictly monotone in both arguments,
- every decreasing family  $(u_n)_{n \in \mathbb{N}}$  of parallel 1-cells of  $\Sigma^*$  is stationary,
- for every 2-cell  $\alpha$  of  $\Sigma$ , the strict inequality  $s(\alpha) > t(\alpha)$  holds.

As a direct consequence of the definition, if  $\Sigma$  admits a termination order, then  $\Sigma$  terminates.

A useful example of termination order is the *left degree-wise lexicographic order* (or *deglex* for short) generated by a given order on the 1-cells of  $\Sigma$ . It is defined by the following strict inequalities, where each  $x_i$  and  $y_j$  is a 1-cell of  $\Sigma$ :

$$x_1 \cdots x_p < y_1 \cdots y_q \quad \text{if } p < q$$

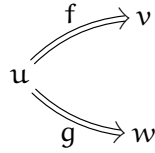
$$x_1 \cdots x_{k-1} x_k \cdots x_p < x_1 \cdots x_{k-1} y_k \cdots y_p \quad \text{if } x_k < y_k.$$

The deglex order is total if, and only if, the original order on 1-cells of  $\Sigma$  is total.

## 2.3. Squier's completion for convergent presentations

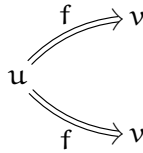
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**2.3.3. Branchings.** A *branching* of  $\Sigma$  is a pair  $(f, g)$  of 2-cells of  $\Sigma^*$  with a common source, as in the diagram



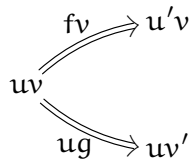
The 1-cell  $u$  is the *source* of this branching and the pair  $(v, w)$  is its *target*. We do not distinguish the branchings  $(f, g)$  and  $(g, f)$ . A branching  $(f, g)$  is *local* if  $f$  and  $g$  are rewriting steps. Local branchings belong to one of the three following families:

- *aspherical* branchings have shape



where  $f : u \Rightarrow v$  is a rewriting step of  $\Sigma$ ,

- *Peiffer* branchings have shape



where  $f : u \Rightarrow u'$  and  $g : v \Rightarrow v'$  are rewriting steps of  $\Sigma$ ,

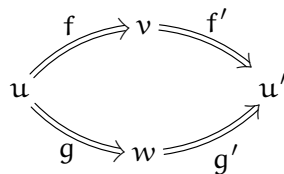
- *overlapping* branchings are the remaining local branchings.

Local branchings are compared by the order  $\preceq$  generated by the relations

$$(f, g) \preceq (ufv, ugv)$$

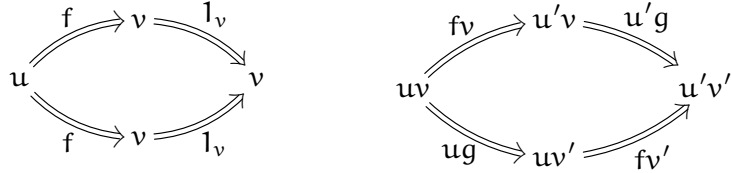
given for any local branching  $(f, g)$  and any possible 1-cells  $u$  and  $v$  of  $\Sigma^*$ . An overlapping local branching that is minimal for the order  $\preceq$  is called a *critical branching*. The terms “aspherical” and “Peiffer” come from the corresponding notions for spherical diagrams in Cayley complexes associated to presentations of groups [27]. The term “critical” comes from rewriting theory [4, 1].

**2.3.4. Confluence.** A branching  $(f, g) : u \Rightarrow (v, w)$  is *confluent* if there exist 2-cells  $f' : v \Rightarrow u'$  and  $g' : w \Rightarrow u'$  in  $\Sigma^*$ , as in the following diagram:

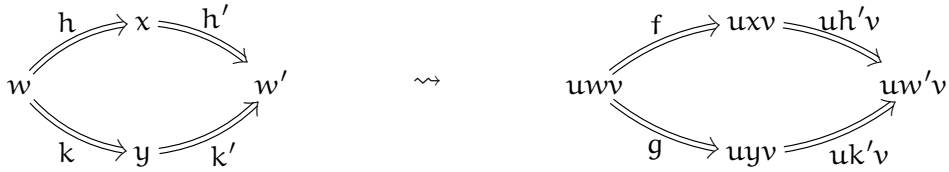


## 2. Coherent presentations of categories and groupoids

We say that  $\Sigma$  is *confluent* (resp. *locally confluent*) if all of its branchings (resp. local branchings) are confluent. In a confluent 2-polygraph, every 1-cell has at most one normal form. A fundamental result of rewriting theory states that local confluence is equivalent to confluence of critical branchings. Indeed, any aspherical or Peiffer branching is confluent:



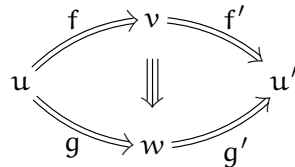
We note that, in the aspherical and Peiffer cases, the 2-cells  $f'$  and  $g'$  can be chosen in such a way that  $f \star_1 f' = g \star_1 g'$  holds. Finally, in the case of an overlapping but not minimal local branching  $(f, g)$ , there exist factorisations  $f = uhv$  and  $g = ukv$  with  $(h, k) : w \Rightarrow (x, y)$  a critical branching of  $\Sigma$ . If  $(h, k)$  is confluent, then so is  $(f, g)$ :



For terminating 2-polygraphs, Newman's lemma, sometimes called the Diamond Lemma, ensures that local confluence and confluence are equivalent properties [31, Theorem 3].

**2.3.5. Convergent polygraphs.** We say that  $\Sigma$  is *convergent* if it terminates and it is confluent. Such a  $\Sigma$  is called a *convergent presentation of  $\bar{\Sigma}$* , and of any category that is isomorphic to  $\bar{\Sigma}$ . In that case, every 1-cell  $u$  of  $\Sigma^*$  has a unique normal form, denoted by  $\hat{u}$ , so that we have  $\bar{u} = \bar{v}$  in  $\bar{\Sigma}$  if, and only if,  $\hat{u} = \hat{v}$  holds in  $\Sigma^*$ . This extends to a section  $\bar{\Sigma} \rightarrow \Sigma^*$  of the canonical projection, sending a 1-cell  $u$  of  $\bar{\Sigma}$  to the unique normal form of its representative 1-cells in  $\Sigma^*$ , still denoted by  $\hat{u}$ . As a consequence, a finite and convergent 2-polygraph  $\Sigma$  yields generators for the 1-cells of the category  $\bar{\Sigma}$  it presents, together with a decision procedure for the corresponding word problem (the purpose of the finiteness condition is to ensure that one can effectively check that a given 1-cell is a normal form). A  $(3, 1)$ -polygraph is *convergent* if its underlying 2-polygraph is.

**2.3.6. Squier's completion.** Let us assume that  $\Sigma$  is convergent. A *family of generating confluences of  $\Sigma$*  is a cellular extension of  $\Sigma^\top$  that contains exactly one 3-cell



for every critical branching  $(f, g)$  of  $\Sigma$ . We note that, if  $\Sigma$  is confluent, it always admits a family of generating confluences. However, such a family is not necessarily unique, since the 3-cell can be directed

## 2.3. Squier's completion for convergent presentations

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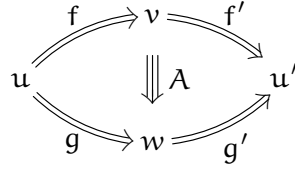
in the reverse way and, for a given branching  $(f, g)$ , we can have several possible 2-cells  $f'$  and  $g'$  with the required shape (see [21, 4.3.2] for a constructive version, based on normalisation strategies).

We call *Squier's completion* of  $\Sigma$  the  $(3, 1)$ -polygraph denoted by  $\mathcal{S}(\Sigma)$  and obtained from  $\Sigma$  by adjunction of a chosen family of generating confluences of  $\Sigma$ . Squier proved the following result (which was extended to  $n$ -polygraphs in [20, Proposition 4.3.4]). We recall the proof in details here, because Theorem 2.3.7 is an essential part of the homotopical completion we introduce in Section 4.

**2.3.7. Theorem ([35, Theorem 5.2]).** *For every convergent presentation  $\Sigma$  of a category  $\mathbf{C}$ , Squier's completion  $\mathcal{S}(\Sigma)$  of  $\Sigma$  is a coherent presentation of  $\mathbf{C}$ .*

*Proof.* We proceed in three steps.

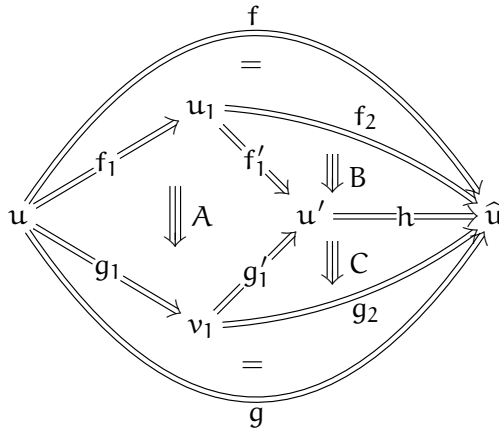
**Step 1.** We prove that, for every local branching  $(f, g) : u \Rightarrow (v, w)$  of  $\Sigma$ , there exist 2-cells  $f' : v \Rightarrow u'$  and  $g' : w \Rightarrow u'$  in  $\Sigma^*$  and a 3-cell  $A : f \star_1 f' \Rrightarrow g \star_1 g'$  in  $\mathcal{S}(\Sigma)^\top$ , as in the following diagram:



As we have seen in the study of confluence of local branchings, in the case of an aspherical or Peiffer branching, we can choose  $f'$  and  $g'$  such that  $f \star_1 f' = g \star_1 g'$ : an identity 3-cell is enough to link them. Moreover, if we have an overlapping branching  $(f, g)$  that is not critical, we have  $(f, g) = (uhv, ukv)$  with  $(h, k)$  critical; we consider the corresponding 3-cell  $\alpha : h \star_1 h' \Rrightarrow k \star_1 k'$  of  $\mathcal{S}(\Sigma)$  and we conclude that  $f' = uh'v$ ,  $g' = uk'v$  and  $A = u\alpha v$  satisfy the required conditions.

**Step 2.** We prove that, for every parallel 2-cells  $f$  and  $g$  of  $\Sigma^*$  whose common target is a normal form, there exists a 3-cell from  $f$  to  $g$  in  $\mathcal{S}(\Sigma)^\top$ . We proceed by Noetherian induction on the common source  $u$  of  $f$  and  $g$ , using the termination of  $\Sigma$ . Let us assume that  $u$  is a normal form: then, by definition, both 2-cells  $f$  and  $g$  must be equal to the identity of  $u$ , so that  $1_u : 1_u \Rrightarrow 1_u$  is a 3-cell of  $\mathcal{S}(\Sigma)^\top$  from  $f$  to  $g$ .

Now, let us fix a 1-cell  $u$  with the following property: for any 1-cell  $v$  such that  $u$  rewrites into  $v$  and for any parallel 2-cells  $f, g : v \Rightarrow \hat{u} = \hat{u}$  of  $\Sigma^*$ , there exists a 3-cell from  $f$  to  $g$  in  $\mathcal{S}(\Sigma)^\top$ . Let us consider parallel 2-cells  $f, g : u \Rightarrow \hat{u}$  and let us prove the result by progressively constructing the following composite 3-cell from  $f$  to  $g$  in  $\mathcal{S}(\Sigma)^\top$ :



## 2. Coherent presentations of categories and groupoids

Since  $u$  is not a normal form, we can decompose  $f = f_1 \star_1 f_2$  and  $g = g_1 \star_1 g_2$  so that  $f_1$  and  $g_1$  are rewriting steps. They form a local branching  $(f_1, g_1)$  and we build the 2-cells  $f'_1$  and  $g'_1$ , together with the 3-cell  $A$  as in the first part of the proof. Then, we consider a 2-cell  $h$  from  $u'$  to  $\hat{u}$  in  $\Sigma^*$ , that must exist by confluence of  $\Sigma$  and since  $\hat{u}$  is a normal form. We apply the induction hypothesis to the parallel 2-cells  $f_2$  and  $f'_1 \star_1 h$  in order to get  $B$  and, symmetrically, to the parallel 2-cells  $g'_1 \star_1 h$  and  $g_2$  to get  $C$ .

**Step 3.** We prove that every 2-sphere of  $\Sigma^\top$  is the boundary of a 3-cell of  $\mathcal{S}(\Sigma)^\top$ . First, let us consider a 2-cell  $f : u \Rightarrow v$  in  $\Sigma^*$ . Using the confluence of  $\Sigma$ , we choose 2-cells

$$\sigma_u : u \Rightarrow \hat{u} \quad \text{and} \quad \sigma_v : v \Rightarrow \hat{v} = \hat{u}$$

in  $\Sigma^*$ . By construction, the 2-cells  $f \star_1 \sigma_v$  and  $\sigma_u$  are parallel and their common target  $\hat{u}$  is a normal form. Thus, there exists a 3-cell in  $\mathcal{S}(\Sigma)^\top$  from  $f \star_1 \sigma_v$  to  $\sigma_u$  or, equivalently, a 3-cell  $\sigma_f$  from  $f$  to  $\sigma_u \star_1 \sigma_v^-$  in  $\mathcal{S}(\Sigma)^\top$ , as in the following diagram:

$$\begin{array}{ccc} & f & \\ & \curvearrowright & \\ u & & v \\ & \sigma_u \searrow & \nearrow \sigma_v^- \\ & \hat{u} & \end{array}$$

Moreover, the  $(3, 1)$ -category  $\mathcal{S}(\Sigma)^\top$  contains a 3-cell  $\sigma_f^-$  from  $f^-$  to  $\sigma_v \star_1 \sigma_u^-$ , given as the following composite:

$$\begin{array}{ccccccc} & & \hat{u} & & & & \\ & \sigma_u \nearrow & & \sigma_v^- \searrow & & & \\ v & \xrightarrow{f^-} & u & & v & \xrightarrow{\sigma_v} & \hat{u} \xrightarrow{\sigma_u^-} & u \\ & & \sigma_f^- & & & & \\ & & \Downarrow & & & & \\ & & f & & & & \end{array}$$

Now, let us consider a general 2-cell  $f : u \Rightarrow v$  of  $\Sigma^\top$ . By construction of  $\Sigma^\top$ , the 2-cell  $f$  can be decomposed (in general in a non-unique way) into a “zig-zag”

$$u \xrightarrow{f_1} v_1 \xrightarrow{g_1^-} u_2 \xrightarrow{f_2} (\dots) \xrightarrow{g_{n-1}^-} u_n \xrightarrow{f_n} v_n \xrightarrow{g_n^-} v$$

where each  $f_i$  and  $g_i$  is a 2-cell of  $\Sigma^*$ . We define  $\sigma_f$  as the following composite 3-cell of  $\mathcal{S}(\Sigma)^\top$ , with source  $f$  and target  $\sigma_u \star_1 \sigma_v^-$ :

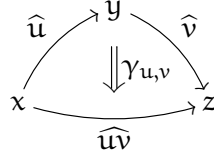
$$\begin{array}{ccccccc} u & \xrightarrow{f_1} & v_1 & \xrightarrow{g_1^-} & (\dots) & \xrightarrow{f_n} & v_n \xrightarrow{g_n^-} & v \\ & \Downarrow \sigma_{f_1} & \sigma_{v_1}^- & \Downarrow \sigma_{g_1^-} & \sigma_{u_2}^- & \Downarrow \sigma_{f_n} & \sigma_{v_n}^- & \Downarrow \sigma_{g_n^-} \\ & \sigma_u \searrow & \hat{u} & \searrow & \hat{u} & \searrow & \hat{u} & \searrow & \hat{u} \\ & & \sigma_{v_1} & & \sigma_{u_2} & & \sigma_{v_n} & & \sigma_v^- \end{array}$$

We proceed similarly for any other 2-cell  $g : u \Rightarrow v$  of  $\Sigma^\top$ , to get a 3-cell  $\sigma_g$  from  $g$  to  $\sigma_u \star_1 \sigma_v^-$  in  $\mathcal{S}(\Sigma)^\top$ . Thus, the composite  $\sigma_f \star_2 \sigma_g^-$  is a 3-cell of  $\mathcal{S}(\Sigma)^\top$  from  $f$  to  $g$ , concluding the proof.  $\square$

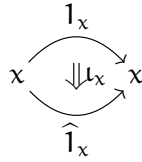
## 2.3. Squier's completion for convergent presentations

**2.3.8. Example (The canonical coherent presentation).** The *canonical presentation* of a category  $\mathbf{C}$  is the 2-polygraph denoted by  $\text{Can}(\mathbf{C})$  and defined as follows:

- the 0-cells and 1-cells of  $\text{Can}(\mathbf{C})$  are the ones of  $\mathbf{C}$ , a 1-cell  $u$  of  $\mathbf{C}$  being denoted by  $\hat{u}$  when seen as a generating 1-cell of  $\text{Can}(\mathbf{C})$ ,
- for every 1-cells  $u : x \rightarrow y$  and  $v : y \rightarrow z$  of  $\mathbf{C}$ , one 2-cell

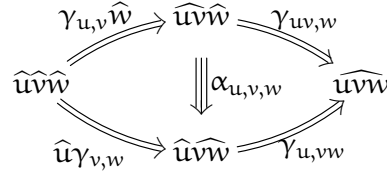


- for every 0-cell  $x$  of  $\mathbf{C}$ , one 2-cell

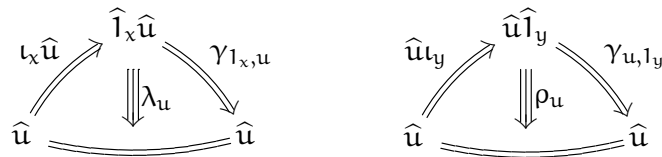


The *canonical coherent presentation* of  $\mathbf{C}$  is the  $(3, 1)$ -polygraph denoted by  $\text{Can}_+(\mathbf{C})$  and obtained by extension of  $\text{Can}(\mathbf{C})$  with the homotopy basis made of the following 3-cells:

- for every 1-cells  $u : x \rightarrow y$ ,  $v : y \rightarrow z$  and  $w : z \rightarrow t$  of  $\mathbf{C}$ , one 3-cell



- for every 1-cell  $u : x \rightarrow y$  of  $\mathbf{C}$ , two 3-cells



Let us prove that  $\text{Can}_+(\mathbf{C})$  is, indeed, a coherent presentation of  $\mathbf{C}$ . The canonical presentation  $\text{Can}(\mathbf{C})$  is not terminating: indeed, for every 0-cell  $x$  of  $\mathbf{C}$ , the 2-cell  $\iota_x$  creates infinite rewriting sequences

$$1_x \Rightarrow \hat{1}_x \Rightarrow \hat{1}_x \hat{1}_x \Rightarrow \hat{1}_x \hat{1}_x \hat{1}_x \Rightarrow \dots$$

However, we get a convergent presentation of  $\mathbf{C}$  by reversing all the 2-cells  $\iota_x$  into  $\iota_x^-$ . Indeed, for termination, we consider the size of the 1-cells (the number of generators they contain) and we check that each 2-cell  $\gamma_{u,v}$  has source of size 2 and target of size 1, while each 2-cell  $\iota_x^-$  has source of size 1 and target of size 0. As a consequence, for every non-identity 2-cell  $f : u \Rightarrow v$  of the free 2-category, the size of  $u$  is strictly greater than the size of  $v$ . For confluence, we study the critical branchings, divided into three families:

## 2. Coherent presentations of categories and groupoids

- for every 1-cells  $u : x \rightarrow y$ ,  $v : y \rightarrow z$  and  $w : z \rightarrow t$ , one critical branching  $(\gamma_{u,v}\widehat{w}, \widehat{u}\gamma_{v,w})$ , giving the 3-cell

$$\begin{array}{ccc}
 & \xrightarrow{\gamma_{u,v}\widehat{w}} & \widehat{u}\widehat{v}\widehat{w} \\
 \widehat{u}\widehat{v}\widehat{w} & & \xrightarrow{\gamma_{uv,w}} \widehat{u}\widehat{v}w \\
 & \Downarrow \gamma_{u,v,w} & \\
 & \xrightarrow{\widehat{u}\gamma_{v,w}} & \widehat{u}\widehat{v}w \\
 & & \xrightarrow{\gamma_{u,vw}} \widehat{u}vw
 \end{array}$$

- for every 1-cell  $u : x \rightarrow y$  of  $\mathbf{C}$ , two critical branchings  $(\gamma_{1_x,u}, \iota_x^- \widehat{u})$  and  $(\gamma_{u,1_y}, \widehat{u}\iota_y^-)$ , producing the 3-cells

$$\begin{array}{ccc}
 & \xrightarrow{\gamma_{1_x,u}} & \widehat{u} \\
 \widehat{1}_x \widehat{u} & & \Downarrow \lambda_u \\
 & \xrightarrow{\iota_x^- \widehat{u}} & \widehat{u}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{\gamma_{u,1_y}} & \widehat{u} \\
 \widehat{u} \widehat{1}_y & & \Downarrow \rho_u \\
 & \xrightarrow{\widehat{u}\iota_y^-} & \widehat{u}
 \end{array}$$

Since considering the 2-cells  $\iota_x$  or  $\iota_x^-$  as generators does not change the generated  $(2, 1)$ -category, we get that those three families of 3-cells form a homotopy basis for  $\text{Can}(\mathbf{C})$ . We replace  $\lambda_u$  by  $\iota_x \widehat{u} \star_1 \lambda_u$  and  $\rho_u$  by  $\widehat{u}\iota_y \star_1 \rho_u$  to get the result.

One can reduce  $\text{Can}_+(\mathbf{C})$  into the smaller *reduced canonical coherent presentation*  $\text{Can}'_+(\mathbf{C})$  of  $\mathbf{C}$ . It is obtained from  $\text{Can}_+(\mathbf{C})$  by removing all the cells about units. This is formalised by a Tietze transformation, as detailed in 4.1. This transformation coherently eliminates the following pairs of cells, preserving the presented category  $\mathbf{C}$  and the homotopy basis:

- the 3-cells  $\gamma_{1_x,u,v}$ ,  $\gamma_{u,1_y,v}$  and  $\gamma_{u,v,1_z}$ , since they are parallel to composites of  $\lambda$ s and  $\rho$ s,
- the 2-cells  $\gamma_{1_x,u}$  and the 3-cells  $\lambda_u$ ,
- the 2-cells  $\gamma_{u,1_x}$  and the 3-cells  $\rho_u$ ,
- the 1-cells  $\widehat{1}_x$  and the 2-cells  $\iota_x$ .

The resulting coherent presentation is detailed in [21, 4.1.6]. If  $\mathbf{M}$  is a monoid, then  $\text{Can}'_+(\mathbf{M})$  gives a notion of non-unital action of  $\mathbf{M}$  on a category  $\mathbf{C}$ . The fact that  $\text{Can}'_+(\mathbf{M})$  is also a coherent presentation of  $\mathbf{M}$  induces that non-unital actions of  $\mathbf{M}$  on  $\mathbf{C}$  are equivalent to (unital) actions of  $\mathbf{M}$  on  $\mathbf{C}$ . This result was proved by Deligne in [13, 1.8] for the special case of semigroups (seen as monoids by formal adjunction of a unit).

### 2.4. Coherent presentations of Artin monoids and Artin groups

We recall standard notions and results about Coxeter groups and Artin monoids and groups, mainly taken from Bourbaki [5], Deligne [12], Brieskorn and Saito [6], Geck and Pfeiffer [19]. We formulate the main result of the article, giving a coherent presentation of Artin monoids and Artin groups.

## 2.4. Coherent presentations of Artin monoids and Artin groups

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**2.4.1. Coxeter groups.** A *Coxeter group* is a group  $\mathbf{W}$  that admits a presentation with a finite set  $S$  of generators and with one relation

$$(st)^{m_{st}} = 1, \quad \text{with } m_{st} \in \mathbb{N} \amalg \{\infty\}, \quad (2)$$

for every  $s$  and  $t$  in  $S$ , with the following requirements and conventions:

- $m_{st} = \infty$  means that there is, in fact, no relation between  $s$  and  $t$ ,
- $m_{st} = 1$  if, and only if,  $s = t$ .

The last requirement implies that  $s^2 = 1$  holds in  $\mathbf{W}$  for every  $s$  in  $S$ . As a consequence, the group  $\mathbf{W}$  can also be seen as the monoid with the same presentation. Let us note that a given Coxeter group can have several generating sets that fit the given scheme, but we always assume that such a set  $S$  has been fixed and comes equipped with a total order.

Following [6, (1.1)], we denote by  $\langle st \rangle^n$  the element of length  $n$  in the free monoid  $S^*$ , obtained by multiplication of alternating copies of  $s$  and  $t$ . Formally, this element is defined by induction on  $n$  as follows:

$$\langle st \rangle^0 = 1 \quad \text{and} \quad \langle st \rangle^{n+1} = s \langle ts \rangle^n.$$

When  $s \neq t$  and  $m_{st} < \infty$ , we use this notation and the relations  $s^2 = t^2 = 1$  to write (2) as a *braid relation*:

$$\langle st \rangle^{m_{st}} = \langle ts \rangle^{m_{st}}. \quad (3)$$

A *reduced expression* of an element  $u$  of  $\mathbf{W}$  is a representative of minimal length of  $u$  in the free monoid  $S^*$ . The *length* of  $u$  is denoted by  $l(u)$  and defined as the length of any of its reduced expressions. The Coxeter group  $\mathbf{W}$  is finite if, and only if, it admits an element of maximal length, [6, Theorem 5.6]; in that case, this element is unique, it is called the *fundamental element* of  $\mathbf{W}$  and it is denoted by  $w_0(S)$ . For  $I \subseteq S$ , the subgroup of  $\mathbf{W}$  spanned by the elements of  $I$  is denoted by  $\mathbf{W}_I$  and it is a Coxeter group with generating set  $I$ . If  $\mathbf{W}_I$  is finite, we denote by  $w_0(I)$  its fundamental element.

**2.4.2. Artin monoids and groups.** The *Artin monoid* and *Artin group* associated to  $\mathbf{W}$  are the monoid denoted by  $\mathbf{B}^+(\mathbf{W})$  and the group  $\mathbf{B}(\mathbf{W})$ , generated by  $S$  and subject to the braid relations (3). This presentation, seen as a 2-polygraph, is denoted by  $\text{Art}(\mathbf{W})$  and called *Artin's presentation*: this is the same as the one of  $\mathbf{W}$ , except for the relations  $s^2 = 1$ . An Artin monoid or group is *spherical* if the corresponding Coxeter group is finite. Let us note that the underlying set of  $\mathbf{W}$  embeds canonically in  $\mathbf{B}^+(\mathbf{W})$  and we abusively denote an element of  $\mathbf{W}$  and its image in  $\mathbf{B}^+(\mathbf{W})$  in the same way ; moreover, the elements of  $\mathbf{B}^+(\mathbf{W})$  that correspond to elements of  $\mathbf{W}$  are exactly the ones whose representatives in  $S^*$  are reduced expressions (that is, they contain no  $s^2$ , for any  $s$  in  $S$ ).

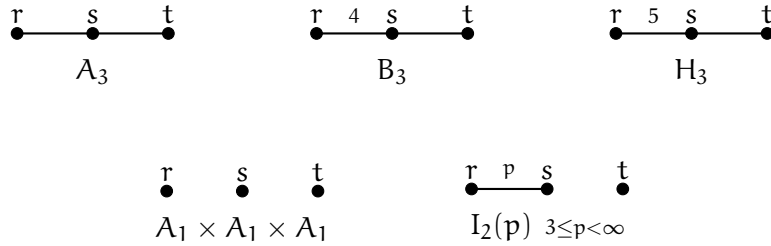
As an example, the monoid  $\mathbf{B}_n^+$  of positive braids on  $n$  strands is the spherical Artin monoid associated to the finite Coxeter group  $\mathbf{S}_n$  of permutations of  $n$  elements. For the latter, we consider the standard set of generators, *i.e.*, the  $n - 1$  generating symmetries  $s_1, \dots, s_{n-1}$ , submitted to the relations  $s_i s_j = s_j s_i$  if  $i \leq j - 2$  and  $s_i s_j s_i = s_j s_i s_j$  if  $i = j - 1$ . This is the reason why Artin groups are also called generalised braid groups.



