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► **To cite this version:**

Stéphane Gaussent, Yves Guiraud, Philippe Malbos. Coherent presentations and actions on categories. 2012. hal-00682233v1

HAL Id: hal-00682233

<https://hal.science/hal-00682233v1>

Preprint submitted on 23 Mar 2012 (v1), last revised 22 May 2015 (v4)

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COHERENT PRESENTATIONS AND ACTIONS ON CATEGORIES

STÉPHANE GAUSSENT

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Abstract – We study Deligne’s notion of action of a monoid on a category and, in particular, the piece of data that corresponds to the coherence relations that such an action should satisfy. We prove that actions of a monoid are equivalent to 2-functors from a 2-categorical cofibrant replacement of the monoid into the 2-category of categories. One way to compute such a cofibrant replacement is to consider the 2-category presented by a coherent presentation of the monoid: this is a presentation extended with a homotopy basis, that is, a set of relations between the relations that identifies any two proofs of the same equality in the monoid. Using higher-dimensional rewriting, in the polygraphic setting, we combine and extend Squier’s theorem and Knuth-Bendix completion procedure into a “reduced homotopical completion” procedure that, when successful, transforms a given presentation into a relatively small coherent presentation. In particular, when used on Deligne’s presentation of Artin-Tits groups of spherical type, the procedure computes the coherence conditions that Deligne finds with geometric methods.

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March 23, 2012

INTRODUCTION

Actions of monoids on categories

Given a monoid \mathbf{M} and a category \mathbf{C} , an action T of \mathbf{M} on \mathbf{C} is a family of endofunctors $T(u)$ of \mathbf{C} , one for every element u in \mathbf{M} , such that T preserves the product of \mathbf{M} . In a direct transposition of the notion of action on a set or a vector space, this could mean that, for every u and v in \mathbf{M} , we have the following equality:

$$T(uv) = T(u)T(v).$$

However, as pointed out by Deligne in [7], one should prefer a weaker compatibility, where each equality $T(uv) = T(u)T(v)$ is replaced by a natural isomorphism

$$T(u)T(v) \xrightarrow{T_{u,v}} T(uv).$$

But, then, one should ask that the action T preserves the next level of relations: the associativity of the product of \mathbf{M} . This is done by asking, for every elements u, v and w of \mathbf{M} , that the following coherence relation be satisfied:

$$\begin{array}{ccc}
 & T_{u,v}T(w) \rightarrow & T(uv)T(w) & \xrightarrow{T_{uv,w}} & T(uvw) \\
 T(u)T(v)T(w) & \xrightarrow{\quad} & & & \\
 & T(u)T_{v,w} \rightarrow & T(u)T(vw) & \xrightarrow{T_{u,vw}} &
 \end{array}
 \quad \textcircled{c}$$

Such a definition requires much data to be specified, one functor for each element of \mathbf{M} and one natural isomorphism for each pair of elements of \mathbf{M} , and many relations to be checked, one for every triple of elements of \mathbf{M} . Thus, in concrete cases, it is natural to search for equivalent but smaller definitions of actions.

In the special case of Artin-Tits groups of spherical type, or “generalised braid groups”, Deligne gives an answer in [7], Theorem 1.5 (more precisely, he studies the case of the monoids of positive braids and explains the link with the actions of groups). For \mathbf{W} a finite Coxeter group with set of generators S , in order to specify an action of the monoid $\mathbf{B}^+(\mathbf{W}, S)$ of positive braids on a category \mathbf{C} , it is sufficient to fix

- an endofunctor $T(u) : \mathbf{C} \rightarrow \mathbf{C}$ for every element u of $\mathbf{W} \setminus \{1\}$,
- a natural isomorphism $T(u)T(v) \simeq T(uv)$ for every elements u and v in $\mathbf{W} \setminus \{1\}$ such that the length of uv is the sum of the lengths of u and v ,

in such a way that these data satisfy the coherence relations corresponding to the associativity, but only for the triples (u, v, w) of elements of $\mathbf{W} \setminus \{1\}$ such that the length of uvw is the sum of the lengths of u, v and w .

It turns out that, in this reduced definition of an action, the endofunctors and the natural isomorphisms are given by an alternative presentation of the monoid $\mathbf{B}^+(\mathbf{W}, S)$, that we call *Deligne’s presentation* by

opposition to the classical *Artin's presentation*: the generators are the elements of $\mathbf{W} \setminus \{1\}$ and the relations are $u|v = uv$, for every elements u and v of $\mathbf{W} \setminus \{1\}$ such that u and v satisfy the length condition, where $|\cdot|$ denotes the product in the free monoid over $\mathbf{W} \setminus \{1\}$.

In this paper, we address the following natural question that arises from this last observation: given a presentation $\langle X \mid R \rangle$ of a monoid \mathbf{M} , a category \mathbf{C} and the data

- an endofunctor $T(x) : \mathbf{C} \rightarrow \mathbf{C}$ for each generator x in X ,
- a natural isomorphism $T(u) \simeq T(v)$ for each relation $u = v$ in R ,

is it possible to determine, in an explicit way, the coherence conditions that must be satisfied so that the given data is an action of \mathbf{M} on \mathbf{C} ?

Cofibrant approximations of monoids

A first step in our analysis of actions of monoids on categories is to note that they are special cases of a more general notion: the 2-representations of 2-categories, as introduced by Elgueta in [8]. Such a 2-representation of a 2-category \mathcal{C} into a 2-category \mathcal{D} is defined as a pseudofunctor from \mathcal{C} to \mathcal{D} , *i.e.*, a suitably weakened 2-functor from \mathcal{C} to \mathcal{D} . The case of actions of monoids on categories is recovered by seeing a monoid \mathbf{M} as a 2-category with one 0-cell and identity 2-cells only, and by taking the target 2-category \mathcal{D} to be \mathbf{Cat} , the 2-category of categories.

In the setting of 2-representations, we can study actions from a homotopical point of view, thanks to the model category structure described by Lack on 2-categories in [15] and [16]. In such a model category, objects admit *cofibrant approximations*: these are *cofibrant* objects (informally, as free as possible) which are *weakly equivalent* to the original object. In the case of a 2-category \mathcal{C} , this is a 2-category $\tilde{\mathcal{C}}$ that is free in dimension 1, together with a 2-functor $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ that induces an equivalence of homotopy.

The main result of the first section is to prove that the notion of 2-representations of 2-categories can be “strictified”, yielding the following result for actions of monoids:

Theorem 1.5.2. *Let \mathbf{M} be a monoid and let $\tilde{\mathbf{M}}$ be a 2-categorical cofibrant approximation of \mathbf{M} . Then the category $\mathbf{Act}(\mathbf{M})$ of actions of \mathbf{M} on categories is equivalent to the one of 2-functors from $\tilde{\mathbf{M}}$ to \mathbf{Cat} .*

As a consequence of this result, our original question can be answered by giving a procedure to compute a cofibrant approximation of a monoid \mathbf{M} from a given presentation of \mathbf{M} .

Coherent presentations of monoids

Let us fix a presentation $\Sigma = (\Sigma_1, \Sigma_2)$ of a monoid \mathbf{M} , where Σ_1 is the set of generators and Σ_2 the set of relations. This data is called a *2-polygraph*, after Burroni, see [5], or a *computad*, after Street, see [24]. From such a 2-polygraph Σ , we can build a 2-category with one 0-cell as follows.

We consider the free category Σ_1^* generated by the elements of Σ_1 , seen as 1-cells with source and target the only 0-cell. We get a surjective functor:

$$\Sigma_1^* \twoheadrightarrow \mathbf{M}.$$

Introduction

The category Σ_1^* is free and, thus, it is a cofibrant 2-category, but the canonical projection is not a weak equivalence in general. Informally, this is because two different words of Σ_1^* that represent the same element of \mathbf{M} are not related in Σ_1^* .

To correct this, we add 2-cells to Σ_1^* , that are freely generated by the relations and their inverses. For that, a relation $u = v$ is seen as a generating 2-cell from the 1-cell u to the 1-cell v of X^* . Then we get a 2-category, which we denote by Σ_2^\top , and a surjective 2-functor:

$$\Sigma_2^\top \twoheadrightarrow \mathbf{M}.$$

Once again, we have a cofibrant 2-category Σ_2^\top but the projection is not a weak equivalence either. Indeed, there may exist several 2-cells from a given word u to another word v , *i.e.*, different computations using the relations of Σ_2 to prove that u and v are equal in \mathbf{M} .

In order to get a weak equivalence, we have to take a quotient of Σ_2^\top by a sufficiently large set of relations between the relations of Σ_2 so that every pair of 2-cells of Σ_2^\top with the same source and the same target are identified in the quotient. Such a set Σ_3 of higher relations is called an *homotopy basis* of Σ_2^\top and gives a cofibrant replacement of \mathbf{M} :

$$\Sigma_2^\top / \Sigma_3 \xrightarrow{\sim} \mathbf{M}.$$

Finally, we get a notion extending the one of presentation of a monoid \mathbf{M} , called a *coherent presentation*. It is defined as an *acyclic (3, 1)-polygraph*, *i.e.*, a triple $\Sigma = (\Sigma_1, \Sigma_2, \Sigma_3)$ where (Σ_1, Σ_2) is a presentation of \mathbf{M} and Σ_3 is a homotopy basis. The corresponding cofibrant replacement $\bar{\Sigma} = \Sigma_2^\top / \Sigma_3$ of \mathbf{M} leads to the main result of the second section:

Theorem 2.3.2. *Let \mathbf{M} be a monoid and let Σ be a coherent presentation of \mathbf{M} . The category $\text{Act}(\mathbf{M})$ is equivalent to the category of 2-functors from $\bar{\Sigma}$ to Cat .*

Thus one way to determine the coherence conditions that should be satisfied in a definition of action based on a presentation is to compute a homotopy basis, *i.e.*, to extend the presentation into a coherent presentation.

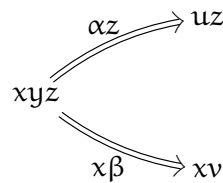
Reduced homotopical completion

Rewriting theory gives a starting point to compute homotopy bases. Indeed, Squier has proved in [23] that a *convergent* presentation of a monoid can be used to determine a homotopy basis. In such a presentation, relations are not considered as equalities but as directed computations and convergence is a property that ensures good computational properties:

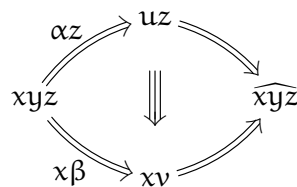
- termination: starting from a given word, there is no way to apply directed relations indefinitely,
- confluence: choices between different directed relations that apply to a given word do not matter.

Convergent presentations are really similar to Gröbner bases. Indeed, one important consequence of having a convergent presentation $\Sigma = (\Sigma_1, \Sigma_2)$ of a monoid \mathbf{M} is that every element u of \mathbf{M} has a canonical representative in Σ_1^* : the only word \hat{u} of Σ_1^* in which every representative of u can be reduced using the directed relations and that cannot be further reduced.

Moreover, if Σ is a convergent presentation of \mathbf{M} , then Squier’s theorem tells us that its *critical branchings* generate a homotopy basis. Such a branching is given by two possible overlapping applications of directed relations on the same word u such that u has minimal size. For example, two relations $\alpha : xy \Rightarrow u$ and $\beta : yz \Rightarrow v$ generate the critical branching



Convergence of the presentation ensures that one can further reduce uz and xv into the same element \widehat{xyz} : one says that the branching is confluent. Squier’s theorem states that, if one fills the following diagram with a 3-cell and repeats the same process with every other critical branching, one gets a homotopy basis:



The problem is that general presentations of monoids have no reason to be convergent. But rewriting theory gives a procedure that can, in many cases, complete a presentation into a convergent presentation of the same monoid, due to Knuth and Bendix [14]. The idea is to consider the critical branchings of a presentation and to formally add, for the ones that are not confluent, new relations that make them confluent; then, one starts again with the new critical branchings generated by the new relations, until a stable state is reached. If this happens, one gets a convergent presentation of the same monoid.

Here, we extend this computational tool into a *homotopical completion* procedure that, given a presentation of a monoid and when successful, generates a coherent presentation of the same monoid. However, the result is generally bigger than one could expect: more relations than the original presentation and a homotopy basis that may contain superfluous elements. To correct this, we introduce a *homotopical reduction* procedure. Essentially, it uses the “critical triple branchings” to compute relations between the elements of the homotopy basis and, potentially, eliminates some of them.

At the end, we get the *reduced homotopical completion* $\text{RHC}(\Sigma)$ of a presentation Σ , whose properties are given in the main result of the third section:

Theorem 3.6.1. *Let Σ be a terminating presentation of a monoid \mathbf{M} . When it exists, the reduced homotopical completion $\text{RHC}(\Sigma)$ of Σ is a coherent presentation of \mathbf{M} whose underlying presentation is Σ .*

Actions of Artin-Tits groups

In the case of an Artin-Tits group $\mathbf{B}(\mathbf{W}, S)$ or monoid $\mathbf{B}^+(\mathbf{W}, S)$, where \mathbf{W} is a Coxeter group and S a set of generators of \mathbf{W} , the classical Artin’s presentation is given by the elements of S as generators and the braid relations

$$st \cdots = ts \cdots$$

1. Actions of monoids on categories

Tits, in [25], and Ronan, in [21], prove that the fundamental group of a complex associated to the group $\mathbf{B}(\mathbf{W}, S)$ is generated by two families of loops. Once reinterpreted in the polygraphic language, the first family is made of trivial loops, while the second one contains what we call *Tits-Zamolodchikov relations*.

There is one such relation $Z_{s,t,u}$ for each triple (s, t, u) of elements of S whose generated subgroup $\mathbf{W}_{\{s,t,u\}}$ of \mathbf{W} is a finite Coxeter group of rank 3. The shape of the cell $Z_{s,t,u}$ is entirely determined by the type of the Coxeter group $\mathbf{W}_{\{s,t,u\}}$: for example, we get the Yang-Baxter relation for the type $A_1 \times A_1 \times A_1$ and the Zamolodchikov equation for A_3 , *i.e.*, for the type of the group of symmetries S_4 . This is summarised in the following result of the fourth section:

Theorem 4.2.3. *Artin’s presentation and the Tits-Zamolodchikov relations form a finite coherent presentation of the monoid $\mathbf{B}^+(\mathbf{W}, S)$ of positive braids.*

In the spherical case, *i.e.*, when \mathbf{W} is finite, we can also consider Deligne’s presentation of $\mathbf{B}^+(\mathbf{W}, S)$. His result on the actions of $\mathbf{B}^+(\mathbf{W}, S)$ on categories, in terms of this special presentation, leads us to believe that the corresponding set of coherence relations comes from a homotopy basis.

This is indeed the case and we end the fourth section with an algorithmic proof of this fact. Deligne’s presentation is not convergent, so that we cannot apply Squier’s theorem, but it is terminating. Hence we can compute the reduced homotopical completion of Deligne’s presentation, which gives exactly the piece of information that corresponds to the coherence relations of Deligne’s result:

Theorem 4.3.6. *The monoid $\mathbf{B}^+(\mathbf{W}, S)$ admits a coherent presentation with a 1-cell for every element of $\mathbf{W} \setminus \{1\}$, a 2-cell*

$$\alpha_{u,v} : u|v \Rightarrow uv,$$

for every u, v in $\mathbf{W} \setminus \{1\}$ with $l(uv) = l(u) + l(v)$ and a 3-cell

$$\begin{array}{ccc} & \alpha_{u,v|w} & \\ & \curvearrowright & \\ u|v|w & & uv|w \\ & \Downarrow A_{u,v,w} & \\ & \curvearrowleft & \\ u|v & & uvw \\ & \alpha_{u,v,w} & \\ & \curvearrowright & \\ & & u|vw \\ & \alpha_{u,vw} & \end{array}$$

for every u, v, w in $\mathbf{W} \setminus \{1\}$ with $l(uvw) = l(u) + l(v) + l(w)$.

1. ACTIONS OF MONOIDS ON CATEGORIES

We present Deligne’s actions of a monoid \mathbf{M} on a category \mathbf{C} as a special case of 2-representations, a notion introduced by Elgueta, [8]. More precisely, an action of \mathbf{M} on \mathbf{C} is exactly a 2-representation (*i.e.*, a pseudofunctor) from \mathbf{M} , seen as a 2-category with one 0-cell and identity 2-cells only, into \mathbf{Cat} , sending the 0-cell to \mathbf{C} . Here and in the following, \mathbf{Cat} is the 2-category of categories, functors and natural transformations (note that, for foundational correctness, we should say the *large* 2-category of *small* categories, where sizes refer to some fixed universe, and similarly later for the 2-category of 2-categories).

In this section, we prove that, up to equivalence, an action of a monoid \mathbf{M} on a category is the same as a 2-functor from a 2-category $\tilde{\mathbf{M}}$ to \mathbf{Cat} , where $\tilde{\mathbf{M}}$ is any cofibrant approximation of \mathbf{M} : informally, this is a 2-category that is as free as possible while being “homotopically equivalent” to \mathbf{M} .

1.1. 2-representations of 2-categories

1.1.1. 2-representations. We recall from [8] that, given 2-categories \mathcal{C} and \mathcal{D} , a *2-representation of \mathcal{C} in \mathcal{D}* is a pseudofunctor $F : \mathcal{C} \rightarrow \mathcal{D}$, which is a suitably weakened notion of 2-functor. More precisely, such a pseudofunctor is specified by the following data:

- for every 0-cell x of \mathcal{C} , a 0-cell $F(x)$ of \mathcal{D} ,
- for every 1-cell $u : x \rightarrow y$ of \mathcal{C} , a 1-cell $F(u) : F(x) \rightarrow F(y)$ of \mathcal{D} ,
- for every 2-cell $f : u \Rightarrow v$ of \mathcal{C} , a 2-cell $F(f) : F(u) \Rightarrow F(v)$ of \mathcal{D} .

As for 2-functors, the data are required to be compatible with vertical composition, in a strict way:

- for every 2-cells $f : u \Rightarrow v : x \rightarrow y$ and $g : v \Rightarrow w : x \rightarrow y$ of \mathcal{C} ,

$$F \left(\begin{array}{ccc} & u & \\ & \Downarrow f & \\ x & \xrightarrow{v} & y \\ & \Downarrow g & \\ & w & \end{array} \right) = \begin{array}{ccc} & F(u) & \\ & \Downarrow F(f) & \\ F(x) & \xrightarrow{F(v)} & F(y) \\ & \Downarrow F(g) & \\ & F(w) & \end{array}$$

- for every 1-cell u of \mathcal{C} , we have $F(1_u) = 1_{F(u)}$.

The data is compatible to horizontal composition, but only up to coherent isomorphisms, which is formalised by the following extra data:

- for every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathcal{C} , an invertible 2-cell of \mathcal{D} , natural in u and v ,

$$\begin{array}{ccccc} & & F(y) & & \\ & F(u) \nearrow & & \searrow F(v) & \\ F(x) & & \Downarrow \cong F_{u,v} & & F(z) \\ & \xrightarrow{F(uv)} & & & \end{array}$$

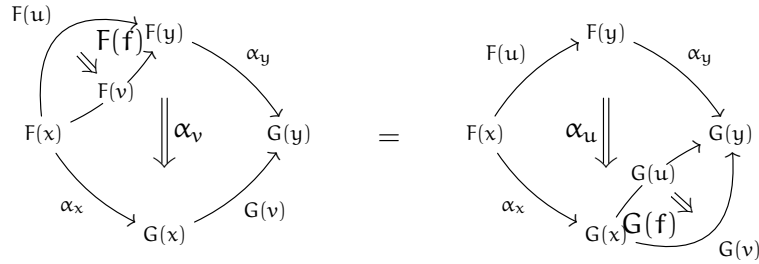
- for every 0-cell x of \mathcal{C} , an invertible 2-cell of \mathcal{D}

$$\begin{array}{ccc} & 1_{F(x)} & \\ & \curvearrowright & \\ F(x) & & F(x) \\ & \Downarrow \cong F_x & \\ & \curvearrowleft & \\ & F(1_x) & \end{array}$$

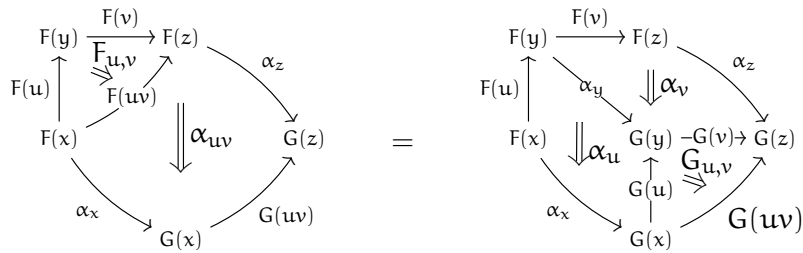
Finally, these 2-cells are required to satisfy the following monoidal coherence relations in \mathcal{D} :

This data must satisfy several coherence relations:

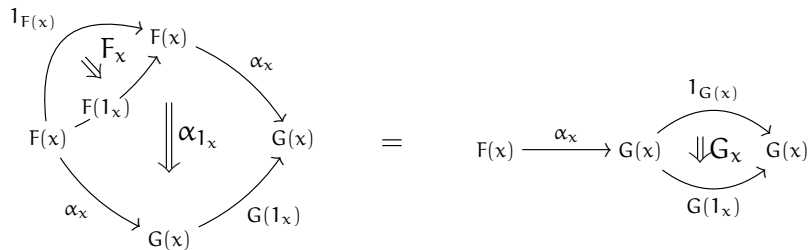
- for every 2-cell $f : u \Rightarrow v : x \rightarrow y$ of \mathcal{C} ,



- for every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ in \mathcal{C} ,



- for every 0-cell x in \mathcal{C} ,



1.1.3. Categories of 2-representations. If $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ are 2-representations and if $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are morphisms of 2-representations, their composition is the morphism $\alpha \star \beta : F \Rightarrow H$ defined by:

- if x is a 0-cell of \mathcal{C} , the 1-cell $(\alpha \star \beta)_x : F(x) \rightarrow H(x)$ of \mathcal{D} is defined as the composite

$$F(x) \xrightarrow{\alpha_x} G(x) \xrightarrow{\beta_x} H(x)$$

1. Actions of monoids on categories

- if $u : x \rightarrow y$ is a 1-cell of \mathcal{C} , the invertible 2-cell $(\alpha \star \beta)_u$ of \mathcal{D} is defined by

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & F(y) & \\
 F(u) \nearrow & & \searrow (\alpha \star \beta)_y \\
 F(x) & & H(y) \\
 (\alpha \star \beta)_x \searrow & & \nearrow H(u) \\
 & H(x) &
 \end{array} & = &
 \begin{array}{ccc}
 & F(y) & \\
 F(u) \nearrow & & \searrow \alpha_y \\
 F(x) & & G(y) \\
 \alpha_x \searrow & & \nearrow \beta_y \\
 & G(x) & \\
 & \nearrow G(u) & \searrow \beta_u \\
 & & H(y) \\
 & \searrow \beta_x & \nearrow H(u) \\
 & & H(x)
 \end{array}
 \end{array}$$

One checks that $\alpha \star \beta$ satisfies the coherence conditions that makes it a morphism of 2-representations and, then, that the composition \star is associative and unitary.

This gives rise to a category of 2-representations of \mathcal{C} into \mathcal{D} , which is denoted by $2\text{Rep}(\mathcal{C}, \mathcal{D})$. The full subcategory of $2\text{Rep}(\mathcal{C}, \mathcal{D})$ whose objects are the 2-functors is denoted by $2\text{Cat}(\mathcal{C}, \mathcal{D})$. Let us note that $2\text{Rep}(\mathcal{C}, \mathcal{D})$ is a hom-category in the bicategory of 2-categories, pseudofunctors and pseudonatural transformations, while $2\text{Cat}(\mathcal{C}, \mathcal{D})$ is a hom-category in the bicategory of 2-categories, 2-functors and pseudonatural transformations.