## 2. Coherent presentations of categories and groupoids

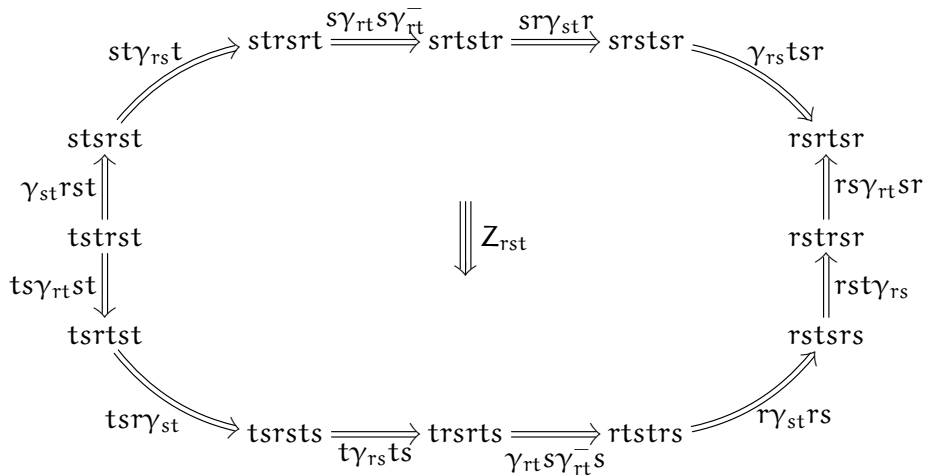
**2.4.3. Classification in the spherical case.** The *Coxeter diagram* of  $\mathbf{W}$  is the finite non-directed graph with one vertex for each element in  $S$ , and with one edge with label  $m_{st}$  between  $s$  and  $t$  if  $m_{st} \geq 3$  (with the convention that the labels 3 are omitted). In particular, there is no edge between two different vertices  $s$  and  $t$  if, and only if, they commute in  $\mathbf{W}$ . The Coxeter group  $\mathbf{W}$  is *irreducible* if its Coxeter diagram is connected, that is, if  $\mathbf{W}$  admits no factorisation  $\mathbf{W} = \mathbf{W}_1 \times \mathbf{W}_2$  where  $\mathbf{W}_1$  and  $\mathbf{W}_2$  are non-trivial Coxeter groups.

The classification of finite and irreducible Coxeter groups is due to Coxeter, [8], who proved that a finite Coxeter group is irreducible if, and only if, its Coxeter diagram belongs to a precise list, see [5, Chapter VI, § 4, Theorem 1] or [19, Theorem 1.1]. In this article, we use the conventions of notation of the second reference and we are interested in the finite Coxeter groups of rank 3, which fall in one of the following cases (up to isomorphism):



This classification extends to spherical Artin monoids and groups, using the direct observation that, for any  $I \subseteq S$ , the submonoid of  $\mathbf{B}^+(\mathbf{W})$  and the subgroup of  $\mathbf{B}(\mathbf{W})$  spanned by  $I$  are the Artin monoid  $\mathbf{B}^+(\mathbf{W}_I)$  and the Artin group  $\mathbf{B}(\mathbf{W}_I)$ .

**2.4.4. Artin's coherent presentation.** Let  $\mathbf{W}$  be a Coxeter group with a fixed totally ordered set of generators  $S$ . We call *Artin's coherent presentation* the  $(3, 1)$ -polygraph  $\text{Art}_+(\mathbf{W})$  obtained from Artin's presentation  $\text{Art}(\mathbf{W})$  by adjunction of one 3-cell  $Z_{rst}$  for every pairwise distinct elements  $r, s$  and  $t$  of  $S$  such that  $\mathbf{W}_{\{r,s,t\}}$  is finite. The 3-cell  $Z_{rst}$  has a shape that depends only on the Coxeter type of  $\mathbf{W}_{\{r,s,t\}}$ , as follows. If  $\mathbf{W}_{\{r,s,t\}}$  is of type  $A_3$ :





## 2. Coherent presentations of categories and groupoids

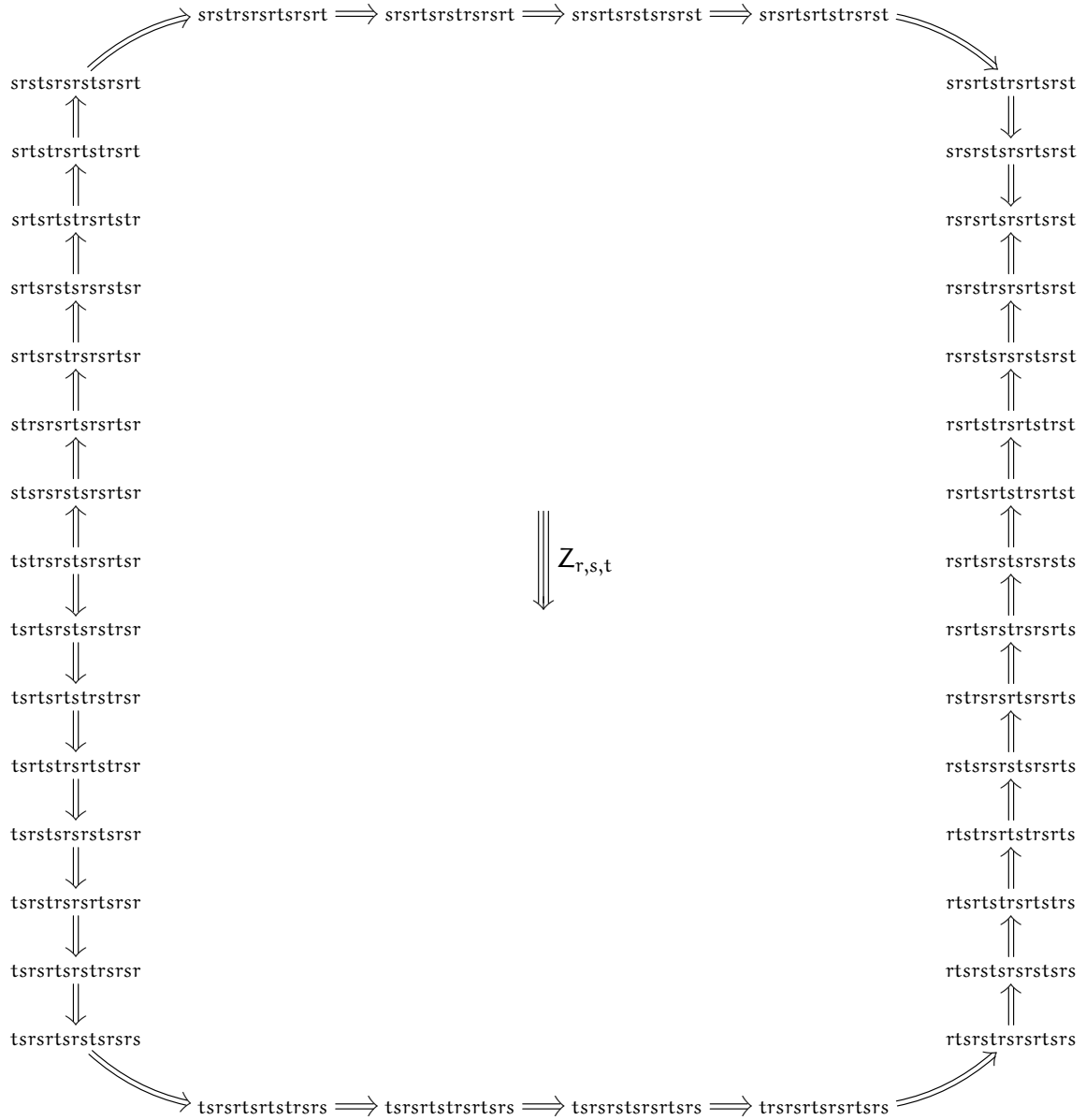


Figure 1: The 3-cell  $Z_{r,s,t}$  for  $W_{\{r,s,t\}}$  of Coxeter type H3

**2.4.5. Theorem.** *For every Coxeter group  $\mathbf{W}$ , the Artin monoid  $\mathbf{B}^+(\mathbf{W})$  and the Artin group  $\mathbf{B}(\mathbf{W})$  admit  $\text{Art}_+(\mathbf{W})$  as a coherent presentation.*

The proof of Theorem 2.4.5 is conducted in Sections 5 and 6. It relies on the generalisation of Squier’s completion, called *homotopical completion-reduction* and introduced in Section 4: this is a procedure, based on rewriting methods, that extends a presentation of a monoid into a relatively compact coherent presentation. In Section 5, we apply homotopical completion-reduction to Deligne’s presentation  $\text{Del}(\mathbf{W})$  of the Artin monoid  $\mathbf{B}^+(\mathbf{W})$ , to get a coherent presentation  $\text{Del}_+(\mathbf{W})$  of  $\mathbf{B}^+(\mathbf{W})$ . As a side result, when applied to  $\text{Del}_+(\mathbf{W})$ , Theorem 3.4.3 extends Deligne’s result on actions to every Artin monoids [13, Theorem 1.5]. Then, in Section 6, we apply homotopical reduction to the coherent presentation  $\text{Del}_+(\mathbf{W})$ . The result is  $\text{Art}_+(\mathbf{W})$ , thus proving Theorem 2.4.5 for Artin monoids. Finally, in 6.4, we prove that, if a monoid  $\mathbf{M}$  embeds in the group  $\mathbf{G}$  with the same presentation, then a coherent presentation of  $\mathbf{M}$  is also a coherent presentation of  $\mathbf{G}$ . This condition is satisfied by Artin monoids and Artin groups [32], thus concluding the proof of Theorem 2.4.5.

### 3. COHERENT PRESENTATIONS AND ACTIONS ON CATEGORIES

Deligne’s actions of a monoid  $\mathbf{M}$  on categories are a special case of 2-representations of 2-categories, as defined by Elgueta in [14]. We prove that, up to equivalence, actions of  $\mathbf{M}$  on categories are the same as 2-functors from  $\bar{\Sigma}$  to  $\text{Cat}$ , where  $\Sigma$  is any coherent presentation of  $\mathbf{M}$ . The constructions are described in the homotopical setting of the canonical model structure on 2-categories [24, 25].

#### 3.1. 2-representations of 2-categories

**3.1.1. 2-representations.** We recall from [14] that, given 2-categories  $\mathcal{C}$  and  $\mathcal{D}$ , a *2-representation of  $\mathcal{C}$  in  $\mathcal{D}$*  is a pseudofunctor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . This is a weakened notion of 2-functor, specified by:

- for every 0-cell  $x$  of  $\mathcal{C}$ , a 0-cell  $F(x)$  of  $\mathcal{D}$ ,
- for every 1-cell  $u : x \rightarrow y$  of  $\mathcal{C}$ , a 1-cell  $F(u) : F(x) \rightarrow F(y)$  of  $\mathcal{D}$ ,
- for every 2-cell  $f : u \Rightarrow v$  of  $\mathcal{C}$ , a 2-cell  $F(f) : F(u) \Rightarrow F(v)$  of  $\mathcal{D}$ .

As for 2-functors, the data are required to be compatible with vertical composition, in a strict way:

- for every 2-cells  $f : u \Rightarrow v : x \rightarrow y$  and  $g : v \Rightarrow w : x \rightarrow y$  of  $\mathcal{C}$ ,

$$F \left( \begin{array}{ccc} & u & \\ \curvearrowright & & \curvearrowleft \\ x & \xrightarrow{v} & y \\ \curvearrowleft & & \curvearrowright \\ & w & \end{array} \right) = \begin{array}{ccc} & F(u) & \\ \curvearrowright & & \curvearrowleft \\ F(x) & \xrightarrow{F(v)} & F(y) \\ \curvearrowleft & & \curvearrowright \\ & F(w) & \end{array}$$

- for every 1-cell  $u$  of  $\mathcal{C}$ , we have  $F(1_u) = 1_{F(u)}$ .

### 3. Coherent presentations and actions on categories

The data is also compatible with horizontal composition, but only up to coherent isomorphisms:

- for every 1-cells  $u : x \rightarrow y$  and  $v : y \rightarrow z$  of  $\mathcal{C}$ , an invertible 2-cell of  $\mathcal{D}$ , natural in  $u$  and  $v$ ,

$$\begin{array}{ccc}
 & F(y) & \\
 F(u) \nearrow & & \searrow F(v) \\
 F(x) & \xrightarrow{\quad} & F(z) \\
 & F(uv) & \\
 & \simeq \Downarrow F_{u,v} & 
 \end{array}$$

- for every 0-cell  $x$  of  $\mathcal{C}$ , an invertible 2-cell of  $\mathcal{D}$

$$\begin{array}{ccc}
 & 1_{F(x)} & \\
 F(x) \nearrow & & \searrow F(x) \\
 & \simeq \Downarrow F_x & \\
 & F(1_x) & 
 \end{array}$$

Finally, these 2-cells are required to satisfy the following monoidal coherence relations in  $\mathcal{D}$ :

- for every 1-cells  $u : x \rightarrow y$ ,  $v : y \rightarrow z$  and  $w : z \rightarrow t$  of  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 \begin{array}{ccc}
 F(u) \nearrow & F(y) \xrightarrow{F(v)} & F(z) \\
 F(x) \xrightarrow{F(uv)} & & F(t) \\
 & \Downarrow F_{u,v} & \\
 & F(uv) & \\
 & \Downarrow F_{uv,w} & \\
 & F(uvw) & 
 \end{array}
 & = &
 \begin{array}{ccc}
 F(u) \nearrow & F(y) \xrightarrow{F(v)} & F(z) \\
 F(x) \xrightarrow{F(uvw)} & & F(t) \\
 & \Downarrow F_{u,vw} & \\
 & F(uvw) & \\
 & \Downarrow F_{v,w} & \\
 & F(vw) & \\
 & \Downarrow F_{vw} & 
 \end{array}
 \end{array}$$

- for every 1-cell  $u : x \rightarrow y$  of  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 \begin{array}{ccc}
 1_{F(x)} \nearrow & F_x \nearrow & F(x) \\
 F(x) \xrightarrow{F(1_x)} & & F(y) \\
 & \Downarrow F_{1_x,u} & \\
 & F(1_x) & \\
 & \Downarrow F_{1_x,u} & \\
 & F(u) & 
 \end{array}
 & = &
 \begin{array}{ccc}
 F(u) \nearrow & & F(y) \\
 F(x) \xrightarrow{F(u)} & & F(y) \\
 & \Downarrow 1_{F(u)} & \\
 & F(u) & 
 \end{array}
 & = &
 \begin{array}{ccc}
 F(u) \nearrow & F(y) \xrightarrow{F_y} & 1_{F(y)} \\
 F(x) \xrightarrow{F(u)} & & F(y) \\
 & \Downarrow F_{u,1_y} & \\
 & F(u) & \\
 & \Downarrow F_{1_y} & \\
 & F(u) & 
 \end{array}
 \end{array}$$

As usual with monoidal coherence relations, this implies that, for every sequence  $(u_1, \dots, u_n)$  of pairwise composable 1-cells in  $\mathcal{C}$ , there exists a unique invertible 2-cell

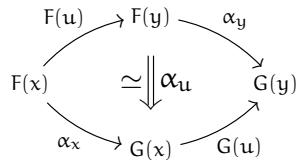
$$F_{u_1, \dots, u_n} : F(u_1) \cdots F(u_n) \Longrightarrow F(u_1 \cdots u_n)$$

in  $\mathcal{D}$  built from the coherence isomorphisms of  $F$ . A 2-functor is just a pseudofunctor whose coherence 2-cells are identities: it can be seen as a *strict* 2-representation.

The notion of 2-representation has been introduced by Elgueta for 2-groups in [14]. It is also studied by Ganter and Kapranov in [15] in the special case of groups. In [34], Rouquier considers the more general case of 2-representations of bicategories. Among concrete target 2-categories for 2-representations, natural choices are the 2-categories of 2-vector spaces, either from Kapranov and Voevodsky [22] or from Baez and Crans [3], of 2-Hilbert spaces [2] or of categories [13].

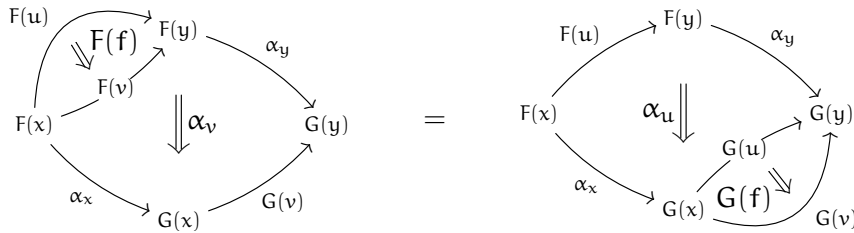
**3.1.2. Morphisms of 2-representations.** If  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  are 2-representations of  $\mathcal{C}$  into  $\mathcal{D}$ , a *morphism of 2-representations from  $F$  to  $G$*  is a pseudonatural transformation  $\alpha : F \Rightarrow G$  between the corresponding pseudofunctors:

- for every 0-cell  $x$  of  $\mathcal{C}$ , a 1-cell  $\alpha_x : F(x) \rightarrow G(x)$  of  $\mathcal{D}$ ,
- for every 1-cell  $u : x \rightarrow y$  of  $\mathcal{C}$ , an invertible 2-cell of  $\mathcal{D}$

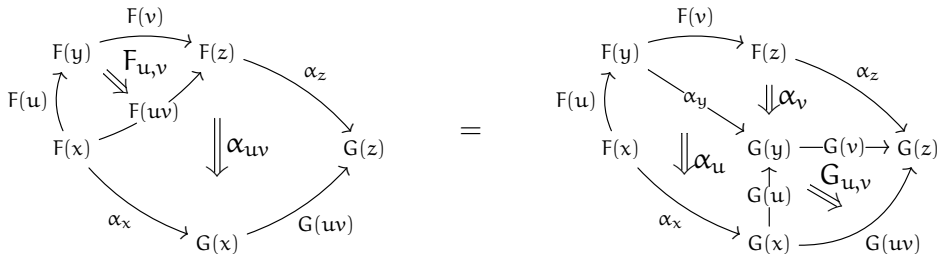


This data must satisfy several coherence relations:

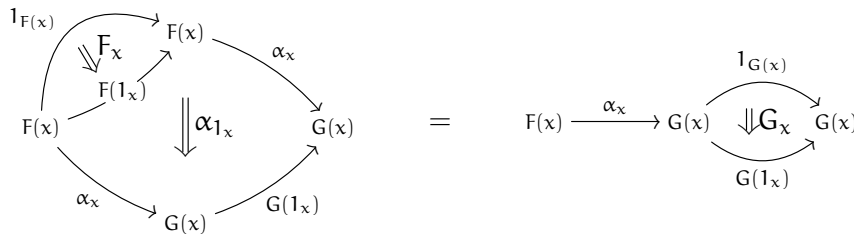
- for every 2-cell  $f : u \Rightarrow v : x \rightarrow y$  of  $\mathcal{C}$ ,



- for every 1-cells  $u : x \rightarrow y$  and  $v : y \rightarrow z$  of  $\mathcal{C}$ ,



- for every 0-cell  $x$  of  $\mathcal{C}$ ,



### 3. Coherent presentations and actions on categories

**3.1.3. Categories of 2-representations.** If  $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$  are 2-representations and if  $\alpha : F \Rightarrow G$  and  $\beta : G \Rightarrow H$  are morphisms of 2-representations, the composition  $\alpha \star \beta : F \Rightarrow H$  is defined by:

- if  $x$  is a 0-cell of  $\mathcal{C}$ , the 1-cell  $(\alpha \star \beta)_x : F(x) \rightarrow H(x)$  of  $\mathcal{D}$  is the composition

$$F(x) \xrightarrow{\alpha_x} G(x) \xrightarrow{\beta_x} H(x)$$

- if  $u : x \rightarrow y$  is a 1-cell of  $\mathcal{C}$ , the invertible 2-cell  $(\alpha \star \beta)_u$  of  $\mathcal{D}$  is defined by

One checks that  $\alpha \star \beta$  satisfies the coherence conditions of a morphism of 2-representations and, then, that the composition  $\star$  is associative and unitary. The category of 2-representations of  $\mathcal{C}$  into  $\mathcal{D}$  is denoted by  $2\text{Rep}(\mathcal{C}, \mathcal{D})$ . The full subcategory of  $2\text{Rep}(\mathcal{C}, \mathcal{D})$  whose objects are the 2-functors is denoted by  $2\text{Cat}(\mathcal{C}, \mathcal{D})$ .

**3.1.4. Actions of monoids on categories.** If  $\mathbf{M}$  is a monoid, we see it as a 2-category with exactly one 0-cell, with the elements of  $\mathbf{M}$  as 1-cells and with identity 2-cells only. We define the category of *actions of  $\mathbf{M}$  on categories* as the category of 2-representations of  $\mathbf{M}$  in  $\text{Cat}$ :

$$\text{Act}(\mathbf{M}) = 2\text{Rep}(\mathbf{M}, \text{Cat}).$$

Expanding the definition, we get that an action  $T$  of  $\mathbf{M}$  is specified by a category  $\mathbf{C}$ , which is the image through  $T$  of the unique 0-cell of  $\mathbf{M}$ , an endofunctor  $T(u) : \mathbf{C} \rightarrow \mathbf{C}$  for every element  $u$  of  $\mathbf{M}$ , a natural isomorphism  $T(u, v) : T(u)T(v) \Rightarrow T(uv)$  for every pair  $(u, v)$  of elements of  $\mathbf{C}$  and a natural isomorphism  $T_\bullet : 1_{\mathbf{C}} \Rightarrow T(1)$ . This data is required to satisfy the following coherence conditions:

- for every triple  $(u, v, w)$  of elements of  $\mathbf{M}$ , the following diagram commutes:

- for every element  $u$  of  $\mathbf{M}$ , the following two diagrams commute:

This definition corresponds to the notion of *unital* action of  $\mathbf{M}$  on  $\mathbf{C}$  that Deligne considers in [13]. For semigroups, he proves that unital actions are equivalent to non-unital actions. We have recovered this fact, for general monoids, in Example 2.3.8.

If  $S$  is an action of  $\mathbf{M}$  on a category  $\mathbf{C}$  and  $T$  is an action of  $\mathbf{M}$  on a category  $\mathbf{D}$ , by expanding the definition, we get that a morphism of actions  $\alpha$  from  $S$  to  $T$  is specified by a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ , corresponding to the component of  $\alpha$  at the unique 0-cell of  $\mathbf{M}$ , and, for every element  $u$  of  $\mathbf{M}$ , a natural isomorphism  $\alpha_u : S(u)F \Rightarrow FT(u)$ . This data must satisfy the coherence conditions of a pseudonatural transformation.

*Remark.* Those morphisms of actions of monoids on categories differ from the ones of Deligne in [13]. Indeed, he considers morphisms between actions of  $\mathbf{M}$  on the same category  $\mathbf{C}$ , such that the functor  $F$  is the identity of  $\mathbf{C}$ , but where the natural transformation  $\alpha_u$  is not necessarily an isomorphism: those are the *icons* between the corresponding pseudofunctors, as introduced by Lack in [26] as a special case of *oplax* natural transformations (defined as pseudonatural transformations whose component 2-cells are not necessarily invertible). In this article, we choose to follow Elgueta and consider pseudonatural transformations, but the results and proofs can be adapted to icons or generalised to oplax natural transformations.

### 3.2. Cofibrant approximations of 2-categories

**3.2.1. Elements of the canonical model category structure on  $2\text{Cat}$ .** We recall a few notions from the model category structure on  $2\text{Cat}$  introduced by Lack in [24] and [25]. A 2-category is *cofibrant* if its underlying 1-category is free. A 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a *weak equivalence* if it satisfies the following two conditions:

- Every 0-cell  $y$  of  $\mathcal{D}$  is equivalent to a 0-cell  $F(x)$  for  $x$  in  $\mathcal{C}$ , *i.e.*, there exists 1-cells  $u : F(x) \rightarrow y$  and  $v : y \rightarrow F(x)$  and invertible 2-cells  $f : u \star_1 v \Rightarrow 1_{F(x)}$  and  $g : v \star_1 u \Rightarrow 1_y$  in  $\mathcal{D}$ .
- For every 0-cells  $x$  and  $x'$  in  $\mathcal{C}$ , the induced functor  $F(x, x') : \mathcal{C}(x, x') \rightarrow \mathcal{D}(F(x), F(x'))$  is an equivalence of categories.

In that case, we say that  $\mathcal{C}$  and  $\mathcal{D}$  are *weakly equivalent*. In particular, an equivalence of 2-categories, that is, a 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that there exists a 2-functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  and pseudonatural isomorphisms  $GF \simeq 1_{\mathcal{C}}$  and  $FG \simeq 1_{\mathcal{D}}$ , is a weak equivalence. If  $\mathcal{C}$  is a 2-category, a *cofibrant approximation* of  $\mathcal{C}$  is a cofibrant 2-category  $\tilde{\mathcal{C}}$  that is weakly equivalent to  $\mathcal{C}$ .

Let us note that a weak equivalence  $F : \mathcal{C} \rightarrow \mathcal{D}$  in  $2\text{Cat}$  is exactly a 2-functor that is an equivalence in the category  $2\text{Rep}$ . Indeed, if  $F$  is a weak equivalence, we define a quasi-inverse  $G$  for  $F$  as follows. For every 0-cell  $y$  of  $\mathcal{D}$ , we choose a 0-cell  $G(y)$  in  $\mathcal{C}$  such that  $FG(y)$  is equivalent to  $y$ , *i.e.*, there exist 1-cells and invertible 2-cells

$$\sigma_y : GF(y) \rightarrow y \quad \tau_y : y \rightarrow GF(y) \quad \alpha_y : \sigma_y \star_1 \tau_y \Rightarrow 1_{GF(y)} \quad \beta_y : \tau_y \star_1 \sigma_y \Rightarrow 1_y$$

in  $\mathcal{D}$ . For every 0-cells  $y$  and  $y'$  of  $\mathcal{D}$ , we choose a quasi-inverse  $G_0(y, y')$  to  $F(F(y), F(y'))$  and we define the functor  $G(y, y')$  as  $G_0(\sigma_y, \sigma'_{y'})$ . The isomorphisms  $G_{v, v'}$  and  $G_y$  come from the 2-cells  $\alpha_y$  and  $\beta_y$ , respectively. By construction, the pseudofunctor  $G$  is a quasi-inverse for  $F$  in  $2\text{Rep}$  and, in general, it cannot be chosen to be strict (unless  $F$  is an equivalence in  $2\text{Cat}$ ). The converse direction is straightforward.



### 3. Coherent presentations and actions on categories

**3.2.2. Example (The canonical cofibrant approximation [24]).** Let  $\mathcal{C}$  be a 2-category. We denote by  $\widehat{\mathcal{C}}$  the cofibrant 2-category with the same 0-cells as  $\mathcal{C}$  and the following higher cells:

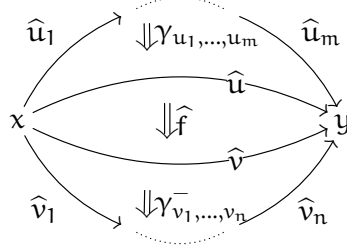
- the 1-cells of  $\widehat{\mathcal{C}}$  are freely generated by the ones of  $\mathcal{C}$ , with  $u$  in  $\mathcal{C}$  denoted by  $\widehat{u}$  when seen as a generator of  $\widehat{\mathcal{C}}$ ,
- the 2-cells from  $\widehat{u}_1 \cdots \widehat{u}_m$  to  $\widehat{v}_1 \cdots \widehat{v}_n$  in  $\widehat{\mathcal{C}}$  are the 2-cells from  $u_1 \cdots u_m$  to  $v_1 \cdots v_n$  in  $\mathcal{C}$ , with the same compositions as in  $\mathcal{C}$ .

By definition, every 2-cell  $f : u \Rightarrow v$  of  $\mathcal{C}$  has several copies in  $\widehat{\mathcal{C}}$ . We denote by  $\widehat{f}$  the one with source  $\widehat{u}$  and target  $\widehat{v}$ . For each pair of composable 1-cells  $(u, v)$  we denote by  $\gamma_{u,v} : \widehat{uv} \Rightarrow \widehat{u}\widehat{v}$  the 2-cell corresponding the identity of  $uv$  in  $\mathcal{C}$ . This 2-cell is invertible and satisfies monoidal coherence relations, so that there exists exactly one invertible 2-cell

$$\gamma_{u_1, \dots, u_n} : \widehat{u}_1 \cdots \widehat{u}_n \Longrightarrow \widehat{u_1 \cdots u_n}$$

for every family  $(u_1, \dots, u_n)$  of composable 1-cells.

Let us consider 1-cells  $u, v : x \rightarrow y$  in  $\mathcal{C}$  such that  $u = u_1 \cdots u_m$  and  $v = v_1 \cdots v_n$  hold. If  $f : u \Rightarrow v$  is a 2-cell of  $\mathcal{C}$ , then it has exactly one copy in  $\widehat{\mathcal{C}}$  that goes from  $\widehat{u}_1 \cdots \widehat{u}_m$  to  $\widehat{v}_1 \cdots \widehat{v}_n$ , which is equal, by definition of the composition in  $\widehat{\mathcal{C}}$ , to the following composite



The canonical projection 2-functor  $\widehat{\mathcal{C}} \rightarrow \mathcal{C}$  is the identity on 0-cells and maps each generating 1-cell  $\widehat{u}$  to  $u$  and each 2-cell to itself: by construction of  $\widehat{\mathcal{C}}$  and definition of  $\pi$ , it follows that  $\pi$  is a weak equivalence and that the 2-category  $\widehat{\mathcal{C}}$  is a cofibrant approximation of  $\mathcal{C}$ , called the *standard cofibrant approximation* of  $\mathcal{C}$ .

Let us note that, if  $\mathcal{C} = \mathbf{C}$  is a category (a monoid, for example), seen as a 2-category with identity 2-cells only, the 2-category  $\widehat{\mathbf{C}}$  has exactly one 2-cell from  $\widehat{u}_1 \cdots \widehat{u}_m$  to  $\widehat{v}_1 \cdots \widehat{v}_n$  if, and only if, the relation

$$u_1 \cdots u_m = v_1 \cdots v_n$$

holds in  $\mathbf{C}$ : this 2-cell is the composite of  $\gamma_{u_1, \dots, u_m}$  followed by  $\gamma_{v_1, \dots, v_n}^{-1}$ . As a consequence, the canonical cofibrant approximation  $\widehat{\mathbf{C}}$  of  $\mathbf{C}$  is exactly the  $(2, 1)$ -category presented by the canonical coherent presentation  $\text{Can}(\mathbf{C})$  of  $\mathbf{C}$ , as given in Example 2.3.8.

**3.2.3. Theorem.** *Let  $\mathbf{C}$  be a category and let  $\Sigma$  be an extended presentation of  $\mathbf{C}$ . The following assertions are equivalent:*

- i) *The  $(3, 1)$ -polygraph  $\Sigma$  is a coherent presentation of  $\mathbf{C}$ .*

### 3.2. Cofibrant approximations of 2-categories

ii) The 2-category  $\bar{\Sigma}$  presented by  $\Sigma$  is a cofibrant approximation of  $\mathbf{C}$ .

*Proof.* Let us assume that  $\Sigma_3$  is a homotopy basis of  $\Sigma_2^\top$ . By definition, the 2-category  $\bar{\Sigma}$  is cofibrant. Let us check that it is weakly equivalent to  $\mathbf{C}$ . We consider the canonical projection  $\pi : \Sigma^\top \rightarrow \mathbf{C}$  that sends every 0-cell to itself, every 1-cell to its equivalence class and every 2-cell and 3-cell to the corresponding identity. This is well-defined since two 1-cells of  $\Sigma_2^\top$  have the same equivalence class in  $\mathbf{C}$  if, and only if, there exists a 2-cell between them in  $\Sigma_2^\top$  and since parallel 2-cells of  $\Sigma^\top$  are sent to the same (identity) 2-cell of  $\mathbf{C}$ .

Since  $\pi$  is the identity on 0-cells, it is sufficient to check that it induces an equivalence of categories between  $\bar{\Sigma}(x, y)$  and  $\mathbf{C}(x, y)$  for every 0-cells  $x$  and  $y$  in  $\mathbf{C}$ . We define a quasi-inverse  $\iota$  by choosing, for each 1-cell  $u : x \rightarrow y$  in  $\mathbf{C}$ , an arbitrary representative 1-cell  $\iota(u)$  in  $\bar{\Sigma}$ . By construction, we have that  $\pi\iota$  is the identity of  $\mathbf{C}(x, y)$  and that  $\iota\pi(u)$  is a 1-cell from  $x$  to  $y$  that has the same equivalence class as  $u$ : we choose an arbitrary 2-cell  $\alpha_u : u \Rightarrow \iota\pi(u)$  in  $\bar{\Sigma}$ . Since every parallel 2-cells of  $\bar{\Sigma}$  are equal, we get the following commutative diagram for every 2-cell  $f$  of  $\bar{\Sigma}$ :

$$\begin{array}{ccc}
 & \alpha_u \rightarrow & \iota\pi(u) \\
 u & \searrow & \swarrow \iota(\pi(f)) \\
 & = & \\
 & \nearrow f & \nwarrow \alpha_v \\
 & v & 
 \end{array}$$

This proves that  $\alpha$  is a natural isomorphism between  $\iota\pi$  and the identity of  $\text{cl}\Sigma(x, y)$ , yielding that  $\pi$  is a weak equivalence and, as a consequence, that  $\bar{\Sigma}$  is a cofibrant approximation of  $\mathbf{C}$ .

Conversely, let us assume that  $\bar{\Sigma}$  is a cofibrant approximation of  $\mathbf{C}$ . Let  $F : \bar{\Sigma} \rightarrow \mathbf{C}$  be a weak equivalence and let  $f, g : u \Rightarrow v : x \rightarrow y$  be parallel 2-cells of  $\Sigma^\top$ . Since  $F$  is a 2-functor and  $\mathbf{C}$  has identity 2-cells only, we must have  $F(u) = F(v)$  and  $F(f) = F(g) = 1_{F(u)}$ . By hypothesis, the 2-functor  $F$  induces an equivalence of categories between  $\bar{\Sigma}(x, y)$  and  $\mathbf{C}(x, y)$ : we choose a quasi-inverse  $G$  and a natural isomorphism  $\alpha$  between  $GF$  and the identity of  $\bar{\Sigma}(x, y)$ . We write the naturality conditions for  $f$  and  $g$  and, using  $GF(f) = GF(g) = 1_{GF(u)}$ , we conclude that they are equal in  $\bar{\Sigma}$ :

$$\begin{array}{ccc}
 \alpha_u \rightarrow & GF(u) & GF(f) \\
 u & \searrow & \swarrow \\
 & = & \\
 & \nearrow f & \nwarrow \alpha_v \\
 & v & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \alpha_u \rightarrow & GF(u) & GF(g) \\
 u & \searrow & \swarrow \\
 & = & \\
 & \nearrow g & \nwarrow \alpha_v \\
 & v & 
 \end{array}$$

Thus  $\Sigma$  is a coherent presentation of  $\mathbf{C}$ . □

*Remark.* The cofibrant approximations of a category  $\mathbf{C}$  form, in general, a strictly larger class than the 2-categories presented by coherent presentations of  $\mathbf{C}$ . Indeed, let  $\mathbf{C}$  be the terminal category: it contains one 0-cell and the corresponding identity 1-cell only. Then  $\mathbf{C}$  is cofibrant and, as a consequence, it is a cofibrant approximation of itself: this corresponds to the coherent presentation of  $\mathbf{C}$  given by the  $(3, 1)$ -polygraph with one 0-cell and no higher-dimensional cells. But  $\mathbf{C}$  also admits, as a cofibrant approximation, the ‘‘equivalence’’ 2-category with two 0-cells  $x$  and  $y$ , two 1-cells  $u : x \rightarrow y$  and  $v : y \rightarrow x$  and two invertible 2-cells  $f : uv \Rightarrow 1_x$  and  $g : vu \Rightarrow 1_y$ , and this 2-category is not presented by a coherent presentation of  $\mathbf{C}$ , since it does not have the same 0-cells as  $\mathbf{C}$ .

### 3. Coherent presentations and actions on categories

#### 3.3. 2-representations of cofibrant 2-categories

Let us fix 2-categories  $\mathcal{C}$  and  $\mathcal{D}$ , with  $\mathcal{C}$  cofibrant. Our objective is to define a “strictification” functor

$$\widehat{\phantom{x}} : 2\text{Rep}(\mathcal{C}, \mathcal{D}) \longrightarrow 2\text{Cat}(\mathcal{C}, \mathcal{D})$$

and to prove that it is a quasi-inverse for the canonical inclusion functor of  $2\text{Cat}(\mathcal{C}, \mathcal{D})$  into  $2\text{Rep}(\mathcal{C}, \mathcal{D})$ .

**3.3.1. Strictification of 2-representations.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a 2-representation. Let us define the 2-functor  $\widehat{F} : \mathcal{C} \rightarrow \mathcal{D}$ , dimension after dimension. On 0-cells,  $\widehat{F}$  takes the same values as  $F$ . Since  $\mathcal{C}$  is cofibrant, its underlying 1-category is free: on generating 1-cells,  $\widehat{F}$  is equal to  $F$  and, then, it is extended by functoriality on every 1-cell. Hence, if  $u = a_1 \cdots a_n$  is a 1-cell of  $\mathcal{C}$ , where the  $a_i$ s are generating 1-cells, we have:

$$\widehat{F}(u) = F(a_1) \cdots F(a_n).$$

From the monoidal coherence relations satisfied by  $F$ , there is a unique invertible 2-cell in  $\mathcal{D}$

$$\widehat{F}(u) = F(a_1) \cdots F(a_n) \xrightarrow{F_{a_1, \dots, a_n}} F(a_1 \cdots a_n) = F(u)$$

from  $\widehat{F}(u)$  to  $F(u)$ , built from the coherence 2-cells of  $F$ . Since the decomposition of  $u$  in generators is unique, we simply denote this 2-cell by  $F_u$ . Let  $f : u \Rightarrow v : x \rightarrow y$  be a 2-cell of  $\mathcal{C}$ . We define  $\widehat{F}(f)$  as the following composite 2-cell of  $\mathcal{D}$ , where the double arrows, which always go from top to bottom, have been omitted for readability:

$$\begin{array}{ccc} \begin{array}{ccc} & \widehat{F}(u) & \\ & \curvearrowright & \\ F(x) & \widehat{F}(f) & F(y) \\ & \curvearrowleft & \\ & \widehat{F}(v) & \end{array} & = & \begin{array}{ccc} & \widehat{F}(u) & \\ & \curvearrowright & \\ F(x) & \begin{array}{c} F(u) \\ \widehat{F}(f) \\ F(v) \end{array} & F(y) \\ & \curvearrowleft & \\ & F_v^- & \\ & \widehat{F}(v) & \end{array} \end{array}$$

As a direct consequence, we get that  $\widehat{F}$  is compatible with vertical composition and identities of 1-cells. Hence, we have defined a 2-functor  $\widehat{F}$  from  $\mathcal{C}$  to  $\mathcal{D}$ . We note that the monoidal coherence relations satisfied by  $F$  imply that the 2-cells  $F_u : \widehat{F}(u) \Rightarrow F(u)$  satisfy the following relations with respect to composition and identities. If  $u : x \rightarrow y$  and  $v : y \rightarrow z$  are 1-cells of  $\mathcal{C}$ , we have:

$$\begin{array}{ccc} \begin{array}{ccc} & \widehat{F}(uv) & \\ & \curvearrowright & \\ F(x) & F_{uv} & F(z) \\ & \curvearrowleft & \\ & F(uv) & \end{array} & = & \begin{array}{ccc} & \widehat{F}(u) & \widehat{F}(v) \\ & \curvearrowright & \curvearrowright \\ F(x) & \begin{array}{c} F_u \\ F(u) \\ F(v) \\ F_v^- \end{array} & F(z) \\ & \curvearrowleft & \\ & F_{u,v} & \\ & F(uv) & \end{array} \end{array}$$

Moreover, if  $x$  is a 0-cell of  $\mathcal{C}$ , we have  $F_{1_x} = F_x$ .

**3.3.2. Strictification of morphisms of 2-representations.** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be 2-representations and let  $\alpha : F \Rightarrow G$  be a morphism between them. Let us define a pseudonatural transformation  $\widehat{\alpha} : \widehat{F} \Rightarrow \widehat{G}$ . For a 0-cell  $x$  of  $\mathcal{C}$ , we take  $\widehat{\alpha}_x = \alpha_x$ . If  $u : x \rightarrow y$  is a 1-cell of  $\mathcal{C}$ , we define  $\widehat{\alpha}_u$  as the following invertible 2-cell of  $\mathcal{D}$ :

$$\begin{array}{ccc}
 \begin{array}{ccc} \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\ \uparrow & & \uparrow \alpha_y \\ F(x) & & G(y) \\ \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\ G(x) & & \end{array} & \widehat{\alpha}_u & \\
 \end{array} = \begin{array}{ccc} \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\ \uparrow F_u & & \uparrow \alpha_y \\ F(x) & \xrightarrow{F(u)} & G(y) \\ \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\ G(x) & \xrightarrow{G(u)} & G(y) \\ \uparrow G_u^- & & \uparrow \widehat{G}(u) \\ \end{array}$$

Then  $\widehat{\alpha} : \widehat{F} \Rightarrow \widehat{G}$  is a pseudonatural transformation. Indeed, if  $x$  is a 0-cell of  $\mathcal{C}$ , we have:

$$\begin{array}{ccc}
 \begin{array}{ccc} 1_{F(x)} & \xrightarrow{\quad} & F(x) \\ \uparrow & & \uparrow \alpha_x \\ F(x) & & G(x) \\ \downarrow \alpha_x & & \downarrow 1_{G(x)} \\ G(x) & & \end{array} & \widehat{\alpha}_{1_x} & \\
 \end{array} = \begin{array}{ccc} 1_{F(x)} & \xrightarrow{\quad} & F(x) \\ \uparrow F_x & & \uparrow \alpha_x \\ F(x) & \xrightarrow{F(1_x)} & G(x) \\ \downarrow \alpha_x & & \downarrow 1_{G(x)} \\ G(x) & \xrightarrow{G(1_x)} & G(x) \\ \uparrow G_x^- & & \uparrow 1_{G(x)} \\ \end{array} = \begin{array}{ccc} & \xrightarrow{\alpha_x} & \\ F(x) & & G(x) \\ & \xleftarrow{\alpha_x} & \\ \end{array}$$

Then, if  $u : x \rightarrow y$  and  $v : y \rightarrow z$  are 1-cells of  $\mathcal{C}$ , we get:

$$\begin{array}{ccc}
 \begin{array}{ccc} \widehat{F}(uv) & \xrightarrow{\quad} & F(z) \\ \uparrow & & \uparrow \alpha_z \\ F(x) & & G(z) \\ \downarrow \alpha_x & & \downarrow \widehat{G}(uv) \\ G(x) & & \end{array} & \widehat{\alpha}_{uv} & \\
 \end{array} = \begin{array}{ccc} \widehat{F}(v) & \xrightarrow{\quad} & F(z) \\ \uparrow F_v & & \uparrow \alpha_z \\ F(y) & \xrightarrow{F(v)} & G(z) \\ \uparrow F_u & & \uparrow \alpha_y \\ \widehat{F}(u) & \xrightarrow{F(u)} & G(y) \\ \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\ G(x) & \xrightarrow{G(u)} & G(y) \\ \uparrow G_{u,v}^- & & \uparrow \widehat{G}(v) \\ \end{array}$$

$$= \begin{array}{ccc} \widehat{F}(v) & \xrightarrow{\quad} & F(z) \\ \uparrow F_v & & \uparrow \alpha_z \\ F(y) & \xrightarrow{F(v)} & G(z) \\ \uparrow F_u & & \uparrow \alpha_y \\ \widehat{F}(u) & \xrightarrow{F(u)} & G(y) \\ \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\ G(x) & \xrightarrow{G(u)} & G(y) \\ \uparrow G_u^- & & \uparrow \widehat{G}(v) \\ \end{array} = \begin{array}{ccc} \widehat{F}(v) & \xrightarrow{\quad} & F(z) \\ \uparrow & & \uparrow \alpha_z \\ F(y) & \xrightarrow{F(v)} & G(z) \\ \uparrow & & \uparrow \alpha_y \\ \widehat{F}(u) & \xrightarrow{F(u)} & G(y) \\ \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\ G(x) & \xrightarrow{G(u)} & G(y) \\ \uparrow & & \uparrow \widehat{G}(v) \\ \end{array}$$

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Finally, if  $f : u \Rightarrow v : x \rightarrow y$  is a 2-cell of  $\mathcal{C}$ :

With similar computations, we check that strictification is compatible with the composition of morphisms of 2-representations and with identities, so that it is a functor from  $2\text{Rep}(\mathcal{C}, \mathcal{D})$  to  $2\text{Cat}(\mathcal{C}, \mathcal{D})$ .

**3.3.3. Proposition.** *Let  $\mathcal{C}$  be a cofibrant 2-category. For every 2-category  $\mathcal{D}$ , the canonical inclusion*

$$2\text{Cat}(\mathcal{C}, \mathcal{D}) \longrightarrow 2\text{Rep}(\mathcal{C}, \mathcal{D})$$

*is an equivalence of categories, with quasi-inverse given by the strictification functor.*

*Proof.* It is sufficient to check that, for every 2-representation  $F : \mathcal{C} \rightarrow \mathcal{D}$ , there exists a pseudonatural isomorphism  $\varphi_F : \widehat{F} \Rightarrow F$  that is itself natural in  $F$ . We define  $\varphi_F$  as follows:

- if  $x$  is a 0-cell of  $\mathcal{C}$ , then  $\widehat{F}(x) = F(x)$  and we take  $(\varphi_F)_x = 1_x$ ,
- if  $u : x \rightarrow y$  is a 1-cell of  $\mathcal{C}$ , then  $(\varphi_F)_u : \widehat{F}(u) \Rightarrow F(u)$  is defined as the invertible coherence 2-cell  $F_u : \widehat{F}(u) \Rightarrow F(u)$ .

This data satisfies the required coherence properties: the compatibility with the 2-cells of  $\mathcal{C}$  is exactly the definition of  $\widehat{F}$  and the compatibility with horizontal composition and identities comes from the monoidal coherence relations of  $F$ , as already checked. Moreover, if  $\alpha : F \Rightarrow G$  is a morphism of 2-representations, the naturality condition

corresponds, on each 1-cell  $u$  of  $\mathcal{C}$ , to the definition of  $\widehat{\alpha}$ . □

### 3.4. 2-representations and cofibrant approximations

Let us recall that, for a 2-category  $\mathcal{C}$ , we denote by  $\widehat{\mathcal{C}}$  its standard cofibrant replacement. We note that the definition of a 2-functor from  $\widehat{\mathcal{C}}$  to a 2-category  $\mathcal{D}$  is exactly the same as the one of a pseudofunctor from  $\mathcal{C}$  to  $\mathcal{D}$ , yielding the following isomorphism of categories:

$$2\text{Rep}(\mathcal{C}, \mathcal{D}) \simeq 2\text{Cat}(\widehat{\mathcal{C}}, \mathcal{D}).$$

In particular, for every monoid  $\mathbf{M}$ , we get an isomorphism of categories:

$$\text{Act}(\mathbf{M}) \simeq 2\text{Cat}(\widehat{\mathbf{M}}, \text{Cat}).$$

In what follows, we prove that weak versions of these isomorphisms exist for all cofibrant approximations. More precisely, the category of 2-representations of a 2-category  $\mathcal{C}$  into a 2-category  $\mathcal{D}$  is equivalent to the one of 2-functors from any cofibrant approximation  $\widetilde{\mathcal{C}}$  of  $\mathcal{C}$  into  $\mathcal{D}$ . This result specialises to the actions of a monoid on categories.

**3.4.1. Lemma.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be 2-categories. The following assertions are equivalent:*

- i) *The 2-categories  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent in  $2\text{Rep}$ .*
- ii) *For every 2-category  $\mathcal{E}$ , there is an equivalence of categories*

$$2\text{Rep}(\mathcal{C}, \mathcal{E}) \approx 2\text{Rep}(\mathcal{D}, \mathcal{E})$$

*that is natural in  $\mathcal{E}$ .*

*Proof.* Let us assume that  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent in  $2\text{Rep}$ , *i.e.*, that there exist pseudofunctors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that

$$GF \simeq 1_{\mathcal{C}} \quad \text{and} \quad FG \simeq 1_{\mathcal{D}}.$$

As a consequence, for every pseudofunctors  $H : \mathcal{C} \rightarrow \mathcal{E}$  and  $K : \mathcal{D} \rightarrow \mathcal{E}$ , we have:

$$HGF \simeq H \quad \text{and} \quad KFG \simeq K.$$

Thus the functors  $2\text{Rep}(F, \mathcal{E})$  and  $2\text{Rep}(G, \mathcal{E})$ , respectively sending a pseudofunctor  $K : \mathcal{D} \rightarrow \mathcal{E}$  to  $KF$  and a pseudofunctor  $H : \mathcal{C} \rightarrow \mathcal{E}$  to  $HG$ , form the required equivalence of categories.

Conversely, let us assume that, for every 2-category  $\mathcal{E}$ , we have  $2\text{Rep}(\mathcal{C}, \mathcal{E}) \approx 2\text{Rep}(\mathcal{D}, \mathcal{E})$  natural in  $\mathcal{E}$ . We denote by

$$\Phi_{\mathcal{E}} : 2\text{Rep}(\mathcal{C}, \mathcal{E}) \rightarrow 2\text{Rep}(\mathcal{D}, \mathcal{E}) \quad \text{and} \quad \Psi_{\mathcal{E}} : 2\text{Rep}(\mathcal{D}, \mathcal{E}) \rightarrow 2\text{Rep}(\mathcal{C}, \mathcal{E})$$

the functors that constitute the equivalence. The naturality of the equivalence means, in particular, that for every pseudofunctors  $H : \mathcal{C} \rightarrow \mathcal{E}$  and  $K : \mathcal{D} \rightarrow \mathcal{E}$ , we have the following isomorphisms:

$$\Psi_{\mathcal{E}}\Phi_{\mathcal{E}}(H) \simeq H \quad \text{and} \quad \Phi_{\mathcal{E}}\Psi_{\mathcal{E}}(K) \simeq K.$$

Let us define the pseudofunctors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  as follows:

$$F = \Psi_{\mathcal{D}}(1_{\mathcal{D}}) \quad \text{and} \quad G = \Phi_{\mathcal{C}}(1_{\mathcal{C}}).$$

#### 4. Homotopical completion and homotopical reduction

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Thus we have, using the properties of  $\Phi$  and  $\Psi$ :

$$GF = \Phi_{\mathcal{C}}(1_{\mathcal{C}}) \circ \Psi_{\mathcal{D}}(1_{\mathcal{D}}) = \Psi_{\mathcal{C}}\Phi_{\mathcal{C}}(1_{\mathcal{C}}) \simeq 1_{\mathcal{C}}.$$

We get  $FG \simeq 1_{\mathcal{D}}$  in a symmetric way to conclude that  $F$  and  $G$  form an equivalence in  $2\text{Rep}$ .  $\square$

A combination of Proposition 3.3.3 and of Lemma 3.4.1 gives the following result.

**3.4.2. Proposition.** *Let  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  be 2-categories. The following assertions are equivalent:*

- i) *The 2-category  $\tilde{\mathcal{C}}$  is a cofibrant approximation of  $\mathcal{C}$ .*
- ii) *For every 2-category  $\mathcal{D}$ , there is an equivalence of categories*

$$2\text{Rep}(\mathcal{C}, \mathcal{D}) \approx 2\text{Cat}(\tilde{\mathcal{C}}, \mathcal{D})$$

*that is natural in  $\mathcal{D}$ .*

*In particular, if  $\mathbf{M}$  is a monoid and  $\tilde{\mathbf{M}}$  is a cofibrant approximation of  $\mathbf{M}$ , then we have an equivalence of categories*

$$\text{Act}(\mathbf{M}) \approx 2\text{Cat}(\tilde{\mathbf{M}}, \text{Cat}).$$

Finally, an application of Theorem 3.2.3 gives the following result, relating the coherent presentations of a category to its 2-representations. In particular, when applied to Deligne's coherent presentation of an Artin monoid  $\mathbf{B}^+(\mathbf{W})$ , as obtained in 5.1.4, Theorem 3.4.3 extends Deligne's Theorem 1.5 of [13] to the non-spherical case.

**3.4.3. Theorem.** *Let  $\mathbf{C}$  be a category, let  $\Sigma$  be an extended presentation of  $\mathbf{C}$ . The following assertions are equivalent:*

- i) *The  $(3, 1)$ -polygraph  $\Sigma$  is a coherent presentation of  $\mathbf{C}$ .*
- ii) *For every 2-category  $\mathcal{C}$ , there is an equivalence of categories*

$$2\text{Rep}(\mathbf{C}, \mathcal{C}) \approx 2\text{Cat}(\bar{\Sigma}, \mathcal{C})$$

*that is natural in  $\mathcal{C}$ .*

*In particular, if  $\mathbf{M}$  is a monoid and if  $\Sigma$  is a coherent presentation of  $\mathbf{M}$ , we have an equivalence of categories*

$$\text{Act}(\mathbf{M}) \approx 2\text{Cat}(\bar{\Sigma}, \text{Cat}).$$

#### 4. HOMOTOPICAL COMPLETION AND HOMOTOPICAL REDUCTION

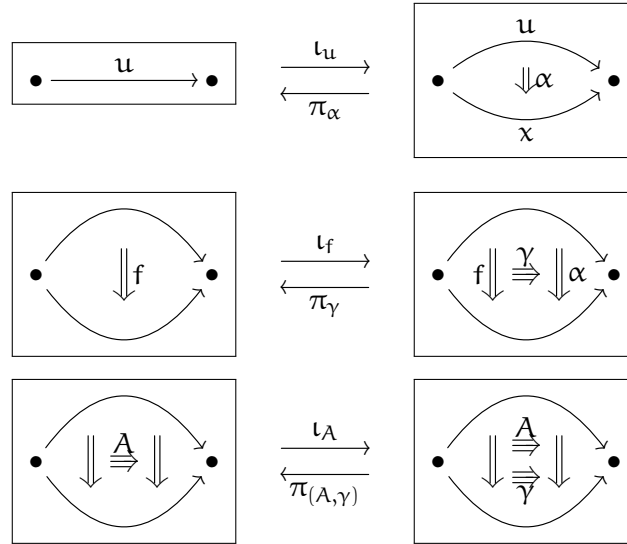
In this section, we introduce the *homotopical completion-reduction procedure*: an algorithmic method to extend a presentation  $\Sigma$  into a coherent presentation, by computing a homotopy basis of  $\Sigma^{\top}$ . It is based on rewriting techniques, mainly Squier's completion and Knuth-Bendix's completion, adapted to the setting of coherent presentations and formulated in terms of Tietze transformations of  $(3, 1)$ -polygraphs.

### 4.1. Tietze transformations of (3, 1)-polygraphs

We adapt the notion of Tietze transformations of presentations of groups [27] to (3, 1)-polygraphs: they consist in coherent adjunctions or eliminations of 1-cells, 2-cells and 3-cells that preserve the presented 2-category, up to equivalence. In particular, we get that two (3, 1)-polygraphs are coherent presentations of the same category if, and only if, they are related by a Tietze transformation.