1.2. Actions of monoids on categories

If \mathbf{M} is a monoid, we can see it as a 2-category with exactly one 0-cell and only identity 2-cells, hence as a discrete 2-category. In particular, we introduce the category of *actions of \mathbf{M} on categories* as the category of 2-representations of \mathbf{M} in Cat :

$$\text{Act}(\mathbf{M}) = 2\text{Rep}(\mathbf{M}, \text{Cat}).$$

1.2.1. Explicit definition of actions. Expanding the definition, we get that an action T of \mathbf{M} is specified by the following data:

- a category \mathbf{C} , which is the image through T of the unique 0-cell of \mathbf{M} ,
- for every element u of \mathbf{M} , an endofunctor $T(u) : \mathbf{C} \rightarrow \mathbf{C}$,
- for every pair (u, v) of elements of \mathbf{M} , a natural isomorphism

$$\begin{array}{ccc}
 & \mathbf{C} & \\
 T(u) \nearrow & & \searrow T(v) \\
 \mathbf{C} & & \mathbf{C} \\
 & \simeq \Downarrow T_{u,v} & \\
 & \mathbf{C} & \\
 & \xrightarrow{T(uv)} &
 \end{array}$$

- a natural isomorphism

$$\begin{array}{ccc}
 & \text{Id}_{\mathbf{C}} & \\
 & \curvearrowright & \\
 \mathbf{C} & \simeq \Downarrow \mathbf{T}_{\bullet} & \mathbf{C} \\
 & \curvearrowleft & \\
 & \text{T}(1) &
 \end{array}$$

This data is required to satisfy the following coherence conditions:

- for every triple (u, v, w) of elements of \mathbf{M} , the following diagram commutes:

$$\begin{array}{ccccc}
 & & \text{T}(uv)\text{T}(w) & & \\
 & \nearrow \text{T}_{u,v}\text{T}(w) & & \searrow \text{T}_{uv,w} & \\
 \text{T}(u)\text{T}(v)\text{T}(w) & & \text{T}(u)\text{T}(vw) & & \text{T}(uvw) \\
 & \searrow \text{T}(u)\text{T}_{v,w} & & \nearrow \text{T}_{u,vw} & \\
 & & \text{T}(u)\text{T}(vw) & &
 \end{array}
 \quad \textcircled{\text{C}}$$

- for every element u of \mathbf{M} , the following two diagrams commute:

$$\begin{array}{ccc}
 \text{T}_{\bullet}\text{T}(u) \nearrow & \text{T}(1)\text{T}(u) & \searrow \text{T}_{1,u} \\
 \text{T}(u) & \text{T}(u) & \\
 \text{Id}_{\text{T}(u)} \searrow & & \nearrow
 \end{array}
 \quad \textcircled{\text{C}}
 \qquad
 \begin{array}{ccc}
 \text{T}(u)\text{T}_{\bullet} \nearrow & \text{T}(u)\text{T}(1) & \searrow \text{T}_{u,1} \\
 \text{T}(u) & \text{T}(u) & \\
 \text{Id}_{\text{T}(u)} \searrow & & \nearrow
 \end{array}
 \quad \textcircled{\text{C}}$$

This corresponds to the notion of *unital action of \mathbf{M} on \mathbf{C}* that Deligne considers in [7]. He proves that unital actions are equivalent to non-unital actions, his main object of study, and we recover this fact later.

1.2.2. Explicit definition of morphisms of actions. If S is an action of \mathbf{M} on a category \mathbf{C} and T is an action of \mathbf{M} on a category \mathbf{D} , a morphism of actions α from S to T is specified as follows:

- a functor $F : \mathbf{C} \rightarrow \mathbf{D}$, corresponding to the component of α at the 0-cell of \mathbf{M} ,
- for every element u of \mathbf{M} , a natural isomorphism α_u

$$\begin{array}{ccc}
 S(u) \rightarrow & \mathbf{C} & \xrightarrow{F} \\
 \mathbf{C} & \simeq \Downarrow \alpha_u & \mathbf{D} \\
 \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\
 & & \text{T}(u)
 \end{array}$$

This data must satisfy the coherence conditions from the definition of pseudonatural transformation.

1. Actions of monoids on categories

1.2.3. A note on Deligne’s morphisms. The notion of morphisms we consider differs from the one of Deligne in [7]. Indeed, he only considers morphisms between actions of \mathbf{M} on the same category \mathbf{C} , such that the functor F is the identity of \mathbf{C} , but where the natural transformation α_u is not necessarily an isomorphism. This kind of morphisms are exactly *icons* between pseudofunctors, as introduced by Lack in [17] as a special case of *oplax* natural transformations.

We choose a different notion of morphisms of actions for several reasons. Firstly, because it is a special case of morphism of 2-representations, so that actions of monoids fit nicely in Elgueta’s setting, being 2-representations of monoids into \mathbf{Cat} . Secondly, because it generalises usual notions of morphisms of actions of a monoid (or a group) on objects of a category, such as sets, vector spaces, Hilbert spaces. Indeed, with Elgueta’s morphisms, if we replace \mathbf{Cat} by \mathbf{Set} or \mathbf{Vect} , considered as 2-categories with identity 2-cells only, then all the 2-cells α_u must be equalities, so that each diagram gives a relation $F(S(u)) = T(F(u))$, which is the usual definition of a morphism from S to T . In Deligne’s case, since F is the identity, this gives $S(u) = T(u)$ instead, so that the only morphism from S to T is the identity, when $S = T$.

However, our results can be adapted to Deligne’s morphisms, *i.e.*, to the case where $2\mathbf{Rep}(\mathcal{C}, \mathcal{D})$ and $2\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ have icons and not pseudonatural transformations as morphisms. Also, one could get similar results with extensions of $2\mathbf{Rep}(\mathcal{C}, \mathcal{D})$ and $2\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ with all the oplax natural transformations as morphisms: this has the same definition as a pseudonatural natural transformation, except that the component 2-cells α_u are not required to be invertible.

1.3. Cofibrant approximations of 2-categories

In this section, we consider the category of 2-categories and 2-functors, which we denote by $2\mathbf{Cat}$.

1.3.1. Elements of the model category structure on $2\mathbf{Cat}$. We recall a few notions from the model category structure on $2\mathbf{Cat}$ introduced by Lack in [15] and [16]. A 2-category is *cofibrant* when its underlying 1-category is free. A 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *weak equivalence* when it is an equivalence in the bicategory $2\mathbf{Rep}$, meaning that there exists a pseudofunctor $G : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms

$$GF \simeq 1_{\mathcal{C}} \quad \text{and} \quad FG \simeq 1_{\mathcal{D}}.$$

In that case, we say that \mathcal{C} and \mathcal{D} are *weakly equivalent* and that G is a *quasi-inverse for* F . In general, this quasi-inverse cannot be chosen to be strict, *i.e.*, to be a 2-functor. However, when G is a 2-functor, we say that F is an *equivalence* and that \mathcal{C} and \mathcal{D} are *equivalent*.

If \mathcal{C} is a 2-category, a *cofibrant approximation of* \mathcal{C} is a cofibrant 2-category $\tilde{\mathcal{C}}$ such that there exists a weak equivalence $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$.

1.3.2. Example: the standard cofibrant approximation [15]. Let \mathcal{C} be a 2-category. We denote by $\hat{\mathcal{C}}$ the cofibrant 2-category with the same 0-cells as \mathcal{C} and the following higher cells:

- the 1-cells of $\hat{\mathcal{C}}$ are freely generated by the ones of \mathcal{C} , with u in \mathcal{C} denoted by \hat{u} when seen as a generator of $\hat{\mathcal{C}}$,
- the 2-cells from $\hat{u}_1 \cdots \hat{u}_m$ to $\hat{v}_1 \cdots \hat{v}_n$ in $\hat{\mathcal{C}}$ are the 2-cells from $u_1 \cdots u_m$ to $v_1 \cdots v_n$ in \mathcal{C} , with the same compositions as in \mathcal{C} .

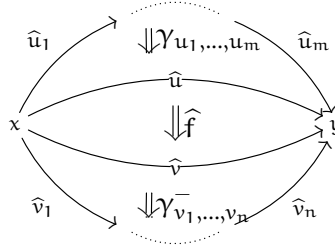
1.3. Cofibrant approximations of 2-categories

By definition, every 2-cell $f : u \Rightarrow v$ of \mathcal{C} has several lifts in $\widehat{\mathcal{C}}$. We denote by \widehat{f} the one with source \widehat{u} and target \widehat{v} . For each pair of composable 1-cells (u, v) we denote by $\gamma_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{uv}$ the 2-cell lifting the identity of the 1-cell uv in \mathcal{C} . This 2-cell is invertible and satisfies monoidal coherence relations, so that there exists exactly one invertible 2-cell

$$\gamma_{u_1, \dots, u_n} : \widehat{u}_1 \cdots \widehat{u}_n \Longrightarrow \widehat{u_1 \cdots u_n}$$

for every family (u_1, \dots, u_n) of composable 1-cells.

Let us consider 1-cells $u, v : x \rightarrow y$ in \mathcal{C} such that $u = u_1 \dots u_m$ and $v = v_1 \dots v_n$ hold. If $f : u \Rightarrow v$ is a 2-cell of \mathcal{C} , then it has exactly one copy in $\widehat{\mathcal{C}}$ that goes from $\widehat{u}_1 \cdots \widehat{u}_m$ to $\widehat{v}_1 \cdots \widehat{v}_n$, which is equal, by definition of the composition in $\widehat{\mathcal{C}}$, to the following composite



where $\gamma_{v_1, \dots, v_n}^{-1}$ denotes the inverse of the 2-cell γ_{v_1, \dots, v_n} . As a consequence, if $\mathcal{C} = \mathbf{C}$ is a category (a monoid, for example), seen as a discrete 2-category (*i.e.*, a 2-category with identity 2-cells only), the 2-category $\widehat{\mathbf{C}}$ has exactly one 2-cell from $\widehat{u}_1 \cdots \widehat{u}_m$ to $\widehat{v}_1 \cdots \widehat{v}_n$ if, and only if, the relation

$$u_1 \dots u_m = v_1 \dots v_n$$

holds in \mathbf{C} : this 2-cell is the composite of γ_{u_1, \dots, u_m} followed by $\gamma_{v_1, \dots, v_n}^{-1}$.

Now, let $\pi : \widehat{\mathcal{C}} \rightarrow \mathcal{C}$ be the canonical projection 2-functor, sending each generating 1-cell \widehat{u} to u and each 2-cell to itself. Let $\iota : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be the pseudofunctor sending each 1-cell u to \widehat{u} and each 2-cell $f : u \Rightarrow v$ to $\widehat{f} : \widehat{u} \Rightarrow \widehat{v}$. One can check that π is a weak equivalence with quasi-inverse ι . Hence, the 2-category $\widehat{\mathcal{C}}$ is a cofibrant approximation of \mathcal{C} , called the *standard cofibrant approximation of \mathcal{C}* .

Actually, as noticed by Lack in [15], Proposition 4.2 and its proof, the assignment $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is a left adjoint to the inclusion functor $2\text{Cat} \hookrightarrow 2\text{Rep}$. On morphisms this adjunction says the following:

1.3.3. Proposition. *For every 2-categories \mathcal{C} and \mathcal{D} , we have the following isomorphism of categories:*

$$2\text{Rep}(\mathcal{C}, \mathcal{D}) \simeq 2\text{Cat}(\widehat{\mathcal{C}}, \mathcal{D}).$$

In particular, for every monoid \mathbf{M} , we have the following isomorphism of categories:

$$\text{Act}(\mathbf{M}) \simeq 2\text{Cat}(\widehat{\mathbf{M}}, \text{Cat}).$$

In what follows, we prove that weak versions of these isomorphisms exist for all cofibrant approximations. More precisely, the category of 2-representations of a 2-category \mathcal{C} into a 2-category \mathcal{D} is equivalent to the one of 2-functors from any cofibrant approximation $\widehat{\mathcal{C}}$ of \mathcal{C} into \mathcal{D} . This result specialises to the actions of a monoid on categories.

1. Actions of monoids on categories

1.4. 2-representations of cofibrant 2-categories

Let us fix 2-categories \mathcal{C} and \mathcal{D} , with \mathcal{C} cofibrant. Our objective is to define a “strictification” functor

$$\widehat{} : 2\text{Rep}(\mathcal{C}, \mathcal{D}) \longrightarrow 2\text{Cat}(\mathcal{C}, \mathcal{D})$$

and to prove that it is a quasi-inverse for the canonical inclusion functor of $2\text{Cat}(\mathcal{C}, \mathcal{D})$ into $2\text{Rep}(\mathcal{C}, \mathcal{D})$.

1.4.1. Strictification of 2-representations. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a 2-representation. Let us define the 2-functor $\widehat{F} : \mathcal{C} \rightarrow \mathcal{D}$, dimension after dimension.

On 0-cells, \widehat{F} takes the same values as F . Since \mathcal{C} is cofibrant, its underlying 1-category is free: on generating 1-cells, \widehat{F} is equal to F and, then, it is extended by functoriality on every 1-cell. Hence, if $u = a_1 \dots a_n$ is a 1-cell of \mathcal{C} , where the a_i s are generating 1-cells, we have:

$$\widehat{F}(u) = F(a_1) \cdots F(a_n).$$

As noted earlier, from the monoidal coherence relations satisfied by F , there is a unique invertible 2-cell in \mathcal{D}

$$\widehat{F}(u) = F(a_1) \cdots F(a_n) \xrightarrow{F_{a_1, \dots, a_n}} F(a_1 \cdots a_n) = F(u)$$

from $\widehat{F}(u)$ to $F(u)$, built from the coherence 2-cells of F . Since the decomposition of u in generators is unique, we simply denote this 2-cell by F_u .

Let $f : u \Rightarrow v : x \rightarrow y$ be a 2-cell of \mathcal{C} . We define $\widehat{F}(f)$ as the following composite 2-cell of \mathcal{D} , where the double arrows have been omitted for clarity:

$$\begin{array}{ccc} \widehat{F}(u) & & \widehat{F}(u) \\ \curvearrowright & & \curvearrowright \\ & \widehat{F}(f) & \\ \curvearrowleft & & \curvearrowleft \\ \widehat{F}(v) & & \widehat{F}(v) \end{array} = \begin{array}{ccc} & & \widehat{F}(u) \\ & & \curvearrowright \\ & F_u & \\ & \curvearrowright & \\ F(x) & \xrightarrow{F(u)} & F(y) \\ & \curvearrowleft & \\ & F(v) & \\ & \curvearrowleft & \\ & & \widehat{F}(v) \end{array}$$

As a direct consequence, we get that \widehat{F} is compatible with vertical composition and identities of 1-cells. Hence, we have defined a 2-functor \widehat{F} from \mathcal{C} to \mathcal{D} .

Before moving on to morphisms of 2-representations, we note that the monoidal coherence relations satisfied by F imply that the 2-cells $F_u : \widehat{F}(u) \Rightarrow F(u)$ satisfy the following relations with respect to composition and identities. If $u : x \rightarrow y$ and $v : y \rightarrow z$ are 1-cells of \mathcal{C} , we have:

$$\begin{array}{ccc} \widehat{F}(uv) & & \widehat{F}(u) \quad \widehat{F}(v) \\ \curvearrowright & & \curvearrowright \quad \curvearrowright \\ & F_{uv} & \\ \curvearrowleft & & \curvearrowleft \quad \curvearrowleft \\ \widehat{F}(uv) & & \widehat{F}(uv) \end{array} = \begin{array}{ccc} & & \widehat{F}(u) \quad \widehat{F}(v) \\ & & \curvearrowright \quad \curvearrowright \\ & & F_u \quad F_v \\ & & \curvearrowright \quad \curvearrowright \\ F(x) & \xrightarrow{F(u)} & F(y) \xrightarrow{F(v)} & F(z) \\ & & \curvearrowleft & \\ & & F_{u,v} & \\ & & \curvearrowleft & \\ & & \widehat{F}(uv) & \end{array}$$

1.4. 2-representations of cofibrant 2-categories

Moreover, if x is a 0-cell of \mathcal{C} , we have $F_{1_x} = F_x$.

1.4.2. Strictification of morphisms of 2-representations. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be 2-representations and let $\alpha : F \Rightarrow G$ be a morphism between them. Let us define a pseudonatural transformation $\hat{\alpha} : \hat{F} \Rightarrow \hat{G}$.

On a 0-cell x of \mathcal{C} , we take $\hat{\alpha}_x = \alpha_x$. If $u : x \rightarrow y$ is a 1-cell of \mathcal{C} , we define $\hat{\alpha}_u$ as the following invertible 2-cell of \mathcal{D} :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\
 \uparrow & & \searrow \alpha_y \\
 F(x) & & G(y) \\
 \downarrow \alpha_x & & \uparrow \widehat{G}(u) \\
 G(x) & &
 \end{array} & \hat{\alpha}_u & \\
 & & \\
 \begin{array}{ccc}
 \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\
 \uparrow F_u & & \searrow \alpha_y \\
 F(x) & \xrightarrow{F(u)} & G(y) \\
 \downarrow \alpha_x & & \uparrow G(u) \\
 G(x) & \xrightarrow{G(u)} & G(y) \\
 & & \downarrow G_u^- \\
 & & \widehat{G}(u)
 \end{array}
 \end{array}$$

Let us check that $\hat{\alpha}$ satisfies the relations that makes it a pseudonatural transformation from \hat{F} to \hat{G} . First, let us consider a 2-cell $f : u \Rightarrow v : x \rightarrow y$ in \mathcal{C} and let us compute in \mathcal{D} :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\
 \uparrow \widehat{F}(f) & & \searrow \alpha_y \\
 F(x) & \xrightarrow{\widehat{F}(v)} & G(y) \\
 \downarrow \alpha_x & & \uparrow \widehat{G}(v) \\
 G(x) & \xrightarrow{\widehat{G}(v)} & G(y)
 \end{array} & \hat{\alpha}_v & \\
 & & \\
 \begin{array}{ccc}
 \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\
 \uparrow F_u & & \searrow \alpha_y \\
 F(x) & \xrightarrow{F(v)} & G(y) \\
 \downarrow \alpha_x & & \uparrow G(v) \\
 G(x) & \xrightarrow{G(v)} & G(y) \\
 & & \downarrow G_v^- \\
 & & \widehat{G}(v)
 \end{array}
 \end{array}
 =
 \begin{array}{ccc}
 \begin{array}{ccc}
 \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\
 \uparrow F_u & & \searrow \alpha_y \\
 F(x) & \xrightarrow{F(u)} & G(y) \\
 \downarrow \alpha_x & & \uparrow G(u) \\
 G(x) & \xrightarrow{G(u)} & G(y) \\
 & & \downarrow G_u^- \\
 & & \widehat{G}(v)
 \end{array}
 \end{array}
 =
 \begin{array}{ccc}
 \begin{array}{ccc}
 \widehat{F}(u) & \xrightarrow{\quad} & F(y) \\
 \uparrow & & \searrow \alpha_y \\
 F(x) & \xrightarrow{\widehat{G}(u)} & G(y) \\
 \downarrow \alpha_x & & \uparrow \widehat{G}(f) \\
 G(x) & \xrightarrow{\widehat{G}(f)} & G(y)
 \end{array}
 \end{array}$$

1. Actions of monoids on categories

Then, if $u : x \rightarrow y$ and $v : y \rightarrow z$ are 1-cells in \mathcal{C} , we get:

$$\begin{array}{c}
 \begin{array}{ccc}
 \widehat{F}(uv) & \xrightarrow{\quad} & F(z) \\
 \uparrow & & \downarrow \alpha_z \\
 F(x) & \xrightarrow{\quad} & G(z) \\
 \downarrow \alpha_x & & \uparrow \widehat{G}(uv) \\
 & \widehat{\alpha}_{uv} &
 \end{array} \\
 = \\
 \begin{array}{ccc}
 & \widehat{F}(v) & \\
 & \downarrow F_v & \\
 \widehat{F}(u) & \xrightarrow{F_u} F(y) \xrightarrow{-F(v)} F(z) & \xrightarrow{\alpha_z} G(z) \\
 \uparrow & \uparrow F_{u,v} & \uparrow \alpha_{uv} \\
 F(x) & \xrightarrow{F(u)} & G(x) \xrightarrow{-G(u)} G(y) \xrightarrow{\widehat{G}(v)} G(z) \\
 \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\
 & & G(x) \xrightarrow{G_u^-} G(y)
 \end{array} \\
 = \\
 \begin{array}{ccc}
 & \widehat{F}(v) & \\
 & \downarrow F_v & \\
 \widehat{F}(u) & \xrightarrow{F_u} F(y) \xrightarrow{-F(v)} F(z) & \xrightarrow{\alpha_z} G(z) \\
 \uparrow & \uparrow F_{u,v} & \uparrow \alpha_v \\
 F(x) & \xrightarrow{F(u)} & G(x) \xrightarrow{-G(u)} G(y) \xrightarrow{\widehat{G}(v)} G(z) \\
 \downarrow \alpha_x & & \downarrow \widehat{G}(u) \\
 & & G(x) \xrightarrow{G_u^-} G(y)
 \end{array} \\
 = \\
 \begin{array}{ccc}
 F(y) & \xrightarrow{\widehat{F}(v)} & F(z) \\
 \uparrow \widehat{F}(u) & & \downarrow \alpha_z \\
 F(x) & \xrightarrow{F(u)} & G(x) \xrightarrow{\widehat{G}(u)} G(y) \xrightarrow{\widehat{G}(v)} G(z) \\
 \downarrow \alpha_x & & \downarrow \widehat{\alpha}_u \\
 & & G(x) \xrightarrow{\alpha_y} G(y)
 \end{array}
 \end{array}$$

Finally, for x a 0-cell of \mathcal{C} , we have:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 1_{F(x)} & \xrightarrow{\quad} & F(x) \\
 \uparrow & & \downarrow \alpha_x \\
 F(x) & \xrightarrow{\quad} & G(x) \\
 \downarrow \alpha_x & & \uparrow 1_{G(x)} \\
 & \widehat{\alpha}_{1_x} &
 \end{array} \\
 = \\
 \begin{array}{ccc}
 1_{F(x)} & \xrightarrow{F_x} & F(x) \\
 \uparrow & \uparrow F_{1_x} & \downarrow \alpha_x \\
 F(x) & \xrightarrow{\quad} & G(x) \\
 \downarrow \alpha_x & & \downarrow 1_{G(x)} \\
 & & G(x) \xrightarrow{G_{1_x}^-} G(x)
 \end{array} \\
 = 1_{\alpha_x}
 \end{array}$$

1.5. 2-representations and cofibrant approximations

With similar computations, we check that strictification is compatible with the composition of morphisms of 2-representations and with identities, so that it is a functor from $2\mathbf{Rep}(\mathcal{C}, \mathcal{D})$ to $2\mathbf{Cat}(\mathcal{C}, \mathcal{D})$.

1.4.3. Proposition. *Let \mathcal{C} be a cofibrant 2-category. For every 2-category \mathcal{D} , the canonical inclusion functor*

$$2\mathbf{Cat}(\mathcal{C}, \mathcal{D}) \longrightarrow 2\mathbf{Rep}(\mathcal{C}, \mathcal{D})$$

is an equivalence of categories, with quasi-inverse given by the strictification functor.

Proof. It is sufficient to check that, for every 2-representation $F : \mathcal{C} \rightarrow \mathcal{D}$, there exists a pseudonatural isomorphism $\varphi_F : \widehat{F} \Rightarrow F$ that is itself natural in F . We define φ_F as follows:

- if x is a 0-cell of \mathcal{C} , then $\widehat{F}(x) = F(x)$ and we take $(\varphi_F)_x = 1_x$,
- if $u : x \rightarrow y$ is a 1-cell of \mathcal{C} , then $(\varphi_F)_u : \widehat{F}(u) \Rightarrow F(u)$ is the invertible coherence 2-cell $F_u : \widehat{F}(u) \Rightarrow F(u)$.

This data satisfies the required coherence properties:

- the compatibility with 2-cells of \mathcal{C} is exactly the definition of \widehat{F} ,
- the compatibility with composition and identities comes from the monoidal coherence relations of F , as already checked.

Now, if $\alpha : F \Rightarrow G$ is a morphism of 2-representations, the naturality condition

$$\begin{array}{ccccc}
 & & \widehat{F} & \xrightarrow{\varphi_F} & F & \xrightarrow{\alpha} & G \\
 & & \searrow & & \searrow & & \searrow \\
 & & & \widehat{\alpha} & & & \\
 & & \widehat{G} & \xrightarrow{\varphi_G} & G & & \\
 & & \swarrow & & \swarrow & & \swarrow \\
 & & & & & &
 \end{array}$$

corresponds, on each 1-cell u of \mathcal{C} , to the definition of $\widehat{\alpha}$. □

1.5. 2-representations and cofibrant approximations

1.5.1. Lemma. *Let \mathcal{C} and \mathcal{D} be two weakly equivalent 2-categories. Then, for every 2-category \mathcal{E} , we have an equivalence of categories*

$$2\mathbf{Rep}(\mathcal{C}, \mathcal{E}) \simeq 2\mathbf{Rep}(\mathcal{D}, \mathcal{E}).$$

Proof. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a weak equivalence, with quasi-inverse $G : \mathcal{D} \rightarrow \mathcal{C}$. We have the following isomorphisms:

$$GF \simeq \text{Id}_{\mathcal{C}} \quad \text{and} \quad FG \simeq \text{Id}_{\mathcal{D}}.$$

As a consequence, for every 2-representations $H : \mathcal{C} \rightarrow \mathcal{E}$ and $K : \mathcal{D} \rightarrow \mathcal{E}$, we have:

$$HGF \simeq H \quad \text{and} \quad KFG \simeq K.$$

This proves that the functors $2\mathbf{Rep}(F, \mathcal{E})$ and $2\mathbf{Rep}(G, \mathcal{E})$, respectively sending the pseudofunctor $K : \mathcal{D} \rightarrow \mathcal{E}$ to $KF : \mathcal{C} \rightarrow \mathcal{E}$ and the pseudofunctor $H : \mathcal{C} \rightarrow \mathcal{E}$ to $HG : \mathcal{D} \rightarrow \mathcal{E}$ form the required equivalence of categories. □

2. Actions of monoids and coherent presentations

Combining Proposition 1.4.3 and Lemma 1.5.1, we get the main result of this section.