**4.1.1. Tietze equivalence.** An equivalence of 2-categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a *Tietze equivalence* if the quotient categories  $\mathcal{C}_1/\mathcal{C}_2$  and  $\mathcal{D}_1/\mathcal{D}_2$  are isomorphic. Two (3, 1)-polygraphs are *Tietze-equivalent* if the 2-categories they present are Tietze-equivalent. In that case, they have the same 0-cells (up to a bijection). Moreover, two coherent presentations of the same category are Tietze-equivalent.

**4.1.2. Tietze transformations.** Let  $\Sigma$  be a (3, 1)-polygraph. An *elementary Tietze transformation* on  $\Sigma$  is a 3-functor with source  $\Sigma^\top$  that belongs to one of the six families pictured as follows and formally described afterwards:



The coherent adjunctions

$$\iota_u : \Sigma^\top \hookrightarrow \Sigma^\top(x)(\alpha) \quad \iota_f : \Sigma^\top \hookrightarrow \Sigma^\top(\alpha)(\gamma) \quad \iota_A : \Sigma^\top \hookrightarrow \Sigma^\top(\gamma)$$

are the canonical inclusions. Conversely, the coherent eliminations

$$\pi_\alpha : \Sigma^\top \twoheadrightarrow \Sigma^\top/\alpha \quad \pi_\gamma : \Sigma^\top \twoheadrightarrow \Sigma^\top/\gamma \quad \pi_{(A,\gamma)} : \Sigma^\top \twoheadrightarrow \Sigma^\top/(A,\gamma)$$

are the canonical projections defined as follows. If  $\alpha : u \Rightarrow x$  is a 2-cell of  $\Sigma$ , with  $x$  a 1-cell of  $\Sigma$  and  $u$  a 1-cell of  $(\Sigma \setminus \{x\})^*$ , the projection  $\pi_\alpha$  maps  $x$  to  $u$  and  $\alpha$  to  $1_u$ , leaving the other cells unchanged. The (3, 1)-category  $\Sigma^\top/\alpha$  is freely generated by the following (3, 1)-polygraph  $\Sigma/\alpha$ :

$$\Sigma_0 \xleftarrow[t_0]{s_0} (\Sigma_1 \setminus \{x\})^* \xleftarrow[\pi_\alpha \circ t_1]{\pi_\alpha \circ s_1} (\Sigma_2 \setminus \{\alpha\})^\top \xleftarrow[\pi_\alpha \circ t_2]{\pi_\alpha \circ s_2} \Sigma_3^\top.$$



#### 4. Homotopical completion and homotopical reduction

If  $\gamma : f \Rightarrow \alpha$  is a 3-cell of  $\Sigma$ , with  $\alpha$  a 2-cell of  $\Sigma$  and  $f$  a 2-cell of  $(\Sigma \setminus \{\alpha\})^\top$ , the projection  $\pi_\gamma$  maps  $\alpha$  to  $f$  and  $\gamma$  to  $1_f$ , leaving the other cells unchanged. The  $(3, 1)$ -category  $\Sigma^\top/\gamma$  is freely generated by the following  $(3, 1)$ -polygraph  $\Sigma/\gamma$ :

$$\Sigma_0 \xleftarrow[t_0]{s_0} \Sigma_1^* \xleftarrow[t_1]{s_1} (\Sigma_2 \setminus \{\alpha\})^\top \xleftarrow[\pi_\gamma \circ t_2]{\pi_\gamma \circ s_2} (\Sigma_3 \setminus \{\gamma\})^\top.$$

Finally, if  $\gamma$  is a 3-cell of  $\Sigma$  and  $A$  is a 3-cell of  $(\Sigma \setminus \{\gamma\})^\top$ , the projection  $\pi_{(A, \gamma)}$  maps  $\gamma$  to  $A$ . The  $(3, 1)$ -category  $\Sigma^\top/(A, \gamma)$  is freely generated by the following  $(3, 1)$ -polygraph  $\Sigma/(A, \gamma)$ :

$$\Sigma_0 \xleftarrow[t_0]{s_0} \Sigma_1^* \xleftarrow[t_1]{s_1} \Sigma_2^\top \xleftarrow[t_2]{s_2} (\Sigma_3 \setminus \{\gamma\})^\top.$$

If  $\Sigma$  and  $\Upsilon$  are  $(3, 1)$ -polygraphs, a (finite) Tietze transformation from  $\Sigma$  to  $\Upsilon$  is a (finite) composition of elementary Tietze transformations.

**4.1.3. Theorem.** *Two (finite)  $(3, 1)$ -polygraphs are Tietze equivalent if, and only if, there exists a (finite) Tietze transformation between them. In particular, two (finite)  $(3, 1)$ -polygraphs are coherent presentations of the same category if, and only if, there exists a (finite) Tietze transformation between them.*

*Proof.* Let us prove that, if two  $(3, 1)$ -polygraphs are related by a Tietze transformation, then they are Tietze-equivalent. Since isomorphisms of categories and equivalence of 2-categories compose, it is sufficient to check the result for each one of the six types of elementary Tietze transformations on a fixed  $(3, 1)$ -polygraph  $\Sigma$ . By definition, the 3-functors  $\pi \circ \iota$  are all equal to the identity of  $\Sigma^\top$  and the 3-functors  $\iota \circ \pi$  induce identities on the presented category  $\Sigma_1^*/\Sigma_2$ . Moreover, the latter induce the following 2-functors on the presented 2-category  $\bar{\Sigma}$ :

$$\bar{\iota}_u \circ \bar{\pi}_\alpha \simeq 1_{\bar{\Sigma}} \quad \bar{\iota}_f \circ \bar{\pi}_A = 1_{\bar{\Sigma}} \quad \bar{\iota}_A \circ \bar{\pi}_{(A, \gamma)} = 1_{\bar{\Sigma}}.$$

Indeed, the first isomorphism is the identity on every cell, except on  $x$  which is mapped to  $\bar{x}$ . The second and third isomorphisms are, in fact, identities since they do not change the equivalence classes of 2-cells modulo 3-cells.

Conversely, let  $\Sigma$  and  $\Upsilon$  be Tietze-equivalent  $(3, 1)$ -polygraphs. We fix an equivalence  $F : \bar{\Sigma} \rightarrow \bar{\Upsilon}$  of 2-categories that induce an isomorphism on the presented categories. We choose a weak inverse  $G : \bar{\Upsilon} \rightarrow \bar{\Sigma}$  and pseudonatural isomorphisms  $\sigma : GF \Rightarrow 1_{\bar{\Sigma}}$  and  $\tau : FG \Rightarrow 1_{\bar{\Upsilon}}$ , in such a way that the quadruple  $(F, G, \sigma, \tau)$  is an adjoint equivalence, which is always feasible [28, Chap. IV, § 4, Theorem 1]. This means that the following ‘‘triangle identities’’ hold:

$$\begin{array}{ccc} & F\sigma & \\ & \curvearrowright & \\ FGF & = & F \\ & \curvearrowleft & \\ & \tau F & \end{array} \quad \begin{array}{ccc} & G\tau & \\ & \curvearrowright & \\ GFG & = & G \\ & \curvearrowleft & \\ & \sigma G & \end{array}$$

Let us lift the 2-functor  $F$  to a 3-functor  $\hat{F} : \Sigma^\top \rightarrow \Upsilon^\top$ , defined as  $F$  on the 0-cells and 1-cells. For every 2-cell  $\alpha : u \Rightarrow v$  of  $\Sigma$ , we choose a representative  $\hat{F}(\alpha) : F(u) \Rightarrow F(v)$  of  $F(\bar{\alpha})$  in  $\Upsilon^\top$  and, then, we

#### 4.1. Tietze transformations of (3, 1)-polygraphs

extend  $\widehat{F}$  by functoriality to every 2-cell of  $\Sigma^\top$ . For a 3-cell  $\gamma : f \Rightarrow g$  of  $\Sigma$ , we have  $\bar{f} = \bar{g}$  by definition of  $\bar{\phantom{x}}$ , so that  $F(\bar{f}) = F(\bar{g})$  holds in  $\Upsilon$ , meaning that there exists a 3-cell in  $\Upsilon^\top$  from  $\widehat{F}(f)$  to  $\widehat{F}(g)$ : we take it as a value for  $\widehat{F}(\gamma)$  and we extend  $\widehat{F}$  to every 3-cell of  $\Sigma^\top$  by functoriality. We proceed similarly with  $G$  to get a 3-functor  $\widehat{G} : \Upsilon^\top \rightarrow \Sigma^\top$ .

Then, for a 1-cell  $x$  of  $\Sigma$ , we choose a representative  $\widehat{\sigma}_x : GF(x) \Rightarrow x$  of  $\sigma_x$  in  $\Sigma^\top$  and we extend it to every 1-cell by functoriality. If  $\alpha : u \Rightarrow v$  is a 2-cell of  $\Sigma$ , the naturality condition satisfied by  $\sigma$  on  $\bar{\alpha}$  lifts to an arbitrarily chosen 3-cell of  $\Sigma$

$$\begin{array}{ccccc}
 & & \widehat{GF}(\alpha) & \longrightarrow & GF(v) & \xrightarrow{\widehat{\sigma}_v} & v \\
 & \nearrow & & & \Downarrow \widehat{\sigma}_\alpha & & \nearrow \\
 GF(u) & & & & & & \\
 & \searrow & & & & & \searrow \\
 & & \widehat{\sigma}_u & \longrightarrow & u & \xrightarrow{\alpha} & v
 \end{array}$$

We proceed similarly with  $\tau$ . The conditions for the adjoint equivalence also lift to a 3-cell  $\lambda_x$  of  $\Upsilon^\top$  for every 1-cell  $x$  of  $\Sigma$  and to a 3-cell  $\rho_y$  of  $\Sigma^\top$  for every 1-cell  $y$  of  $\Upsilon$ :

$$\begin{array}{ccc}
 \widehat{F}(\widehat{\sigma}_x) & & \widehat{G}(\widehat{\tau}_y) \\
 \begin{array}{ccc}
 \widehat{FGF}(x) & \xrightarrow{\quad} & F(x) \\
 \Downarrow \lambda_x & & \\
 \widehat{\tau}_{F(x)} & \xrightarrow{\quad} & 
 \end{array} & & \begin{array}{ccc}
 GFG(y) & \xrightarrow{\quad} & G(y) \\
 \Downarrow \rho_y & & \\
 \widehat{\sigma}_{G(y)} & \xrightarrow{\quad} & 
 \end{array}
 \end{array}$$

Now, let us build a Tietze transformation from  $\Sigma$  to  $\Upsilon$ . We start by constructing a (3, 1)-polygraph  $\Xi$  that contains both  $\Sigma$  and  $\Upsilon$ , together with coherence cells that correspond to the Tietze-equivalence. The (3, 1)-polygraph  $\Xi$  has the same 0-cells as  $\Sigma$  (and as  $\Upsilon$ ) and it contains the 1-cells, 2-cells and 3-cells of  $\Sigma$  and  $\Upsilon$ , plus the following cells:

- two 2-cells  $\varphi_x : F(x) \Rightarrow x$  and  $\psi_y : G(y) \Rightarrow y$ , for every 1-cells  $x$  of  $\Sigma$  and  $y$  of  $\Upsilon$ ,
- two 3-cells  $\varphi_\alpha$  and  $\psi_\beta$ , for every 2-cells  $\alpha : u \Rightarrow u'$  and  $\beta : v \Rightarrow v'$ , with shapes

$$\begin{array}{ccc}
 \varphi_u^- \nearrow & F(u) \xrightarrow{\widehat{F}(\alpha)} F(u') & \searrow \varphi_{u'} \\
 & \Downarrow \varphi_\alpha & \\
 u \xrightarrow{\quad} & & u'
 \end{array}
 \qquad
 \begin{array}{ccc}
 \psi_v^- \nearrow & G(v) \xrightarrow{\widehat{G}(\beta)} G(v') & \searrow \psi_{v'} \\
 & \Downarrow \psi_\beta & \\
 v \xrightarrow{\quad} & & v'
 \end{array}$$

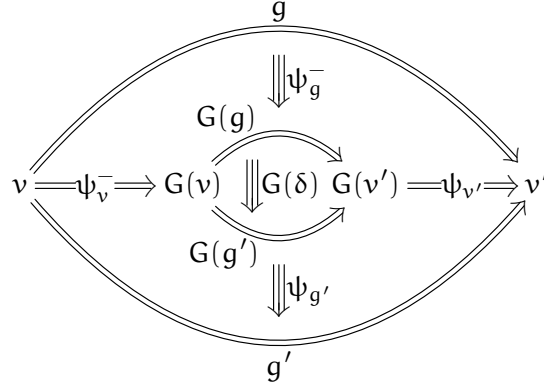
- two 3-cells  $\xi_x$  and  $\eta_y$ , for every 1-cells  $x$  of  $\Sigma$  and  $y$  of  $\Upsilon$ , with shapes

$$\begin{array}{ccc}
 \psi_{F(x)}^- \nearrow & GF(x) \xrightarrow{\widehat{\sigma}_x} & x \\
 & \Downarrow \xi_x & \\
 F(x) \xrightarrow{\quad} & & x
 \end{array}
 \qquad
 \begin{array}{ccc}
 \varphi_{G(y)}^- \nearrow & FG(y) \xrightarrow{\widehat{\tau}_y} & y \\
 & \Downarrow \eta_y & \\
 G(y) \xrightarrow{\quad} & & y
 \end{array}$$

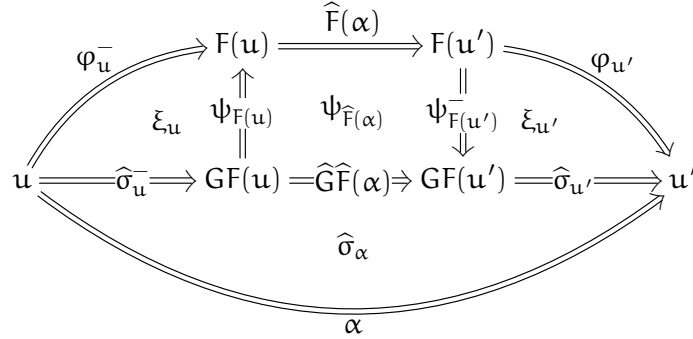
#### 4. Homotopical completion and homotopical reduction

We construct a Tietze-transformation  $\Phi$  from  $\Sigma$  to  $\Xi$  step-by-step, as follows.

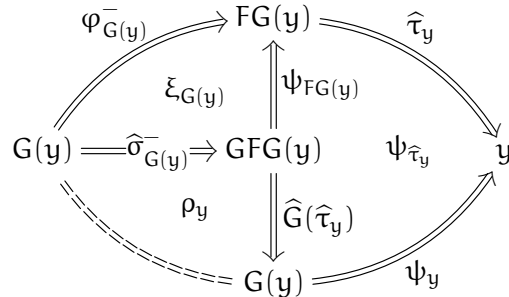
- Adjunction of the cells of  $\Upsilon$ . For every 1-cell  $y$  of  $\Upsilon$ , we apply  $\iota_{G(y)}$  to coherently add  $y$  and  $\psi_y : G(y) \Rightarrow y$ . Then, for every 2-cell  $\beta : v \Rightarrow v'$  of  $\Upsilon$ , we apply  $\iota_{\psi_v^{-1} \star_1 \widehat{G}(\beta) \star_1 \psi_{v'}}$  to coherently add  $\beta$  and  $\psi_\beta$ . Then, we add every 3-cell  $\delta : g \Rightarrow g'$  of  $\Upsilon$  with  $\iota_B$ , where  $B$  is the following 3-cell:



- Adjunction of the coherence cells for  $\Sigma$ . For every 1-cell  $x$ , we apply  $\iota_{\psi_{F(x)}^{-1} \star_1 \widehat{\sigma}_x}$  to coherently add the 2-cell  $\varphi_x$  and the 3-cell  $\xi_x$ . Then, for every 2-cell  $\alpha : u \Rightarrow u'$  of  $\Sigma$ , we add the 3-cell  $\varphi_\alpha$  with  $\iota_A$ , where  $A$  is the following 3-cell (where the triple arrows have been omitted for readability):



- Adjunction of the last coherence cells for  $\Upsilon$ . For every 1-cell  $y$  of  $\Upsilon$ , we add the 3-cell  $\eta_y$  by  $\iota_C$ , where  $C$  is the following 3-cell:



As a result, we get a Tietze transformation  $\Phi : \Sigma^\top \rightarrow \Xi^\top$ . Since the construction and the result are totally symmetric in  $\Sigma$  and  $\Upsilon$ , and since the Tietze transformation  $\Phi$  contains coherent adjunctions only, we also get a Tietze transformation  $\Psi : \Xi^\top \rightarrow \Upsilon^\top$ . By composition, we get a Tietze transformation from  $\Sigma^\top$  to  $\Upsilon^\top$ . To conclude, we note that both  $\Phi$  and  $\Psi$  are finite when both  $\Sigma$  and  $\Upsilon$  are.  $\square$

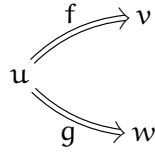
### 4.2. Homotopical completion

On the one hand, Squier's completion extends a convergent 2-polygraph  $\Sigma$  into a coherent convergent presentation  $\mathcal{S}(\Sigma)$  of  $\bar{\Sigma}$ . On the other hand, Knuth-Bendix's completion procedure [23] transforms a terminating 2-polygraph  $\Sigma$  into a convergent presentation of  $\bar{\Sigma}$ . The homotopical completion procedure interleaves both completion procedures to extend Squier's completion to terminating but not necessarily confluent 2-polygraphs.

**4.2.1. Homotopical completion.** Let  $\Sigma$  be a terminating 2-polygraph, equipped with a total termination order  $\leq$ . The *homotopical completion* of  $\Sigma$  is the  $(3, 1)$ -polygraph

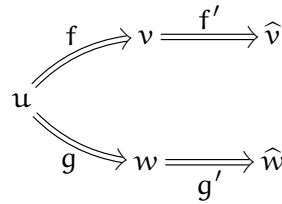
$$\mathcal{S}(\Sigma) = (\check{\Sigma}, \Gamma \amalg \Delta)$$

where  $\check{\Sigma}$  is the 2-polygraph and  $\Gamma$  and  $\Delta$  are the cellular extensions of  $\check{\Sigma}^\top$  obtained by the following procedure. It starts with  $\check{\Sigma}$  equal to  $\Sigma$ , with  $B$  equal to the set of critical branchings of  $\Sigma$  and with  $\Gamma$  and  $\Delta$  equal to the empty cellular extension of  $\Sigma^\top$ . If  $B$  is empty, then the procedure stops. Otherwise, it picks a branching

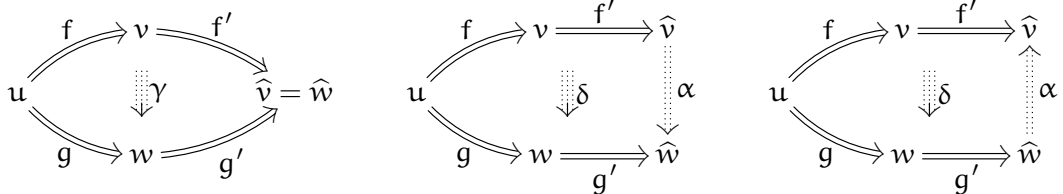


in  $B$  and it performs the following operations:

1. It computes 2-cells  $f' : v \Rightarrow \hat{v}$  and  $g' : w \Rightarrow \hat{w}$  of  $\check{\Sigma}^*$ , where  $\hat{v}$  and  $\hat{w}$  are normal forms for  $v$  and  $w$ , respectively, as in the following diagram:



2. It tests which (in)equality  $\hat{v} = \hat{w}$  or  $\hat{v} > \hat{w}$  or  $\hat{v} < \hat{w}$  holds, corresponding to the following three situations, respectively:



#### 4. Homotopical completion and homotopical reduction

If  $\hat{u} = \hat{v}$ , then the procedure adds the dotted 3-cell  $\gamma$  of the leftmost diagram to  $\Gamma$ . Otherwise, it adds the dotted 2-cell  $\alpha$  and 3-cell  $\delta$  of the corresponding situation to  $\check{\Sigma}$  and  $\Delta$ , respectively; moreover, it adds all the new critical branchings created by  $\delta$  to  $B$ .

3. It removes  $(f, g)$  from  $B$  and restarts from the beginning.

If the procedure stops, it returns the 2-polygraph  $\check{\Sigma}$  and the cellular extensions  $\Gamma$  and  $\Delta$  of  $\check{\Sigma}^\top$ . Otherwise, it builds increasing sequences of 2-polygraphs and of cellular extensions, whose limits are denoted by  $\check{\Sigma}$ ,  $\Gamma$  and  $\Delta$ . Note that, if the starting 2-polygraph  $\Sigma$  is already convergent, then the homotopical completion of  $\Sigma$  coincides with Squier's completion  $\mathcal{S}(\Sigma)$ , making notations consistent. By construction, the underlying 2-polygraph  $\check{\Sigma}$  of  $\mathcal{S}(\Sigma)$  is convergent: it is the result of Knuth-Bendix's completion procedure. Theorem 2.3.7 yields the following theorem.

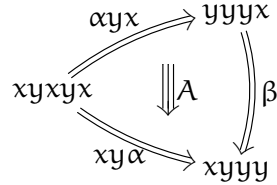
**4.2.2. Theorem.** *Let  $\Sigma$  be a terminating 2-polygraph.*

- i) *The  $(3, 1)$ -polygraph  $(\check{\Sigma}, \Delta)$  is Tietze-equivalent to  $(\Sigma, \emptyset)$ .*
- ii) *The homotopical completion  $\mathcal{S}(\check{\Sigma})$  of  $\Sigma$  is a coherent convergent presentation of  $\bar{\Sigma}$ .*
- iii) *The  $(3, 1)$ -polygraph  $(\Sigma, \pi_\Delta(\Gamma))$  is a coherent presentation of  $\bar{\Sigma}$ .*

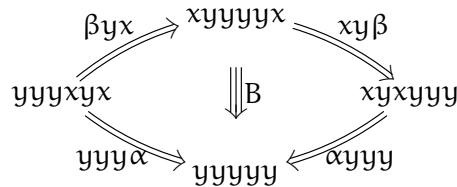
**4.2.3. Example.** Let us consider the monoid  $\mathbf{M}$  presented by the 2-polygraph

$$\Sigma = (x, y; xyx \xrightarrow{\alpha} yy).$$

We prove that  $\Sigma$  terminates with the deglex order generated by  $x < y$ . Let us apply the homotopical completion procedure to  $\Sigma$ , which has one, non confluent critical branching  $(\alpha yx, xy\alpha)$ . The procedure coherently adds the 2-cell  $\beta$  and the 3-cell  $A$  as follows, where the direction of  $\beta$  is given by the inequality  $yyyx > xyyy$ :



The 2-cell  $\beta$  creates a new, confluent critical branching, resulting in the adjunction of the 3-cell  $B$ :



No 2-cell was added, so that the homotopical completion of  $\Sigma$  is

$$\mathcal{S}(\check{\Sigma}) = (x, y; \alpha, \beta; A, B).$$

and it gives a coherent convergent presentation of  $\mathbf{M}$ . By application of the Tietze transformation  $\pi_B$ , we coherently remove the 2-cell  $\beta$  and the 3-cell  $A$  to get the coherent presentation  $(x, y; \alpha; \pi_A(B))$  of  $\mathbf{M}$ . By definition, the 3-cell  $\pi_A(B)$  is obtained from  $B$  by replacing all the occurrences of  $\beta$  in the boundary of  $B$  by the rest of the boundary of  $A$ , namely  $xy\alpha^{-} \star_1 \alpha yx$ . One can prove that the source and the target of  $\pi_A(B)$  are both equal to  $yyy\alpha$ , so that  $\pi_A(B)$  is parallel to  $1_{yyy\alpha}$  and, as a consequence, can be coherently eliminated by a Tietze transformation: the monoid  $\mathbf{M}$  admits  $(\Sigma, \emptyset)$  as coherent presentation.

The homotopical reduction procedure proposes a more systematic way to eliminate unnecessary 3-cells from a coherent convergent presentation.

### 4.3. Homotopical reduction in dimensions 1 and 2

The homotopical completion  $\mathcal{S}(\Sigma)$  of a 2-polygraph  $\Sigma$  is a coherent convergent presentation of  $\bar{\Sigma}$ . However, the  $(3, 1)$ -polygraph  $\mathcal{S}(\Sigma)$  has usually more cells than one could expect for a coherent presentation of  $\bar{\Sigma}$ : for example, the pairs of extra 2-cells and 3-cells added during homotopical completion for non-confluent critical branchings can be coherently removed from  $\mathcal{S}(\Sigma)$ . In this paragraph, we present the homotopical reduction procedures in dimension 1 and dimension 2, as a systematic way to coherently eliminate 1-cells and 2-cells from a  $(3, 1)$ -polygraph.

**4.3.1. Homotopical 1-reduction.** Let  $\Sigma$  be a  $(3, 1)$ -polygraph. The *homotopical 1-reduction* of  $\Sigma$  is the  $(3, 1)$ -polygraph denoted by  $\mathcal{R}_1(\Sigma)$  and obtained by the following procedure. It starts with  $\tilde{\Sigma} = \Sigma$  and  $\Gamma = \Sigma_2$ . If  $\Gamma$  is empty, then the procedure stops. Otherwise, it picks a 2-cell  $\alpha$  in  $\Gamma$  and determines if  $\alpha$  has one of the two shapes

$$(1) \quad u \xRightarrow{\alpha} x \qquad (2) \quad x \xRightarrow{\alpha} u$$

where  $x$  is a 1-cell of  $\Sigma$  and  $u$  is a 1-cell of  $(\Sigma \setminus \{x\})^*$ . If so, the procedure performs the following operations, depending on the case:

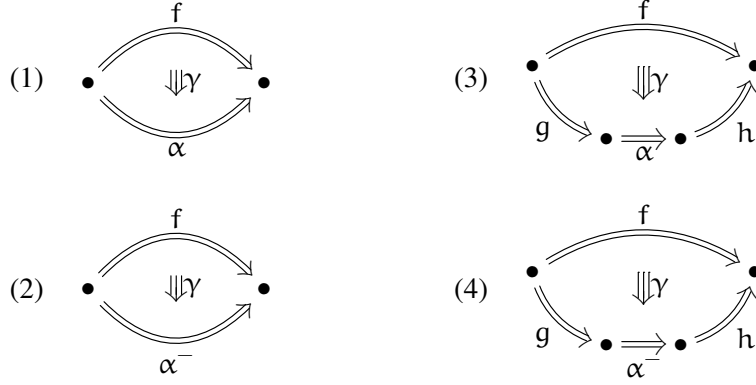
- (1) The procedure coherently eliminates  $x$  and  $\alpha$  from  $\tilde{\Sigma}$  by the Tietze transformation  $\pi_\alpha$ . Then, it restarts with  $\Gamma$  replaced by  $\pi_\alpha(\Gamma \setminus \{\alpha\})$ .
- (2) The procedure replaces  $\alpha$  with a 2-cell  $\tilde{\alpha} : u \Rightarrow x$  in  $\tilde{\Sigma}$ , corresponding to  $\alpha^-$ , by the following sequence of Tietze transformations:
  - the coherent adjunction of the 2-cell  $\tilde{\alpha} : u \Rightarrow x$  and a 3-cell  $\gamma : \alpha^- \Rightarrow \tilde{\alpha}$  by  $\iota_{\alpha^-}$ ,
  - the coherent adjunction of a 3-cell  $\tilde{\gamma} : \tilde{\alpha}^- \Rightarrow \alpha$  by  $\iota_{\alpha \star_1 A^- \star_1 \tilde{\alpha}^-}$ ,
  - the coherent elimination of the 3-cell  $\gamma$  by  $\pi_{(\tilde{\alpha} \star_1 \tilde{\gamma}^- \star_1 \alpha^-, \gamma)}$ ,
  - the coherent elimination of  $\alpha$  and  $\tilde{\gamma}$  by  $\pi_{\tilde{\gamma}}$ .

Then, the procedure applies the same operation as in case (1), with  $\alpha$  replaced by  $\tilde{\alpha}$ .

If  $\Sigma_2$  is finite, the procedure ends and we define  $\mathcal{R}_1(\Sigma)$  as the final value of  $\tilde{\Sigma}$ . Otherwise, we define  $\mathcal{R}_1(\Sigma)$  as the limit of the decreasing sequence formed by the successive values of  $\tilde{\Sigma}$ . This  $(3, 1)$ -polygraph is defined only up to Tietze equivalence, since the order of examination of the 2-cells can change the result and since a 2-cell can fall in both cases, inducing a choice between the two possible induced Tietze transformation. Nevertheless, by construction, the  $(3, 1)$ -polygraph  $\mathcal{R}(\Sigma)$  is Tietze-equivalent to  $\Sigma$ .

## 4. Homotopical completion and homotopical reduction

**4.3.2. Homotopical 2-reduction.** For a  $(3, 1)$ -polygraph  $\Sigma$ , the *homotopical 2-reduction* of  $\Sigma$  is the  $(3, 1)$ -polygraph denoted by  $\mathcal{R}_2(\Sigma)$  and obtained by a procedure that is almost identical to the case of dimension 1. The difference consists in the examination of the 3-cells  $\gamma$  of  $\Sigma$  that may induce a Tietze transformation  $\pi_\gamma$ . We first consider the following four shapes for  $\gamma$ :



where  $\alpha$  is a 2-cell of  $\Sigma$  and  $f$ ,  $g$  and  $h$  are 2-cells of  $(\Sigma_2 \setminus \{\alpha\})^\top$ .

(1) We apply the Tietze transformation  $\pi_\gamma$ .

(2) We replace  $\gamma$  with a 3-cell  $\tilde{\gamma} : f^- \Rightarrow \alpha$  by the following sequence of Tietze transformations:

- the coherent adjunction of  $\tilde{\gamma}$  by  $\iota_{f^- \star_1 \gamma^- \star_1 \alpha}$ ,
- the coherent elimination of  $\gamma$  by  $\pi_{(f \star_1 \tilde{\gamma}^- \star_1 \alpha^-, \gamma)}$ .

Then, we go to case (1) with  $\gamma$  replaced by  $\tilde{\gamma}$ .

(3) We replace  $\gamma$  with a 3-cell  $\tilde{\gamma} : g^- \star_1 f \star_1 h^- \Rightarrow \alpha$  by the following sequence of Tietze transformations:

- the coherent adjunction of  $\tilde{\gamma}$  by  $\iota_{g^- \star_1 \gamma \star_1 h^-}$ ,
- the coherent elimination of  $\gamma$  by  $\pi_{(g \star_1 \tilde{\gamma} \star_1 h, \gamma)}$ .

Then, we go to case (1) with  $\gamma$  replaced by  $\tilde{\gamma}$ .

(4) We apply the same transformations as in case (3) to replace  $\gamma$  with  $\tilde{\gamma} : g^- \star_1 f \star_1 h^- \Rightarrow \alpha^-$  and, then, we go to case (2) with  $\gamma$  replaced by  $\tilde{\gamma}$ .

Secondly, if  $\gamma$  does not fall in one of the four previous cases, we check if  $\gamma^-$  does. If so, we replace  $\gamma$  with a 3-cell  $\tilde{\gamma} : t(\gamma) \Rightarrow s(\gamma)$  by the following sequence of Tietze transformations:

- the coherent adjunction of  $\tilde{\gamma}$  by  $\iota_{s(\gamma)^- \star_1 \gamma \star_1 t(\gamma)^-}$ ,
- the coherent elimination of  $\gamma$  by  $\pi_{(s(\gamma) \star_1 \tilde{\gamma} \star_1 t(\gamma), \gamma)}$ .

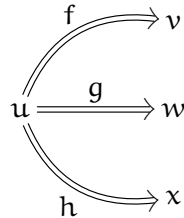
Then, we go to the corresponding case (1), (2), (3) or (4) with  $\gamma$  replaced by  $\tilde{\gamma}$ .

The  $(3, 1)$ -polygraph  $\mathcal{R}_2(\Sigma)$  is defined as the result of the procedure or as the limit of the decreasing sequence formed by the successively built  $(3, 1)$ -polygraphs. Once again, the  $(3, 1)$ -polygraph  $\mathcal{R}_2(\Sigma)$  is only defined up to Tietze equivalence but, by construction, it is Tietze-equivalent to  $\Sigma$ .

4.4. Homotopical reduction in dimension 3

The  $(3, 1)$ -polygraph  $\mathcal{S}(\Sigma)$  obtained by homotopical completion can also contain 3-cells that are not necessary to have a homotopy basis. One could define an abstract procedure that examines, in a similar way to the homotopical reduction in dimensions 1 and 2, all the 3-spheres of  $\mathcal{S}(\Sigma)^\top$  to eliminate the redundant 3-cells. However, this induces a practical difficulty: the effective computation of the 3-spheres. But, in the case of a coherent convergent presentation, the *critical triple branchings* [21] give a way to compute some 3-spheres that, in the examples we consider here, are sufficient to eliminate all the unnecessary 3-cells.

**4.4.1. Triple branchings.** A *triple branching* of a 2-polygraph  $\Sigma$  is a triple  $(f, g, h)$  of 2-cells of  $\Sigma_2^*$  with a common source, as in the diagram



A triple branching  $(f, g, h)$  is *local* when  $f, g$  and  $h$  are rewriting steps. Local branchings belong to one of the following three families:

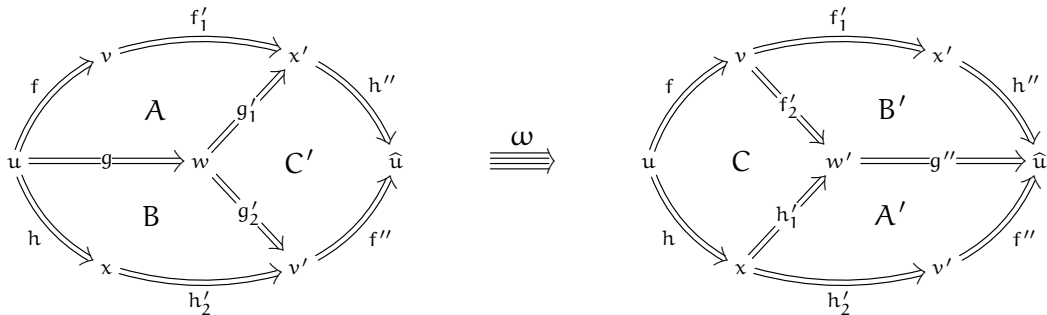
- *aspherical* triple branchings have two of their 2-cells equal,
- *Peiffer* triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two,
- *overlapping* triple branchings are the remaining local triple branchings.

Local triple branchings are compared by the order  $\preceq$  generated by the relations

$$(f, g, h) \preceq (ufv, ugv, uhv)$$

given for any local triple branching  $(f, g, h)$  and any possible 1-cells  $u$  and  $v$  of  $\Sigma^*$ . An overlapping local triple branching that is minimal for the order  $\preceq$  is called a *critical triple branching*.

**4.4.2. Generating triple confluences.** Let  $\Sigma$  be a coherent convergent  $(3, 1)$ -polygraph. A *family of generating triple confluences* of  $\Sigma$  is a cellular extension of  $\Sigma^\top$  that contains exactly one 3-sphere

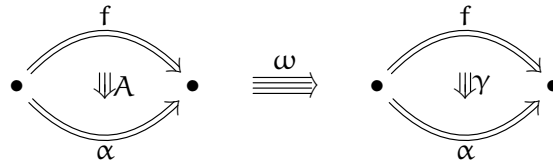




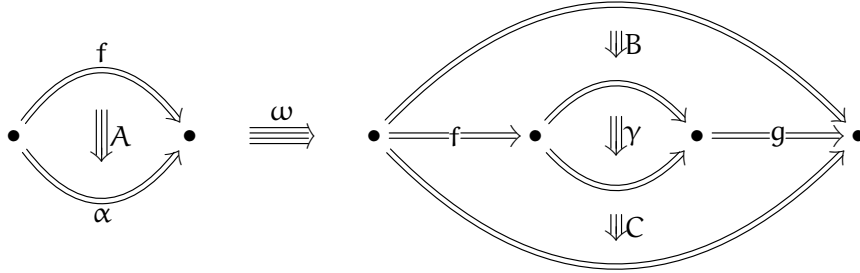
#### 4. Homotopical completion and homotopical reduction

for every critical triple branching  $(f, g, h)$  of  $\Sigma$ , built as follows. First, we consider the branching  $(f, g)$ : we use confluence to get  $f'_1$  and  $g'_1$  and coherence to build the 3-cell  $A$ . We repeat this step with the branchings  $(g, h)$  and  $(f, h)$ . Then, we consider the branching  $(f'_1, f'_2)$  and we use convergence to get  $g''$  and  $h''$  with  $\hat{u}$  as common target, plus the 3-cell  $B'$  by coherence. We do the same operation with  $(h'_1, h'_2)$  to get  $A'$ . Finally, we build the 3-cell  $C'$  to relate the parallel 2-cells  $g'_1 \star_1 h''$  and  $g'_2 \star_1 f''$ .

**4.4.3. Homotopical reduction in dimension 3.** For  $\Sigma$  a coherent convergent  $(3, 1)$ -polygraph, the *homotopical 3-reduction* of  $\Sigma$  is the  $(3, 1)$ -polygraph  $\mathcal{R}_3(\Sigma)$  obtained by a procedure that is similar to the case of homotopical 2-reduction, applied by considering the 3-spheres of a family of generating triple confluences of  $\Sigma$ . If such a 3-sphere  $\omega$  has shape



where  $\gamma$  is a 3-cell of  $\Sigma$  and  $A$  is a 3-cell of  $(\Sigma \setminus \{\gamma\})^\top$ , then one applies the Tietze transformation  $\pi_\omega$  directly. Otherwise, the procedure considers different possible shapes of  $\omega$ , such as:



In that case, the procedure considers the 3-sphere

$$\tilde{\omega} = f^- \star_1 (B^- \star_2 \omega \star_2 C^-) \star_1 g^-$$

instead of  $\omega$  and applies the Tietze transformation  $\pi_{\tilde{\omega}}$ . Once again, by construction, the  $(3, 1)$ -polygraph  $\mathcal{R}_3(\Sigma)$  is Tietze-equivalent to  $\Sigma$ .

#### 4.5. Homotopical completion-reduction

If  $\Sigma$  is a terminating 2-polygraph, equipped with a total termination order, the *homotopical completion-reduction* of  $\Sigma$  is the  $(3, 1)$ -polygraph defined by

$$\mathcal{R}(\Sigma) = \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \mathcal{S}(\Sigma).$$

**4.5.1. Theorem.** *Let  $\Sigma$  be a terminating 2-polygraph.*

- i) *The  $(3, 1)$ -polygraph  $\mathcal{R}_3 \mathcal{S}(\Sigma)$  is a coherent convergent presentation of  $\bar{\Sigma}$ .*
- ii) *The homotopical completion-reduction of  $\Sigma$  is a coherent presentation of  $\bar{\Sigma}$ .*

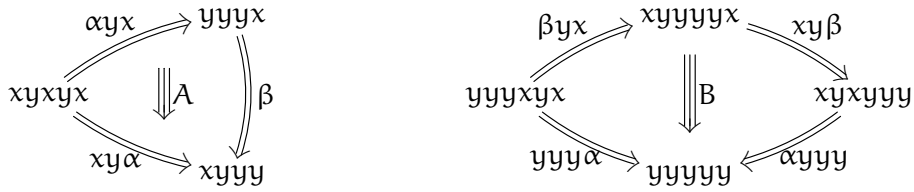
**4.5.2. Example.** We have seen in Example 4.2.3 that the monoid  $\mathbf{M}$  presented by

$$\Sigma = (x, y; xyx \xrightarrow{\alpha} yy)$$

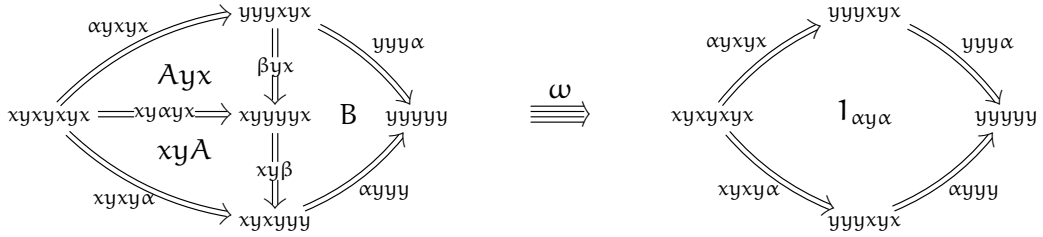
admits the coherent convergent presentation

$$\mathcal{S}(\Sigma) = (x, y; xyx \xrightarrow{\alpha} yy, yyyx \xrightarrow{\beta} xyyy; A, B)$$

where  $A$  and  $B$  are the 3-cells



Homotopical 3-reduction considers the 3-spheres associated to the critical triple branchings. Here, we only need to consider  $(\alpha y x y x, x y \alpha y x, x y x y \alpha)$ , with source  $x y x y x y x$ , giving the following 3-sphere:

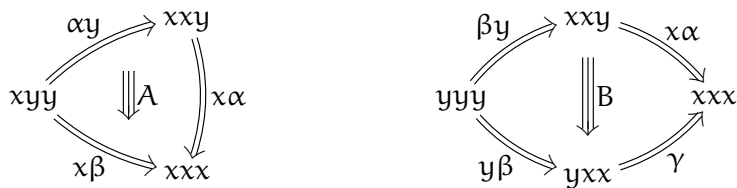


The Tietze transformation  $\pi_\omega$  coherently eliminates  $B$ , proving that  $(x, y; \alpha, \beta; A)$  is a coherent convergent presentation of  $\mathbf{M}$ . Then, homotopical 2-reduction coherently removes  $\beta$  and  $A$ , so that the homotopical completion-reduction of  $\Sigma$  is  $\mathcal{R}(\Sigma) = (\Sigma, \emptyset)$  and it is a coherent presentation of  $\mathbf{M}$ .

**4.5.3. Example.** Let us consider the monoid  $\mathbf{M}$  presented by the 2-polygraph

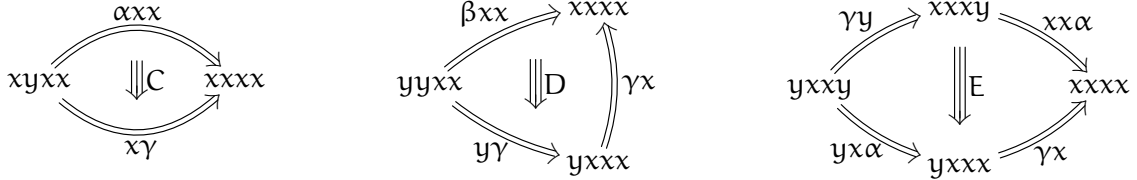
$$\Sigma = (x, y; xy \xrightarrow{\alpha} xx, yy \xrightarrow{\beta} xx).$$

This 2-polygraph terminates, with a total termination order given by the deglex order generated by  $x < y$ . It has two critical branchings, one of them being confluent and the other one requiring the coherent adjunction of a new 2-cell  $\gamma : yxx \Rightarrow xxx$ , together with the 3-cells

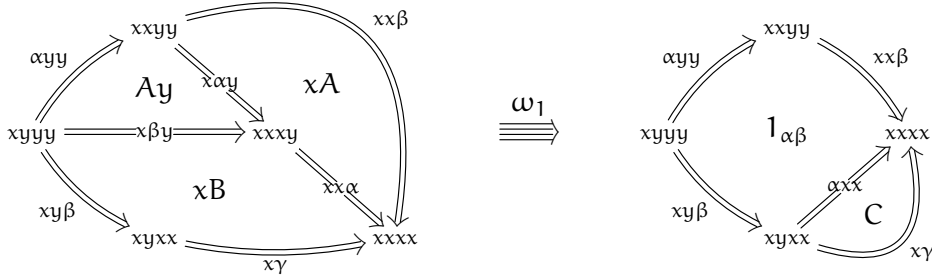


#### 4. Homotopical completion and homotopical reduction

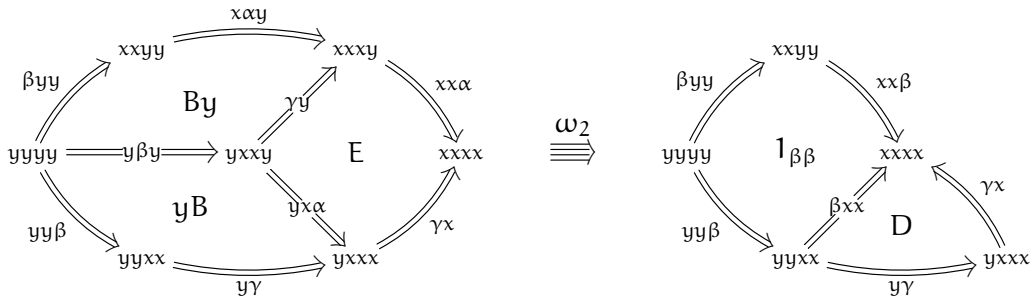
The adjunction of  $\gamma$  generates three new critical branchings, all of them being confluent:



Thus, the homotopical completion of  $\Sigma$  is  $\mathcal{S}(\Sigma) = (x, y; \alpha, \beta, \gamma; A, B, C, D, E)$ . Let us study the critical triple branchings of  $\Sigma$ . We note that, if  $\omega$  is a 3-sphere associated to a critical triple branching that is generated by  $\gamma$ , then all the 1-cells occurring in the boundary of  $\omega$  have size 5 or more: as a consequence, this  $\omega$  cannot be used to eliminate any 3-cell. There remain two critical triple branchings to consider. The first one gives the following 3-sphere:



Thus, the Tietze transformation  $\pi_{\omega_1}$  coherently eliminates C. The other critical triple branching generates the following 3-sphere:



The 4-cell  $\omega_2$  generates two possible different Tietze transformations, eliminating D or E: we choose to keep D. As a consequence, homotopical 3-reduction yields the coherent convergent presentation  $(x, y; \alpha, \beta, \gamma; A, B, D)$ . Then, homotopical 2-reduction coherently removes B and  $\gamma$ , so that the homotopical completion-reduction of  $\Sigma$  is

$$\mathcal{R}(\Sigma) = (x, y; \alpha, \beta; A, \pi_B(D))$$

where  $\pi_B(D)$  is D with both occurrences of  $\gamma$  replaced by  $y\beta^- \star_1 \beta y \star_1 x\alpha$ .

## 5. DELIGNE'S COHERENT PRESENTATION

In this section, we apply the homotopical completion procedure and a part of the homotopical reduction procedure to Deligne's presentation of Artin monoids, given in [13, 1.4.5] in the spherical case and in [29, Proposition 1.1] in the general case. We get a coherent presentation that generalises Deligne's result [13, Theorem 1.5] on the actions of spherical Artin monoids to general Artin monoids. We fix a Coxeter group  $\mathbf{W}$  with a totally ordered set of generators  $S$ .

### 5.1. Preliminaries on Artin monoids

We introduce notations for products in  $\mathbf{W}$  that preserve or not the length and, then, we recall some arithmetic properties on Artin monoids, observed by Garside for braid monoids [16] and generalised by Brieskorn and Saito [6].

**5.1.1. Length notations.** We recall that the length  $l(u)$  of an element  $u$  of  $\mathbf{W}$  is the one of its reduced expressions, that is, its representatives in  $S^*$  of minimal length. Hence, for every  $u$  and  $v$  in  $\mathbf{W}$ , we have  $l(uv) \leq l(u) + l(v)$  and we use distinct graphical notations depending on whether the equality holds or not:

$$\begin{aligned} u \widehat{\vee} v &\Leftrightarrow l(uv) = l(u) + l(v), \\ u \times v &\Leftrightarrow l(uv) < l(u) + l(v). \end{aligned}$$

When  $w = uv$  holds in  $\mathbf{W}$  with  $u \widehat{\vee} v$ , we write  $w \doteq uv$ . We generalise these notations for a greater number of elements of  $\mathbf{W}$ . For example, in the case of three elements  $u, v$  and  $w$  of  $\mathbf{W}$ , we write  $u \widehat{\vee} v \widehat{\vee} w$  when both equalities  $l(uv) = l(u) + l(v)$  and  $l(vw) = l(v) + l(w)$  hold. This case splits in the following two mutually exclusive subcases:

$$\begin{aligned} u \widehat{\vee} v \widehat{\vee} w &\Leftrightarrow \begin{cases} u \widehat{\vee} v \widehat{\vee} w \\ l(uvw) = l(u) + l(v) + l(w), \end{cases} \\ u \times v \times w &\Leftrightarrow \begin{cases} u \widehat{\vee} v \widehat{\vee} w \\ l(uvw) < l(u) + l(v) + l(w). \end{cases} \end{aligned}$$

**5.1.2. Arithmetic properties of Artin monoids.** If  $u$  and  $v$  are two elements of  $\mathbf{B}^+(\mathbf{W})$ , we say that  $u$  is a divisor of  $v$  and that  $v$  is a multiple of  $u$  if there exists an element  $u'$  in  $\mathbf{B}^+(\mathbf{W})$  such that  $uu' = v$ . In that case, the element  $u'$  is uniquely defined and called the *complement of  $u$  in  $v$*  [6, Proposition 2.3]. Moreover, if  $v$  is in  $\mathbf{W}$ , seen as an element of  $\mathbf{B}^+(\mathbf{W})$  by the canonical embedding, then we also have  $u$  and  $u'$  in  $\mathbf{W}$  and  $v \doteq uu'$ . A *common multiple* of a family  $(x_1, \dots, x_n)$  of elements of  $\mathbf{B}^+(\mathbf{W})$  is an element  $y$  in  $\mathbf{B}^+(\mathbf{W})$  such that each  $x_i$  is a divisor of  $y$ . A *least common multiple (lcm for short)* is a common multiple that is a divisor of every common multiple.

If a family of elements of  $\mathbf{B}^+(\mathbf{W})$  has a common multiple, then it has a lcm [6, Proposition 4.1]. However, in general, any family of elements does not admit a common multiple: indeed, we have the existence of common multiples for any family if, and only if, the Coxeter group  $\mathbf{W}$  is finite [6, Proposition 5.5]. In particular, if  $I$  is a subset of  $S$ , then the family of elements  $(s_i)_{i \in I}$  has a common right-multiple if, and only if, the Coxeter group  $\mathbf{W}_I$  is finite. In that case, the lcm of the family  $(s_i)_{i \in I}$  is the

## 5. Deligne's coherent presentation

fundamental element  $w_0(I)$  of  $\mathbf{W}_I$ . This implies that, if an element  $u$  of  $\mathbf{W}$  admits reduced expressions  $s_1 u_1, \dots, s_n u_n$  where  $s_1, \dots, s_n$  are in  $S$ , then the subgroup  $W_{\{s_1, \dots, s_n\}}$  is finite and its fundamental element  $w_0(s_1, \dots, s_n)$  is a divisor of  $u$ . As a consequence, the element  $u$  has a unique reduced expression of the shape  $w_0(s_1, \dots, s_n)u'$ , which is a slight generalisation of [37, Lemma 4].

**5.1.3. Deligne's presentation.** The braid monoid  $\mathbf{B}^+(\mathbf{W})$  admits a presentation, built from Artin's presentation by adjunction of redundant generators and relations, that is considered in the spherical case in [13, (1.4.5)] and in the general case in [29, Proposition 1.1] and [19, Proposition 4.1.3]. *Deligne's presentation of  $\mathbf{B}^+(\mathbf{W})$*  is the 2-polygraph  $\text{Del}(\mathbf{W})$  whose set of 1-cells is  $\mathbf{W} \setminus \{1\}$  and with a 2-cell

$$\alpha_{u,v} : u|v \Rightarrow uv$$

whenever  $u \widehat{v}$  holds, where  $\cdot| \cdot$  denotes the product in the free monoid over  $\mathbf{W} \setminus \{1\}$ , to avoid confusion with the product in  $\mathbf{W}$ .

For example, Deligne's presentation of the braid monoid  $\mathbf{B}_3^+$  has five 1-cells

$$s \quad t \quad st \quad ts \quad sts$$

and six 2-cells

$$\begin{array}{ccccc} s|t \xrightarrow{\alpha_{s,t}} st & s|ts \xrightarrow{\alpha_{s,ts}} sts & st|s \xrightarrow{\alpha_{st,s}} sts \\ t|s \xrightarrow{\alpha_{t,s}} ts & t|st \xrightarrow{\alpha_{t,st}} sts & ts|t \xrightarrow{\alpha_{ts,t}} sts. \end{array}$$

We denote by  $\text{Del}_+(\mathbf{W})$  the extended presentation of  $\mathbf{B}^+(\mathbf{W})$  obtained from  $\text{Del}(\mathbf{W})$  by adjunction of one 3-cell

$$\begin{array}{ccc} \alpha_{u,v|w} & \xrightarrow{\quad} & uv|w \\ \alpha_{u,v|w} & \xrightarrow{\quad} & \Downarrow A_{u,v,w} \\ u|v|w & & uvw \\ \alpha_{u,v|w} & \xrightarrow{\quad} & u|vw \\ \alpha_{u,v|w} & \xrightarrow{\quad} & \alpha_{u,vw} \end{array}$$

for every  $u, v$  and  $w$  of  $\mathbf{W} \setminus \{1\}$  with  $u \widehat{v} w$ . After the main theorem of this section, we call *Deligne's coherent presentation* the  $(3, 1)$ -polygraph  $\text{Del}_+(\mathbf{W})$ .

**5.1.4. Theorem.** *For every Coxeter group  $\mathbf{W}$ , the Artin monoid  $\mathbf{B}^+(\mathbf{W})$  admits  $\text{Del}_+(\mathbf{W})$  as a coherent presentation.*

*Proof.* The proof of Theorem 5.1.4 is conducted in two parts: in 5.2, we compute the homotopical completion of  $\text{Del}(\mathbf{W})$  to get a coherent convergent presentation of  $\mathbf{B}^+(\mathbf{W})$  and, then, we apply a part of homotopical 3-reduction in 5.3 to get the result.  $\square$

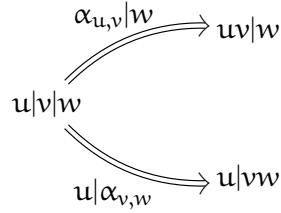
In the spherical case, this result is a consequence of Theorem 1.5 and Remark 2.6 of [13]. There, Deligne proves that, if  $\mathbf{W}$  is finite, then the category  $\text{Act}(\mathbf{B}^+(\mathbf{W}))$  of 2-representations of  $\mathbf{B}^+(\mathbf{W})$  in a 2-category  $\mathcal{C}$  is equivalent to the category of 2-functors from a  $\overline{\text{Del}}_+(\mathbf{W})$  into  $\mathcal{D}$ . Moreover, the given proof makes this equivalence natural in  $\mathcal{C}$ : by Theorem 3.4.3, we get that  $\text{Del}_+(\mathbf{W})$  is a coherent presentation of  $\mathbf{B}^+(\mathbf{W})$ .

### 5.2. Homotopical completion of Deligne's presentation

We compute the homotopical completion of  $\text{Del}(\mathbf{W})$  to get a coherent convergent presentation of  $\mathbf{B}^+(\mathbf{W})$ .

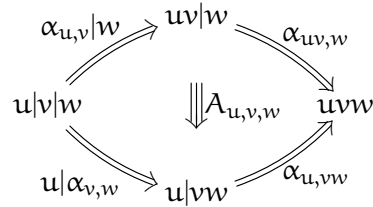
**5.2.1. Termination of  $\text{Del}(\mathbf{W})$ .** We equip the 1-cells of  $\text{Del}(\mathbf{W})^*$  with the order given by the interpretation mapping  $u_1|u_2|\cdots|u_n$  to the family  $(l(u_1), \dots, l(u_n))$  of natural numbers, compared by the right deglex order generated by the natural order. Let us note that this order is not total, but we will not encounter pairs of incomparable 1-cells during homotopical completion. For every 2-cell  $\alpha_{u,v}$  of  $\text{Del}(\mathbf{W})$ , the strict inequality  $s(\alpha_{u,v}) > t(\alpha_{u,v})$  holds since, for the considered order, the pair  $(l(u), l(v))$  is strictly greater than the singleton  $l(uv)$ . Hence we get that  $\text{Del}(\mathbf{W})$  terminates.