1.5.2. Theorem. *Let \mathcal{C} be a 2-category and let $\tilde{\mathcal{C}}$ be a cofibrant approximation of \mathcal{C} . Then, for every 2-category \mathcal{D} , we an following equivalence of categories*

$$2\text{Rep}(\mathcal{C}, \mathcal{D}) \approx 2\text{Cat}(\tilde{\mathcal{C}}, \mathcal{D}).$$

In particular, if \mathbf{M} is a monoid and $\tilde{\mathbf{M}}$ is a cofibrant approximation of \mathbf{M} , then we have an equivalence of categories

$$\text{Act}(\mathbf{M}) \approx 2\text{Cat}(\tilde{\mathbf{M}}, \text{Cat}).$$

Remark. This result can also be proved by using the standard cofibrant approximation. We have seen that the above equivalences are isomorphisms in the case of the standard cofibrant approximation $\widehat{\mathcal{C}}$ of \mathcal{C} . By definition, if $\tilde{\mathcal{C}}$ is a cofibrant approximation of a 2-category \mathcal{C} , there exists a weak equivalence $F : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$, with quasi-inverse $H : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$. If we denote by $G : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ the canonical projection 2-functor and $K : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ its quasi-inverse, we get pseudofunctors $KF : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$ and $HG : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ that form an equivalence in 2Rep .

Since $\tilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ are cofibrant, we apply the strictification functor to KF and HG to get the 2-functors $\widehat{KF} : \tilde{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ and $\widehat{HG} : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$. One can check that, since KF and HG form an equivalence in 2Rep , we get that \widehat{KF} and \widehat{HG} form an equivalence in 2Cat between $\tilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$.

Then, for any 2-category \mathcal{D} , we apply the 2-functor $2\text{Cat}(\cdot, \mathcal{D}) : 2\text{Cat} \rightarrow \text{Cat}$ to get the following equivalence of categories:

$$2\text{Cat}(\tilde{\mathcal{C}}, \mathcal{D}) \approx 2\text{Cat}(\widehat{\mathcal{C}}, \mathcal{D}).$$

Composed with the isomorphisms between $2\text{Cat}(\widehat{\mathcal{C}}, \mathcal{D})$ and $2\text{Rep}(\mathcal{C}, \mathcal{D})$, we get the result.

2. ACTIONS OF MONOIDS AND COHERENT PRESENTATIONS

In this section, we restrict to the case of 2-representations of categories and monoids. We study a notion of higher-dimensional presentations of categories, called *coherent presentations*, that consist of generators, relations and relations between relations. We prove that, for every coherent presentation Σ of a category \mathbf{C} , the 2-category $\bar{\Sigma}$ presented by Σ is a cofibrant approximation of \mathbf{C} .

This gives the main result of this section: for every coherent presentation Σ of \mathbf{C} and every 2-category \mathcal{C} , we have an equivalence of categories

$$2\text{Rep}(\mathbf{C}, \mathcal{C}) \approx 2\text{Rep}(\bar{\Sigma}, \mathcal{C}).$$

All this material is formulated in the language of *polygraphs*, a notion of presentations of higher-dimensional categories introduced by Burroni, [5], and, under the name of *computads*, by Street, [24].

2.1. Higher-dimensional categories

We recall some notations about n -categories from [10] and [11]. If \mathcal{C} is an n -category (we always consider strict, globular n -categories), we denote by \mathcal{C}_k the set (and the k -category) of k -cells of \mathcal{C} . If f

is a k -cell of \mathcal{C} , then $s_i(f)$ and $t_i(f)$ respectively denote the i -source and i -target of f ; we drop the suffix i when $i = k - 1$. The source and target maps satisfy the *globular relations*:

$$s_i \circ s_{i+1} = s_i \circ t_{i+1} \quad \text{and} \quad t_i \circ s_{i+1} = t_i \circ t_{i+1}.$$

We respectively denote by $f : u \rightarrow v$, $f : u \rightrightarrows v$, $f : u \Rrightarrow v$ a 1-cell, a 2-cell, a 3-cell f with source u and target v .

If f and g are i -composable k -cells, that is when $t_i(f) = s_i(g)$, we denote by $f \star_i g$ their i -composite; we simply use fg when $i = 0$. The compositions satisfy the *exchange relations* given, for every $i \neq j$ and every possible cells f, g, h and k , by:

$$(f \star_i g) \star_j (h \star_i k) = (f \star_j h) \star_i (g \star_j k).$$

If f is a k -cell, we denote by 1_f its identity $(k + 1)$ -cell. When 1_f is composed with cells of dimension $k + 1$ or higher, we simply denote it by f . An n -category is *discrete* when it has identity n -cells only.

2.1.1. $(n, 1)$ -categories. In an n -category \mathcal{C} , a k -cell f , with source x and target y , is *invertible* when there exists a (necessarily unique) k -cell f^- in \mathcal{C} , with source y and target x in \mathcal{C} , called the *inverse of f* , that satisfies

$$f \star_{k-1} f^- = 1_x \quad \text{and} \quad g \star_{k-1} f = 1_y.$$

An $(n, 1)$ -category is a category enriched in $(n - 1)$ -groupoids, *i.e.*, an n -category whose k -cells are invertible for every $k > 1$.

2.1.2. Spheres. Let \mathcal{C} be an n -category. A *0-sphere of \mathcal{C}* is a pair $\gamma = (f, g)$ of 0-cells of \mathcal{C} and, for $1 \leq k \leq n$, a *k -sphere of \mathcal{C}* is a pair $\gamma = (f, g)$ of parallel k -cells of \mathcal{C} , *i.e.*, with $s(f) = s(g)$ and $t(f) = t(g)$; we call f the *source* of γ and g its *target*. If f is a k -cell of \mathcal{C} , for $1 \leq k \leq n$, the *boundary of f* is the $(k - 1)$ -sphere $(s(f), t(f))$. A k -sphere of \mathcal{C} is *degenerate* when its source and target coincide.

2.2. Presentations by polygraphs

We recall the notion of $(n, 1)$ -polygraph, introduced in [11], and use it to define *coherent presentations* of categories.

2.2.1. Cellular extensions. Let \mathcal{C} be an n -category. A *cellular extension of \mathcal{C}* is a set Γ equipped with a map ∂ from Γ to the set of n -spheres of \mathcal{C} . By considering all the formal compositions of elements of Γ , seen as $(n + 1)$ -cells with source and target in \mathcal{C} , one builds the *free $(n + 1)$ -category generated by Γ over \mathcal{C}* , denoted by $\mathcal{C}[\Gamma]$. The *size* of an $(n + 1)$ -cell f of $\mathcal{C}[\Gamma]$ is the number of $(n + 1)$ -cells of Γ it contains.

The *quotient of \mathcal{C} by Γ* , denoted by \mathcal{C}/Γ , is the n -category one gets from \mathcal{C} by identification of the n -cells $s(\gamma)$ and $t(\gamma)$, for every n -sphere γ of Γ . If \mathcal{C} is an $(n, 1)$ -category and Γ is a cellular extension of \mathcal{C} , then the *free $(n + 1, 1)$ -category generated by Γ over \mathcal{C}* is denoted by $\mathcal{C}(\Gamma)$ and defined as follows:

$$\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \Gamma^-] / \text{Inv}(\Gamma)$$

where Γ^- contains the same $(n + 1)$ -cells as Γ , with source and target reversed, and $\text{Inv}(\Gamma)$ is the cellular extension made of two $(n + 2)$ -cells

$$\gamma \star_{n+1} \gamma^- \rightarrow 1_f \quad \text{and} \quad \gamma^- \star_{n+1} \gamma \rightarrow 1_g$$

2. Actions of monoids and coherent presentations

for each $(n + 1)$ -cell γ from f to g in Γ .

2.2.2. Contexts. Let \mathcal{C} be an n -category. A *context* of \mathcal{C} is a composition C of (partial) maps

$$f \mapsto f \star_i g \quad \text{and} \quad f \mapsto g \star_i f$$

on n -cells of \mathcal{C} , where g is a j -cell of \mathcal{C} and with $0 \leq i < j$. We denote by $C[f]$ the image of an n -cell f by C . Up to exchange relations, such a context can always be decomposed as follows

$$C[f] = g_n \star_{n-1} (g_{n-1} \star_{n-2} \cdots (g_1 \star_1 g_0 f h_0 \star_1 h_1) \cdots \star_{n-2} h_{n-1}) \star_{n-1} h_n$$

where each g_i and h_i is an i -cell of \mathcal{C} .

2.2.3. Homotopy bases. Let \mathcal{C} be an $(n, 1)$ -category. A *homotopy basis* of \mathcal{C} is a cellular extension Γ of \mathcal{C} such that the $(n, 1)$ -category \mathcal{C}/Γ is aspherical, *i.e.*, such that, for every n -sphere γ of \mathcal{C} , there exists an $(n + 1)$ -cell with boundary γ in the $(n + 1, 1)$ -category $\mathcal{C}(\Gamma)$. For example, the n -spheres of \mathcal{C} form a cellular extension which is a homotopy basis of \mathcal{C} .

2.2.4. $(n, 1)$ -polygraphs. Let n be a natural number. An n -*polygraph* is a family $\Sigma = (\Sigma_0, \dots, \Sigma_n)$ made of a set Σ_0 and, for every $0 \leq k \leq n - 1$, a cellular extension Σ_{k+1} of the free k -category

$$\Sigma_k^* = \Sigma_0[\Sigma_1] \cdots [\Sigma_k].$$

An $(n, 1)$ -*polygraph* is a family $\Sigma = (\Sigma_0, \dots, \Sigma_n)$ made of a set Σ_0 and, for every $0 \leq k \leq n - 1$, a cellular extension Σ_{k+1} of the free $(k, 1)$ -category

$$\Sigma_k^\top = \Sigma_0[\Sigma_1](\Sigma_2) \cdots (\Sigma_k).$$

Let us note that 2-polygraphs and $(2, 1)$ -polygraphs are identical notions.

For an $(n, 1)$ -polygraph Σ , an element of Σ_k is a k -cell of Σ and Σ is *finite* when it has finitely many cells in every dimension. An $(n, 1)$ -polygraph Σ is *acyclic* when, for every $1 < k < n$, the cellular extension Σ_{k+1} is a homotopy basis of the $(k, 1)$ -category Σ_k^\top .

Remark. An $(n, 1)$ -polygraph yields a diagram which is similar to the one given in the original definition of n -polygraphs, [5]:

$$\begin{array}{ccccccc}
 \Sigma_0^* & \longleftarrow & \Sigma_1^* & \longleftarrow & \Sigma_2^\top & \longleftarrow & \Sigma_3^\top & \longleftarrow & (\dots) & \longleftarrow & \Sigma_{n-1}^\top \\
 \parallel & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & & \swarrow & \uparrow & \swarrow \\
 \Sigma_0 & & \Sigma_1 & & \Sigma_2 & & \Sigma_3 & & (\dots) & & \Sigma_{n-1} & & \Sigma_n
 \end{array}$$

This diagram contains the source and target attachment maps of generating $(k + 1)$ -cells on composite k -cells, their extension to composite $(k + 1)$ -cells and the inclusion of generating k -cells into composite k -cells.

2.2.5. Presentations by polygraphs. Let Σ be an $(n, 1)$ -polygraph with $n \geq 2$. We denote by $\bar{\Sigma}$ the $(n - 1, 1)$ -category it presents: if $n = 2$, this is the category

$$\bar{\Sigma} = \Sigma_1^*/\Sigma_2$$

and, if $n \geq 3$, this is $(n - 1, 1)$ -category

$$\bar{\Sigma} = \Sigma_{n-1}^\top/\Sigma_n.$$

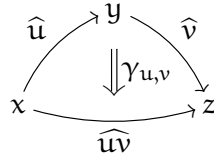
We usually denote by \bar{f} the image of a k -cell f of Σ^* or Σ^\top through the canonical projection onto $\bar{\Sigma}$.

The *category presented by Σ* is the category presented by the 2-polygraph Σ_2 underlying Σ . For a category \mathbf{C} we define:

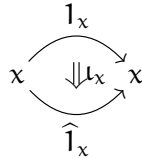
- a *presentation of \mathbf{C}* as a 2-polygraph whose presented category is isomorphic to \mathbf{C} ,
- a *coherent presentation of \mathbf{C}* as an acyclic $(3, 1)$ -polygraph whose presented category is isomorphic to \mathbf{C} , that is, a 2-polygraph extended with a homotopy basis,
- an *n -dimensional coherent presentation of \mathbf{C}* as an acyclic $(n, 1)$ -polygraph whose presented category is isomorphic to \mathbf{C} .

2.2.6. Example (The standard coherent presentation). The *standard presentation* of a category \mathbf{C} is the $(2, 1)$ -polygraph given as follows:

- its 0-cells and 1-cells are the ones of \mathbf{C} , a 1-cell u of \mathbf{C} being denoted by \hat{u} when seen as a generating 1-cell of its standard coherent presentation,
- for every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathbf{C} , one 2-cell

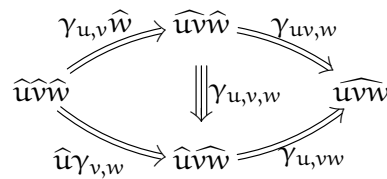


- for every 0-cell x of \mathbf{C} , one 2-cell



The *standard coherent presentation of \mathbf{C}* is the $(3, 1)$ -polygraph made of its standard presentation extended with the homotopy basis made of the following 3-cells:

- for every 1-cells $u : x \rightarrow y$, $v : y \rightarrow z$ and $w : z \rightarrow t$ of \mathbf{C} , one 3-cell



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- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , two 3-cells

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \widehat{1}_x \widehat{u} & \\
 \widehat{\iota}_x \widehat{u} \nearrow & \Downarrow \lambda_u & \searrow \gamma_{1_x, u} \\
 \widehat{u} & & \widehat{u} \\
 & \widehat{1}_{\widehat{u}} &
 \end{array}
 &
 &
 \begin{array}{ccc}
 & \widehat{u} \widehat{1}_y & \\
 \widehat{u} \widehat{\iota}_y \nearrow & \Downarrow \rho_u & \searrow \gamma_{u, 1_y} \\
 \widehat{u} & & \widehat{u} \\
 & \widehat{1}_{\widehat{u}} &
 \end{array}
 \end{array}$$

We will see later that these 3-cells indeed form a homotopy basis and that the $(2, 1)$ -category presented by the standard coherent presentation of \mathbf{C} is, as a 2-category, its standard cofibrant approximation $\widehat{\mathbf{C}}$.

2.3. Cofibrant approximations and coherent presentations

2.3.1. Lemma. *Let \mathbf{C} be a category. If Σ is a coherent presentation of \mathbf{C} , then the $(2, 1)$ -category $\overline{\Sigma}$ presented by Σ is a cofibrant approximation of \mathbf{C} .*

Proof. By definition, $\overline{\Sigma}$ is a cofibrant 2-category. Let us construct a weak equivalence $\pi : \overline{\Sigma} \rightarrow \mathbf{C}$.

First, using the fact that Σ_2 is a presentation of \mathbf{C} , we define $\pi : \Sigma_2^\top \rightarrow \mathbf{C}$ as the canonical projection: it is the identity on 0-cells, it sends each 1-cell u of Σ_1^* to its equivalence class \overline{u} in \mathbf{C} and each 2-cell $f : u \Rightarrow v$ to the identity of \overline{u} (which is also \overline{v} since, by definition of the quotient Σ_1^*/Σ_2 , we have $u = v$ if, and only if, there exists a 2-cell from u to v in Σ_2^\top). Then, let us check that π factors through $\overline{\Sigma}$, defining a 2-functor still denoted by π from $\overline{\Sigma}$ to \mathbf{C} . Since Σ_3 is, by hypothesis, a homotopy basis of Σ_2^\top , this amounts at checking that, for any parallel 2-cells f and g , we have $\pi(f) = \pi(g)$. And this last equality holds because each member is sent to the identity of \overline{u} , if u denotes the common source of f and g .

To prove that π is a weak equivalence, we define a pseudofunctor $\iota : \mathbf{C} \rightarrow \overline{\Sigma}$ that is a quasi-inverse for π . It is the identity on 0-cells and, for every 1-cell u in \mathbf{C} , we arbitrarily choose a representative 1-cell \widehat{u} in Σ_1^* . Then, for every composable 1-cells u and v in \mathbf{C} , we have an invertible 2-cell $\iota_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{uv}$ in $\overline{\Sigma}$: both $\widehat{u}\widehat{v}$ and \widehat{uv} represent the same 1-cell of \mathbf{C} , namely uv , so that there is an invertible 2-cell between them in Σ_2^\top , whose image in $\overline{\Sigma}$ we take for $\iota_{u,v}$. For the same reason, we have an invertible 2-cell $\iota_x : 1_x \Rightarrow \widehat{1}_x$ in $\overline{\Sigma}$. Finally, the monoidal coherence conditions that ι must satisfy hold because Σ_3 is a homotopy basis of Σ_2^\top and, as a consequence, any two parallel 2-cells of $\overline{\Sigma}$ are equal.

Finally, we check that both composites of π and ι are isomorphic to identities. We have $\pi\iota = \text{Id}_{\mathbf{C}}$. Conversely, $\iota\pi$ is the identity on 0-cells and, if $u : x \rightarrow y$ is a 1-cell of $\overline{\Sigma}$, then $\iota(\pi(u)) = \widehat{u}$: this 1-cell is in the same equivalence class as u , so that we get an invertible 2-cell $\alpha_u : u \Rightarrow \widehat{u}$ in $\overline{\Sigma}$. Now, if $f : u \Rightarrow v$ is a 2-cell of $\overline{\Sigma}$, then $\iota(\pi(f)) = 1_{\widehat{u}}$. Since $\overline{u} = \overline{v}$ by hypothesis, we get the following two parallel, hence equal, 2-cells in $\overline{\Sigma}$:

$$\begin{array}{ccc}
 & \widehat{u} & \iota(\pi(f)) \\
 \alpha_u \nearrow & & \searrow \\
 u & & \widehat{v} \\
 f \searrow & & \nearrow \alpha_v \\
 & v &
 \end{array}$$

This proves that α is a natural isomorphism from $\iota\pi$ to the identity, thus yielding that π is a weak equivalence and, finally, we get that $\overline{\Sigma}$ is a cofibrant approximation of \mathbf{C} . \square

2.4. Tietze equivalences and Tietze transformations

2.3.2. Theorem. *Let \mathbf{C} be a category and let Σ be a coherent presentation of \mathbf{C} . Then, for every 2-category \mathcal{C} , we have the following equivalences of categories:*

$$2\text{Rep}(\mathbf{C}, \mathcal{C}) \approx 2\text{Cat}(\overline{\Sigma}, \mathcal{C}).$$

In particular, if \mathbf{M} is a monoid and if Σ is a coherent presentation of \mathbf{M} , we have the following equivalence of categories:

$$\text{Act}(\mathbf{M}) \approx 2\text{Cat}(\overline{\Sigma}, \text{Cat}).$$

The last equivalence formalises the problem that Deligne’s formulates in [7]: how to give a definition of an action of a monoid \mathbf{M} on a category \mathcal{C} in terms of a presentation by generators and relations of \mathbf{M} ? More precisely, what coherence relations should one impose on the natural isomorphisms associated to the relations?

The equivalence says that these relations can be given by a homotopy basis. More precisely, considering a coherent presentation Σ of \mathbf{M} , an action of \mathbf{M} on \mathcal{C} is the same (“*il revient au même*” in Deligne’s words) as the following data:

- An endofunctor

$$\mathcal{C} \xrightarrow{T(x)} \mathcal{C}$$

for each generating 1-cell x in Σ ; from this, one computes $T(u) : \mathcal{C} \rightarrow \mathcal{C}$ for every 1-cell u in Σ^\top by functoriality.

- A natural isomorphism

$$\begin{array}{ccc} & T(u) & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow T(\varphi) \\ \curvearrowleft \end{array} & \mathcal{C} \\ & T(v) & \end{array}$$

for each generating 2-cell $\varphi : u \Rightarrow v$ in Σ ; from this, one computes $T(f) : T(u) \Rightarrow T(v)$ for every 2-cell $f : u \Rightarrow v$ in Σ^\top by functoriality.

This data must satisfy, for every 3-cell $\alpha : f \Rightarrow g : u \Rightarrow v$ in Σ , the following coherence relation:

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow T(f) \\ \curvearrowleft \end{array} & \mathcal{C} \\ & T(v) & \end{array} = \begin{array}{ccc} \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow T(g) \\ \curvearrowleft \end{array} & \mathcal{C} \\ & T(v) & \end{array}$$

As a conclusion, the effective computation of a homotopy basis of a presentation Σ of a monoid \mathbf{M} gives a concrete way to determine the coherence relations of the definition of action of \mathbf{M} corresponding to Σ .

2.4. Tietze equivalences and Tietze transformations

In order to formalise a procedure to compute a homotopy basis from a $(2, 1)$ -polygraph, we introduce a way to compare $(n, 1)$ -polygraphs according to the category or the 2-category they present: the Tietze equivalences. Then, we give elementary transformations of $(n, 1)$ -polygraphs that correspond to Tietze equivalences: the Tietze transformations.

2. Actions of monoids and coherent presentations

2.4.1. Tietze equivalences. Two $(n, 1)$ -polygraphs Σ and Υ are *1-Tietze-equivalent* when the categories

$$\Sigma_1^*/\Sigma_2 \quad \text{and} \quad \Upsilon_2^*/\Upsilon_2$$

they present are isomorphic. In that case, Σ and Υ have the same 0-cells, up to a bijection.

The $(n, 1)$ -polygraphs Σ and Υ are *2-Tietze-equivalent* when they are 1-Tietze-equivalent and when the $(2, 1)$ -categories

$$\Sigma_2^\top/\Sigma_3 \quad \text{and} \quad \Upsilon_2^\top/\Upsilon_3$$

they present are equivalent when seen as 2-categories. In particular, we note that two coherent presentations of the same category are 2-Tietze-equivalent.

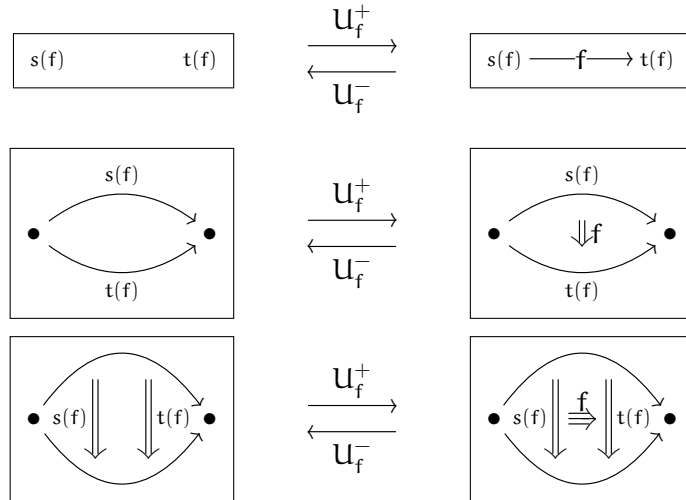
Tietze equivalences can be used to compare a $(p, 1)$ -polygraph to an $(n, 1)$ -polygraph, with $p < n$, by seeing the $(p, 1)$ -polygraph as an $(n, 1)$ -polygraph with no k -cells for $k > p$. With an appropriate notion of equivalence of higher-dimensional categories, one could introduce p -Tietze equivalence of $(n, 1)$ -polygraphs; this would be useful, for example, to study the actions of monoids on higher-dimensional categories.

2.4.2. Elementary polygraphic transformations. We introduce the *elementary polygraphic transformations* on an $(n, 1)$ -polygraph Σ , for $n \geq 2$, divided into three families:

Adjunction of a k -cell, $1 \leq k \leq n$. If f is a $(k-1)$ -sphere of Σ , the transformation U_f^+ formally adds f to Σ as a generating k -cell.

Elimination of a k -cell, $1 \leq k \leq n$. If f is a k -cell of Σ such that f does not appear in the sources and targets of any j -cell, for $j > k$, the transformation U_f^- removes f from Σ .

For $k \in \{1, 2, 3\}$, the adjunctions and eliminations of k -cells correspond to the following maps, with U_f^- being the inverse of U_f^+ :



Replacement of a k -cell, $2 \leq k \leq n$. If f is a k -cell of Σ and C is an invertible context of $(\Sigma_k \setminus \{f\})^\top$ and ε is in $\{-, +\}$, the transformation $U_{f,C[f^\varepsilon]}$ replaces f by $C[f^\varepsilon]$ in Σ by performing the following sequence of operations:

2.4. Tietze equivalences and Tietze transformations

- adjunction of a k -cell $x_{C[f^\varepsilon]}$, parallel to $C[f^\varepsilon]$,
- replacement of f with $C^{-}[x_{C[f^\varepsilon]}^\varepsilon]$ in the sources and targets of every j -cell, for $j > k$,
- elimination of the k -cell f .

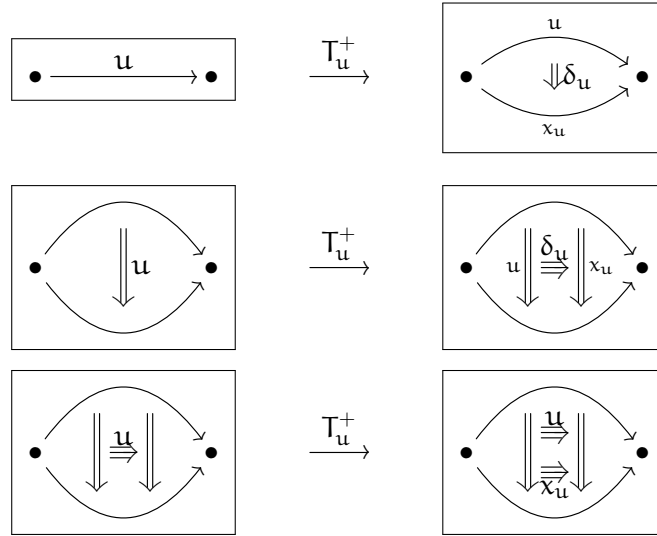
A (finite) *polygraphic transformation* is a (finite) composite of elementary polygraphic transformations. We note that every polygraphic transformation is invertible (up to the renaming of the generating cells). A polygraphic transformation is *positive* when it only contains adjunctions and replacements, *negative* when it only contains eliminations and replacements, *neutral* when it only contains replacements.

2.4.3. Coherent polygraphic transformations. From the elementary polygraphic transformations, we build the following two families of composite transformations:

Coherent adjunction of a k -cell, $1 \leq k < n$. If u is a k -cell of Σ^\top , the transformation T_u^+ adds to Σ the following cells:

- a generating k -cell x_u that is parallel to u ,
- if $k < n$, a generating $(k + 1)$ -cell δ_u from f to x_u .

For $n = 3$ and $k \in \{1, 2, 3\}$, the coherent adjunctions of k -cells correspond to the following inclusions:

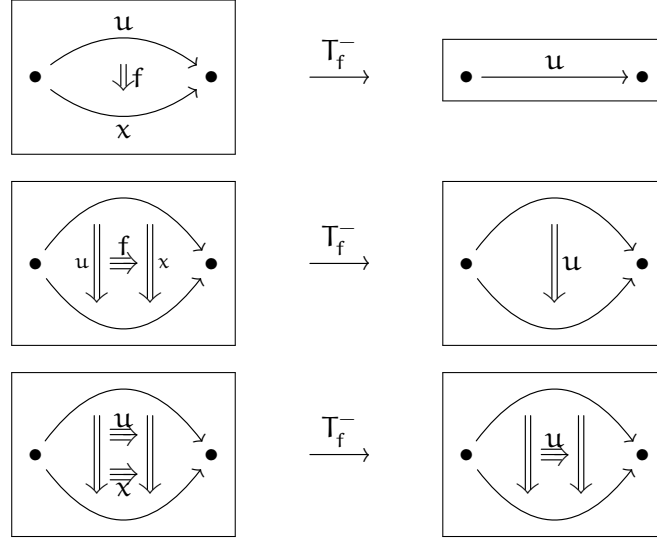


Coherent elimination of a k -cell, $1 \leq k \leq n$. If f is a $(k + 1)$ -cell of Σ , or an n -sphere of Σ^\top if $k = n$, with source u in Σ^\top and target x in Σ , the transformation T_f^- successively performs the following sequence of operations:

- replacements of j -cells, for $j > k$, so that u is replaced by x in their sources and targets,
- replacements of j -cells, for $j > k + 1$, so that f is replaced by 1_u in their sources and targets,
- if $k < n$, elimination of the $(k + 1)$ -cell f ,
- elimination of the k -cell x .

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When $n = 3$ and $k \in \{1, 2, 3\}$, the coherent eliminations of k -cells correspond to the following projections:



2.5. 1-Tietze transformations

A *1-Tietze transformation* is a polygraphic transformation that is a composition of

- replacements of k -cells, for $k \geq 2$,
- coherent adjunctions and coherent eliminations of 1-cells and 2-cells,
- adjunctions and eliminations of k -cells, for $k \geq 3$.

The terminology is borrowed from combinatorial group theory, see [18], in such a way that Tietze transformations of presentations of monoids correspond to 1-Tietze transformations of $(2, 1)$ -polygraphs. As in the original case, we get that 1-Tietze-equivalent $(n, 1)$ -polygraphs are exactly the ones that are related by 1-Tietze transformations.

2.5.1. Lemma. *The 1-Tietze transformations preserve 1-Tietze equivalence.*

Proof. First, we note that transformations that act on k -cells, for $k \geq 3$, have no influence on the category presented by an $(n, 1)$ -polygraph. Then, replacements of 2-cells also preserve the category presented by a $(n, 1)$ -polygraph: they replace a generating relation by a composition of relations that are equivalent to it.

In terms of presentations of categories, a coherent adjunction or coherent elimination of a 1-cell adds or removes a generator, together with a relation that defines it in terms of the other generators. Thus, these transformations induce isomorphisms between the corresponding presented categories.

Coherent adjunctions and coherent eliminations of 2-cells amount at adding or removing generating relations that are consequences of the other generating relations. In that case, we get equalities between the presented categories. \square

2.5.2. Proposition. *Two $(2, 1)$ -polygraphs are 1-Tietze-equivalent if, and only if, there exists a 1-Tietze transformation between them. Moreover, if both $(2, 1)$ -polygraphs are finite, then the 1-Tietze transformation can be chosen to be finite.*

Proof. We have seen that 1-Tietze transformations preserve 1-Tietze equivalence. Conversely, let us consider $(2, 1)$ -polygraphs Σ and Υ that present the same category (up to isomorphism). We define the $(2, 1)$ -polygraph Ξ with the following cells:

- the same 0-cells as Σ (and Υ),
- the 1-cells of Σ plus the 1-cells of Υ ,
- the 2-cells of Σ plus the 2-cells of Υ ,
- for each 1-cell x of Σ , a 2-cell $\varphi_x : v_x \Rightarrow x$, where v_x is a 1-cell of Υ^\top such that $\overline{v_x} = \bar{x}$ in the quotient,
- for each 1-cell y of Υ , a 2-cell $\psi_y : u_y \Rightarrow y$, where u_y is a 1-cell of Σ^\top such that $\overline{u_y} = \bar{y}$ in the quotient.

Let us prove that there exists a 1-Tietze transformation from Σ to Ξ . First, we use a coherent adjunction to add y and $\psi_y : u_y \Rightarrow y$ for each 1-cell y in Υ . We denote by $T_{\Upsilon_1}^+(\Sigma)$ the resulting $(2, 1)$ -polygraph.

Then, we consider a 2-cell $\gamma : v \Rightarrow v'$ in Υ . By definition of $\overline{\Upsilon}$, we have $\bar{v} = \bar{v}'$. Thus, if $v = y_1 \dots y_m$ and $v' = y'_1 \dots y'_n$ with each y_i and y'_j in Υ , we have

$$\overline{u_{y_1}} \dots \overline{u_{y_m}} = \overline{u_{y'_1}} \dots \overline{u_{y'_n}}.$$

As a consequence, there exists a 2-cell

$$f : u_{y_1} \dots u_{y_m} \Longrightarrow u_{y'_1} \dots u_{y'_n}$$

in Σ^\top . Hence, we get the following 2-cell in $T_{\Upsilon_1}^+(\Sigma)^\top$:

$$\psi_{y_1}^- \dots \psi_{y_m}^- \star_1 f \star_1 \psi_{y'_1} \dots \psi_{y'_n} : y_1 \dots y_m \Longrightarrow y'_1 \dots y'_n.$$

We use a coherent adjunction to add the 2-cell $\gamma : v \Rightarrow v'$ to $T_{\Upsilon_1}^+(\Sigma)$, and repeat the same process for every 2-cell of Υ . We denote by $T_\Upsilon^+(\Sigma)$ the resulting $(2, 1)$ -polygraph.

Let x be a 1-cell of Σ . We decompose the corresponding 1-cell v_x of Υ^\top into $y_1 \dots y_n$, with each y_i in Υ . Then we have

$$\bar{x} = \overline{u_{y_1}} \dots \overline{u_{y_n}},$$

so that, there exists a 2-cell

$$f : u_{y_1} \dots u_{y_n} \Longrightarrow x$$

in Σ^\top . Hence we have the following 2-cell in $T_\Upsilon^+(\Sigma)^\top$:

$$\psi_{y_1}^- \dots \psi_{y_n}^- \star_1 f : v_x = y_1 \dots y_n \Longrightarrow x.$$

We apply a coherent adjunction to add the 2-cell $\varphi_x : v_x \Rightarrow x$ to $T_\Upsilon^+(\Sigma)$, and repeat the same process for every 1-cell of Σ . This results in Ξ and, by symmetry, we get the following 1-Tietze transformations, which are finite whenever Σ and Υ are finite:

$$\Sigma \xrightarrow{S} \Xi \xrightarrow{T^-} \Upsilon. \quad \square$$

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2.5.3. Lemma. *An $(n, 1)$ -polygraph is 1-Tietze-equivalent to its underlying $(2, 1)$ -polygraph, by eliminations of k -cells, $k \geq 3$.*

Proof. Let Σ be an $(n, 1)$ -polygraph. Starting with the n -cells of Σ , we can apply eliminations so that we get 1-Tietze equivalence between Σ and its underlying $(n - 1, 1)$ -polygraph. By induction, we progressively eliminate all the k -cells of Σ , for $k \geq 3$, to get the result. \square

2.5.4. Theorem. *Two $(n, 1)$ -polygraphs are 1-Tietze-equivalent if, and only if, there exists a 1-Tietze transformation between them. Moreover, if both $(n, 1)$ -polygraphs are finite, the 1-Tietze transformation can be chosen to be finite.*

2.6. 2-Tietze transformations

A 2-Tietze transformation is a polygraphic transformation that is a composition of

- replacements of k -cells, for ≥ 2 ,
- coherent adjunctions and coherent eliminations of 1-cells, 2-cells and 3-cells,
- adjunctions and eliminations of k -cells, for $k \geq 4$.

2.6.1. Lemma. *The 2-Tietze transformations preserve 2-Tietze equivalence.*

Proof. First, we note that transformations that act on k -cells, for $k \geq 4$, preserve the 2-category presented by an $(n, 1)$ -polygraph. This is also the case of replacements of 2-cells or 3-cells.

In terms of presentations of 2-categories, coherent adjunctions and coherent eliminations of 2-cells and 3-cells correspond to adding or removing superfluous generators or relations, yielding isomorphic presented 2-categories.

Let Σ be a $(3, 1)$ -polygraph, let u be a 1-cell of Σ^* and let T_u^+ be the coherent adjunction that adds a 1-cell x and a 2-cell $\varphi : u \Rightarrow x$ to Σ . We want to prove that the resulting $(3, 1)$ -polygraph $T_u^+(\Sigma)$ is 2-Tietze-equivalent to Σ . First, we note that they are 1-Tietze-equivalent since T_u^+ is a 1-Tietze transformation. Then, let us prove that the 2-categories presented by Σ and $T_u^+(\Sigma)$ are equivalent. We consider the 2-functors

$$\overline{\Sigma} \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\pi} \end{array} \overline{T_u^+(\Sigma)}$$

where ι is induced by the inclusion of Σ_2 into $T_u^+(\Sigma)$ and π is induced by the canonical projection sending x to u and φ to 1_u , leaving the other generating cells unchanged. Let us prove that this pair of 2-functors is an equivalence of 2-categories. First, we have $\pi\iota = \text{Id}_{\overline{\Sigma}}$. Then, let us define a natural isomorphism σ from the composite $\overline{\iota\pi}$ to the identity of $\overline{T_u^+(\Sigma)}$ as follows:

- on 0-cells, σ is the identity,
- on 1-cells, we take $\sigma_x = \varphi$, which is invertible in $\overline{T_u^+(\Sigma)}$, and $\sigma_y = 1_y$ for every other generating 1-cell y of Σ .

The only non-trivial diagram we have to check is the one for the 2-cell φ , and it is commutative:

$$\begin{array}{ccc} \iota\pi(u) = u & \xrightarrow{\sigma_u = 1_u} & u \\ \iota\pi(\varphi) = 1_u \downarrow & \circlearrowleft & \downarrow \varphi \\ \iota\pi(x) = u & \xrightarrow{\sigma_x = \varphi} & x \end{array}$$

For the case of coherent eliminations of 1-cells, we use the fact that, up to replacements of k -cells, for $k \geq 2$, we can consider that the 1-cell and the 2-cell removed do not appear in the sources and targets of other cells. Then, the coherent elimination is the inverse transformation of a coherent adjunction and, as such, it is a 2-Tietze transformation. \square

2.6.2. Proposition. *Two $(3, 1)$ -polygraphs are 2-Tietze-equivalent if, and only if, there exists a 2-Tietze transformation between them. Moreover, if both $(3, 1)$ -polygraphs are finite, then the 2-Tietze transformation can be chosen to be finite.*

Proof. We have seen that 2-Tietze transformations preserve 2-Tietze equivalence. Conversely, let us consider $(3, 1)$ -polygraphs Σ and Υ that are 2-Tietze-equivalent. First, we define $(3, 1)$ -polygraphs Σ' and Υ' such that Σ' is 2-Tietze-equivalent to Σ , Υ' is 2-Tietze-equivalent to Υ and Σ' and Υ' have the same 1-cells.

Let us define Σ' as the $(3, 1)$ -polygraph obtained from Σ by application of a sequence of coherent adjunctions of 1-cells. For each 1-cell y of Υ , we choose a 1-cell u_y of Σ^\top such that $\bar{u}_y = \bar{y}$ in the quotient category Σ_1^*/Σ_2 . Then we apply a coherent adjunction for each 1-cell y , thus adding y and a 2-cell $\varphi_y : u_y \Rightarrow y$, and get Σ' as a result. We define Υ' from Υ in a symmetric way.

Now, we have two $(3, 1)$ -polygraphs Σ' and Υ' that are 2-Tietze-equivalent with the same 1-cells. As a consequence, the 2-categories $\bar{\Sigma}'$ and $\bar{\Upsilon}'$ they present are not only equivalent, but isomorphic. From them, we proceed in a similar way to the case of 1-Tietze-equivalent $(2, 1)$ -polygraphs in order to construct a $(3, 1)$ -polygraph Ξ that contains all the 2-cells and 3-cells of Σ' and Υ' , plus coherence 3-cells. Then we check that Ξ can be obtained from both Σ' and Υ' by coherent adjunctions of 2-cells and 3-cells, thus proving that there exists a 2-Tietze transformation from Σ' to Υ' , hence from Σ to Υ . \square

2.6.3. Lemma. *An $(n, 1)$ -polygraph is 2-Tietze-equivalent to its underlying $(3, 1)$ -polygraph, by eliminations of k -cells, $k \geq 4$.*

Proof. Let Σ be an $(n, 1)$ -polygraph. Starting with the n -cells of Σ , we can apply eliminations so that we get 2-Tietze equivalence between Σ and its underlying $(n - 1, 1)$ -polygraph. By induction, we progressively eliminate all the k -cells of Σ , for $k \geq 4$, to get the result. \square

2.6.4. Theorem. *Two $(n, 1)$ -polygraphs are 2-Tietze equivalent if, and only if, there exists a 2-Tietze transformation between them. Moreover, if both $(n, 1)$ -polygraphs are finite, the 2-Tietze transformation can be chosen to be finite.*

3. Rewriting methods for computing coherent presentations

3. REWRITING METHODS FOR COMPUTING COHERENT PRESENTATIONS

In this section, we give algorithmic methods that can extend a presentation Σ into a coherent presentation, by computing a homotopy basis of Σ^\top . They rely on using rewriting techniques, mainly Squier's theorem and Knuth-Bendix completion procedure, adapted to the setting of coherent presentations and formulated using Tietze transformations.

3.1. Convergent 2-polygraphs

Here we recall notions and results of rewriting theory for 2-polygraphs, already contained in [10] for the general case of n -polygraphs. We fix a 2-polygraph Σ .

3.1.1. Rewriting and normal forms. A *rewriting step* of Σ is a 2-cell of the free 2-category Σ^* with shape

$$\begin{array}{c}
 y \xrightarrow{v} x \begin{array}{c} \xrightarrow{u} \\ \Downarrow \varphi \\ \xrightarrow{u'} \end{array} x' \xrightarrow{w} y'
 \end{array}$$

where φ is a 2-cell of Σ and v and v' are 1-cells of Σ^* .

In what follows, we could assume that, given a 1-cell u of Σ^* , the set of rewriting steps with source u has a distinguished element. Indeed, in our examples, Σ is finite, so that there exist only finitely many rewriting steps with source u , making such a choice possible, at least in an arbitrary way. In practice, we often have a “leftmost” or “rightmost” rewriting step on a given u , *i.e.*, a reduction step $v\varphi w$ with v or w of minimal size. We refer the reader to [11] for a more thorough investigation on this subject, where canonical rewriting steps are given in terms of “normalisation strategies”.

A *rewriting sequence* of Σ is a finite or infinite sequence

$$u_1 \xrightarrow{f_1} u_2 \xrightarrow{f_2} (\dots) \xrightarrow{f_{n-1}} u_n \xrightarrow{f_n} (\dots)$$

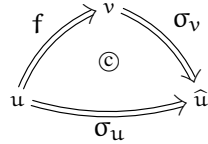
of rewriting steps. If Σ has a non-empty rewriting sequence from u to v , we say that u *rewrites into* v . Let us note that every 2-cell f of Σ^* decomposes into a finite rewriting sequence of Σ , this decomposition being unique up to exchange relations.

We say that a 1-cell u of Σ^* is a *normal form* when Σ has no rewriting step with source u . A *normal form* of u is a 1-cell v that is a normal form and such that u rewrites into v .

3.1.2. Termination. We say that Σ *terminates* when it has no infinite rewriting sequence. In that case, every 1-cell has at least one normal form. Moreover, *Noetherian induction* allows definitions and proofs of properties of 1-cells by induction on the maximum size of the 2-cells leading to normal forms.

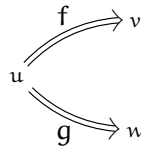
As an example of how Noetherian induction is used, let us define a canonical normal form \hat{u} for every 1-cell u of Σ^* , together with a 2-cell $\sigma_u : u \Rightarrow \hat{u}$ in Σ^* . If u is a normal form, then we define \hat{u} as u and σ_u as the identity. Let us assume that u is not a normal form and that, for every v in which u reduces, we have built \hat{v} and σ_v . In that case, we choose a rewriting step $f : u \Rightarrow v$ and we define \hat{u} to

be \hat{v} and σ_u as the following composite:



Such a family of 2-cells $\sigma_u : u \Rightarrow \hat{u}$ forms the first dimension of a *normalisation strategy* for Σ . As already mentioned, this notion was introduced in [11], to which we refer the reader for a deeper study.

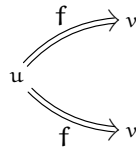
3.1.3. Branchings. A *branching* of Σ is a pair (f, g) of 2-cells of Σ^* with a common source, as in the diagram



The 1-cell u is the *source* of this branching and the pair (v, w) is its *target*. We do not distinguish the branchings (f, g) and (g, f) .

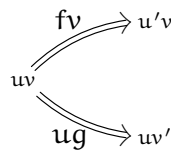
A branching (f, g) is *local* when f and g are rewriting steps. Local branchings belong to one of the three following families:

- *Aspherical* branchings have shape



with $f : u \Rightarrow v$ a rewriting step of Σ .

- *Peiffer* branchings have shape



with $f : u \Rightarrow v$ and $g : u' \Rightarrow v'$ rewriting steps of Σ .

- *Overlapping* branchings are the remaining local branchings.

Local branchings are compared by the order \preceq generated by the relations

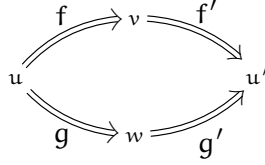
$$(f, g) \preceq (ufv, ugv)$$

given for any local branching (f, g) and any possible 1-cells u and v of Σ^* . An overlapping local branching that is minimal for the order \preceq is called a *critical branching*.

The terms “aspherical” and “Peiffer” come from the corresponding notions for spherical diagrams in Cayley complexes associated to presentations of groups, see [18]. The term “critical” comes from rewriting theory, see [4, 1].

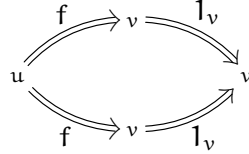
3. Rewriting methods for computing coherent presentations

3.1.4. Confluence. A branching $(f, g) : u \Rightarrow (v, w)$ is *confluent* when there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ^* , as in the following diagram:

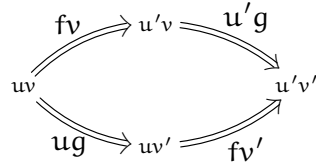


We say that Σ is *confluent* (resp. *locally confluent*) when all of its branchings (resp. local branchings) are confluent.

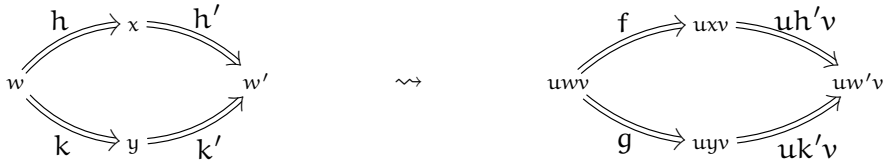
In a confluent 2-polygraph, every 1-cell has at most one normal form. An usual result in rewriting states that local confluence is equivalent to confluence of critical branchings, [1, 10]. Indeed, any aspherical branching is confluent:



We also have confluence of any Peiffer local branching:



Finally, in the case of an overlapping but not minimal local branching (f, g) , there exist factorisations $f = uhv$ and $g = ukv$ with $(h, k) : w \Rightarrow (x, y)$ a critical branching of Σ . Moreover, if (h, k) is confluent, then so is (f, g) :



For terminating 2-polygraphs, Newman's lemma, sometimes called the Diamond Lemma, ensures that local confluence and confluence are equivalent properties, [20]. Its proof is contained in the one of Squier's theorem, which we recall later.

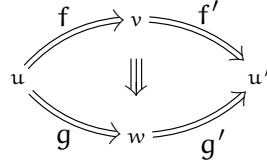
3.1.5. Convergent polygraphs. We say that Σ is *convergent* when it terminates and it is confluent. Such a Σ is called a *convergent presentation* of $\bar{\Sigma}$, and of any category that is isomorphic to $\bar{\Sigma}$. In that case, every 1-cell u of Σ^* has a unique normal form, denoted by \hat{u} , so that we have $\bar{u} = \bar{v}$ in $\bar{\Sigma}$ if, and only if, $\hat{u} = \hat{v}$ holds in Σ^* . This extends to a section $\bar{\Sigma} \rightarrow \Sigma^*$ of the canonical projection, sending a 1-cell u of $\bar{\Sigma}$ to the unique normal form of its representative 1-cells in Σ^* , still denoted by \hat{u} .

As a consequence, a finite and convergent 2-polygraph Σ yields a concrete way to describe the morphisms of the category $\bar{\Sigma}$ it presents, as elements of a free category, together with a decision procedure for the corresponding word problem. The purpose of the finiteness condition is to ensure that one can effectively check that a given 1-cell is a normal form.

3.2. Squier's theorem

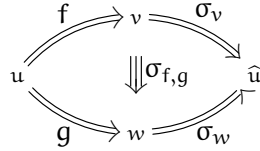
Here we recall Squier's theorem, giving a way to construct a homotopy basis for a convergent 2-polygraph. We fix a 2-polygraph Σ .