**5.2.2. The critical branchings of  $\text{Del}(\mathbf{W})$ .** The 2-polygraph  $\text{Del}(\mathbf{W})$  has exactly one critical branching for every  $u, v$  and  $w$  of  $\mathbf{W} \setminus \{1\}$  with  $\widehat{u} \widehat{v} \widehat{w}$ :



Then, given such a critical branching, there are two cases, depending on the length of  $uvw$  with respect to the sum of the lengths of  $u, v$  and  $w$ .

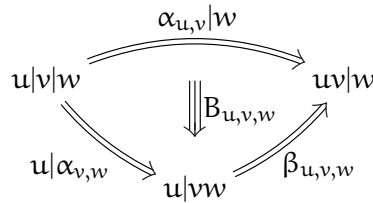
- If  $\widehat{u} \widehat{v} \widehat{w}$ , the critical branching is confluent, resulting in the adjunction of the following 3-cell:



- Otherwise, if  $\widehat{u} \widehat{v^x} \widehat{w}$ , then both  $uv|w$  and  $u|vw$  are normal forms. Since  $l(vw) > l(w)$ , we have  $u|vw > uv|w$ . Thus, homotopical completion coherently adds the new 2-cell

$$u|vw \xrightarrow{\beta_{u,v,w}} uv|w,$$

together with the following 3-cell:



## 5. Deligne's coherent presentation

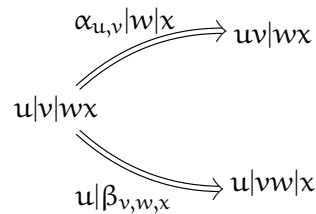
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After this first part of homotopical completion, we get a terminating  $(3, 1)$ -polygraph that is Tietze-equivalent to  $(\text{Del}(\mathbf{W}), \emptyset)$ . However, the adjunction of the family  $\beta$  of 2-cells creates new critical branchings that we need to consider.

**5.2.3. The new critical branchings.** The sources of all the 2-cells  $\alpha$  and  $\beta$  have size 2 in the free monoid over  $\mathbf{W} \setminus \{1\}$ . This leaves two main cases for the critical branchings that involve at least one 2-cell  $\beta$ .

The first case occurs when the sources of the 2-cells of  $\text{Del}(\mathbf{W})$  that generate the branching overlap on one element of  $\mathbf{W} \setminus \{1\}$ . The source of such a branching has size 3, with one 2-cell of the branching reducing the leftmost two generating 1-cells and the other one reducing the rightmost two. That case subdivides as follows, depending on the type  $\alpha$  or  $\beta$  of the involved 2-cells.

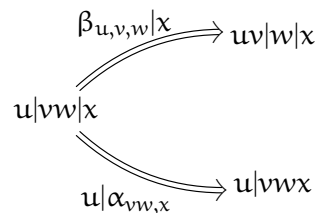
- One critical branching when  $u \widehat{v} \widehat{w}^x x$ :



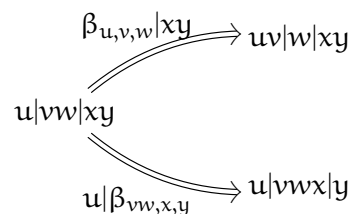
This case splits in the following two disjoint subcases:



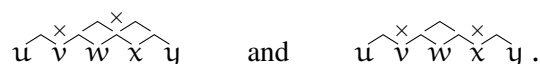
- One critical branching when  $u \widehat{v}^x \widehat{w}^x x$ :



- One critical branching when  $u \widehat{v}^x \widehat{w}^x x y$ :



This case splits in the following two disjoint subcases:



## 5.2. Homotopical completion of Deligne's presentation

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The second main case occurs when the 2-cells of  $\text{Del}(\mathbf{W})$  that generate the branching totally overlap, *i.e.*, they have the same source. Since one of those 2-cells must be a  $\beta$ , the source has shape  $u|vw$  with  $\widehat{u} \overset{x}{\wedge} \widehat{v} \widehat{w}$ , preventing the other 2-cell to be an  $\alpha$ . The only remaining possibility is to have a different decomposition  $vw = v'w'$ , with  $\widehat{u} \overset{x}{\wedge} \widehat{v}' \widehat{w}'$ , so that the branching is:

$$\begin{array}{ccc}
 & \beta_{u,v,w} & \rightarrow uv|w \\
 & \curvearrowright & \\
 u|vw & & \\
 \equiv & & \\
 u|v'w' & & \\
 & \curvearrowleft & \\
 & \beta_{u,v',w'} & \rightarrow uv'|w'
 \end{array}$$

**5.2.4. Confluence of the new critical branchings.** We now proceed to the examination of each individual case, proving that the corresponding critical branching is confluent, which induces the adjunction of 3-cells to produce a homotopy basis.

– Case  $\widehat{u} \overset{x}{\wedge} \widehat{v} \overset{x}{\wedge} \widehat{w} \widehat{x}$ :

$$\begin{array}{ccccc}
 & \alpha_{u,v|wx} & \rightarrow & uv|wx & \xrightarrow{\beta_{uv,w,x}} \\
 & \curvearrowright & & & \searrow \\
 u|v|wx & & & \Downarrow C_{u,v,w,x} & uvw|x \\
 & \curvearrowleft & & & \nearrow \\
 & u|\beta_{v,w,x} & \rightarrow & u|vw|x & \xrightarrow{\alpha_{u,vw|x}}
 \end{array}$$

– Case  $\widehat{u} \overset{x}{\wedge} \widehat{v} \overset{x}{\wedge} \widehat{w}' \widehat{x}$ :

$$\begin{array}{ccccc}
 & \alpha_{u,v|wx} & \rightarrow & uv|wx & \\
 & \curvearrowright & & & \nearrow \\
 u|v|wx & & & \Downarrow D_{u,v,w,x} & uv|wx \\
 & \curvearrowleft & & & \nearrow \\
 & u|\beta_{v,w,x} & \rightarrow & u|vw|x & \xrightarrow{\beta_{u,v,w|x}} uv|w|x \xrightarrow{uv|\alpha_{w,x}}
 \end{array}$$

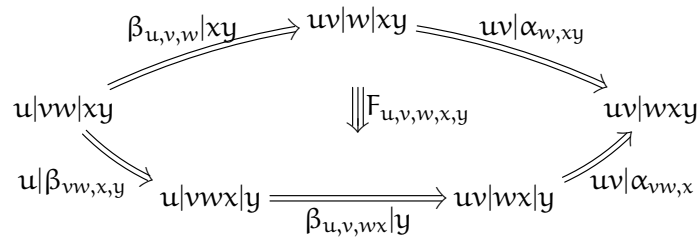
– Case  $\widehat{u} \overset{x}{\wedge} \widehat{v}' \overset{x}{\wedge} \widehat{w} \widehat{x}$ :

$$\begin{array}{ccccc}
 & \beta_{u,v,w|x} & \rightarrow & uv|w|x & \xrightarrow{uv|\alpha_{w,x}} \\
 & \curvearrowright & & & \searrow \\
 u|vw|x & & & \Downarrow E_{u,v,w,x} & uv|wx \\
 & \curvearrowleft & & & \nearrow \\
 & u|\alpha_{vw,x} & \rightarrow & u|vw|x & \xrightarrow{\beta_{u,v,w|x}}
 \end{array}$$

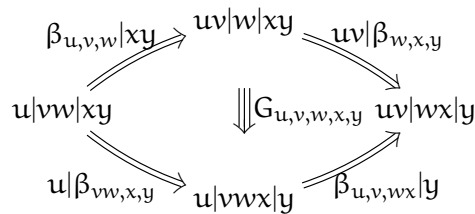


## 5. Deligne's coherent presentation

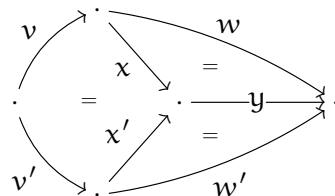
– Case  $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x \overset{x}{\curvearrowright} y$  :



– Case  $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x \overset{x}{\curvearrowright} y$  :



– Case  $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w$  and  $u \overset{x'}{\curvearrowright} v' \overset{x'}{\curvearrowright} w'$ , with  $vw = v'w'$ . Here we use the properties satisfied by Artin monoids, seen in 5.1.2, to get the following relations in  $\mathbf{B}^+(\mathbf{W})$ :



Indeed, we note that the elements  $v$  and  $v'$  have a common multiple since  $vw = v'w'$ . Hence, they admit a lcm. The elements  $x$  and  $x'$  are respectively defined as the complements of  $v$  and  $v'$  in their lcm. The element  $y$  is the complement of the lcm  $vx = v'x'$  of  $v$  and  $v'$  in their common multiple  $vw = v'w'$ . By uniqueness of the complements of  $v$  and  $v'$  in  $vw = v'w'$ , we get  $w = xy$  and  $w' = x'y$ . Moreover, we have:

$$v \overset{x}{\curvearrowright} x \overset{x}{\curvearrowright} y \quad \text{and} \quad v' \overset{x'}{\curvearrowright} x' \overset{x'}{\curvearrowright} y.$$

Finally, from the hypothesis  $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w$ , we get that  $y \neq 1$ .

Then, there are two possible subcases for the confluence diagram, depending on  $x$  and  $x'$ . The first subcase is when we have either  $x = 1$  or  $x' = 1$ . We note that both cannot happen at the same time, otherwise  $v = v'$  and  $w = w'$ , so that the branching would be aspherical and not critical.

### 5.3. Homotopical reduction of Deligne's presentation

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For example, let us assume that  $x' = 1$ , inducing  $v' = vx$ ,  $w = xy$  and  $w' = y$ :

$$\begin{array}{ccc}
 & uv|xy & \\
 \beta_{u,v,xy} \nearrow & & \searrow \beta_{uv,x,y} \\
 u|vxy & \Downarrow H_{u,v,x,y} & uvx|y \\
 & \beta_{u,vx,y} \searrow & 
 \end{array}$$

In the second subcase, when  $x \neq 1$  and  $x' \neq 1$ , we have:

$$\begin{array}{ccc}
 \beta_{u,v,w} \nearrow & uv|w = uv|xy & \searrow \beta_{uv,x,y} \\
 u|vw & & uvx|y \\
 = & \Downarrow I_{u,v,w,v',w'} & = \\
 u|v'w' & & uv'|x'y \\
 \beta_{u,v',w'} \searrow & uv'|w' = uv'|x'y & \nearrow \beta_{uv',x',y}
 \end{array}$$

Since all the critical branchings are confluent, the homotopical completion procedure ends. As an instance of Theorem 4.2.2, we get the following result.

**5.2.5. Proposition.** *For every Coxeter group  $\mathbf{W}$ , the Artin monoid  $\mathbf{B}^+(\mathbf{W})$  admits, as a coherent convergent presentation, the  $(3, 1)$ -polygraph  $\mathcal{S}(\text{Del}(\mathbf{W}))$  with one 0-cell, one 1-cell for every element of  $\mathbf{W} \setminus \{1\}$ , one 2-cell*

$$u|v \xrightarrow{\alpha_{u,v}} uv,$$

for every  $u$  and  $v$  of  $\mathbf{W} \setminus \{1\}$  with  $u \hat{\ } v$ , one 2-cell

$$u|vw \xrightarrow{\beta_{u,v,w}} uv|w,$$

for every  $u$ ,  $v$  and  $w$  of  $\mathbf{W} \setminus \{1\}$  with  $u \hat{\ }^x v \hat{\ } w$  and the nine families of 3-cells A, B, C, D, E, F, G, H and I previously listed.

### 5.3. Homotopical reduction of Deligne's presentation

We obtain the coherent presentation  $\text{Del}_+(\mathbf{W})$  by application of coherent eliminations on the coherent presentation  $\mathcal{S}(\text{Del}(\mathbf{W}))$  of Proposition 5.2.5.

**5.3.1. The triple critical branchings of  $\mathcal{S}(\text{Del}(\mathbf{W}))$ .** We examine the possible overlaps of the sources of three 2-cells of  $\mathcal{S}(\text{Del}(\mathbf{W}))$ , in a similar way to the study of its critical branchings. There are four different cases, depending on the generating 2-cells forming the triple branching, and, then, different subcases depending on the 2-cells that close it. We only list the subcases used in the next paragraph.

## 5. Deligne's coherent presentation

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- One critical triple branching when  $u \widehat{v} \widehat{w} \widehat{x}$  :

$$\begin{array}{ccc}
 & \alpha_{u,v|w|x} & \rightarrow uv|w|x \\
 & \nearrow & \\
 u|v|w|x & \xrightarrow{u|\alpha_{v,w|x}} & u|vw|x \\
 & \searrow & \\
 & u|v|\alpha_{w,x} & \rightarrow u|v|wx
 \end{array}$$

This case splits into the five subcases:

$$u \widehat{v} \widehat{w} \widehat{x} \quad u \widehat{v} \widehat{x} \widehat{w} \widehat{x} \quad u \widehat{v} \widehat{w} \widehat{x} \quad u \widehat{v} \widehat{x} \widehat{w} \widehat{x} \quad u \widehat{v} \widehat{x} \widehat{w} \widehat{x}$$

- One critical triple branching when  $u \widehat{v} \widehat{w} \widehat{x} \widehat{y}$  :

$$\begin{array}{ccc}
 & \alpha_{u,v|w|xy} & \rightarrow uv|w|xy \\
 & \nearrow & \\
 u|v|w|xy & \xrightarrow{u|\alpha_{v,w|xy}} & u|vw|xy \\
 & \searrow & \\
 & u|v|\beta_{w,x,y} & \rightarrow u|v|wx|y
 \end{array}$$

We only consider the subcase  $u \widehat{v} \widehat{w} \widehat{x} \widehat{y}$  .

- One critical triple branching when  $u \widehat{v} \widehat{x} \widehat{w} \widehat{x} \widehat{y}$  :

$$\begin{array}{ccc}
 & \beta_{u,v,w|x|y} & \rightarrow uv|w|x|y \\
 & \nearrow & \\
 u|vw|x|y & \xrightarrow{u|\alpha_{vw,x|y}} & u|vwx|y \\
 & \searrow & \\
 & u|vw|\alpha_{x,y} & \rightarrow u|vw|xy
 \end{array}$$

We only consider the subcase  $u \widehat{v} \widehat{x} \widehat{w} \widehat{x} \widehat{y}$  .

### 5.3. Homotopical reduction of Deligne's presentation

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- One critical triple branching when  $u \overset{x}{\vee} v w$ ,  $u \overset{x}{\vee} v' w'$  and  $u \overset{x}{\vee} v'' w''$  with  $vw = v'w' = v''w''$ :

$$\begin{array}{ccc}
 & \beta_{u,v,w} \curvearrowright & uv|w \\
 u|vw & & \\
 \equiv & \beta_{u,v',w'} \curvearrowright & uv'|w' \\
 u|v'w' & & \\
 \equiv & & \\
 u|v''w'' & & \\
 & \beta_{u,v'',w''} \curvearrowright & uv''|w''
 \end{array}$$

The subcases are determined by the value of the lcm of  $v$ ,  $v'$  and  $v''$ , and it is sufficient to examine the situation where  $v''$  is the lcm of  $v$  and  $v'$ .

**5.3.2. A family of generating triple confluences of  $\mathcal{S}(\text{Del}(\mathbf{W}))$ .** We now proceed to the examination of all the cases we have noted in the previous paragraph.

- In the case  $u \overset{\vee}{\vee} v w x$ , we get the 3-sphere  $\omega_{u,v,w,x}^A$ :

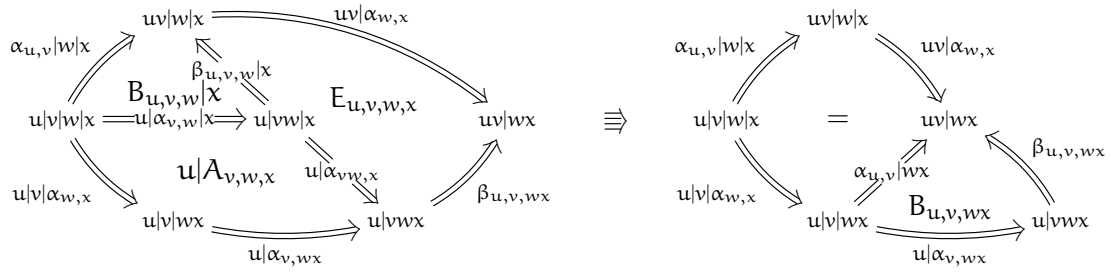
$$\begin{array}{ccc}
 \begin{array}{ccc}
 \alpha_{u,v,w|x} \curvearrowright & & \alpha_{uv,w|x} \curvearrowright \\
 \alpha_{u,v|w|x} \curvearrowright & \begin{array}{c} A_{u,v,w|x} \\ \alpha_{u,vw|x} \end{array} & \alpha_{uvw,x} \curvearrowright \\
 u|v|w|x \equiv u|\alpha_{v,w}|x \Rightarrow u|vw|x & & uvw|x \\
 u|v|\alpha_{w,x} \curvearrowright & \begin{array}{c} u|A_{v,w,x} \\ \alpha_{u,vw,x} \end{array} & \alpha_{u,vwx} \curvearrowright \\
 u|v|wx & & u|vwx \\
 u|\alpha_{v,wx} \curvearrowright & &
 \end{array}
 & \equiv &
 \begin{array}{ccc}
 \alpha_{u,v,w|x} \curvearrowright & & \alpha_{uv,w|x} \curvearrowright \\
 \alpha_{u,v|w|x} \curvearrowright & \begin{array}{c} A_{uv,w,x} \\ uv|\alpha_{w,x} \end{array} & \alpha_{uvw,x} \curvearrowright \\
 u|v|w|x \equiv u|\alpha_{v,w}|x \Rightarrow u|vw|x & & uvw|x \\
 u|v|\alpha_{w,x} \curvearrowright & \begin{array}{c} u|A_{v,w,x} \\ \alpha_{u,v|wx} \end{array} & \alpha_{u,vwx} \curvearrowright \\
 u|v|wx & & u|vwx \\
 u|\alpha_{v,wx} \curvearrowright & &
 \end{array}
 \end{array}$$

- In the case  $u \overset{x}{\vee} v w x$ , we get the 3-sphere  $\omega_{u,v,w,x}^H$ :

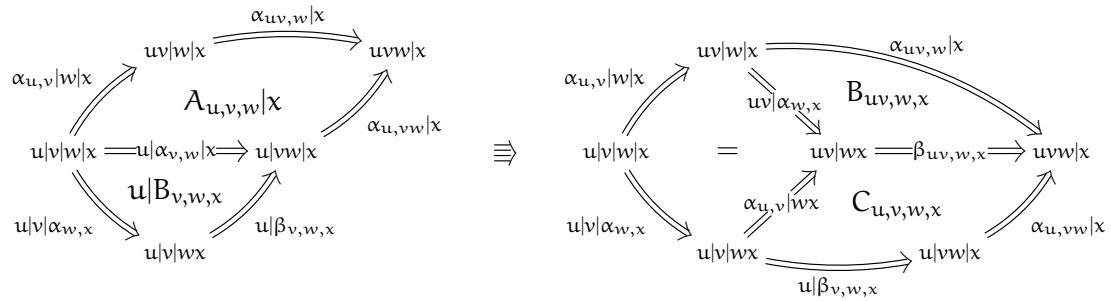
$$\begin{array}{ccc}
 \begin{array}{ccc}
 \alpha_{u,v,w|x} \curvearrowright & & \alpha_{uv,w|x} \curvearrowright \\
 \alpha_{u,v|w|x} \curvearrowright & \begin{array}{c} A_{u,v,w|x} \\ \alpha_{u,vw|x} \end{array} & \alpha_{uvw,x} \curvearrowright \\
 u|v|w|x \equiv u|\alpha_{v,w}|x \Rightarrow u|vw|x & & uvw|x \\
 u|v|\alpha_{w,x} \curvearrowright & \begin{array}{c} u|A_{v,w,x} \\ \alpha_{u,vw,x} \end{array} & \beta_{u,vw,x} \curvearrowright \\
 u|v|wx & & u|vwx \\
 u|\alpha_{v,wx} \curvearrowright & &
 \end{array}
 & \equiv &
 \begin{array}{ccc}
 \alpha_{u,v,w|x} \curvearrowright & & \alpha_{uv,w|x} \curvearrowright \\
 \alpha_{u,v|w|x} \curvearrowright & \begin{array}{c} B_{uv,w,x} \\ uv|\alpha_{w,x} \end{array} & \alpha_{uvw,x} \curvearrowright \\
 u|v|w|x \equiv u|\alpha_{v,w}|x \Rightarrow u|vw|x & & uvw|x \\
 u|v|\alpha_{w,x} \curvearrowright & \begin{array}{c} u|A_{v,w,x} \\ \alpha_{u,v|wx} \end{array} & \beta_{u,v,w,x} \curvearrowright \\
 u|v|wx & & u|vwx \\
 u|\alpha_{v,wx} \curvearrowright & &
 \end{array}
 \end{array}$$

## 5. Deligne's coherent presentation

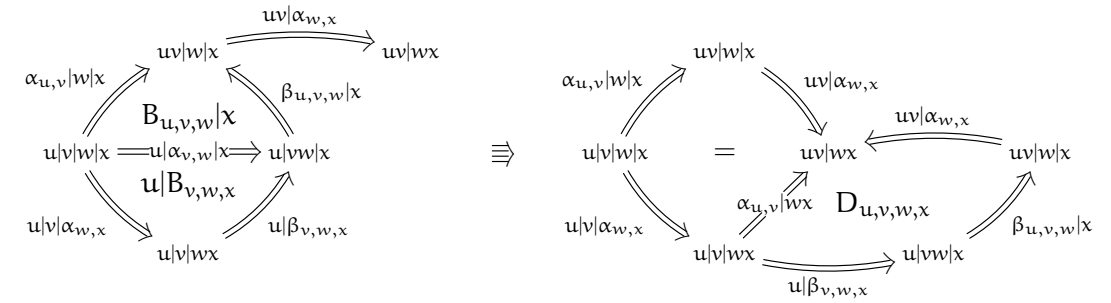
– In the case  $u \overset{x}{\curvearrowright} v \overset{w}{\curvearrowright} x$ , we get the 3-sphere  $\omega_{u,v,w,x}^E$ :



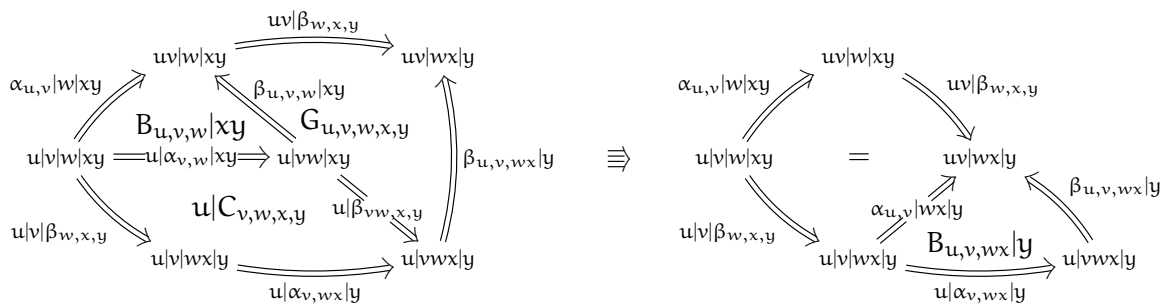
– In the case  $u \overset{v}{\curvearrowright} w \overset{x}{\curvearrowright} x$ , we get the 3-sphere  $\omega_{u,v,w,x}^C$ :



– In the case  $u \overset{v}{\curvearrowright} w \overset{x}{\curvearrowright} x$ , we get the 3-sphere  $\omega_{u,v,w,x}^D$ :

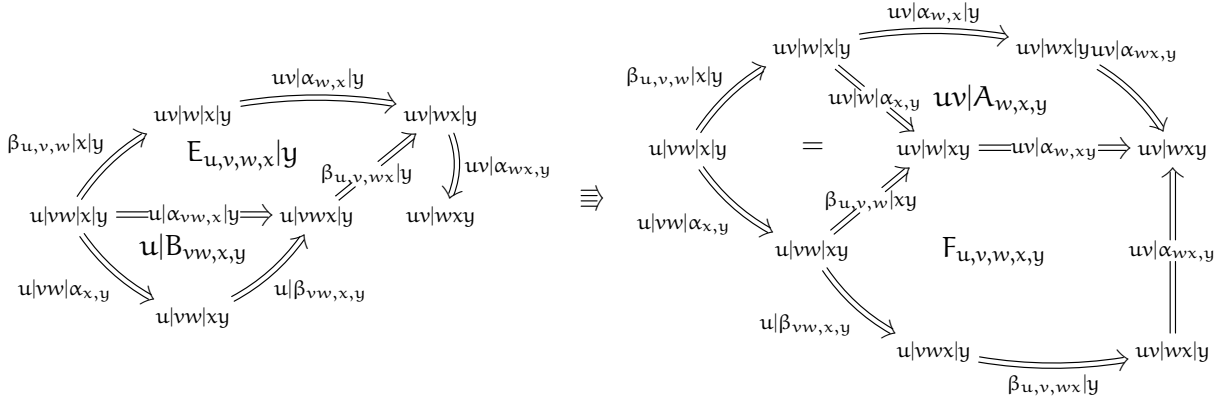


– In the case  $u \overset{v}{\curvearrowright} w \overset{x}{\curvearrowright} y$ , we get the 3-sphere  $\omega_{u,v,w,x,y}^G$ :



### 5.3. Homotopical reduction of Deligne's presentation

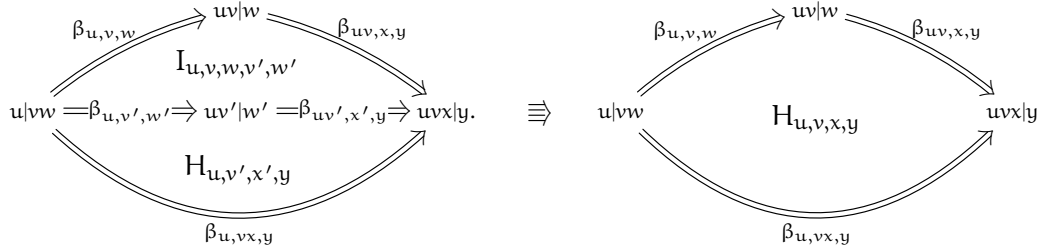
- In the case  $u \overset{x}{\widehat{v}} \overset{w}{\widehat{x}} y$ , we get the 3-sphere  $\omega_{u,v,w,x,y}^F$ :



- Finally, let us consider the case  $u \overset{x}{\widehat{v}} \widehat{w}$ ,  $u \overset{x'}{\widehat{v'}} \widehat{w'}$  and  $u \overset{x''}{\widehat{v''}} \widehat{w''}$  with  $vw = v'w' = v''w''$ . We proceed in the same way as for the critical branching inducing the 3-cells H and I, obtaining the unique elements  $x, x', x''$  and  $y$  of  $\mathbf{W}$  such that:

$$w = xy, \quad w' = x'y, \quad w'' = x''y, \quad vx = v'x' = v''x'', \quad y \neq 1.$$

Moreover, at most one of  $x, x'$  and  $x''$  can be equal to 1: indeed, otherwise, the triple branching would be aspherical and not critical. In the case where  $x'' = 1$ , corresponding to  $v''$  being the lcm of  $v$  and  $v'$ , we get the 3-sphere  $\omega_{u,v,w,v',w'}^I$ :



**5.3.3. Homotopical reduction of  $\mathcal{S}(\text{Del}(\mathbf{W}))$ .** We get Theorem 5.1.4 by successive application of two Tietze transformations:

- a homotopical 3-reduction that considers in sequence the 3-spheres of type  $\omega^I, \omega^H, \dots, \omega^C$  and coherently eliminates the corresponding 3-cell of type I, H,  $\dots$ , C, respectively.
- a homotopical 2-reduction that coherently eliminates every 3-cell of type B and the corresponding 2-cell of type  $\beta$ .