3.2.1. Generating confluences. A family of generating confluences of Σ is a cellular extension of Σ^\top that contains exactly one 3-cell



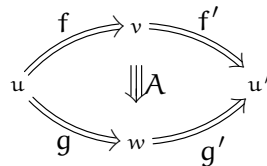
for every critical branching (f, g) of Σ .

We note that, if Σ is confluent, such a family always exists. However, such a family is not unique, since the 3-cell can be directed in the reverse way and, for a given branching (f, g) , we can have several possible 2-cells f' and g' with the required shape. One way to determine a precise family is to fix a normalisation strategy σ and the corresponding family of generating confluences, made of one 3-cell



for every critical branching (f, g) of Σ . Either way, we still get Squier's theorem for 2-polygraphs (a general version for n -polygraphs is given in [10]). Its proof is decomposed in three steps.

3.2.2. Lemma. Let Σ be a convergent 2-polygraph and let Γ be a family of generating confluences of Σ . Then, for every local branching $(f, g) : u \Rightarrow (v, w)$ of Σ , there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ^* and a 3-cell $A : f \star_1 f' \Rightarrow g \star_1 g'$ in Γ^\top , as in



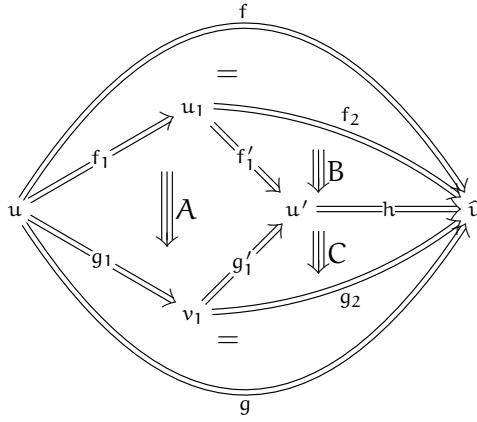
Proof. As we have seen in the study of confluence of local branchings, in the case of an aspherical or Peiffer branching, we can choose f' and g' such that $f \star_1 f' = g \star_1 g'$: an identity 3-cell is enough to link them. Moreover, if we have an overlapping branching (f, g) that is not critical, we have $(f, g) = (uhv, ukv)$ with (h, k) critical; we consider the corresponding 3-cell $\alpha : h \star_1 h' \Rightarrow k \star_1 k'$ of Γ and conclude that $f' = uh'v$, $g' = kuk'v$ and $A = u\alpha v$ satisfy the required conditions. \square

3. Rewriting methods for computing coherent presentations

3.2.3. Lemma. *Let Σ be a convergent 2-polygraph and let Γ be a family of generating confluences of Σ . Then, for every parallel 2-cells f and g of Σ^* with target a normal form, there exists a 3-cell from f to g in Γ^\top .*

Proof. The proof works by Noetherian induction on the common source u of f and g , using the termination of Σ . Let us assume that u is a normal form: then, by definition, the 2-cells f and g are the identity of u , so that $1_{1_u} : 1_u \Rightarrow 1_u$ is a 3-cell of Γ^\top from f to g .

Now, let us fix a 1-cell u with the following property: for any 1-cell v in which u reduces and for any parallel 2-cells $f, g : v \Rightarrow \hat{v} = \hat{u}$ of Σ^* , there exists a 3-cell from f to g in Γ^\top . Let us consider parallel 2-cells $f, g : u \Rightarrow \hat{u}$ and prove the result by constructing the following 3-cell from f to g in Γ^\top :



Since u is not a normal form, we can decompose $f = f_1 \star_1 f_2$ and $g = g_1 \star_1 g_2$ so that f_1 and g_1 are rewriting steps. They form a local branching (f_1, g_1) and we build the 2-cells f'_1 and g'_1 , together with the 3-cell A as in the first part of the proof. Then, we consider a 2-cell h from u' to \hat{u} in Σ^* , that must exist by confluence of Σ and since \hat{u} is a normal form. We apply the induction hypothesis to the parallel 2-cells f_2 and $f'_1 \star_1 h$ in order to get B and, symmetrically, to the parallel 2-cells $g'_1 \star_1 h$ and g_2 to get C . \square

3.2.4. Theorem (Squier's theorem, [23, 10]). *Let Σ be a convergent 2-polygraph. Every family of generating confluences of Σ is a homotopy basis of Σ^\top .*

Proof. Let us consider a 2-cell $f : u \Rightarrow v$ in Σ^\top . By construction of Σ^\top , the 2-cell f can be decomposed (in a general in a non-unique way) into a “zig-zag”

$$f = f_1 \star_1 g_1^- \star_1 \cdots \star_1 f_n \star_1 g_n^-$$

where each f_i and g_i is a 2-cell of Σ^* . For each i , we use the convergence of Σ to choose 2-cells of Σ^*

$$h_i : s(f_i) \Rightarrow \hat{u} \quad \text{and} \quad k_i : t(f_i) \Rightarrow \hat{u}.$$

We also choose a 2-cell of Σ^*

$$h_{n+1} : v \Rightarrow \hat{u}.$$

Hence, for each i , we get parallel 2-cells of Σ^* whose targets are normal forms, so that we can consider 3-cells of Σ^\top with the following shape:

$$A_i : f_i \star_1 k_i \Rightarrow h_i \quad \text{and} \quad B_i : g_i \star_1 k_i \Rightarrow h_{i+1}.$$

Thus, the 3-cells

$$A'_i = A_i \star_1 k_i^- \quad \text{and} \quad B'_i = g_i^- \star_1 B_i^- \star_1 h_{i+1}^-,$$

of Σ^\top have the following sources and targets:

$$A'_i : f_i \Rightarrow h_i \star_1 k_i^- \quad \text{and} \quad B'_i : g_i^- \Rightarrow k_i \star_1 h_{i+1}.$$

By composing all these 3-cells, we get

$$A'_1 \star_1 B'_1 \star_1 \cdots \star_1 A'_n \star_1 B'_n : f \Rightarrow h_1 \star_1 h_{n+1}.$$

We can proceed similarly for another 2-cell $g : u \Rightarrow v$ of Σ^\top , to get a 3-cell from $h_1 \star_1 h_{n+1}$ to g in Σ^\top . By composing the resulting 3-cells, we get a 3-cell from f to g in Γ^\top . We conclude that Γ is a homotopy basis of Σ^\top . \square

3.2.5. Squier's completion. Given a convergent 2-polygraph Σ , we denote by $\mathcal{S}(\Sigma)$ the $(3, 1)$ -polygraph obtained from Σ by adjunction of a chosen family of generating confluences of Σ . We note that this $(3, 1)$ -polygraph is only defined up to that choice but we still have the following result, so that, in particular, two different families of generating confluences give 2-Tietze-equivalent $(3, 1)$ -polygraphs:

3.2.6. Corollary. *For every convergent presentation Σ of a category \mathbf{C} , the $(3, 1)$ -polygraph $\mathcal{S}(\Sigma)$ is a coherent presentation of \mathbf{C} .*

3.2.7. Example (The standard coherent presentation). We recall that, given a category \mathbf{C} , the standard presentation of \mathbf{C} is the 2-polygraph with the following cells

- the same 0-cells as \mathbf{C} ,
- one 1-cell $\widehat{u} : x \rightarrow y$ for every 1-cell $u : x \rightarrow y$ of \mathbf{C} ,
- one 2-cell $\gamma_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{uv}$ for every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathbf{C} ,
- one 2-cell $\iota_x : 1_x \Rightarrow \widehat{1}_x$ for every 0-cell x of \mathbf{C} .

This presentation is not terminating: indeed, for every 0-cell x of \mathbf{C} , the 2-cell ι_x creates infinite rewriting sequences

$$1_x \Rightarrow \widehat{1}_x \Rightarrow \widehat{1}_x \widehat{1}_x \Rightarrow \widehat{1}_x \widehat{1}_x \widehat{1}_x \Rightarrow \cdots$$

But the standard presentation is 1-Tietze-equivalent to the 2-polygraph we get by reversing all the 2-cells ι_x , and this 2-polygraph is convergent. Indeed, for termination, one checks that each 2-cell $\gamma_{u,v}$ has source of size 2 and target of size 1, while each 2-cell ι_x has source of size 1 and target of size 0. As a consequence, for every non-degenerate 2-cell $f : u \Rightarrow v$ of the free 2-category, the size of u is strictly greater than the size of v .

For confluence, we study the critical branchings, divided into three families:

3. Rewriting methods for computing coherent presentations

- for every 1-cells $u : x \rightarrow y$, $v : y \rightarrow z$ and $w : z \rightarrow t$, one critical branching

$$(\gamma_{u,v}\widehat{w}, \widehat{u}\gamma_{v,w})$$

which is confluent and gives the 3-cell

$$\begin{array}{ccc} & \xrightarrow{\gamma_{u,v}\widehat{w}} & \widehat{u}\widehat{v}\widehat{w} \\ \widehat{u}\widehat{v}\widehat{w} & & \xrightarrow{\gamma_{uv,w}} \\ & \Downarrow \gamma_{u,v,w} & \\ & & \widehat{u}\widehat{v}\widehat{w} \\ \widehat{u}\widehat{v}\widehat{w} & & \xrightarrow{\gamma_{u,vw}} \\ & \xrightarrow{\widehat{u}\gamma_{v,w}} & \widehat{u}\widehat{v}\widehat{w} \end{array}$$

- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , one critical branching

$$(\gamma_{1,u}, \iota_x^- \widehat{u})$$

which is confluent and gives the following 3-cell, written with ι_x instead of ι_x^- ,

$$\begin{array}{ccc} & \xrightarrow{\widehat{1}_x \widehat{u}} & \widehat{1}_x \widehat{u} \\ \iota_x \widehat{u} & & \xrightarrow{\gamma_{1_x,u}} \\ & \Downarrow \lambda_u & \\ & & \widehat{u} \\ \widehat{u} & & \xrightarrow{1_{\widehat{u}}} \\ & \xrightarrow{\widehat{u}} & \widehat{u} \end{array}$$

- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , one critical branching

$$(\gamma_{u,1}, \widehat{u}\iota_y^-)$$

which is confluent and gives the following 3-cell, written with ι_y instead of ι_y^- ,

$$\begin{array}{ccc} & \xrightarrow{\widehat{u}\widehat{1}_y} & \widehat{u}\widehat{1}_y \\ \widehat{u}\iota_y & & \xrightarrow{\gamma_{u,1_y}} \\ & \Downarrow \rho_u & \\ & & \widehat{u} \\ \widehat{u} & & \xrightarrow{1_{\widehat{u}}} \\ & \xrightarrow{\widehat{u}} & \widehat{u} \end{array}$$

As a consequence, we get that these three families of 3-cells form a homotopy basis of the $(2, 1)$ -category freely generated by the standard presentation of \mathbf{C} . Thus, the standard coherent presentation of \mathbf{C} is, indeed, a coherent presentation of \mathbf{C} .

3.2.8. Example (The reduced standard coherent presentation). Given a category \mathbf{C} , the reduced standard presentation of \mathbf{C} is the 2-polygraph with the following cells

- the same 0-cells as \mathbf{C} ,
- one 1-cell $\widehat{u} : x \rightarrow y$ for every non-identity 1-cell $u : x \rightarrow y$ of \mathbf{C} ,
- one 2-cell $\gamma_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{uv}$ for every non-identity 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathbf{C} such that uv is not an identity,

- one 2-cell $\gamma_{u,v} : \widehat{uv} \Rightarrow 1_x$ for every non-identity 1-cells $u : x \Rightarrow y$ and $v : y \Rightarrow x$ of \mathbf{C} such that $uv = 1_x$.

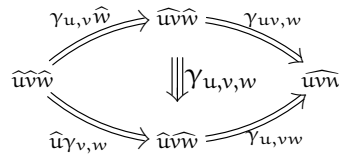
This 2-polygraph is convergent. Indeed, for termination, one checks that each generating 2-cell $\gamma_{u,v}$ has source of size 2 and target of size 0 or 1. As a consequence, for every non-degenerate 2-cell $f : u \Rightarrow v$ of the free 2-category, the size of u is strictly greater than the size of v .

For confluence, we note that this 2-polygraph has one critical branching

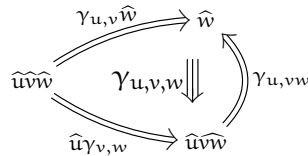
$$(\gamma_{u,v}\widehat{w}, \widehat{u}\gamma_{v,w})$$

for every non-identity 1-cells $u : x \rightarrow y$, $v : y \rightarrow z$ and $w : z \rightarrow t$. All these critical branchings are confluent. To check this fact, we distinguish four cases, depending on whether uv and vw are identities or not.

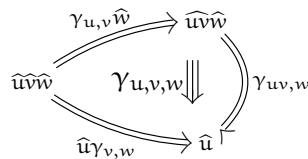
- if neither uv or vw is an identity:



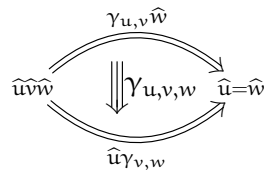
- if uv is an identity, but not vw :



- if uv is not an identity, but vw is:



- if uv and vw are identities, and thus $u = uvw = w$:



3. Rewriting methods for computing coherent presentations

The *reduced standard coherent presentation* of \mathbf{C} is the $(3, 1)$ -polygraph obtained from the reduced standard presentation of \mathbf{C} by adjunction of all these 3-cells $\gamma_{u,v,w}$ and, by 3.2.6, it is a coherent presentation of \mathbf{C} .

This result can also be obtained by proving that the reduced standard coherent presentation of \mathbf{C} is 2-Tietze-equivalent to the standard coherent presentation of \mathbf{C} . Indeed, we can apply the following sequence of 2-Tietze transformations to the reduced version to get the non-reduced one:

- for every 0-cell x of \mathbf{C} , the coherent adjunction of the 1-cell $\widehat{1}_x$ and the 2-cell ι_x ,
- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , the coherent adjunction of the 2-cell $\gamma_{1_x, u}$ and the 3-cell λ_u ,
- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , the coherent adjunction of the 2-cell $\gamma_{u, 1_y}$ and the 3-cell ρ_u ,
- for every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathbf{C} , the coherent adjunctions of the 3-cells $\gamma_{1_x, u, v}$, $\gamma_{u, 1_y, v}$ and $\gamma_{u, v, 1_z}$ that correspond to the associativity condition (they are parallel to 3-cells made of λ s and ρ s),
- the replacements of the $\gamma_{u,v}$ and $\gamma_{u,v,w}$ by the adequate 3-cells when uv or vw is an identity.

As a side remark, we get that, in the case of monoids, the notions of unital and non-unital actions give equivalent categories, as noted by Deligne in [7].

3.3. Knuth-Bendix completion procedure

Thanks to Squier's theorem, we can extend a convergent presentation into a coherent presentation. However, when the presentation is not convergent, we cannot apply this construction directly. For that, Knuth-Bendix completion procedure, [14], gives a way to transform some non-convergent 2-polygraphs into convergent ones.

3.3.1. Termination orders and direction of 2-cells. Let Σ be a 2-polygraph. A *reduction order* for Σ is an order relation \leq on parallel 1-cells of Σ^* such that the following properties are satisfied:

- The composition of 1-cells of Σ^* is strictly monotone in both arguments.
- Every decreasing family $(u_n)_{n \in \mathbb{N}}$ of parallel 1-cells of Σ^* is stationary.

A *termination order* for Σ is a reduction order \leq such that, for every 2-cell φ of Σ , the strict inequality $s(\varphi) > t(\varphi)$ holds.

As a direct consequence of the definition, a 2-polygraph that admits a termination order is terminating. In fact, the converse is also true: for a terminating 2-polygraph Σ , we define, for every 1-cell u in Σ^* , the natural number $\xi(u)$ as the maximal size of the 2-cells of Σ^* with source u . The order defined by $u \leq v$ when $\xi(u) \leq \xi(v)$ is a termination order for Σ .

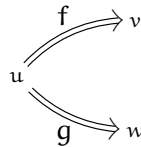
3.3.2. The idea of the procedure. Knuth-Bendix completion procedures considers a 2-polygraph Σ equipped with a termination order (hence Σ must be terminating) and builds an increasing sequence of terminating 2-polygraphs that progressively correct the obstructions to confluence:

$$\Sigma = c^0(\Sigma) \rightsquigarrow c^1(\Sigma) \rightsquigarrow c^2(\Sigma) \rightsquigarrow (\dots)$$

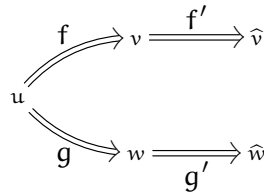
3.3. Knuth-Bendix completion procedure

To build $c^{k+1}(\Sigma)$ from $c^k(\Sigma)$, the procedure checks the critical branchings of $c^k(\Sigma)$, where potential obstructions to confluence lie. Given a critical branching (f, g) of $c^k(\Sigma)$ with source u , one progressively reduce their respective targets, say v and w , for example by using a normalisation strategy. Since $c^k(\Sigma)$ terminates, one will reach normal forms \hat{v} and \hat{w} . If $\hat{v} = \hat{w}$, then the branching is confluent (the converse is not necessarily true since $c^k(\Sigma)$ is not confluent). Otherwise, the completion procedure tries to add a generating 2-cell between \hat{v} and \hat{w} , progressively building $c^{k+1}(\Sigma)$. Let us detail this procedure.

3.3.3. The completion procedure. Let Σ be a 2-polygraph with a termination order \leq . For every critical branching

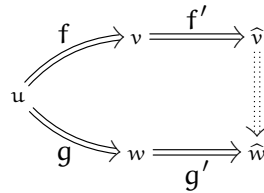


of Σ , we compute 2-cells $f' : v \Rightarrow \hat{v}$ and $g' : w \Rightarrow \hat{w}$ in Σ^* , where \hat{v} and \hat{w} are some normal forms for v and w , respectively, as in

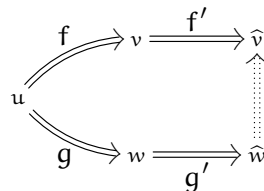


There are four possibilities:

- if $\hat{v} = \hat{w}$, then the critical branching is already confluent,
- if $\hat{v} > \hat{w}$, we apply a coherent adjunction to add the dotted 2-cell



- if $\hat{v} < \hat{w}$, we apply a coherent adjunction to add the dotted 2-cell



- otherwise, the completion fails: we do not know how to make the branching (f, g) confluent while preserving termination.

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After doing this operation for each branching of Σ , we get a terminating 2-polygraph $c^1(\Sigma)$. It is equal to Σ if, and only if, Σ is confluent and finite if, and only if, Σ is finite. Moreover, $c^1(\Sigma)$ is obtained from Σ by coherent adjunctions of 2-cells, so that $c^1(\Sigma)$ is 1-Tietze-equivalent to Σ .

If 2-cells have been added to Σ to get $c^1(\Sigma)$, the completion procedure repeats the operations on the critical branchings of $c^1(\Sigma)$ that are generated by at least one of the new 2-cells, and so on. If no failure occurs, the procedure either reaches a stable state, hence a confluent 2-polygraph, or can forever, otherwise:

$$\Sigma = c^0(\Sigma) \rightsquigarrow c^1(\Sigma) \rightsquigarrow c^2(\Sigma) \rightsquigarrow (\dots)$$

In the non-failing case, the *completion of Σ* is the 2-polygraph

$$C(\Sigma) = \bigcup_{k=0}^{\infty} c^k(\Sigma).$$

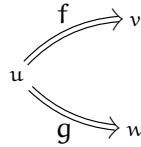
Remark. Let us note that $C(\Sigma)$ depends on several choices. First, for a given critical branching (f, g) , the 2-cells f' and g' have no reason to be unique. In fact, it is even possible that the branching (f, g) is confluent but that the 2-cells f' and g' we choose do not reach the same normal form, since Σ is not necessarily confluent as a whole. To solve this problem in practice, the 2-cells f' and g' can be built according to a normalisation strategy, as described in [11]. Also, the choice of another termination order can lead to different outcomes of the completion procedure: even in the case of a failure, a different choice of termination order can lead to a successful computation.

3.3.4. Proposition. *Let Σ be a terminating 2-polygraph that presents a category \mathbf{C} . The completion $C(\Sigma)$ of Σ , when it exists, is a convergent presentation of \mathbf{C} . Moreover, $C(\Sigma)$ is finite if, and only if, Σ is finite and the increasing sequence $(c^k(\Sigma))_{k \in \mathbb{N}}$ is stationary.*

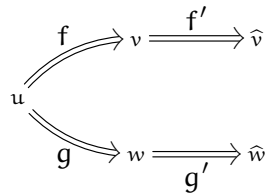
3.4. Homotopical completion

We extend Knuth-Bendix completion procedure to the case of $(3, 1)$ -polygraphs. Homotopical completion functions the same way as completion, by looking to critical branchings one by one; the procedure potentially adds 2-cells, but also 3-cells that tend towards a homotopy basis.

3.4.1. The procedure. Let Σ be a 2-polygraph, seen as a $(3, 1)$ -polygraph with no 3-cell, equipped with a termination order \leq . For every critical branching

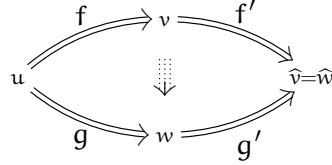


of Σ , we compute 2-cells $f' : v \Rightarrow \hat{v}$ and $g' : w \Rightarrow \hat{w}$ in Σ^* , where \hat{v} and \hat{w} are some normal forms for v and w , respectively, as in

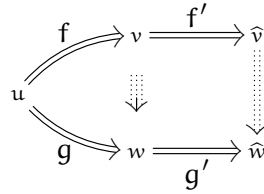


There are four possibilities:

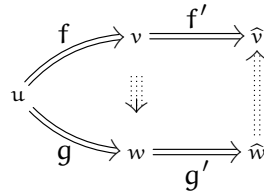
- if $\hat{v} = \hat{w}$, we apply an adjunction to add the dotted 3-cell



- if $\hat{v} > \hat{w}$, we apply a coherent adjunction to add the dotted 2-cell and 3-cell



- if $\hat{v} < \hat{w}$, we apply a coherent adjunction to add the dotted 2-cell and 3-cell



- otherwise, the completion fails: we do not know how to make the branching (f, g) confluent while preserving termination.

After performing these operations for each branching of (f, g) , we get a $(3, 1)$ -polygraph $hc^1(\Sigma)$, whose underlying 2-polygraph is $c^1(\Sigma)$. Homotopical completion repeats this construction (only exploring the new critical branchings each time) until it reaches a stable state, which may or may not happen. The operations can be summarised as follows:

$$\Sigma = hc^0(\Sigma) \rightsquigarrow hc^1(\Sigma) \rightsquigarrow hc^2(\Sigma) \rightsquigarrow (\dots)$$

Each arrow is made of 1-Tietze transformations (adjunctions of 3-cells for confluent critical branchings) and 2-Tietze transformations (coherent adjunctions of 2-cells and 3-cells for non-confluent critical branchings).

In the non-failing case, the *homotopical completion of Σ* is the $(3, 1)$ -polygraph

$$HC(\Sigma) = \bigcup_{k=0}^{\infty} hc^k(\Sigma).$$

3. Rewriting methods for computing coherent presentations

Remark. We note that applying Squier's theorem to the usual completion $C(\Sigma)$ of Σ , which is the underlying 2-polygraph of $HC(\Sigma)$, yields a $(3, 1)$ -polygraph that is 2-Tietze equivalent to $HC(\Sigma)$: in what follows, we consider that $HC(\Sigma)$ and $\mathcal{S}(C(\Sigma))$ are equal. But, from a computational point of view, computing $\mathcal{S}(C(\Sigma))$ would require to compute again all the critical branchings of $C(\Sigma)$, whereas homotopical completion builds the homotopy basis in parallel to the convergent presentation.

Using the properties of the usual completion and Squier's theorem, we get:

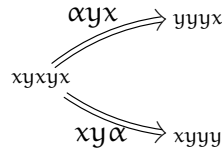
3.4.2. Theorem. *Let Σ be a terminating 2-polygraph that presents a category \mathbf{C} . The homotopical completion $HC(\Sigma)$ of Σ is, when it exists, a coherent presentation of \mathbf{C} , whose underlying 2-polygraph is a convergent presentation of \mathbf{C} . Moreover, $HC(\Sigma)$ is finite if, and only if, Σ is finite and the increasing sequence $(hc^k(\Sigma))_{k \in \mathbb{N}}$ is stationary.*

3.4.3. Example. Let us consider the monoid \mathbf{M} presented by the 2-polygraph Σ with two 1-cells, x and y , and one 2-cell

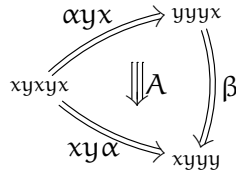
$$xyx \xrightarrow{\alpha} yy.$$

The monoid \mathbf{M} is a Garside group, with Garside element y^3 , see [13]. To prove that the 2-polygraph Σ terminates we can consider, for example, the order given by the number of copies of x in a word.

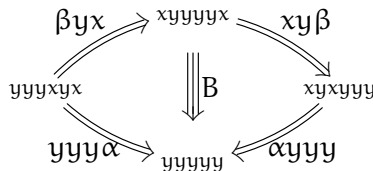
Let us apply the homotopical completion procedure to Σ . It has one critical branching, which is not confluent:



With the chosen order, based on the number of copies of x , the completion procedure would fail. We extend the order by considering, as a secondary condition, the degree-wise left lexicographic order given by $x < y$. Then, the completion procedure coherently adds the following 2-cell β and 3-cell A :



Since we have added a new 2-cell, we must explore the critical branchings it creates. There is only one such branching, which is confluent, yielding a new 3-cell B :



This time, we have added no 2-cell: the procedure stops with a coherent presentation of \mathbf{M} with two 2-cells α and β , which are a convergent presentation of \mathbf{M} , and two 3-cells A and B , forming a homotopy basis.

3.5. Homotopical reduction

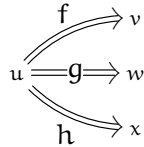
We have seen that the homotopical completion can be used to compute a coherent presentation of a category from a convergent presentation. However, the resulting coherent presentation is in general bigger than we could expect. Here we present a procedure that can be used towards that goal, homotopical reduction, divided in two parts.

3.5.1. Homotopical reduction in dimension 2. Let Σ be a terminating 2-polygraph. When successful, the homotopical completion procedure yields a $(3, 1)$ -polygraph $\text{HC}(\Sigma)$ built on Σ by adjunctions of 2-cells and 3-cells. All the adjunctions of 2-cells are coherent ones, made after examination of a non-confluent critical branching. Thus, we can reverse these coherent adjunctions at the end of the procedure, to coherently remove the extra 2-cells and 3-cells.

The resulting $(3, 1)$ -polygraph, is still a coherent presentation of $\bar{\Sigma}$, with the original presentation Σ as underlying 2-polygraph. But we can also postpone these coherent eliminations after a simplification of the homotopy basis we have computed, such the homotopical reduction in dimension 3.

3.5.2. The idea of homotopical reduction in dimension 3. It may happen that some 3-cells of a coherent presentation Σ are superfluous, in the sense that they are not required to relate parallel 2-cells. For example, given a 3-cell $A : f \Rightarrow g$ in Σ , there might also exist a 3-cell $B : f \Rightarrow g$ in $(\Sigma \setminus \{A\})^\top$. In that case, removing A from Σ is a 2-Tietze transformation and, thus, this operation produces a smaller coherent presentation of the same category. The difficulty here is to compute such a 3-cell B . Here we describe one way to do so, by means of critical triple branchings.

3.5.3. Triple branchings. Let us recall from [11] that, for a 2-polygraph Σ , a *triple branching* of Σ is a triple (f, g, h) of 2-cells of Σ^* with a common source, as in the diagram



The triple branchings are classified in the same way as branchings. A triple branching (f, g, h) is *local* when f, g and h are rewriting steps. Local branchings belong to one of the three following families:

- *Aspherical* triple branchings have two of their 2-cells equal.
- *Peiffer* triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two.
- *Overlapping* triple branchings are the remaining local branchings.

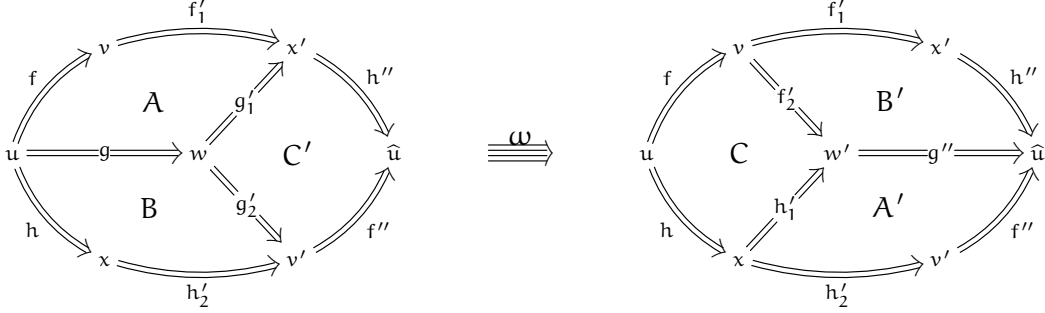
Local triple branchings are compared by the order \preceq generated by the relations

$$(f, g, h) \preceq (ufv, ugv, uhv)$$

given for any local triple branching (f, g, h) and any possible 1-cells u and v of Σ^* . An overlapping local triple branching that is minimal for the order \preceq is called a *critical triple branching*.

3. Rewriting methods for computing coherent presentations

3.5.4. Generating triple confluences. Let Σ be a convergent 2-polygraph. A family of generating triple confluences of Σ is a cellular extension of $\mathcal{S}(\Sigma)^\top$ that contains exactly one 4-cell with shape



for every critical triple branching (f, g, h) of Σ .

Such a family always exist. Indeed, we can build such a 4-cell ω as follows. First, we consider the branching (f, g) , we use the confluence of Σ to get f'_1 and g'_1 and the coherence of Σ to build the 3-cell A . We repeat this step with the branchings (g, h) and (f, h) . Then, we consider the branching (f'_1, f'_2) and we use the convergence of Σ to get g'' and h'' with \hat{u} as common target, plus the 3-cell B' by coherence. We do the same operation with (h'_1, h'_2) to get A' . Finally, we build C' to relate the parallel 2-cells $g'_1 \star_1 h''$ and $g'_2 \star_1 f''$.

This construction includes many choices. In [11], we have explained how to use a higher-dimensional normalisation strategies to make these constructions in a coherent way. In that case, constructing a 4-cell for each critical triple branching of Σ yields a homotopy basis of the $(3, 1)$ -category $\mathcal{S}(\Sigma)^\top$. However, we do not need this property here and we stick with arbitrary choices.

3.5.5. Homotopical reduction in dimension 3. Let Σ be a convergent 2-polygraph and let Γ be family of generating triple confluences. For each 4-cell ω of Γ , we apply the following operation.

The source and the target of a 4-cell ω of Γ is made \star_2 -compositions of 3-cells of $\mathcal{S}(\Sigma)^\top$ with shape

$$f \star_1 u A^\varepsilon v \star_1 g$$

where A is a 3-cell of $\mathcal{S}(\Sigma)$, ε is $+$ or $-$, f and g are 2-cells of Σ^\top and u and v are 1-cells of Σ^\top . Then, if there is such a 3-cell for which u and v are identities, this 3-cell appears in an invertible context in the boundary of ω . In that case, up to composition by 2-cells and 3-cells of its boundary, we may assume that ω has the following shape

$$\omega : B \rightrightarrows A,$$

where B is a 3-cell of $\mathcal{S}(\Sigma)^\top$.

Now, if the 3-cell A does not appear in B , then we can apply a coherent elimination of A and ω to $\mathcal{S}(\Sigma)$, so that $\mathcal{S}(\Sigma)$ is 2-Tietze-equivalent to $\mathcal{S}(\Sigma) \setminus \{B\}$. Let us note that there may be several possible choices for the 3-cell A , since other ones can occur in an invertible context in the boundary of ω . Also, the order of examination of the 4-cells can change the result.

After doing this operation once, if possible, for each 4-cell of Γ , we get a $(3, 1)$ -polygraph, denoted by $\mathcal{R}(\Sigma)$ and which is, by construction, 2-Tietze-equivalent to $\mathcal{S}(\Sigma)$. Again, due to the potential choices in the procedure, this $(3, 1)$ -polygraph is defined up to 2-Tietze equivalence only. Nevertheless, we get:

3.5.6. Proposition. *If Σ is a convergent presentation of a category \mathbf{C} , then the $(3, 1)$ -polygraph $\mathcal{R}(\Sigma)$ is a coherent presentation of \mathbf{C} .*

3.6. Reduced homotopical completion

Let Σ be a terminating 2-polygraph. The *reduced homotopical completion* of Σ is the $(3, 1)$ -polygraph $\text{RHC}(\Sigma)$ obtained as the result of the following successive transformations on Σ , when no failure occurs:

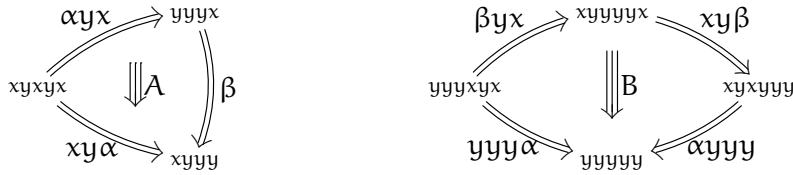
- homotopical completion,
- homotopical reduction in dimension 3,
- homotopical reduction in dimension 2.

3.6.1. Theorem. *Let Σ be a terminating presentation of a category \mathbf{C} . When it exists, the reduced homotopical completion $\text{RHC}(\Sigma)$ of Σ is a coherent presentation of \mathbf{C} whose underlying 2-polygraph is Σ .*

3.6.2. Example. Thanks to homotopical completion, we have seen that the monoid \mathbf{M} admits a coherent presentation by the $(3, 1)$ -polygraph with two 1-cells x and y , two 2-cells

$$\alpha : xyx \Rightarrow yy \quad \text{and} \quad \beta : yyyx \Rightarrow xyyy$$

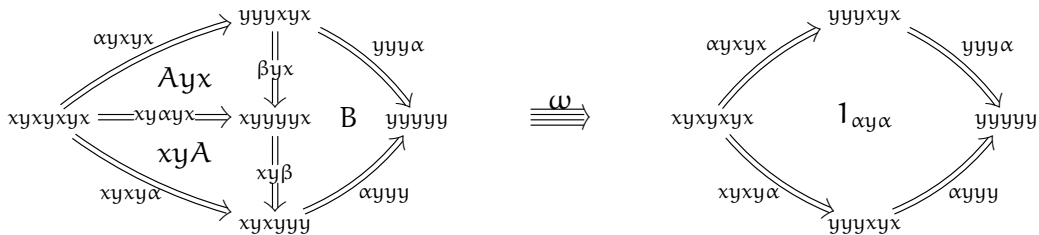
and two 3-cells



To apply homotopical reduction, we compute 4-cells associated to critical triple branchings. Here, we only need to consider the following one, with source $xyxyxyx$:

$$(\alpha xyx, xy\alpha x, xyxy\alpha).$$

The corresponding 4-cell is:



As a consequence, homotopical reduction proves that B is a superfluous 3-cell, *i.e.*, that A alone forms a homotopy basis. Then, we coherently eliminate β and A , which have been added by homotopical completion.

We get that the original 2-polygraph with the 2-cell α only is, when seen as a $(3, 1)$ -polygraph with no 3-cell, a coherent presentation of the monoid \mathbf{M} .

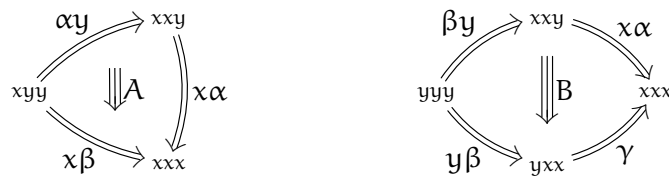
3. Rewriting methods for computing coherent presentations

3.6.3. Example. Let us consider the monoid \mathbf{M} with presented by the 2-polygraph Σ with two 1-cells x and y and two 2-cells

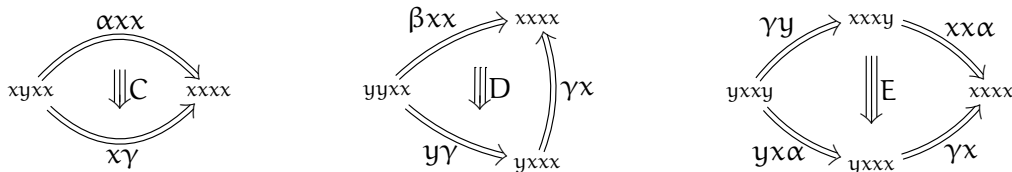
$$\alpha : xy \Rightarrow xx \quad \text{and} \quad \beta : yy \Rightarrow xx.$$

This 2-polygraph terminates. To prove this fact, we can, for example, count the number of occurrences of the 1-cell y . However, in prevision for completion, we prefer to consider the degree-wise left lexicographic order generated by $y > x$.

Let us apply the reduced homotopical completion procedure to Σ . It has two critical branchings, one of them being confluent and the other one requiring the coherent adjunction of a new 2-cell γ , resulting in:



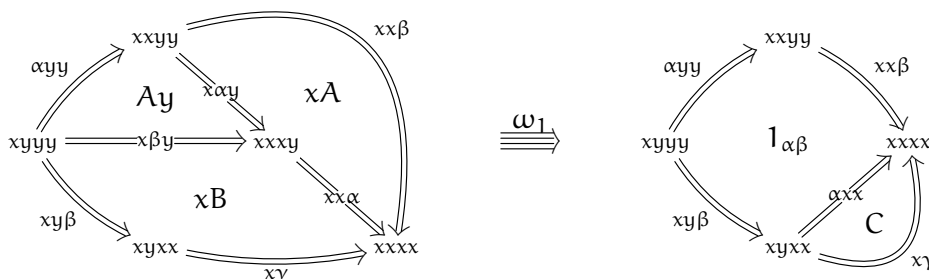
The adjunction of γ generates three new critical branchings, all of them being confluent:



Thus, the homotopical completion $\text{HC}(\Sigma)$ of Σ is a coherent presentation of \mathbf{M} with two 2-cells, three 2-cells and five 3-cells.

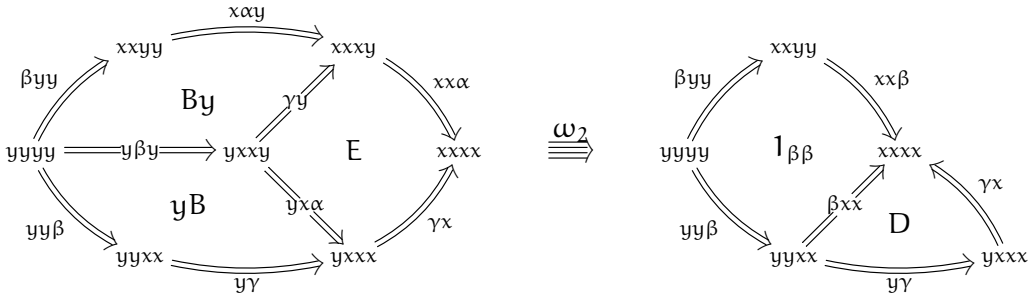
Now, let us apply the homotopical reduction procedure to this $(3, 1)$ -polygraph. We start with the study of the critical triple branchings of Σ . We note that, if γ is involved in a critical triple branching, then the source of the corresponding 2-cells has size at least 5. Moreover, the three 2-cells are homogeneous: the source and target of each one have the same size. As a consequence, the 1-cells that appear in a triple generating confluence associated to a critical triple branching that contains γ have size at least 5. This excludes the possibility to find one of the 3-cells of $\text{HC}(\Sigma)$ in an invertible context in the source or target of the corresponding 4-cell.

There remains two critical triple branchings to study. The first one gives the following 4-cell:



Thus, the 3-cell C appears in an invertible context in the boundary of ω_1 : it is a superfluous 3-cell in the homotopy basis of $\text{HC}(\Sigma)$, leaving A , B , D and E . The last critical triple branching generates the

following 4-cell



Here, the 4-cell ω_2 can be used to coherently remove one of the 3-cells D or E , since both appear in an invertible context in the boundary of ω_2 . We get a homotopy basis made of the 3-cells A , B and D (or E).

Finally, the homotopical reduction of 2-cells coherently eliminates the 2-cell γ and the 3-cell B , that were adjoined during homotopical completion to make the critical branching $(\beta y, y\beta)$ confluent. This transformation also replaces each occurrence of γ in D (or E) by its “definition”, given by B :

$$\gamma = y\beta^- \star_1 \beta y \star_1 x\alpha.$$

As a result, we get a coherent presentation of \mathbf{M} with the reduced homotopical completion $\text{RHC}(\Sigma)$ made of Σ and the homotopy basis whose 3-cells are A and D (or E).

3.7. Métivier-Squier reduction revisited

In a 2-polygraph, a case of potential simplifications occurs when some 2-cells can be applied, as rewriting rules, to other 2-cells. What follows is not used in the rest of this paper but we present it because it is a classical result in rewriting that can be extended to coherent presentations by using the formalism of Tietze transformations. The original version is due to Métivier for term rewriting systems, [19], and to Squier for word rewriting systems, [23]. In fact, the proof works for any type of rewriting systems, including n -polygraphs for any n .

3.7.1. Reduced convergent presentations. A 2-polygraph Σ is *left-reduced* when the source of each of its 2-cells is only reducible by that 2-cell. It is *right-reduced* when the target of each of its 2-cells is a normal form. It is *reduced* when it is both left-reduced and right-reduced.

Let us note that, in that case, for every 1-cell u of Σ^* , there exists finitely many 2-cells with size 1 and source u in Σ^* : indeed, we have at most one such 2-cell for every possible decomposition $u = v w v'$ and the number of those decompositions is finite in a free category.

3.7.2. Lemma. *Every convergent 2-polygraph Σ is 1-Tietze-equivalent to a reduced and convergent 2-polygraph $\hat{\Sigma}$. Moreover, if Σ is finite, then $\hat{\Sigma}$ can be chosen to be finite.*

Proof. Let Σ be a convergent 2-polygraph. We successively transform Σ as follows:

3. Rewriting methods for computing coherent presentations

- First, we define Σ' as the 2-polygraph with the same 2-cells as Σ but, for each one $\varphi : u \Rightarrow v$, we replace its target v by its normal form, hence the normal form of u :

$$\begin{array}{ccc}
 u \xrightarrow{\varphi} v & \mapsto & u \xrightarrow{\varphi} v \\
 \downarrow & & \downarrow \\
 \hat{u} & & \hat{u}
 \end{array}$$

This is a 1-Tietze transformation and, by construction, the 2-polygraph Σ' is right-reduced and convergent, and it is finite when Σ is.

- Then, we define Σ'' as the 2-polygraph with exactly one 2-cell from u to \hat{u} when Σ' has at least one 2-cell from u to \hat{u} :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \varphi_1 & \\
 u & \curvearrowright & \hat{u} \\
 & \vdots & \\
 & \varphi_n & \\
 & \curvearrowleft & \\
 u & & \hat{u}
 \end{array} & \mapsto & u \xrightarrow{\varphi} \hat{u}
 \end{array}$$

This is a 1-Tietze transformation and, by construction, the 2-polygraph Σ'' is still right-reduced and convergent, and finite when Σ is.

- Finally, we define $\hat{\Sigma}$ as the 2-polygraph Σ'' minus the 2-cells whose source can be reduced by another 2-cell of Σ'' :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 v w v' & \xrightarrow{\varphi} & \widehat{v w v'} \\
 & \searrow & \uparrow \\
 & v \psi v' & v \widehat{w v'} \\
 & & \uparrow \\
 & & \widehat{v w v'}
 \end{array} & \mapsto & \begin{array}{ccc}
 v w v' & \xrightarrow{\varphi} & \widehat{v w v'} \\
 & \searrow & \uparrow \\
 & v \psi v' & v \widehat{w v'} \\
 & & \uparrow \\
 & & \widehat{v w v'}
 \end{array}
 \end{array}$$

This is still a 1-Tietze transformation and the 2-polygraph $\hat{\Sigma}$ has the required properties. \square

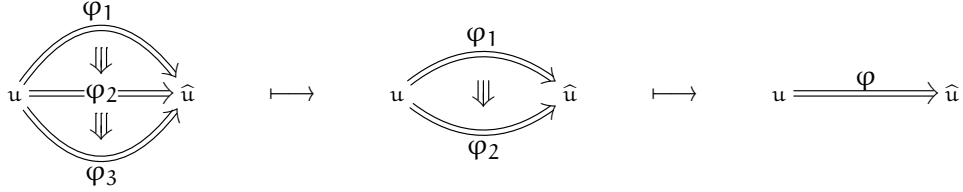
3.7.3. Reduction for coherent presentations. This procedure extends to the case of coherent presentations. Indeed, let us consider a coherent presentation Σ of a category, whose underlying 2-polygraph is convergent. For example, Σ can be the result of a homotopical completion. Then the three kinds of operations we have seen lift to 2-Tietze transformations:

- The first operation is the same, but seen as a composition of 2-Tietze transformations, a coherent adjunction followed by a coherent elimination:

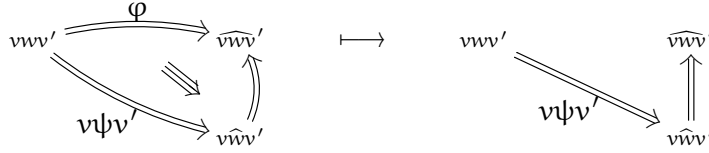
$$\begin{array}{ccc}
 u \xrightarrow{\varphi} v & \mapsto & \begin{array}{ccc}
 & \varphi & \\
 u & \curvearrowright & v \\
 & \searrow & \downarrow \\
 & \varphi' & \hat{u}
 \end{array} & \mapsto & u \xrightarrow{\varphi} v \\
 \downarrow & & & & \downarrow \\
 \hat{u} & & & & \hat{u}
 \end{array}$$

- The second operation is a composite of coherent eliminations. Indeed, in the case of parallel 2-cells $\varphi_1, \dots, \varphi_n : u \Rightarrow \hat{u}$, since Σ is a coherent presentation, we have 3-cells of Σ^\top between each

pair of these 2-cells. For example, in the case $n = 3$, we remove φ_2 and φ_3 by the following sequence of coherent eliminations:



- The last case is a coherent elimination. Indeed, the fact that Σ is a coherent presentation ensures that there exists a 3-cell that can be used to coherently eliminate φ :



3.7.4. Proposition. *Every coherent presentation Σ whose underlying 2-polygraph is convergent is 2-Tietze equivalent to a coherent presentation $\widehat{\Sigma}$ whose underlying 2-polygraph is reduced and convergent. Moreover, if Σ is finite, then $\widehat{\Sigma}$ can be chosen to be finite.*

3.8. Homotopy bases transfer theorem

In [10], it was proved that, given an n -polygraph with a finite homotopy basis, then another n -polygraph that presents the same $(n - 1)$ -category admits a finite homotopy basis. The proof contained an implicit description of a way to transfer the first homotopy basis to the second n -polygraph. Here we give an explicit description of this construction in the case $n = 2$.

3.8.1. Lemma. *Let \mathbf{C} and \mathbf{D} be categories and let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor. For every presentations Σ of \mathbf{C} and Υ of \mathbf{D} , the functor F lifts to a 2-functor \widetilde{F} , i.e., such that the following diagram commutes:*

$$\begin{array}{ccc} \Sigma^\top & \twoheadrightarrow & \mathbf{C} \\ \widetilde{F} \downarrow & \text{\textcircled{C}} & \downarrow F \\ \Upsilon^\top & \twoheadrightarrow & \mathbf{D} \end{array}$$

Proof. We denote by $\pi : \Sigma^\top \twoheadrightarrow \mathbf{C}$ and $\rho : \Upsilon^\top \twoheadrightarrow \mathbf{D}$ the canonical projections.

On a 0-cell x , we take $\widetilde{F}(x) = F(x)$. If $a : x \rightarrow y$ is a generating 1-cell of Σ , we choose, in an arbitrary way, a 1-cell $\widetilde{F}(a) : F(x) \rightarrow F(y)$ in Υ^\top such that $\rho\widetilde{F}(a) = F\pi(a)$; then, we extend \widetilde{F} by functoriality to every 1-cell of Σ^\top .