This results in the  $(3, 1)$ -polygraph  $\text{Del}_+(\mathbf{W})$ , proving that it is a coherent presentation of  $\mathbf{B}^+(\mathbf{W})$ .

## 5. Deligne's coherent presentation

### 5.4. Deligne's coherent presentation for Garside monoids

Garside monoids have been introduced as a generalisation of spherical Artin monoids by Dehornoy and Paris [11, 9] to abstract the arithmetic properties observed by Garside on braid monoids [16] and by Brieskorn-Saito and Deligne on spherical Artin monoids [6, 12]. We fix a Garside monoid  $\mathbf{M}$  and we follow [17] for most of the terminology and notations.

**5.4.1. Recollections on Garside monoids.** In the monoid  $\mathbf{M}$ , every pair  $(u, v)$  of elements admits a lcm  $u \wedge v$ . Moreover,  $\mathbf{M}$  has a fundamental element, denoted by  $w_0$ , such that the set  $\mathbf{W}$  of its divisors generates  $\mathbf{M}$ . The complement of an element  $u$  of  $\mathbf{W}$  in  $w_0$  is denoted by  $\partial(u)$ . A pair  $(u, v)$  of elements of  $\mathbf{W}$  is *left-weighted* if we have  $\partial(u) \wedge v = 1$ . There exists a unique left-weighted pair  $(u', v')$  of elements of  $\mathbf{W}$  such that  $uv = u'v'$  holds in  $\mathbf{M}$ : we take  $u' = u(\partial(u) \wedge v)$  and  $v'$  to be the complement of  $\partial(u) \wedge v$  in  $v$ . The operation transforming  $(u, v)$  into  $(u', v')$  is called *local sliding*. It induces a computational process that transforms any element  $u$  of  $\mathbf{W}^*$  into its (*left*) *normal form* by a finite sequence of local slidings, thereafter represented by dashed arrows:

$$u \dashrightarrow (\dots) \dashrightarrow \widehat{u}.$$

Moreover, two elements  $u$  and  $v$  of  $\mathbf{W}^*$  represent the same element of  $\mathbf{M}$  if, and only if, they have the same normal form, so that they are linked by a finite sequence of local slidings and their inverses:

$$u \dashrightarrow \widehat{u} \dashleftarrow v.$$

**5.4.2. Deligne's presentation for Garside monoids.** First, let us note that, since the set  $\mathbf{W}$  of divisors of the fundamental element  $w_0$  generates  $\mathbf{M}$ , then so does  $\mathbf{W} \setminus \{1\}$ . Given two elements  $u$  and  $v$  of  $\mathbf{W} \setminus \{1\}$ , we use the notations  $u \widehat{v}$  and  $u \times v$  to mean

$$\begin{aligned} u \widehat{v} &\Leftrightarrow \partial(u) \wedge v = 1, \\ u \times v &\Leftrightarrow \partial(u) \wedge v \neq 1. \end{aligned}$$

We define *Deligne's presentation of  $\mathbf{M}$*  as the 2-polygraph  $\text{Del}(\mathbf{M})$  with one 0-cell, one 1-cells for every element of  $\mathbf{W} \setminus \{1\}$  and one 2-cell

$$u|v \xrightarrow{\alpha_{u,v}} uv$$

for every  $u$  and  $v$  in  $\mathbf{W} \setminus \{1\}$  such that  $u \widehat{v}$  holds.

Let us check that Deligne's presentation is, indeed, a presentation of the monoid  $\mathbf{M}$ . If  $u \widehat{v}$  holds, transforming  $u|v$  into  $uv$  is a local sliding since  $uv$  is the normal form of  $u|v$ , so that each 2-cell  $\alpha_{u,v}$  is an instance of local sliding. Conversely, if  $u|vw$  is transformed into  $uv|w$  by local sliding, this implies, in particular, that both  $u \widehat{v}$  and  $v \widehat{w}$  hold. Thus, the composite 2-cell

$$\begin{array}{ccc} & \alpha_{u,v|w} & \\ & \nearrow & \\ & u|v|w & \\ & \searrow & \\ & u|\alpha_{v,w} & \\ & \searrow & \\ u|vw & \dashrightarrow & uv|w \end{array}$$

corresponds to the local sliding transformation applied to  $u|vw$ .

The proof of Theorem 5.1.4 adapts in a straightforward way to the case of  $\text{Del}(\mathbf{M})$ , yielding a coherent presentation  $\text{Del}_+(\mathbf{M})$  of  $\mathbf{M}$ .

## 6. ARTIN'S COHERENT PRESENTATION

Let  $\mathbf{W}$  be a Coxeter group with a totally ordered set  $S$  of generators. In this section, we continue the homotopical reduction of  $\mathcal{S}(\text{Del}(\mathbf{W}))$ , to get that the homotopical completion-reduction of  $\text{Del}(\mathbf{W})$  is Artin's coherent presentation  $\text{Art}_+(\mathbf{W})$  of  $\mathbf{B}^+(\mathbf{W})$ . This gives Theorem 2.4.5 for Artin monoids. Then, we use the fact that Artin monoids embed in the corresponding Artin group to get Theorem 2.4.5 for Artin groups.

### 6.1. Example: the case of $\mathbf{B}_3^+$

Deligne's coherent presentation  $\text{Del}_+(\mathcal{A}_2)$  of  $\mathbf{B}_3^+ = \mathbf{B}^+(\mathcal{A}_2)$  has five 1-cells

$$s \quad t \quad st \quad ts \quad sts$$

six 2-cells

$$\begin{array}{ccc} s|t \xrightarrow{\alpha_{s,t}} st & s|ts \xrightarrow{\alpha_{s,ts}} sts & st|s \xrightarrow{\alpha_{st,s}} sts \\ t|s \xrightarrow{\alpha_{t,s}} ts & t|st \xrightarrow{\alpha_{t,st}} sts & ts|t \xrightarrow{\alpha_{ts,t}} sts \end{array}$$

and two 3-cells

$$\begin{array}{ccc} \alpha_{s,t|s} & st|s & \alpha_{st,s} \\ \swarrow & \Downarrow \mathcal{A}_{s,t,s} & \searrow \\ s|t|s & & sts \\ \searrow & & \swarrow \\ s|\alpha_{t,s} & s|ts & \alpha_{s,ts} \end{array} \quad \begin{array}{ccc} \alpha_{t,s|t} & ts|t & \alpha_{ts,t} \\ \swarrow & \Downarrow \mathcal{A}_{t,s,t} & \searrow \\ t|s|t & & sts \\ \searrow & & \swarrow \\ t|\alpha_{s,t} & t|st & \alpha_{t,st} \end{array}$$

Let us prove that the homotopical reduction of  $\text{Del}_+(\mathcal{A}_2)$  is Artin's coherent presentation  $\text{Art}_+(\mathcal{A}_2)$ , *i.e.*, the  $(3, 1)$ -polygraph

$$(s, t; tst \xrightarrow{\gamma_{st}} sts; \emptyset).$$

The homotopical 3-reduction procedure, applied to  $\text{Del}_+(\mathcal{A}_2)$ , can coherently eliminate each 3-cell with one of two different 2-cells:  $\alpha_{s,ts}$  or  $\alpha_{st,s}$  for  $\mathcal{A}_{s,t,s}$  and  $\alpha_{t,st}$  or  $\alpha_{ts,t}$  for  $\mathcal{A}_{t,s,t}$ . Similarly, the homotopical 2-reduction procedure can coherently eliminate the 1-cell  $sts$  with any one of the 2-cells  $\alpha_{s,ts}$ ,  $\alpha_{t,st}$ ,  $\alpha_{st,s}$  and  $\alpha_{ts,t}$ . In order to make those choices systematic, we use the fixed order  $s < t$  on  $S$  and proceed as follows:

- a 1-cell  $u \doteq s_i u'$ , with  $s_i$  the smallest divisor of  $u$  in  $S$ , is coherently eliminated with the 2-cell  $\alpha_{s_i, u'} : s_i | u' \Rightarrow u$ ,
- a 2-cell  $\alpha_{u,v}$ , such that  $u \doteq s_i u'$  with  $s_i$  the smallest divisor of  $u$  in  $S$ , is coherently eliminated with the 3-cell  $\mathcal{A}_{s_i, u', v}$ .



## 6. Artin's coherent presentation

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Using those choices, we get that the monoid  $\mathbf{B}_3^+$  admits the coherent presentation

$$(s, t; \pi(\alpha_{t,st}); \emptyset)$$

where  $\pi$  is the Tietze-transformation induced by homotopical reduction. This transformation is defined by induction on the 1-cells by  $\pi(s) = s$  if  $s \in S$  and  $\pi(u) = s_i \pi(u')$  if  $u \doteq s_i u'$  with  $s_i$  the smallest divisor of  $u'$  in  $S$ .

There remains to compute the source  $\pi(t|st)$  and the target  $\pi(sts)$  of  $\pi(\alpha_{t,st})$  to conclude. Let us note that the product in  $(\mathbf{W} \setminus \{1\})^*$  is denoted by  $\cdot$  and the product in  $S^*$  is denoted by concatenation, so that we have  $\pi(u|v) = \pi(u)\pi(v)$ . For the source of  $\pi(\alpha_{t,st})$ , we have

$$\pi(t|st) = \pi(t)\pi(st) = ts\pi(t) = tst$$

and, for its target, we get

$$\pi(sts) = s\pi(ts) = st\pi(s) = sts.$$

Thus, the 2-cell  $\pi(\alpha_{t,st})$  is, up to isomorphism, the unique 2-cell  $\gamma_{st}$  of  $\text{Art}(A_2)$ : we recover the fact, predicted by Tits [37, Proposition 4], that  $\text{Art}_+(A_2) = (\text{Art}(A_2), \emptyset)$  is a coherent presentation of  $\mathbf{B}_3^+$ .

### 6.2. Classification of the cells of $\text{Del}_+(\mathbf{W})$

We consider Deligne's coherent presentation  $\text{Del}_+(\mathbf{W})$  of  $\mathbf{B}^+(\mathbf{W})$ . The proof of Theorem 5.1.4 has removed all the 3-spheres coming from the critical triple branchings of  $\mathcal{S}(\text{Del}(\mathbf{W}))$ , except for the ones of type  $\omega^\Lambda$ , thereafter simply denoted by  $\omega$  and collectively forming the cellular extension  $\Omega(\mathbf{W})$  of  $\text{Del}_+(\mathbf{W})^\Gamma$ . We start with a classification of the cells of  $\text{Del}_+(\mathbf{W})$  and the 3-spheres of  $\Omega(\mathbf{W})$ , that produces pairs of cells that can be coherently eliminated to get the homotopical reduction of  $\text{Del}_+(\mathbf{W})$ .

**6.2.1. Smallest divisors.** If  $u$  is an element of  $\mathbf{W} \setminus \{1\}$ , the *smallest divisor* of  $u$  is denoted by  $d_u$  and defined as the smallest element of  $S$  that is a divisor of  $u$ . Let  $(u_1, \dots, u_n)$  be a family of elements of  $\mathbf{W} \setminus \{1\}$  such that

$$l(u_1 \cdots u_n) = l(u_1) + \cdots + l(u_n).$$

For every  $k \in \{1, \dots, n\}$ , we write  $s_k = d_{u_1 \cdots u_k}$ . We note that  $s_1 \geq s_2 \geq \cdots \geq s_n$  since each  $s_k$  divides  $u_1 \cdots u_l$  for  $l \geq k$ . Moreover, the family  $(s_1, \dots, s_k)$  has  $u_1 \cdots u_k$  as common multiple, so that their lcm  $w_0(s_1, \dots, s_k)$  exists and divides  $u_1 \cdots u_k$ , and each subgroup  $\mathbf{W}_{s_1, \dots, s_k}$  is finite. Thus, we have the following diagram, where each arrow  $u \rightarrow v$  means that  $u$  is a divisor of  $v$ :

$$\begin{array}{ccccccc} w_0(s_1) & \longrightarrow & w_0(s_1, s_2) & \longrightarrow & (\cdots) & \longrightarrow & w_0(s_1, \dots, s_{n-1}) & \longrightarrow & w_0(s_1, \dots, s_n) \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ u_1 & \longrightarrow & u_1 u_2 & \longrightarrow & (\cdots) & \longrightarrow & u_1 \cdots u_{n-1} & \longrightarrow & u_1 \cdots u_n \end{array}$$

If every vertical arrow is an equality, we say that  $(u_1, \dots, u_n)$  is of *type I*. Since each  $u_k$  is different from 1, this implies no horizontal arrow is an equality, so that  $s_1 > \cdots > s_n$  holds. Moreover, we have  $u_1 = s_1$  and, by uniqueness of the complement, we get that each  $u_{k+1}$  is the complement of  $w_0(s_1, \dots, s_k)$  in  $w_0(s_1, \dots, s_{k+1})$ . Thus, the family  $(u_1, \dots, u_n)$  is uniquely determined by the elements  $s_1, \dots, s_n$  of  $S$  such that  $s_1 > \cdots > s_n$ .

Otherwise, there exists a maximal  $k$  in  $\{0, \dots, n-1\}$  such that  $(u_1, \dots, u_k)$  is a family of type I, *i.e.*, such that  $u_1 \cdots u_k = w_0(s_1, \dots, s_k)$ . Then there are two possibilities, depending if the equality  $w_0(s_1, \dots, s_k) = w_0(s_1, \dots, s_{k+1})$  holds or not, which is equivalent to the equality  $s_k = s_{k+1}$  since  $s_1 > \dots > s_k \geq s_{k+1}$ . If  $s_k = s_{k+1}$ , we say that  $(u_1, \dots, u_n)$  is of *type II*. Otherwise, we have  $u_{k+1} \doteq vw$ , with  $v$  and  $w$  in  $\mathbf{W} \setminus \{1\}$ , such that  $(u_1, \dots, u_k, v)$  is of type I: we say that  $(u_1, \dots, u_n)$  is of *type III*.

Finally, we define a mapping

$$\Phi(u_1, \dots, u_n) = (l(u_1 \cdots u_n), d_{u_1}, l(u_1), d_{u_1 u_2}, l(u_1 u_2), \dots, d_{u_1 \cdots u_{n-1}}, l(u_1 \cdots u_{n-1}))$$

into  $\mathbb{N} \times (S \times \mathbb{N})^{n-1}$ , equipped with the well-founded lexicographic order generated by the natural order on  $\mathbb{N}$  and the fixed order on  $S$ . We use the mapping  $\Phi$  and the lexicographic order to compare families  $(u_1, \dots, u_n)$  of elements of  $\mathbf{W} \setminus \{1\}$  such that  $l(u_1 \cdots u_n) = l(u_1) + \dots + l(u_n)$ .

**6.2.2. The classification.** Each 1-cell  $u$ , each 2-cell  $\alpha_{u,v}$  and each 3-cell  $A_{u,v,w}$  of  $\text{Del}_+(\mathbf{W})$  is classified according to the type of the family of elements that indexes it. For the 1-cells of  $\text{Del}_+(\mathbf{W})$ , we get:

- Type I:  $s$  in  $S$ ,
- Type III:  $su$  with  $s = d_{su}$ .

For the 2-cells of  $\text{Del}_+(\mathbf{W})$ , we get:

- Type I:  $\alpha_{s,u}$  with  $s > d_{su}$  and  $su = w_0(s, d_{su})$ ,
- Type II:  $\alpha_{s,u}$  with  $s = d_{su}$ ,
- Type III:  $\begin{cases} \text{(a)} & \alpha_{su,v} & \text{with } s = d_{su}, \\ \text{(b)} & \alpha_{s,uv} & \text{with } s > d_{su} = d_{suv} \text{ and } su = w_0(s, d_{su}). \end{cases}$

For the 3-cells of  $\text{Del}_+(\mathbf{W})$ , we get:

- Type I:  $A_{s,u,v}$  with  $s > d_{su} > d_{suv}$  and  $suv = w_0(s, d_{su}, d_{suv})$ ,
- Type II:  $\begin{cases} \text{(a)} & A_{s,u,v} & \text{with } s = d_{su}, \\ \text{(b)} & A_{s,u,v} & \text{with } s > d_{su} = d_{suv} \text{ and } su = w_0(s, d_{su}), \end{cases}$
- Type III:  $\begin{cases} \text{(a)} & A_{su,v,w} & \text{with } s = d_{su}, \\ \text{(b)} & A_{s,uv,w} & \text{with } s > d_{su} = d_{suv} \text{ and } su = w_0(s, d_{su}), \\ \text{(c)} & A_{s,u,vw} & \text{with } s > d_{su} > d_{suv} = d_{suvw} \text{ and } suv = w_0(s, d_{su}, d_{suv}). \end{cases}$

We also consider the following type of 3-sphere  $\omega_{u,v,w,x}$  of  $\Omega(\mathbf{W})$ :

- Type II:  $\begin{cases} \text{(a)} & \omega_{s,u,v,w} & \text{with } s = d_{su}, \\ \text{(b)} & \omega_{s,u,v,w} & \text{with } s > d_{su} = d_{suv} \text{ and } su = w_0(s, d_{su}), \\ \text{(c)} & \omega_{s,u,v,w} & \text{with } s > d_{su} > d_{suv} = d_{suvw} \text{ and } suv = w_0(s, d_{su}, d_{suv}). \end{cases}$

We observe that there exist bijections between the 1-cells of type III and the 2-cells of type II, between the 2-cells of type III and the 3-cells of type II, and between the 3-cells of type III and the 3-spheres of type II. In what follows, we compare the cells of  $\text{Del}_+(\mathbf{W})$  according to the well-founded order defined on their indices.

## 6. Artin's coherent presentation

### 6.3. Homotopical reduction of $\text{Del}_+(\mathbf{W})$

Using the classification of the cells of  $\text{Del}_+(\mathbf{W})$ , we compute the chain of Tietze transformations

$$\text{Del}_+(\mathbf{W})^\top \xrightarrow{\pi_3} \mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top \xrightarrow{\pi_2} \mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top \xrightarrow{\pi_1} \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top$$

and we prove that  $\pi_3$  coherently eliminates the 3-cells of type III and the corresponding 3-spheres of type II, that  $\pi_2$  coherently eliminates the 2-cells of type III and the corresponding 3-cells of type II and, finally, that  $\pi_1$  eliminates the 1-cells of type III and the corresponding 2-cells of type II. We begin with homotopical 3-reduction, proving that the homotopy basis of Theorem 5.1.4 can be further reduced.

**6.3.1. Homotopical 3-reduction.** For each 3-sphere  $\omega_{s,u,v,w}$  of type II

The diagram illustrates the homotopical 3-reduction of a 3-sphere. It consists of two parts connected by an equivalence symbol  $\cong$ .  
 The left part shows a 3-sphere  $\omega_{s,u,v,w}$  with vertices  $s, u, v, w$ . It features several 2-cells:  $A_{s,u,v|w}$ ,  $A_{s,uv,w}$ ,  $A_{s,u,vw}$ ,  $A_{s,u,v,w}$ , and  $A_{s,u,v,w}$ . There are also 1-cells like  $su|v|w$ ,  $s|u|v|w$ ,  $s|u|\alpha_{v,w}$ ,  $su|vw$ ,  $s|uv,w$ ,  $s|uv,w$ ,  $s|uv,w$ , and  $s|\alpha_{u,v,w}$ . 3-cells  $\alpha_{s,u|v|w}$ ,  $\alpha_{s,uv,w}$ ,  $\alpha_{s,u,vw}$ ,  $\alpha_{s,u,v,w}$ , and  $\alpha_{s,u,v,w}$  are shown as curved arrows between the 2-cells.  
 The right part shows the same 3-sphere but with a simplified structure. Many 2-cells are now identities (e.g.,  $su|v|w = s|u|v|w$ ,  $s|u|\alpha_{v,w} = s|u|vw$ ). The 3-cells are also simplified, with some being identities (e.g.,  $\alpha_{s,u|v|w} = \alpha_{s,uv,w}$ ).

we consider the 3-sphere  $\omega'_{s,u,v,w}$  obtained from  $\omega_{s,u,v,w}$  by composition with 2-cells and 3-cells of  $\text{Del}_+(\mathbf{W})^\top$ , so that  $\omega'_{s,u,v,w}$  has the following shape, depending on the subcase of type II:

- (a)  $\omega'_{s,u,v,w} : \bar{A}_{s,u,v,w} \cong A_{s,u,v,w}$ ,
- (b)  $\omega'_{s,u,v,w} : \bar{A}_{s,uv,w} \cong A_{s,uv,w}$ ,
- (c)  $\omega'_{s,u,v,w} : \bar{A}_{s,u,vw} \cong A_{s,u,vw}$ .

By construction, the homotopical 3-reduction of  $\text{Del}_+(\mathbf{W})$  yields the same result if we replace the 3-spheres  $\omega_{s,u,v,w}$  of type II by the 3-spheres  $\omega'_{s,u,v,w}$ .

Let us prove that each  $A_{u,v,w}$  of type III is strictly greater than every 3-cell of  $\text{Del}_+(\mathbf{W})$  that appears in  $\bar{A}_{u,v,w}$ . We observe that  $A_{su,v,w}$ ,  $A_{s,uv,w}$  and  $A_{s,u,vw}$  are always strictly greater than  $A_{s,u,v}$  and  $A_{u,v,w}$  since  $l(suvw) > l(suv)$  and  $l(suvw) > l(uvw)$ . Then, we proceed by case analysis:

- (a)  $A_{su,v,w} > A_{s,uv,w}$  and  $A_{su,v,w} > A_{s,u,vw}$  since  $s = d_{su}$  and  $l(su) > l(s)$ .
- (b)  $A_{s,uv,w} > A_{su,v,w}$  since  $s > d_{su}$  and  $A_{s,uv,w} > A_{s,u,vw}$  since  $d_{suv} = d_{su}$  and  $l(suv) > l(su)$ .
- (c)  $A_{s,u,vw} > A_{su,v,w}$  since  $s > d_{su}$  and  $A_{s,u,vw} > A_{s,uv,w}$  since  $d_{su} > d_{suv}$ .

As a consequence, we can define a 3-functor

$$\text{Del}_+(\mathbf{W})^\top \xrightarrow{\pi_3} \text{Del}_+(\mathbf{W})^\top$$

by induction on the well-founded order on 3-cells of  $\text{Del}_+(\mathbf{W})$  by

$$\pi_3(A_{u,v,w}) = \begin{cases} \pi_3(\bar{A}_{u,v,w}) & \text{if } A_{u,v,w} \text{ is of type III,} \\ A_{u,v,w} & \text{otherwise.} \end{cases}$$

For each 3-sphere  $\omega'_{s,u,v,w}$ , we define the following 3-sphere of  $\text{Del}_+(\mathbf{W})^\top$ :

$$\omega''_{s,u,v,w} : \pi_3(s(\omega'_{s,u,v,w})) \cong t(\omega'_{s,u,v,w}).$$

By construction, the homotopical 3-reduction gives the same result if we replace the 3-spheres  $\omega'_{s,u,v,w}$  by the 3-spheres  $\omega''_{s,u,v,w}$ .

Each 3-sphere  $\omega''_{s,u,v,w}$  generates a Tietze transformation that coherently eliminates its target 3-cell of type III. Moreover, this Tietze transformation leaves the boundary of every other 3-sphere of type  $\omega''$  unchanged. Since there is exactly one 3-sphere for every 3-cell of type III, the homotopical 3-reduction procedure coherently eliminates every 3-cell of type III from  $\text{Del}_+(\mathbf{W})$ .

At the end, by definition of  $\pi_3$ , we have  $\mathcal{R}_3(\text{Del}_+(\mathbf{W})) = \pi_3(\text{Del}_+(\mathbf{W}))$  and the Tietze transformation generated by homotopical 3-reduction is the corresponding factorisation of  $\pi_3$ :

$$\text{Del}_+(\mathbf{W})^\top \xrightarrow{\pi_3} \mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top.$$

**6.3.2. Homotopical 2-reduction.** We proceed in a similar way to homotopical 3-reduction. Firstly, we replace by a Tietze transformation every 3-cell  $A_{s,u,v}$  of type II

$$\begin{array}{ccc} \alpha_{s,u|v} & \xrightarrow{\quad} & su|v & \xrightarrow{\quad} & \alpha_{su,v} \\ & \searrow & \Downarrow A_{s,u,v} & \nearrow & \\ s|u|v & & & & suv \\ & \swarrow & & \nwarrow & \\ s|\alpha_{u,v} & \xrightarrow{\quad} & s|uv & \xrightarrow{\quad} & \alpha_{s,uv} \end{array}$$

with the following 3-cell  $A'_{s,u,v}$ , depending on the subcase:

$$(a) \quad \begin{array}{ccc} & \xrightarrow{\quad \alpha_{su,v} \quad} & \\ su|v & & suv \\ & \searrow \alpha_{s,u|v} & \swarrow \alpha_{s,uv} \\ & s|u|v & \xrightarrow{\quad s|\alpha_{u,v} \quad} & s|uv & \xrightarrow{\quad} & \\ & & \Downarrow A'_{s,u,v} & & & \end{array} \quad (b) \quad \begin{array}{ccc} & \xrightarrow{\quad \alpha_{s,uv} \quad} & \\ s|uv & & suv \\ & \searrow s|\alpha_{u,v} & \swarrow \alpha_{su,v} \\ & s|u|v & \xrightarrow{\quad \alpha_{s,u|v} \quad} & su|v & \xrightarrow{\quad} & \end{array}$$

For  $\alpha_{u,v}$  of type III, we denote by  $\bar{\alpha}_{u,v}$  the target of the corresponding 3-cell  $A'$  of type II. Then, we define by induction a 3-functor

$$\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top \xrightarrow{\pi_2} \mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top$$

by

$$\pi_2(\alpha_{u,v}) = \begin{cases} \pi_2(\bar{\alpha}_{u,v}) & \text{if } \alpha_{u,v} \text{ is of type III,} \\ \alpha_{u,v} & \text{otherwise.} \end{cases}$$

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To ensure that  $\pi_2$  is well-defined, we check that each  $\alpha_{u,v}$  of type III is strictly greater than every 2-cell that appears in  $\bar{\alpha}_{u,v}$ . We observe that  $\alpha_{su,v}$  and  $\alpha_{s,uv}$  are always strictly greater than  $\alpha_{s,u}$  and  $\alpha_{u,v}$  since  $l(suv) > l(su)$  and  $l(suv) > l(uv)$ . Then, we proceed by case analysis:

- (a)  $\alpha_{su,v} > \alpha_{s,uv}$  since  $s = d_{su}$  and  $l(su) > l(s)$ .
- (b)  $\alpha_{s,uv} > \alpha_{su,v}$  since  $s > d_{su}$ .

We conclude the proof with the same arguments as for homotopical 3-reduction. We get that the coherent presentation  $\mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))$  is  $\pi_2(\text{Del}_+(\mathbf{W}))$ , obtained from  $\mathcal{R}_3(\text{Del}_+(\mathbf{W}))$  by coherent elimination of the 2-cells of type III and the 3-cells of type II. We also get that  $\pi_2$  factors through a Tietze transformation

$$\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top \xrightarrow{\pi_2} \mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top.$$

**6.3.3. Homotopical 1-reduction.** Finally, the homotopical 1-reduction procedure coherently eliminates each 2-cell of type II

$$s|u \xrightarrow{\alpha_{s,u}} su$$

with the 1-cell  $su$  of type III, inducing the Tietze transformation

$$\mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top \xrightarrow{\pi_1} \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))^\top$$

where  $\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(\text{Del}_+(\mathbf{W}))$  only contains the cells of  $\text{Del}_+(\mathbf{W})$  of type I. The 3-functor  $\pi_1$  is defined on 1-cells by induction on their length by  $\pi_1(s) = s$  if  $s \in S$  and by  $\pi_1(su) = s|\pi_1(u)$  if  $s = d_{su}$ . Moreover, it sends the 2-cells of type II to identities and it projects the boundaries of the 2-cells and 3-cells of type I accordingly.