Let $\varphi : u \Rightarrow v$ be a generating 2-cell of Σ . By definition of $\widetilde{\Sigma}$, we have $\pi(u) = \pi(v)$, so that $\rho\widetilde{F}(u) = \rho\widetilde{F}(v)$. Thus, by definition of Υ , there exists a 2-cell from $\widetilde{F}(u)$ to $\widetilde{F}(v)$ in Υ^\top . We pick one of these 2-cells for $\widetilde{F}(\varphi)$ and, then, we extend \widetilde{F} to every 2-cell of Σ^\top by functoriality. \square

3. Rewriting methods for computing coherent presentations

3.8.2. Lemma. *Let Σ and Υ be two presentations of the same category \mathbf{C} . Let F and G be liftings of the identity of \mathbf{C} , i.e., such that the following diagram commute:*

$$\begin{array}{ccc} \Sigma^\top & \longrightarrow & \mathcal{C} \\ F \downarrow & \text{\textcircled{C}} & \downarrow \text{Id}_{\mathcal{C}} \\ \Upsilon^\top & \longrightarrow & \mathcal{C} \end{array} \quad \begin{array}{ccc} \Sigma^\top & \longrightarrow & \mathcal{C} \\ G \uparrow & \text{\textcircled{C}} & \uparrow \text{Id}_{\mathcal{C}} \\ \Upsilon^\top & \longrightarrow & \mathcal{C} \end{array}$$

Then, for every 1-cell u in Υ^\top , there exists a 2-cell $\Lambda_u : FG(u) \Rightarrow u$ which is functorial in u , i.e., such that the following relations are satisfied:

$$\Lambda_{1_x} = 1_x \quad \text{and} \quad \Lambda_{uv} = \Lambda_u \Lambda_v$$

Proof. Let a be a generating 1-cell in Υ^\top . By hypothesis on F and G , we have:

$$\rho FG(a) = \pi G(a) = \rho(a).$$

Thus, by definition of $\bar{\Upsilon}$, there exists a 2-cell $\Lambda_a : FG(a) \Rightarrow a$ in Υ^\top . We extend Λ to every 1-cell u in Υ^\top by functoriality. \square

3.8.3. Theorem (Homotopy bases transfer). *Let Σ and Υ be two presentations of the same category \mathbf{C} , let F and G be liftings of the identity of \mathbf{C} and let $\Lambda_u : FG(u) \Rightarrow u$ be functorially chosen in Υ^\top for every 1-cell u of Υ^\top . If Γ is a homotopy basis of Σ^\top , then*

$$\Delta = F(\Gamma) \amalg \Lambda_\Upsilon$$

is a homotopy basis of Υ^\top , where:

- *the cellular extension $F(\Gamma)$ contains one 3-cell*

$$\begin{array}{ccc} & F(f) & \\ \curvearrowright & & \curvearrowleft \\ F(u) & \Downarrow F(\gamma) & F(v) \\ \curvearrowleft & & \curvearrowright \\ & F(g) & \end{array}$$

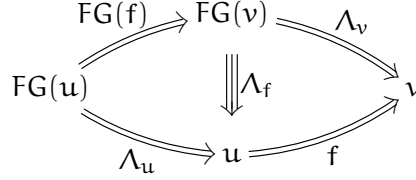
for every 3-cell $\gamma : f \Rightarrow g$ in Γ ,

- *the cellular extension Λ_Υ contains one 3-cell*

$$\begin{array}{ccccc} & FG(\varphi) & & FG(v) & \\ & \curvearrowright & & \curvearrowleft & \\ FG(u) & & \Downarrow \Lambda_\varphi & & v \\ & \curvearrowleft & & \curvearrowright & \\ & \Lambda_u & & \varphi & \end{array}$$

for every 2-cell φ in Υ .

Proof. Let us define a 3-cell Λ_f of Δ^\top , for every 2-cell f of Υ_2 , with the following shape:



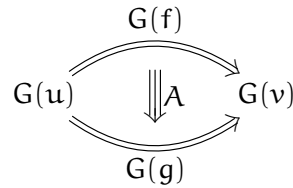
For that, we extend the notation Λ_φ in a functorial way, according to the following formulas:

$$\begin{aligned} \Lambda_{1_u} &= 1_{\Lambda_u} & \Lambda_{fg} &= \Lambda_f \Lambda_g & \Lambda_{f^-} &= FG(f)^- \star_1 \Lambda_f^- \star_1 f^- \\ \Lambda_{f \star_1 g} &= (FG(f) \star_1 \Lambda_g) \star_2 (\Lambda_f \star_1 g). \end{aligned}$$

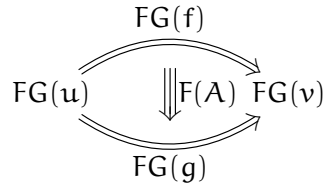
One checks that this is well-defined, *i.e.*, that the definition of Λ_f is compatible with the relations on 2-cells, such as the exchange relation:

$$\Lambda_{fg \star_1 hk} = \Lambda_{(f \star_1 h)(g \star_1 k)}.$$

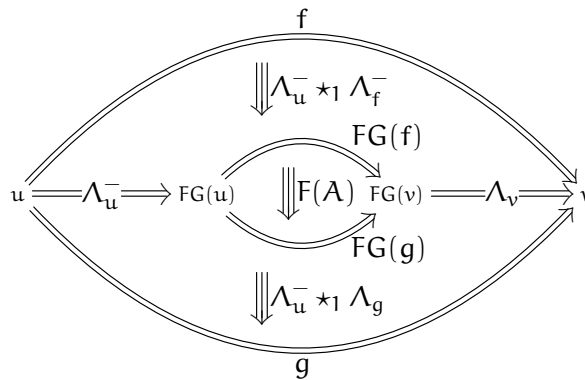
Now, let us consider parallel 2-cells $f : u \Rightarrow v$ and $g \Rightarrow v$ in Υ^\top . The 2-cells $G(f)$ and $G(g)$ are parallel in Σ^\top so that, since Γ is a homotopy basis of Σ^\top , there exists a 3-cell



in Γ^\top . An application of F to A gives the 3-cell



which, by definition of Δ and functoriality of F , is in Δ^\top . Using $F(A)$ and the 3-cells Λ_f and Λ_g , we get the following 3-cell from f to g in Δ^\top :



4. A complete example: Artin-Tits groups of spherical type

This concludes the proof that $\Delta = F(\Gamma) \amalg \Lambda_\gamma$ is a homotopy basis of the $(2, 1)$ -category Υ^\top . \square

4. A COMPLETE EXAMPLE: ARTIN-TITS GROUPS OF SPHERICAL TYPE

In this section, we study the notion of action of the Artin-Tits groups of spherical type, also known as “generalised braid groups”. First, we consider the case of Artin’s presentation, given in terms of generating symmetries and braid relations: there, we use a result by Tits to deduce that a homotopy basis is given by the Tits-Zamolodchikov relations. Then, we explore the case of Deligne’s presentation for Artin-Tits groups of spherical type: there, we apply our machinery of reduced homotopical completion to compute a homotopy basis and recover, and slightly improve, Deligne’s result (Theorem 1.5 of [7]).

4.1. Recollections

4.1.1. Coxeter groups. A *Coxeter group* is a group \mathbf{W} that admits a presentation with a finite set S of generators and the following relations:

$$(st)^{m_{s,t}} = 1, \quad \text{with } m_{s,t} \in \mathbb{N} \amalg \{\infty\}, \quad \text{for every } s \text{ and } t \text{ in } S \quad (1)$$

where $m_{s,t} = \infty$ means that there is, in fact, no relation between s and t and with the requirement that $m_{s,t} = 1$ if, and only if, $s = t$, so that $s^2 = 1$ holds in \mathbf{W} for every s in S . As a consequence, \mathbf{W} can also be seen as the monoid with the same presentation. A Coxeter group may have several presentations that fit the given scheme and several of the following notions depend on a choice of a set of generators: we fix such a set S for \mathbf{W} . According to the parity of $m_{s,t}$, when $s \neq t$ and $m_{s,t} < \infty$, and using $s^2 = t^2 = 1$, relation (1) can also be written as a *braid relation*

$$st \dots = ts \dots \Leftrightarrow \begin{cases} (st)^p = (ts)^p, & \text{when } m_{s,t} = 2p \\ (st)^p s = (ts)^p t, & \text{when } m_{s,t} = 2p + 1. \end{cases} \quad (2)$$

The *length* of an element u of \mathbf{W} is denoted by $l(u)$ and defined as the length of the smallest representative of u in the free monoid S^* . For every u and v in \mathbf{W} , we have $l(uv) \leq l(u) + l(v)$ and we use distinct graphical notations depending on whether the equality holds or not:

$$u \widehat{\vee} v \Leftrightarrow l(uv) = l(u) + l(v)$$

$$u \times v \Leftrightarrow l(uv) < l(u) + l(v)$$

We generalise these graphical notations, such as in the case of a triple $u \widehat{\vee} v \widehat{\vee} w$:

$$u \widehat{\vee} v \widehat{\vee} w \Leftrightarrow l(uvw) = l(u) + l(v) + l(w)$$

$$u \times v \times w \Leftrightarrow l(uvw) < l(u) + l(v) + l(w)$$

4.1.2. Artin-Tits monoids and groups. Given a Coxeter group \mathbf{W} , with generators S , the associated *Artin-Tits monoid* and *Artin-Tits group* are the monoid, denoted by $\mathbf{B}^+(\mathbf{W}, S)$, and the group, denoted by $\mathbf{B}(\mathbf{W}, S)$, generated by S and submitted to the braid relations (2): this is the same presentation as the one of \mathbf{W} , except for the relations $s^2 = 1$. An Artin-Tits group is *of spherical type* when \mathbf{W} is finite. When no confusion may occur on the pair (\mathbf{W}, S) , we simply write \mathbf{B}^+ and \mathbf{B} for $\mathbf{B}^+(\mathbf{W}, S)$ and $\mathbf{B}(\mathbf{W}, S)$, respectively.

For example, the braid group \mathbf{B}_n and the positive braid monoid \mathbf{B}_n^+ are the Artin-Tits group and monoid associated to the Coxeter group \mathbf{S}_n of symmetries: they have $(n - 1)$ generators s_1, \dots, s_{n-1} , submitted to the relations $s_i s_j = s_j s_i$ if $i \leq j - 2$ and $s_i s_j s_i = s_j s_i s_j$ when $i = j - 1$. This is the reason why Artin-Tits groups are also called “generalised braid groups”. This presentation of \mathbf{B} and \mathbf{B}^+ is called *Artin's presentation*.

As shown by Deligne (see [6], Proposition 1.27 and Section 4), the monoid \mathbf{B}^+ satisfies the Öre condition (on the left and on the right), so that we have an equivalence of categories

$$\text{Act}(\mathbf{B}) \approx \text{Act}(\mathbf{B}^+)_{\text{auto}}$$

where the latter is defined as the full subcategory of $\text{Act}(\mathbf{B}^+)$ whose objects are the actions by auto-equivalences: this is an action T on a category \mathbf{C} such that, for each u in \mathbf{B}^+ , the endofunctor T of \mathbf{C} is also an equivalence of categories (see [7], Proposition 1.9). Hence, it is sufficient to study actions of the monoid \mathbf{B}^+ .

4.2. Actions with Artin's presentation

For $I \subset S$, let \mathbf{W}_I be the subgroup of \mathbf{W} spanned by the elements in I . If w is in \mathbf{W} , we denote by $\Gamma(w)$ the 1-dimensional complex defined as follows:

- its vertices are the words

$$\gamma = s_{i_1} \cdots s_{i_r}$$

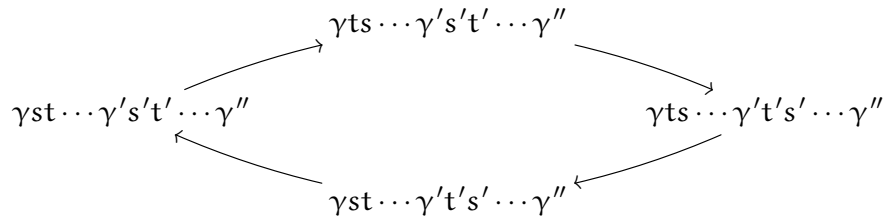
in S^* such that $w = s_{i_1} \cdots s_{i_r}$,

- it has one edge between γ and γ' if one is obtained from the other by a braid relation.

The following result is proved by Ronan in [21], Theorem 2.17, after a result of Tits in [25], Proposition 4.

4.2.1. Theorem. *The fundamental group of $\Gamma(w)$ is “generated” by the loops of the following forms:*

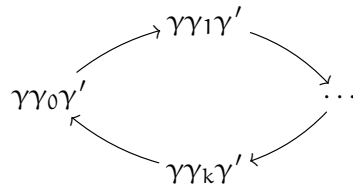
- a) every loop of $\Gamma(w)$ with shape



for $s, t, s',$ and t' in S such that $m_{s,t}$ and $m_{s',t'}$ are finite and $\gamma, \gamma',$ and γ'' in S^* ,

4. A complete example: Artin-Tits groups of spherical type

b) every loop of $\Gamma(w)$ with shape



for $\gamma_0, \dots, \gamma_k$ in some \mathbf{W}_I , for I is a subset of S of cardinality 3 such that \mathbf{W}_I is finite.

In the theorem, “generated” means that, for any vertex δ , the fundamental group $\pi_1(\Gamma(w), \delta)$ is generated by (classes of) paths of the form pfp^{-1} , where f has one of the two forms a) or b), p is a path from δ to the origin of f and p^{-1} is the reverse path.

Set $I = \{s, t, u\}$. Among the loops of the second family, those of the form



correspond to the case where, for instance, u commutes with both s and t : they actually belong to the first case. The ones such that $k > 1$ are associated to reduced expressions of the longest element of \mathbf{W}_I . We denote them by $Z_{s,t,u}$ and call them the *Tits-Zamolodchikov loops*. Since \mathbf{W}_I is a finite Coxeter group, these loops are associated to the longest element in a group of one of the following types:

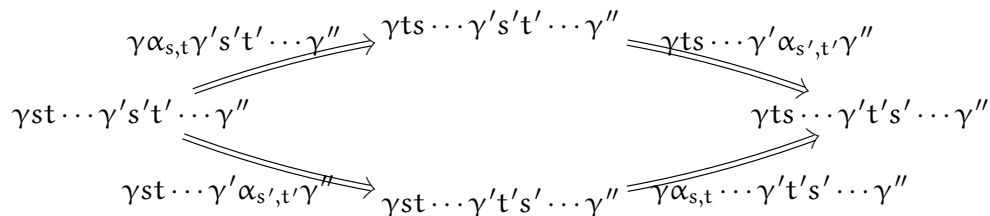
$$A_1 \times A_1 \times A_1 \quad I_n \times A_1 \quad A_3 \quad B_3 \quad H_3$$

4.2.2. The corresponding coherent presentation. The monoid $\mathbf{B}^+ = \mathbf{B}^+(\mathbf{W}, S)$ is presented by the 2-polygraph $\Sigma_{\mathbf{W}, S}$ with one 0-cell, with the elements of S as 1-cells and with one 2-cell

$$\alpha_{s,t} : st \cdots \Rightarrow ts \cdots$$

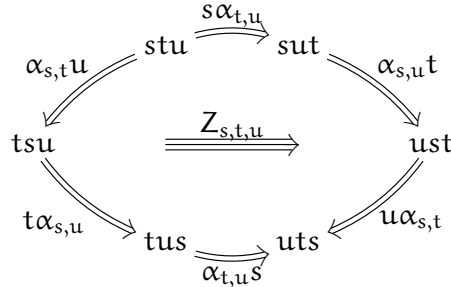
for every braid relation (when $2 \leq m_{s,t} < \infty$). By construction, the 2-cells of the $(2, 1)$ -category $\Sigma_{\mathbf{W}, S}^\top$ are exactly the classes of edges of the graphs $\Gamma(w)$, where w ranges over S^* , modulo the exchange relations. As a consequence, the loops of $\Gamma(w)$ can be transposed as 2-spheres of $\Sigma_{\mathbf{W}, S}^\top$.

The first family is made of degenerate 2-spheres, since they correspond to the exchange relations:

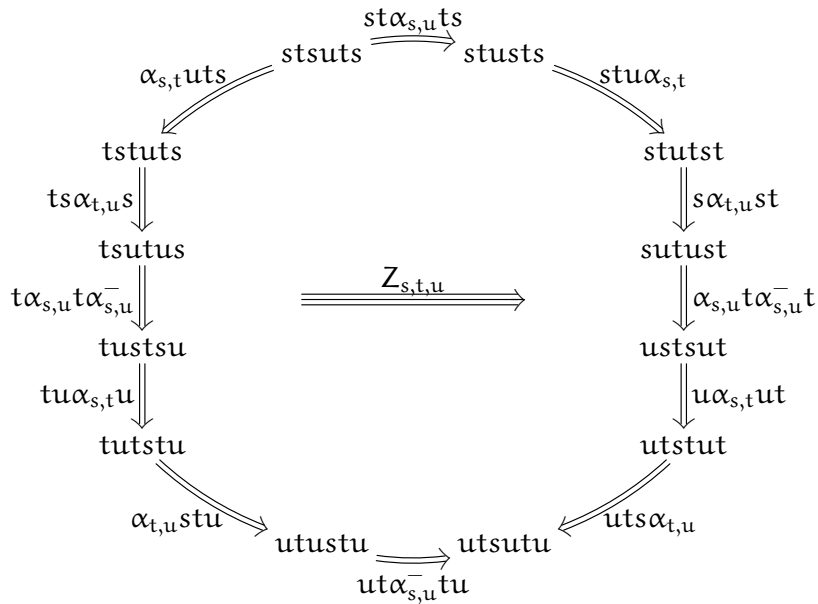


The second family is made of the Tits-Zamolodchikov relations: there is one such 3-cell $Z_{s,t,u}$ for every possible subset $I = \{s, t, u\}$ of cardinality 3 of S such that \mathbf{W}_I is finite. Let us give the two simplest examples. For $A_1 \times A_1 \times A_1$, i.e., when \mathbf{W}_I is isomorphic to \mathbb{Z}^3 , the corresponding Tits-Zamolodchikov

relation is the usual permutohedron also known as Yang-Baxter relation:



And, if \mathbf{W}_I is of type A_3 , i.e., when it is isomorphic to the group S_3 of symmetries, the corresponding Tits-Zamolodchikov relation is:



The other cases are computed by considering a representative (with minimal length) of the longest element of \mathbf{W}_I and computing all the rewriting steps one can perform on it: for example, one can start with $st \cdots u$ for $I_n \times A_1$, $ststutstu$ for B_3 and $ststutsutststutstu$ for H_3 . For the last two cases, B_3 and H_3 , representations in “string diagrams” of the corresponding Tits-Zamolodchikov relations can be found in recent slides by Geordie Williamson, presenting a common work with Ben Elias on a presentation by generators and relations of the monoidal category of Soergel bimodules, [26].

The result of Tits tells us that these two families of 3-spheres form a homotopy basis of $\Sigma_{\mathbf{W},S}^\top$. But, since the first family is degenerate, we get that the Tits-Zamolodchikov relations form a homotopy basis of $\Sigma_{\mathbf{W},S}^\top$.

4.2.3. Theorem. *The $(3, 1)$ -polygraph made of $\Sigma_{\mathbf{W},S}$ and the Tits-Zamolodchikov relations forms a finite coherent presentation of the monoid $\mathbf{B}^+(\mathbf{W}, S)$ of positive braids.*

Thus, as a consequence of Theorem 2.3.2, we get the following result.

4. A complete example: Artin-Tits groups of spherical type

4.2.4. Corollary. *An action of $\mathbf{B}^+(\mathbf{W}, S)$ on a category \mathbf{C} is the same as the following data:*

- for any $s \in S$, an endofunctor $T(s) : \mathbf{C} \rightarrow \mathbf{C}$,
- for any braid relation $st \cdots = ts \cdots$, a natural isomorphism $T(s)T(t) \cdots \simeq T(t)T(s) \cdots$,

such that, for any subset $I = \{s, t, u\}$ of cardinality 3 of S with \mathbf{W}_I finite, the Tits-Zamolodchikov $T(Z_{s,t,u})$ relation holds.

4.3. Actions with Deligne's presentation

Thereafter we assume that \mathbf{W} is a finite Coxeter group, with a fixed set S of generators, *i.e.*, that the Artin-Tits group $\mathbf{B}(\mathbf{W}, S)$ is of spherical type.

4.3.1. Deligne's presentation. The braid monoid $\mathbf{B}^+(\mathbf{W}, S)$ admits another presentation, built from Artin's presentation by adding redundant generators and relations. This presentation turns out to be very useful and is already used by Deligne in [7], Section 1.4, and, implicitly, in [6]. The set of generators of *Deligne's presentation* is $\mathbf{W} \setminus \{1\}$ and there is a relation

$$u|v = uv$$

whenever $u \widehat{v}$ holds, where $\cdot|\cdot$ denotes the product in the free monoid generated by $\mathbf{W} \setminus \{1\}$, to avoid confusion with the product in \mathbf{W} .

For example, the braid monoid \mathbf{B}_3^+ and the group \mathbf{B}_3 are presented by the five generators

$$s_1, \quad s_2, \quad s_1s_2, \quad s_2s_1, \quad s_1s_2s_1,$$

submitted to the six relations

$$s_1|s_2 = s_1s_2, \quad s_2|s_1 = s_2s_1, \quad s_1|s_2s_1 = s_2|s_1s_2 = s_1s_2|s_1 = s_2s_1|s_2 = s_1s_2s_1.$$

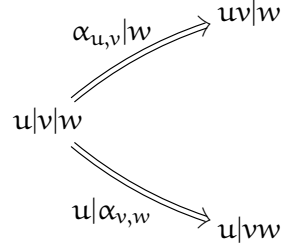
Dropping the reference to \mathbf{W} and S , we denote by Σ the 2-polygraph, corresponding to this presentation, with one 0-cell, with the elements of $\mathbf{W} \setminus \{1\}$ as 1-cells and with one 2-cell

$$\alpha_{u,v} : u|v \Rightarrow uv, \quad \text{for every } u \widehat{v}.$$

We consider the canonical extension of the length map l to the 1-cells of Σ^* as a morphism of monoids, sending $u_1|u_2|\cdots|u_n$ to the family $(l(u_1), \dots, l(u_n))$ of (non-zero) natural numbers. Two such families (p_1, \dots, p_m) and (q_1, \dots, q_n) are compared by the degree-wise right lexicographic strict order, so that we have $(p_1, \dots, p_m) > (q_1, \dots, q_n)$ if $m > n$ or if $m = n$ and there exists i such that $p_n = q_n, \dots, p_{i+1} = q_{i+1}$ and $p_i > q_i$. We compare 1-cells of Σ^* with this strict order, taken back through the length map l . Let us note that this strict order is not total: two distinct 1-cells of the same length are not comparable.

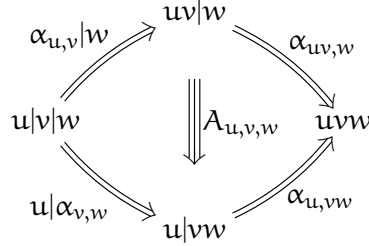
We have that, for every 2-cell $\alpha_{u,v}$ of Σ , the strict inequality $s(\alpha_{u,v}) > t(\alpha_{u,v})$ holds since, for the considered order, the pair $(l(u), l(v))$ is strictly greater than the singleton $l(uv)$. Since the reflexive closure of the considered strict order is a termination order, we get that Σ terminates.

4.3.2. Homotopical completion – the first part. We compute the critical branchings of Σ . There is exactly one for every triple $u \widehat{v} w$:



Then, given such a critical branching, there are two cases, depending on the length of $l(uvw)$.

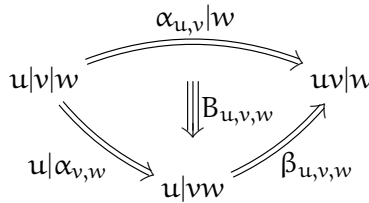
- For $u \widehat{v} w$, the critical branching is confluent, resulting in the adjunction of a 3-cell:



- Otherwise, for $u \widehat{v^x} w$, then both $uv|w$ and $u|vw$ are normal forms. Since $l(vw) > l(w)$, we have $u|vw > uv|w$. Thus, homotopical completion coherently adds a new 2-cell

$$\beta_{u,v,w} : u|vw \Rightarrow uv|w, \quad \text{for every } u \widehat{v^x} w,$$

together with the following 3-cell:



After this first part of homotopical completion, we get a finite and terminating 2-polygraph that is Tietze-equivalent to Σ . But the adjunction of a new family of 2-cells creates new critical branchings that we have to examine.

4.3.3. Homotopical completion – the second part. We now examine the critical branchings involving at least one 2-cell β . We note that the sources of all the 2-cells α and β have size 2 in the free monoid over $\mathbf{W} \setminus \{1\}$. Thus, have the following possibilities:

4. A complete example: Artin-Tits groups of spherical type

- The two 2-cells of Σ that generate the branching overlap on one element of $\mathbf{W} \setminus \{1\}$. In that case, the source of the branching has size 3, with one 2-cell of the branching reducing the leftmost two generating 1-cells and the other one reducing the rightmost two. That case subdivises as follows, depending on the type α or β of the involved 2-cells:

- The source is $u|v|wx$ with $u|v$ reduced by $\alpha_{u,v}$ and $v|wx$ reduced by $\beta_{v,w,x}$. This implies $\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x}$, which happen in two distinct cases:

$$\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x} \quad \text{and} \quad \overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x}$$

- The source is $u|vw|x$ with $u|vw$ reduced by $\beta_{u,v,w}$ and $vw|x$ reduced by $\alpha_{v,w,x}$. This implies $\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x}$, which happens in one case only:

$$\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x}$$

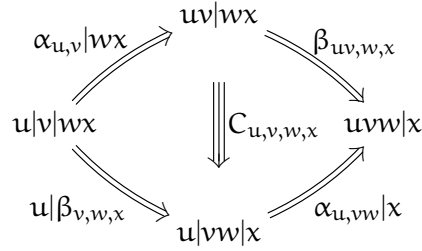
- The source is $u|vw|xy$ with $u|vw$ reduced by $\beta_{u,v,w}$ and $vw|xy$ reduced by $\beta_{v,w,x,y}$. This implies $\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x} \overset{\times}{y}$ and either $\overset{\times}{v} \overset{\times}{w} \overset{\times}{x} \overset{\times}{y}$ or $\overset{\times}{v} \overset{\times}{w} \overset{\times}{x} \overset{\times}{y}$, which happens in two distinct cases:

$$\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x} \overset{\times}{y} \quad \text{and} \quad \overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x} \overset{\times}{y}$$

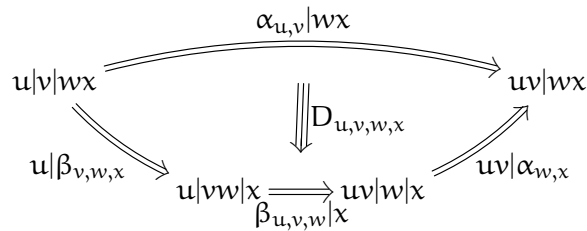
- The two 2-cells of Σ that generate the branching totally overlap, *i.e.*, they have the same source. Since one of those 2-cells must be a β , the source has shape $u|vw$ with $\overset{\times}{u} \overset{\times}{v} \overset{\times}{w}$, preventing the other 2-cell to be an α . The only remaining possibility is to have another decomposition $vw = v'w'$, with $\overset{\times}{u} \overset{\times}{v'} \overset{\times}{w'}$, so that the branching is formed by $\beta_{u,v,w}$ and $\beta_{u,v',w'}$.

We now proceed to the examination of each individual case:

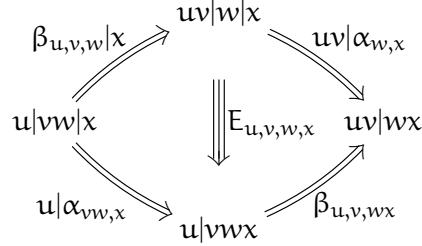
- Case $\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x}$:



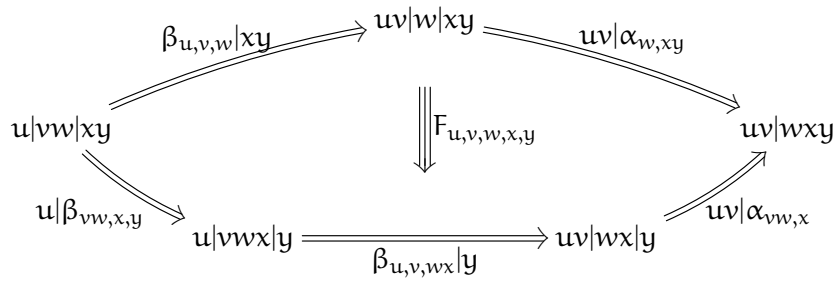
- Case $\overset{\times}{u} \overset{\times}{v} \overset{\times}{w} \overset{\times}{x}$:



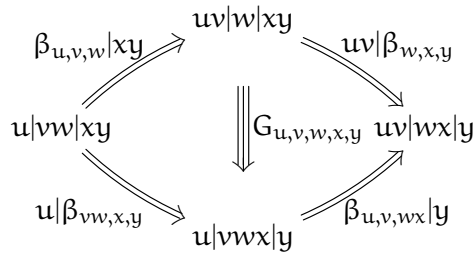
– Case $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x$:



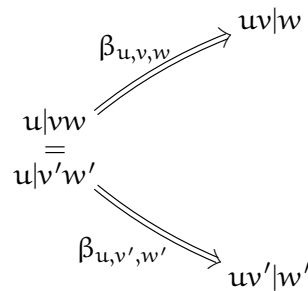
– Case $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x \overset{x}{\curvearrowright} y$:



– Case $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x \overset{x}{\curvearrowright} y$:



– For every u, v, w, v', w' with $vw = v'w'$, $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w$ and $u \overset{x}{\curvearrowright} v' \overset{x}{\curvearrowright} w'$, we have a critical branching

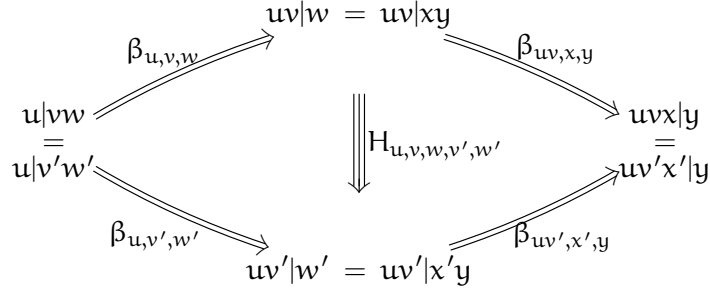


By classical properties of Artin-Tits groups of spherical type, mainly the fact that they are Garside groups, there must exist elements x, x' and y in \mathbf{W} such that

$$w = xy, \quad w' = x'y, \quad vx = v'x', \quad y \neq 1.$$

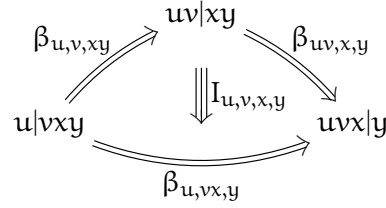
4. A complete example: Artin-Tits groups of spherical type

Then, we have two possible cases for the confluence diagram, depending on x and x' . If $x \neq 1$ and $x' \neq 1$, then we have:



Here, we have chosen the direction of the 3-cell $H_{u,v,w,v',w'}$ in an arbitrary way.

The second case is when either $x = 1$ or $x' = 1$. Let us note that both cannot happen at the same time, otherwise $v = v'$ and $w = w'$, so that the branching is aspherical and not critical. For example, let us assume that $x' = 1$, so that $v' = vx$, $w = xy$ and $w' = y$:



All the critical branchings created during the first part are confluent, ending the homotopical completion procedure. As an instance of Theorem 3.4.2, we get:

4.3.4. Proposition. *The monoid $\mathbf{B}^+(\mathbf{W}, S)$ admits a coherent presentation by the $(3, 1)$ -polygraph with one 0-cell, one 1-cell for every element of $\mathbf{W} \setminus \{1\}$, two 2-cells*

$$\alpha_{u,v} : u|v \Rightarrow uv, \quad \text{for every } u \widehat{\ } v,$$

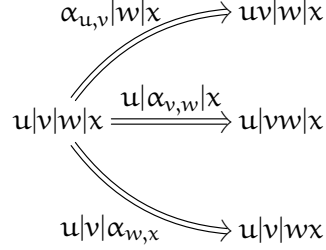
and

$$\beta_{u,v,w} : u|vw \Rightarrow uv|w, \quad \text{for every } u \widehat{\ } \overset{x}{v} \widehat{\ } w,$$

and the nine families of 3-cells A, B, C, D, E, F, G, H and I previously listed.

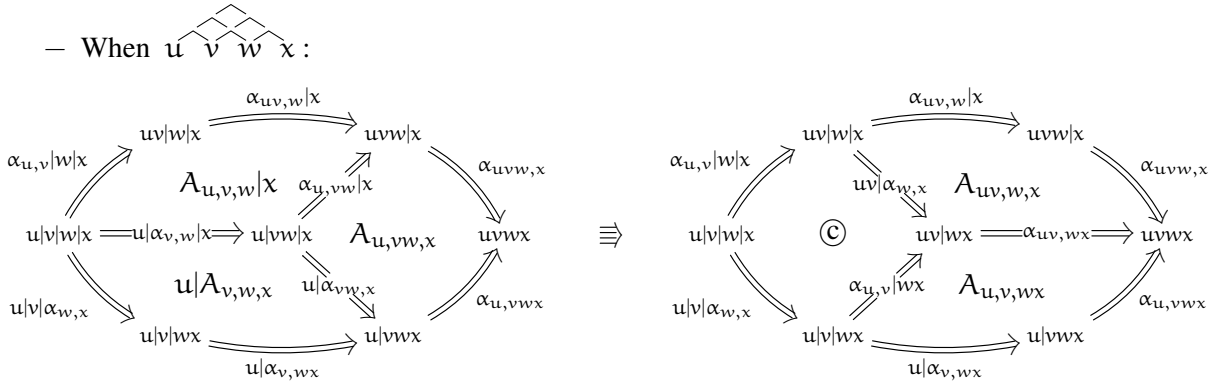
4.3.5. Homotopical reduction. We study the critical triple branchings of the convergent 2-polygraph $\text{HC}(\Sigma)$ produced by the homotopical completion procedure and, for each one, we check if it can be used to perform a coherent elimination of a 3-cell of $\text{HC}(\Sigma)$. There are four different cases, depending on the generating 2-cells forming the branching, and, then, different subcases depending on the 2-cells that close it.

Case 1. There is one critical triple branching for every $\widehat{u} \widehat{v} \widehat{w} \widehat{x}$:

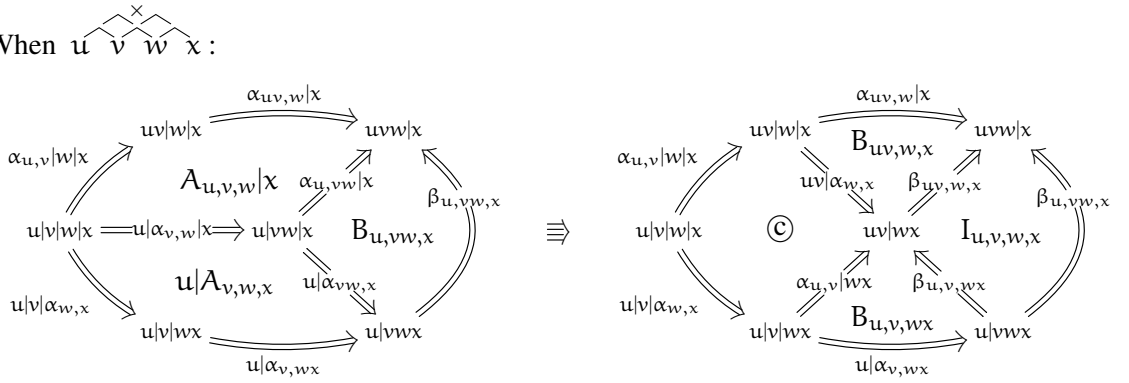


Then, we distinguish five subcases.

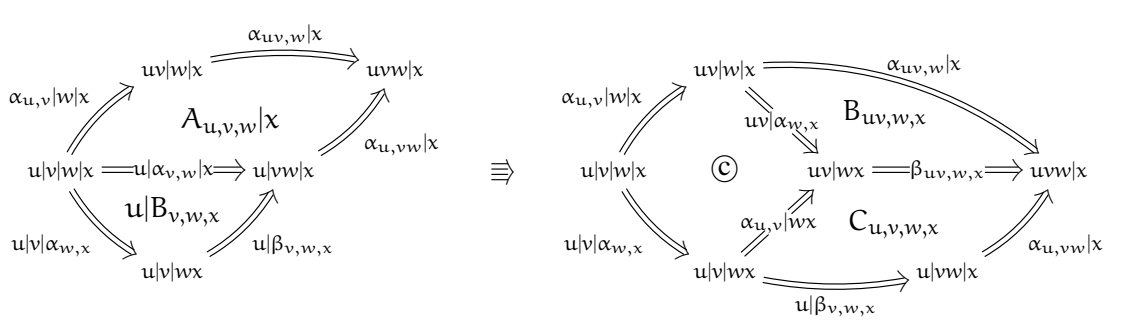
– When $\widehat{u} \widehat{v} \widehat{w} \widehat{x}$:



– When $\widehat{u} \widehat{v} \widehat{w} \widehat{x}$:

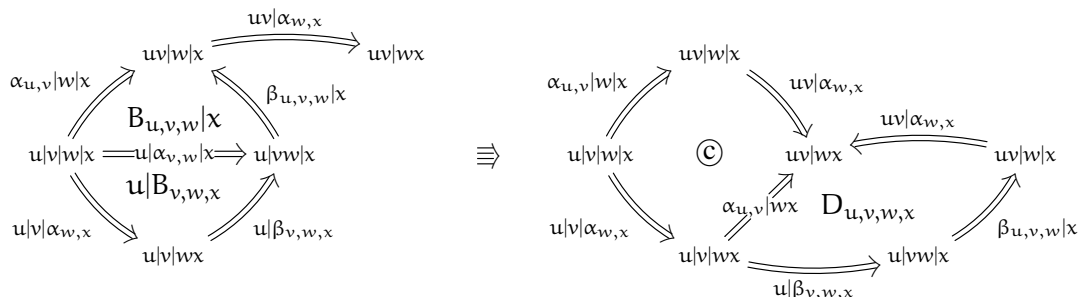


– When $\widehat{u} \widehat{v} \widehat{w} \widehat{x}$:

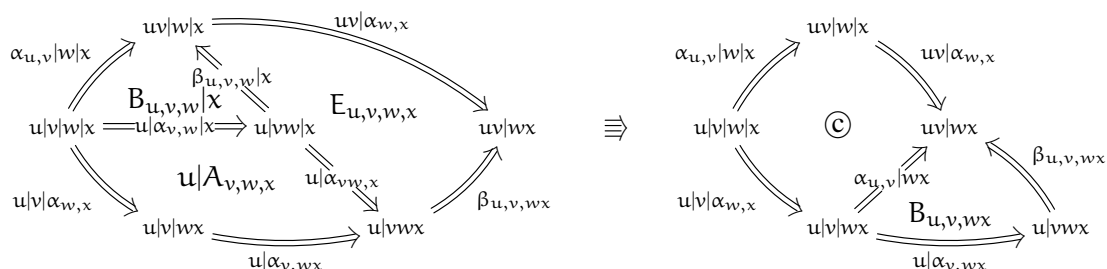


4. A complete example: Artin-Tits groups of spherical type

– When $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x$:



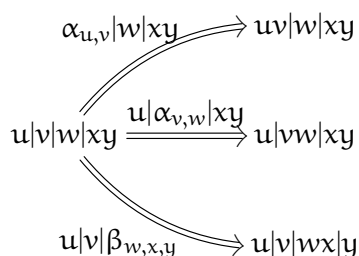
– When $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} x$:



The first family of 4-cells can be used to coherently eliminate all the 3-cells $A_{u,v,w}$, by induction on the size of u , with the exception of the 3-cells $A_{s,v,w}$, where s is a generator of the Coxeter group W . Symmetrically, we could also keep 3-cells with shape $A_{u,s,w}$ or with shape $A_{u,v,s}$.

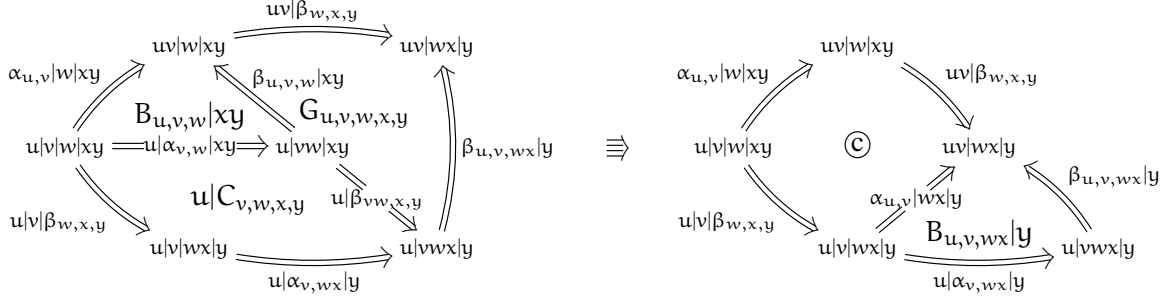
The four other families of 4-cells are used to coherently eliminate the 3-cells $I_{u,v,w,x}$, $C_{u,v,w,x}$, $D_{u,v,w,x}$ and $E_{u,v,w,x}$, in order. Indeed, in each case, the corresponding 4-cell proves that the 3-cell to be removed has the same boundary as a 3-cell made of A s and B s.

Case 2. There is one critical triple branching for every $u \overset{x}{\curvearrowright} v \overset{x}{\curvearrowright} w \overset{x}{\curvearrowright} y$:



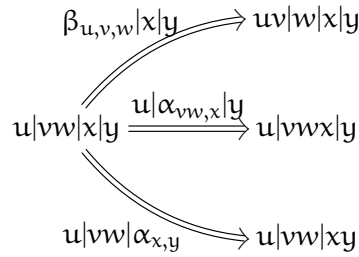
4.3. Actions with Deligne's presentation

As previously, there are several subcases. However, only one of them yields a new relation between 3-cells, namely when $u \overset{x}{\curvearrowright} v \overset{w}{\curvearrowright} x \overset{y}{\curvearrowright}$:

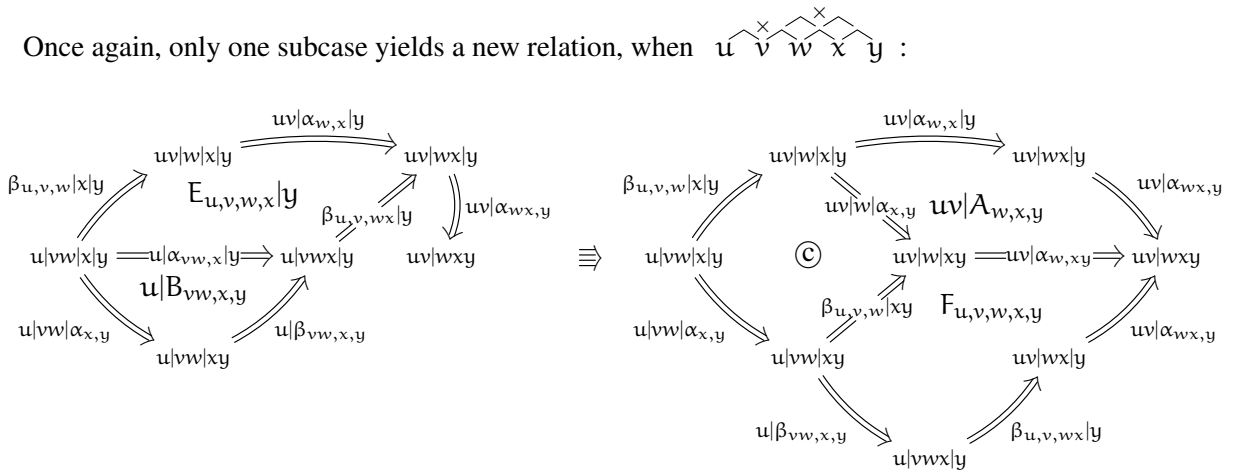


We have seen that the 3-cell $C_{v,w,x,y}$ is superfluous, since a previous computation of a critical triple branching has produced a 4-cell that proved that it can be replaced by a combination of As and Bs. Hence, we can use this 4-cell to coherently eliminate $G_{u,v,w,x,y}$.

Case 3. There is one critical triple branching for every $u \overset{x}{\curvearrowright} v \overset{w}{\curvearrowright} x \overset{y}{\curvearrowright}$:



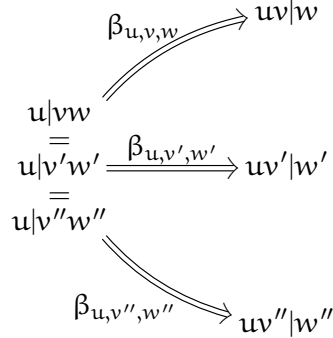
Once again, only one subcase yields a new relation, when $u \overset{x}{\curvearrowright} v \overset{w}{\curvearrowright} x \overset{y}{\curvearrowright}$:



We have seen that the 3-cell $E_{u,v,w,x}$ is superfluous, having the same boundary as a composite of As and Bs, so that the 3-cell $F_{u,v,w,x,y}$ can be coherently eliminated thanks to this 4-cell.

4. A complete example: Artin-Tits groups of spherical type

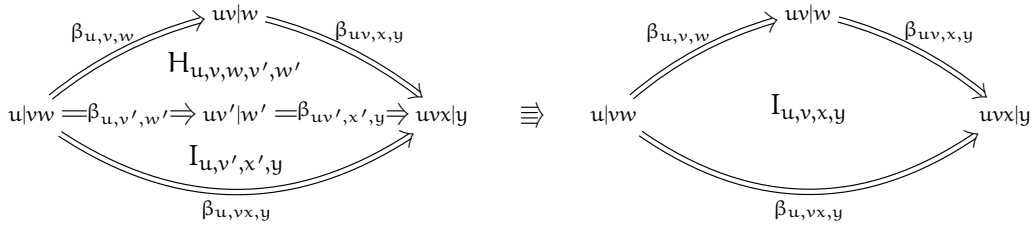
Case 4. Finally, there is one critical triple branching for every $u \widehat{v^x} w$, $u \widehat{v^{x'}} w'$ and $u \widehat{v^{x''}} w''$ with $vw = v'w' = v''w''$:



As a consequence, there exist x, x', x'' and y in \mathbf{W} such that

$$w = xy, \quad w' = x'y, \quad w'' = x''y, \quad vx = v'x' = v''x'', \quad y \neq 1.$$

Moreover, at most one of x, x' and x'' can be equal to 1. Indeed, otherwise, the triple branching would be aspherical and not critical. Thus, in the case where $x'' = 1$, we get:



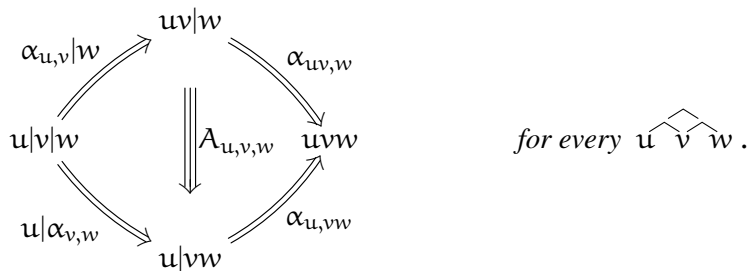
Since the 3-cells I s are superfluous, each one having the same boundary as a composite of A s and B s, we can coherently eliminate the 3-cell $H_{u,v,w,v',w'}$.

As a conclusion of homotopical reduction in dimension 3, we are left with the two families of 2-cells α and β and the two families of 3-cells A and B . Then, since β and B have been coherently adjoined by the homotopical completion, the homotopical reduction in dimension 2 removes them, leaving α and A only in $\text{RHC}(\Sigma)$.

4.3.6. Theorem. *The monoid $\mathbf{B}^+(\mathbf{W}, S)$ admits a coherent presentation by the $(3, 1)$ -polygraph with one 0-cell, one 1-cell for every element of $\mathbf{W} \setminus \{1\}$, one 2-cell*

$$\alpha_{u,v} : u|v \Rightarrow uv, \quad \text{for every } u \widehat{v},$$

and one 3-cell



4.3.7. Corollary (Deligne, [7] Theorem 1.5). *An action of $\mathbf{B}^+(\mathbf{W}, S)$ on a category \mathbf{C} is the same as the following data:*

- for any $u \in \mathbf{W} \setminus \{1\}$, an endofunctor $T(u) : \mathbf{C} \rightarrow \mathbf{C}$,
- for any $u \widehat{\vee} v$, a natural isomorphism $T_{u,v} : T(u)T(v) \simeq T(uv)$,

such that, for any $u \widehat{\vee} v \widehat{\wedge} w$, the relation $T(A_{u,v,w})$ holds:

$$\begin{array}{ccc}
 & T(uv)T(w) & \\
 T_{u,v}T(w) \nearrow & & \searrow T_{uv,w} \\
 T(u)T(v)T(w) & \textcircled{C} & T(uvw) \\
 T(u)T_{v,w} \searrow & & \nearrow T_{u,vw} \\
 & T(u)T(vw) &
 \end{array}$$

Remark. In fact, as mentioned during homotopical reduction, the reduced homotopical completion yields a more compact homotopy basis, since it is sufficient to consider the family $A_{s,v,w}$ of 3-cells, with s in S and v and w in $\mathbf{W} \setminus \{1\}$ or, symmetrically, the family $A_{u,s,w}$ or the family $A_{u,v,s}$. As a consequence, this simplification can also be done in Deligne’s result.

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