**6.3.4. The resulting coherent presentation.** After homotopical reduction, we get a coherent presentation of  $\mathbf{B}^+(\mathbf{W})$  that contains exactly the cells of type I of  $\text{Del}_+(\mathbf{W})$ , with boundary modified by the 3-functor  $\pi = \pi_1\pi_2\pi_3$ . There remains to compute the values of  $\pi$  to obtain Theorem 2.4.5 for Artin monoids.

The 1-cells of type I of  $\text{Del}_+(\mathbf{W})$  are the elements of  $S$ . A 1-cell  $u$  of  $\text{Del}_+(\mathbf{W})$  is mapped through  $\pi$  to the element

$$\pi(u) = s_1 \cdots s_n$$

of  $S^*$ , such that  $u \doteq s_1 \cdots s_n$  and  $s_i = d_{s_1 \cdots s_n}$ .

The 2-cells of type I of  $\text{Del}_+(\mathbf{W})$  are the  $\alpha_{s,u}$  such that  $s > d_{su}$  and  $su = w_0(s, d_{su})$ . Hence, there is one such 2-cell for every  $t > s$  in  $S$  such that  $\mathbf{W}_{\{s,t\}}$  is finite, *i.e.*, such that  $m_{st}$  is finite, and its image through  $\pi$  is precisely

$$\gamma_{st} : \langle ts \rangle^{m_{st}} \Rightarrow \langle st \rangle^{m_{st}}.$$

The image of a 2-cell  $\alpha_{u,v}$  of  $\text{Del}_+(\mathbf{W})$  through  $\pi$  is given by induction as follows:

$$\begin{array}{ll} \pi(\alpha_{s,u}) = \gamma_{rs} & \text{if } u = \langle rs \rangle^{m_{rs}-1} \text{ for } r < s \\ \pi(\alpha_{s,u}) = 1_{su} & \text{if } s = d_{su} \\ \pi(\alpha_{s,u}) = s\pi(\alpha_{v,w}^-) \star_1 \gamma_{rs} w \star_1 \pi(\alpha_{sv,w}) & \text{if } u \doteq vw \text{ and } v = \langle rs \rangle^{m_{rs}-1} \text{ for } r = d_{su} < s \\ \pi(\alpha_{su,v}) = s\pi(\alpha_{u,v}) \star_1 \pi(\alpha_{s,uv}) & \text{if } s = d_{su}. \end{array}$$

### 6.3. Homotopical reduction of $\text{Del}_+(\mathbf{W})$

Finally, the 3-cells of type I of  $\text{Del}_+(\mathbf{W})$  are the  $A_{s,u,v}$  such that  $s > d_{su} > d_{suv}$ ,  $su = w_0(s, d_{su})$  and  $suv = w_0(s, d_{su}, d_{suv})$ . Hence, there is exactly one such 3-cell for every  $t > s > r$  in  $S$  such that the subgroup  $\mathbf{W}_{\{r,s,t\}}$  is finite.

There remains to compute the image of each of those 3-cells through  $\pi$ , depending on the type of the finite Coxeter group  $\mathbf{W}_{\{r,s,t\}}$ :  $A_3$ ,  $B_3$ ,  $H_3$ ,  $A_1 \times A_1 \times A_1$  and  $I_2(p) \times A_1$ . The main technical difficulty is to manipulate elements of  $\mathbf{W}$  and, in particular, the different reduced expressions of the longest element  $w_0(r, s, t)$ . For that, we have used the PyCox package of Geck [18]. Let us detail some of the computations.

For  $t > s > r$  in  $S$  such that  $\mathbf{W}_{\{r,s,t\}}$  is of type  $A_1 \times A_1 \times A_1$ , the corresponding 3-cell of type I of  $\text{Del}_+(\mathbf{W})$  is

$$\begin{array}{ccc}
 & \alpha_{t,s|r} \rightarrow st|r & \xrightarrow{\alpha_{st,r}} \\
 t|s|r & \Downarrow A_{t,s,r} & rst \\
 & t|\alpha_{s,r} \rightarrow t|rs & \xrightarrow{\alpha_{t,rs}}
 \end{array}$$

The image of  $A_{t,s,r}$  through  $\pi$  is given by the inductive application of  $\pi$  to the 2-cells of its boundary. For the source of  $\pi(A_{t,s,r})$ , we get  $\pi(\alpha_{t,s|r}) = \gamma_{str}$  and

$$\begin{aligned}
 \pi(\alpha_{st,r}) &= s\pi(\alpha_{t,r}) \star_1 \pi(\alpha_{s,rt}) \\
 &= s\gamma_{r,t} \star_1 \gamma_{s,r}t \star_1 \pi(\alpha_{rs,t}) \\
 &= s\gamma_{r,t} \star_1 \gamma_{s,r}t.
 \end{aligned}$$

For the target of  $\pi(A_{t,s,r})$ , we get  $\pi(t|\alpha_{s,r}) = t\gamma_{rs}$  and

$$\begin{aligned}
 \pi(\alpha_{t,rs}) &= \gamma_{rt}s \star_1 \pi(\alpha_{rt,s}) \\
 &= \gamma_{rt}s \star_1 r\gamma_{st} \star_1 \pi(\alpha_{r,st}) \\
 &= \gamma_{rt}s \star_1 r\gamma_{st}.
 \end{aligned}$$

Thus, the image of  $A_{t,s,r}$  through  $\pi$  is

$$\begin{array}{ccc}
 \gamma_{str} \rightarrow str & \xrightarrow{s\gamma_{rt}} & srt \xrightarrow{\gamma_{rs}t} \\
 tsr & \Downarrow Z_{r,s,t} & rst \\
 t\gamma_{rs} \rightarrow trs & \xrightarrow{\gamma_{rt}s} & rts \xrightarrow{r\gamma_{st}}
 \end{array}$$

In the case where  $\mathbf{W}_{\{r,s,t\}}$  is of type  $A_3$ , the corresponding 3-cell of type I of  $\text{Del}_+(\mathbf{W})$  is

$$\begin{array}{ccc}
 \alpha_{t,st|rst} \rightarrow sts|rst & \xrightarrow{\alpha_{sts,rst}} & \\
 t|st|rst & \Downarrow A_{t,st,rst} & rst \\
 t|\alpha_{st,rst} \rightarrow t|rsrts & \xrightarrow{\alpha_{t,rsrts}} &
 \end{array}$$

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To get  $\pi(A_{t,st,rst})$ , we compute the image of its source through  $\pi$ . We get  $\pi(\alpha_{t,st}|rst) = \gamma_{st}rst$  and

$$\pi(\alpha_{sts,rst}) = st\pi(\alpha_{s,rst}) \star_1 s\pi(\alpha_{t,rst}) \star_1 \pi(\alpha_{s,rstsr}).$$

Then, we have  $st\pi(\alpha_{s,rst}) = st\gamma_{rs}t$  and  $\pi(\alpha_{s,rstsr}) = \gamma_{rs}tsr$ , together with

$$s\pi(\alpha_{t,rst}) = s\gamma_{rt}sr \star_1 sr\pi(\alpha_{t,srt}) \star_1 s\pi(\alpha_{r,sts}).$$

Finally, we get  $s\pi(\alpha_{r,sts}) = 1_{srstsr}$  and

$$sr\pi(\alpha_{t,srt}) = srt\pi(\alpha_{st,r})^- \star_1 sr\gamma_{st}r \star_1 sr\pi(\alpha_{sts,r}) = srts\gamma_{rt}^- \star_1 sr\gamma_{st}r.$$

A similar sequence of computations for the target of  $A_{t,st,rst}$  gives that  $\pi(A_{t,st,rst})$  is precisely the 3-cell

$$\begin{array}{ccccc}
 & st\gamma_{rs}t & \xrightarrow{\quad} & strst & \xrightarrow{s\gamma_{rt}s\gamma_{rt}^-} & srtstr & \xrightarrow{s\gamma_{st}r} & srstsr & \xrightarrow{\gamma_{rs}tsr} & rsrtsr \\
 & \nearrow & & \uparrow \gamma_{st}rst & & \Downarrow Z_{rst} & & \downarrow & & \uparrow rs\gamma_{rt}sr \\
 & strst & & tstrst & & & & rstrsr & & rstrsr \\
 & \downarrow ts\gamma_{rt}st & & \downarrow & & & & \uparrow rst\gamma_{rs} & & \uparrow rstrsr \\
 & tsrtst & & tsrsts & \xrightarrow{t\gamma_{rs}ts} & trsrts & \xrightarrow{\gamma_{rt}s\gamma_{rt}^-s} & rtstrs & \xrightarrow{\gamma_{st}rs} & rstsrs \\
 & \searrow tsr\gamma_{st} & & \downarrow & & & & \uparrow & & \uparrow rstsrs
 \end{array}$$

Similar computations for the other types  $I_2(p) \times A_1$ ,  $B_3$  and  $H_3$  gives that the homotopical completion-reduction of  $\text{Del}(\mathbf{W})$  is Artin's coherent presentation  $\text{Art}_+(\mathbf{W})$ . This ends the proof of Theorem 2.4.5 for Artin monoids:  $\text{Art}_+(\mathbf{W})$  is, indeed, a coherent presentation of  $\mathbf{B}^+(\mathbf{W})$ .

### 6.4. Artin's coherent presentation of Artin groups

We conclude the proof of Theorem 2.4.5 for Artin groups thanks to the following result proved by Paris in [32]: every Artin monoid  $\mathbf{B}^+(\mathbf{W})$  embeds in the Artin group  $\mathbf{B}(\mathbf{W})$ . Thus, Theorem 6.4.2 applies to Artin monoids to transfer their coherent presentation to Artin groups.

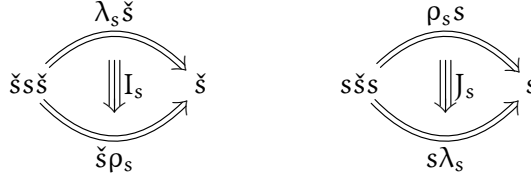
**6.4.1. Notations.** If  $\Sigma$  is a 2-polygraph, the groupoid  $\mathbf{G}$  presented by  $\Sigma$  admits, as a category, the presentation  $\text{Gpd}(\Sigma)$ , obtained from  $\Sigma$  by adjunction, for every 1-cell  $s : x \rightarrow y$  of  $\Sigma$ , of the following cells:

- a 1-cell  $\check{s} : y \rightarrow x$ ,
- two 2-cells  $\lambda_s : \check{s}s \Rightarrow 1_y$  and  $\rho_s : s\check{s} \Rightarrow 1_x$ .

## 6.4. Artin's coherent presentation of Artin groups

If  $\Sigma$  is a  $(3, 1)$ -polygraph, we denote by  $\text{Gpd}_+(\Sigma)$  the extended presentation of the groupoid  $\mathbf{G}$  presented by  $\Sigma$  that is obtained from  $\Sigma$  by adjunction, for every 1-cell  $s : x \rightarrow y$  of  $\Sigma$ , of the following cells:

- the same 1-cell  $\check{s}$  and 2-cells  $\lambda_s$  and  $\rho_s$  as for  $\text{Gpd}(\Sigma)$ ,
- two 3-cells



By definition of a coherent presentation of a groupoid, we have that  $\Sigma$  is a coherent presentation of the groupoid  $\mathbf{G}$  if  $\text{Gpd}(\Sigma)$  is a coherent presentation of  $\mathbf{G}$ , seen as a category.

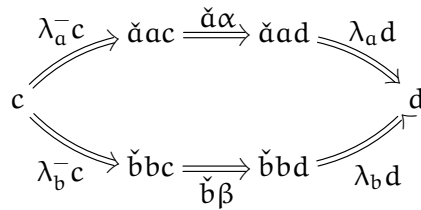
**6.4.2. Theorem.** *Let  $\Sigma$  be a  $(3, 1)$ -polygraph, let  $\mathbf{C}$  be the category presented by  $\Sigma$  and let  $\mathbf{G}$  be the groupoid presented by  $\Sigma$ . If  $\mathbf{C}$  embeds in  $\mathbf{G}$  and if  $\Sigma$  is a coherent presentation of  $\mathbf{C}$ , then  $\Sigma$  is a coherent presentation of  $\mathbf{G}$ .*

The proof of Theorem 6.4.2 is conducted throughout the rest of the section, by homotopical completion-reduction of a specific presentation of  $\mathbf{G}$ , seen as a category. Before this, we give a counterexample that shows that Theorem 6.4.2 does not hold if  $\mathbf{M}$  does not embeds in  $\mathbf{G}$ .

**6.4.3. A counterexample.** Let us consider the monoid  $\mathbf{M}$  presented by

$$\Sigma = (a, b, c, d; \alpha : ac \Rightarrow ad, \beta : bc \Rightarrow bd).$$

This 2-polygraph is convergent with no critical branching, so that  $(\Sigma, \emptyset)$  is a coherent presentation of  $\mathbf{M}$ . But  $\mathbf{M}$  does not embed in the group  $\mathbf{G}$  presented by  $\Sigma$ , since  $c = d$  holds in  $\mathbf{G}$ . Moreover, the two different ways to prove this equality in  $\mathbf{G}$  generate the following 3-sphere  $\gamma$  in  $\text{Gpd}(\Sigma)^\top$ :



This 3-sphere cannot be the boundary of a composite of 3-cells of types I or J. Indeed, neither of those 3-cells contain the 2-cells  $\alpha$  or  $\beta$ . As a consequence, all their composites have the same number of  $\alpha$  and  $\beta$  in their source and target, which is not the case for  $\gamma$ . Thus, the group  $\mathbf{G}$  does not admit  $(\Sigma, \emptyset)$  as a coherent presentation.

**6.4.4. The presentation  $\widetilde{\text{Gpd}}(\Sigma)$ .** In the following, we use the same notations as in Theorem 6.4.2. We denote by  $S$  the set of 1-cells of  $\Sigma$  and we consider the canonical presentation  $\text{Can}(\mathbf{C})$ , with the unit cells  $\iota_x : \widehat{1}_x \Rightarrow 1_x$  going in the terminating direction. We consider the alternative presentation  $\widetilde{\text{Gpd}}(\Sigma)$  of  $\mathbf{G}$ , obtained from  $\text{Can}(\mathbf{C})$  by adjunction of the following cells:



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- a 1-cell  $\check{s} : y \rightarrow x$  for every  $s : x \rightarrow y$  in  $S$ ,
- a 2-cell  $\lambda_{s,u} : \check{s}\widehat{su} \Rightarrow \widehat{u}$  for every  $s : x \rightarrow y$  in  $S$  and  $u : y \rightarrow z$  in  $\mathbf{C}$ ,
- a 2-cell  $\rho_{u,s} : \widehat{us}\check{s} \Rightarrow \widehat{u}$  for every  $u : x \rightarrow y$  in  $S$  and  $s : y \rightarrow z$  in  $S$ .

We observe that  $\widetilde{\text{Gpd}}(\Sigma)$  terminates by considering the length of the 1-cells as termination order.

**6.4.5. Homotopical completion of  $\widetilde{\text{Gpd}}(\Sigma)$ .** We check that  $\widetilde{\text{Gpd}}(\Sigma)$  is confluent and we get the 3-cells of the coherent presentation  $\mathcal{S}(\widetilde{\text{Gpd}}(\Sigma))$  of  $\mathbf{G}$ . First, we have the three families of the canonical coherent presentation  $\text{Can}_+(\mathbf{C})$

$$\begin{array}{ccc}
 \begin{array}{ccc} \widehat{u}\widehat{v}\widehat{w} & \xrightarrow{\alpha_{u,v}\widehat{w}} & \widehat{u}\widehat{v}\widehat{w} \\ & \Downarrow A_{u,v,w} & \\ \widehat{u}\widehat{v}\widehat{w} & \xrightarrow{\widehat{u}\alpha_{v,w}} & \widehat{u}\widehat{v}\widehat{w} \end{array} & 
 \begin{array}{ccc} \widehat{1}\widehat{u} & \xrightarrow{\alpha_{1,u}} & \widehat{u} \\ & \Downarrow L_u & \\ \widehat{1}\widehat{u} & \xrightarrow{\widehat{1}\alpha_{u,1}} & \widehat{u} \end{array} & 
 \begin{array}{ccc} \widehat{u}\widehat{1} & \xrightarrow{\alpha_{u,1}} & \widehat{u} \\ & \Downarrow R_u & \\ \widehat{u}\widehat{1} & \xrightarrow{\widehat{u}\alpha_{1,u}} & \widehat{u} \end{array}
 \end{array}$$

for every possible 1-cells, with the indices of identities omitted. Then, we have the 3-cells induced by the 2-cells of types  $\lambda$  and  $\rho$

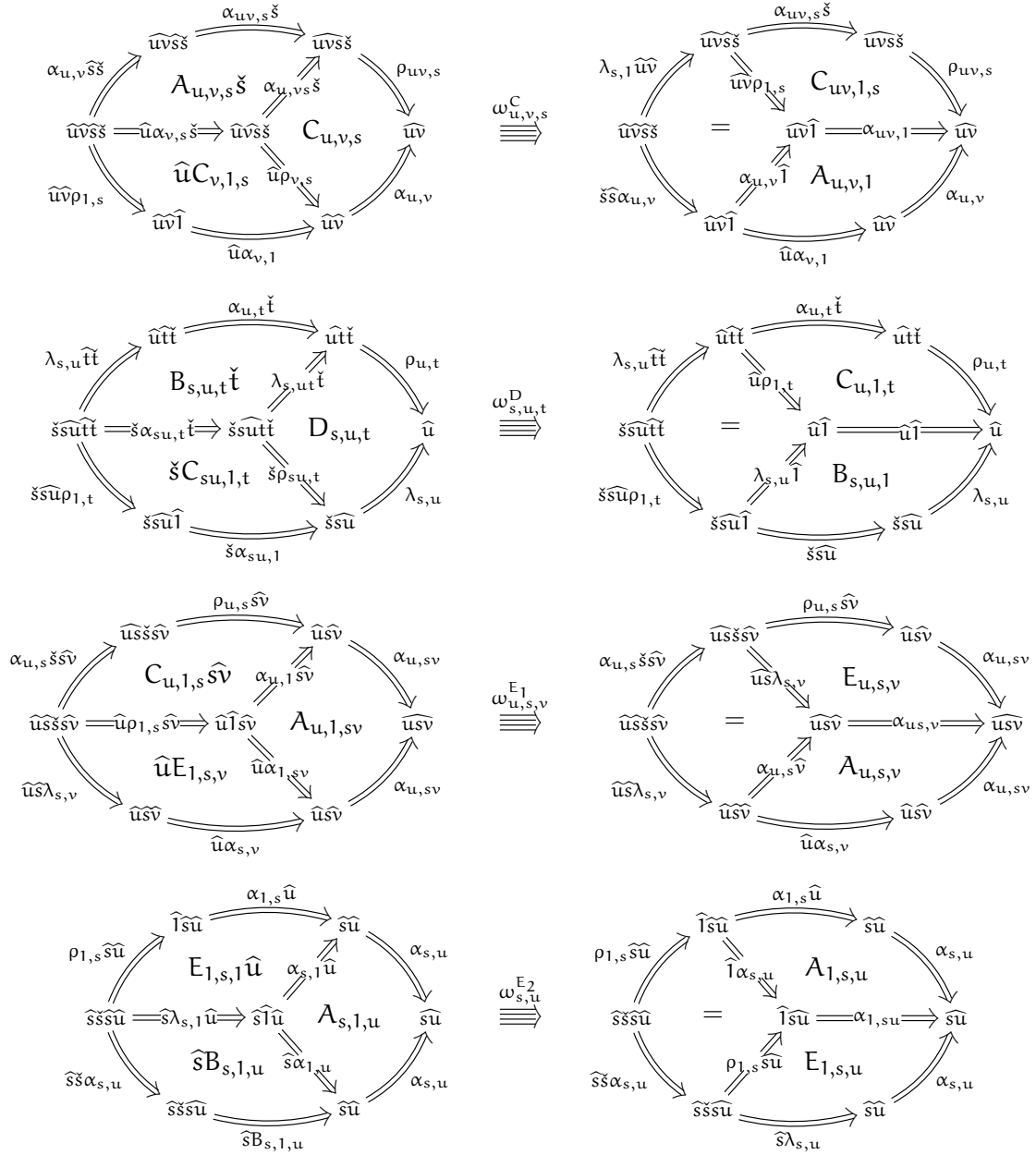
$$\begin{array}{ccc}
 \begin{array}{ccc} \check{s}\widehat{su}\widehat{v} & \xrightarrow{\lambda_{s,u}\widehat{v}} & \widehat{u}\widehat{v} \\ & \Downarrow B_{s,u,v} & \\ \check{s}\widehat{su}\widehat{v} & \xrightarrow{\check{s}\alpha_{su,v}} & \check{s}\widehat{su}\widehat{v} \end{array} & 
 \begin{array}{ccc} \widehat{u}\widehat{v}\check{s}\check{s} & \xrightarrow{\alpha_{u,v}\check{s}} & \widehat{u}\widehat{v}\check{s}\check{s} \\ & \Downarrow C_{u,v,s} & \\ \widehat{u}\widehat{v}\check{s}\check{s} & \xrightarrow{\widehat{u}\rho_{v,s}} & \widehat{u}\widehat{v}\check{s}\check{s} \end{array} & 
 \begin{array}{ccc} \check{s}\widehat{su}\check{t}\check{t} & \xrightarrow{\lambda_{s,u}\check{t}} & \widehat{u}\check{t}\check{t} \\ & \Downarrow D_{s,u,t} & \\ \check{s}\widehat{su}\check{t}\check{t} & \xrightarrow{\check{s}\rho_{u,t}} & \check{s}\widehat{su}\check{t}\check{t} \end{array} \\
 \\ 
 \begin{array}{ccc} \widehat{u}\check{s}\check{s}\widehat{sv} & \xrightarrow{\rho_{u,s}\widehat{sv}} & \widehat{u}\check{s}\widehat{sv} \\ & \Downarrow E_{u,s,v} & \\ \widehat{u}\check{s}\check{s}\widehat{sv} & \xrightarrow{\widehat{u}\check{s}\lambda_{s,v}} & \widehat{u}\check{s}\widehat{sv} \end{array} & 
 \begin{array}{ccc} \check{s}\check{s}\check{s} & \xrightarrow{\lambda_{s,1}\check{s}} & \widehat{1}\check{s} \\ & \Downarrow F_s & \\ \check{s}\check{s}\check{s} & \xrightarrow{\check{s}\rho_{1,s}} & \check{s}\widehat{1} \end{array}
 \end{array}$$

for every possible 1-cells, also with the indices of units omitted. Let us note that  $\widetilde{\text{Gpd}}(\Sigma)$  has no critical branching of shape  $(\lambda_{s,u}, \lambda_{s,v})$  with source  $\check{s}\widehat{su} = \check{s}\widehat{sv}$  because, by hypothesis, the category  $\mathbf{C}$  embeds into the groupoid  $\mathbf{G}$ : as a consequence, an equality  $su = sv$  in  $\mathbf{C}$  implies  $u = v$ . For the same reason, there are no critical branching of shape  $(\rho_{u,s}, \rho_{v,s})$ .

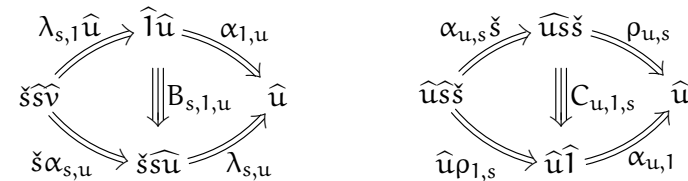
**6.4.6. Homotopical reduction of  $\mathcal{S}(\widetilde{\text{Gpd}}(\Sigma))$ .** Among the critical triple branchings of  $\mathcal{S}(\widetilde{\text{Gpd}}(\Sigma))$ , we consider the following ones:

$$\begin{array}{ccc}
 \begin{array}{ccc} \widehat{1}\widehat{u}\widehat{v} & \xrightarrow{\alpha_{1,u}\widehat{v}} & \widehat{u}\widehat{v} \\ \lambda_{s,1}\widehat{u}\widehat{v} \nearrow & & \searrow \alpha_{u,v} \\ \check{s}\widehat{su}\widehat{v} & \xrightarrow{\check{s}\alpha_{u,v}\widehat{v}} & \check{s}\widehat{su}\widehat{v} \\ \check{s}\widehat{su}\widehat{v} \searrow & & \nearrow \lambda_{s,u}\widehat{v} \\ \check{s}\widehat{su}\widehat{v} & \xrightarrow{\check{s}\alpha_{s,u,v}} & \check{s}\widehat{su}\widehat{v} \end{array} & 
 \omega_{s,u,v}^B & 
 \begin{array}{ccc} \widehat{1}\widehat{u}\widehat{v} & \xrightarrow{\alpha_{1,u}\widehat{v}} & \widehat{u}\widehat{v} \\ \lambda_{s,1}\widehat{u}\widehat{v} \nearrow & \widehat{1}\alpha_{u,v} & \searrow \alpha_{u,v} \\ \check{s}\widehat{su}\widehat{v} & \xrightarrow{\check{s}\alpha_{u,v}\widehat{v}} & \check{s}\widehat{su}\widehat{v} \\ \check{s}\widehat{su}\widehat{v} \searrow & \widehat{1}\widehat{u}\widehat{v} & \nearrow \alpha_{1,u}\widehat{v} \\ \check{s}\widehat{su}\widehat{v} & \xrightarrow{\check{s}\alpha_{s,u,v}} & \check{s}\widehat{su}\widehat{v} \end{array}
 \end{array}$$

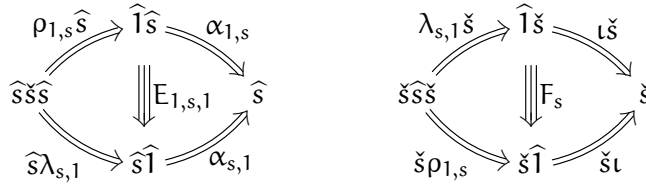
## 6.4. Artin's coherent presentation of Artin groups



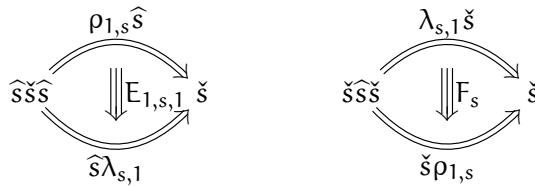
Those 3-spheres generate a Tietze transformation of  $\mathcal{S}(\widetilde{\text{Gpd}}(\Sigma))$  that coherently eliminates every  $B_{s,u,v}$  for  $u \neq 1$ , every  $C_{u,v,s}$  for  $v \neq 1$ , every  $D_{s,u,t}$ , and every  $E_{u,s,v}$  for  $u \neq 1$  or  $v \neq 1$ . Thus, we get a homotopy basis of  $\text{Gpd}(\Sigma)$  made of the 3-cells of  $\text{Can}_+(\mathbf{C})$ , plus the following 3-cells:



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The 3-cells  $B_{s,1,u}$  and  $C_{u,1,s}$  generate a Tietze transformation that coherently eliminates them with the 2-cells  $\lambda_{s,u}$  and  $\rho_{u,s}$  for  $u \neq 1$ . Then, we consider the Tietze transformation from  $\text{Can}_+(\mathbf{C})$  and to the reduced canonical coherent presentation  $\text{Can}'_+(\mathbf{C})$  of  $\mathbf{C}$ . This yields a coherent presentation of  $\mathbf{G}$ , seen as a category, made of  $\text{Can}'_+(\mathbf{C})$  extended with the 3-cells



Finally, we consider a Tietze transformation  $\pi$  from the  $\text{Can}'_+(\mathbf{C})$  to  $\Sigma$  and, then, we identify  $\lambda_{s,1}$  to  $\lambda_s$ ,  $\rho_{1,s}$  to  $\rho_s$ ,  $E_{1,s,1}$  to  $J_s$  and  $F_s$  to  $I_s$ . We obtain exactly  $\text{Gpd}_+(\Sigma)$ , thus proving that it forms a coherent presentation of  $\mathbf{G}$ , seen as a category. This concludes the proof of Theorem 6.4.2 and, as a consequence, of Theorem 2.4.5.

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